Available online at www.sciencedirect.com

SciVerse ScienceDirect

Mathematics and Computers in Simulation 84 (2012) 66–82

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.com/locate/matcom

Original article

A dynamic model of oligopoly with R&D externalities along networks. Part II

Gian Italo Bischi [a,*], Fabio Lamantia [b]

[a] *Department of Economics, Society, Politics, University of Urbino "Carlo Bo", Via Saffi 42, Urbino, Italy*
[b] *Department of Economics, Statistics and Finance, University of Calabria, Via Bucci, Rende, CS, Italy*

Received 2 May 2012; received in revised form 26 July 2012; accepted 20 August 2012
Available online 8 September 2012

Abstract

In Bischi and Lamantia [4] a two-stage oligopoly game has been proposed to describe networks of firms that invest in cost-reducing R&D activity with the possibility of sharing R&D results with partner firms as well as gaining knowledge for free through spillovers, and an adaptive dynamic mechanism is proposed to describe how firms repeatedly update their R&D efforts over time. In that paper existence and stability of equilibria have been analyzed given a fixed structure of the collaboration network, divided into sub-networks. In this paper we analyze the influences of the degree of collaboration and spillovers on profits, social welfare and, more generally, on overall efficiency. We first consider two relevant benchmark cases, for which analytical results are provided, and then numerical experiments are performed to stress the role of the level of connectivity (i.e. the collaboration attitude) inside networks as well as the effects of involuntary knowledge spillovers inside each network and among different competing networks.

Keywords: R&D cooperation; Networks; Repeated games; Knowledge spillovers

1. Introduction

In a companion paper by Bischi and Lamantia [4], Part I henceforth, a two-stage oligopoly game has been proposed to describe networks of firms that invest in cost-reducing R&D activity with the possibility of sharing R&D results with partner firms as well as gaining knowledge for free through spillovers (see [9,6,14,1]). If the system is out of equilibrium, an adaptive dynamic mechanism regulates how firms repeatedly update their R&D efforts over time. In [4], existence and stability of equilibria have been analyzed, given a fixed structure of the collaboration network divided into sub-networks. Moreover, some insights on the global dynamics of the system are provided mainly through numerical simulations.

In this paper we analyze, for the same model, the influences of the degree of collaboration and spillovers on profits, social welfare and, more generally, on overall efficiency. We first consider two relevant benchmark cases, for which analytical results are provided, and then numerical experiments are performed to stress the role of the level of connectivity (i.e. the collaboration attitude) inside networks as well as the effects of involuntary knowledge spillovers inside each network and among different competing networks.

* Corresponding author. Fax: +39 0722 305541.
E-mail addresses: gian.bischi@uniurb.it (G. Italo Bischi), lamantia@unical.it (F. Lamantia).

0378-4754/$36.00
http://dx.doi.org/10.1016/j.matcom.2012.09.001

Although we focus on examples where equilibrium efforts are always achieved, the dynamic formulation helps us in getting many interesting insights. In particular, multiple stable equilibria may exist, i.e. problems of equilibrium selection arise. In these cases, the dynamic adjustment of efforts as well as the initial actions of players are of crucial importance. In this context, we show two important results through the analysis of transverse stability of equilibria. One, when only some unlinked agents invest in R&D and others just produce, the latter firms would be better off by starting their own R&D investments, and, two, the firms in a dominant group could create a barrier preempting fringe firms in the creation of their own R&D investments/network. Moreover, we present a numerical example with two regular networks and a bistability regime induced by the amount of spillovers.

In addition to simulation in the case of two symmetric R&D sub-networks, we also develop similar simulations for the case in which the probability to establish a link between firms in the same sub-network is a random variable. These analysis with random networks confirms the results obtained in the simplified setting of symmetric networks.

The plan of the paper is the following. In Section 2 it is given a short summary of the basic assumptions and the main properties of the model proposed in Part I. Section 3 is devoted to the study of two opposite benchmark cases, namely the competition between two empty or two complete networks, as well as some considerations on private and social optimality. Numerical experiments on the long-run dynamics when the exogenous network structure is regular or stochastic are reported in Sections 4 and 5 respectively. Section 6 concludes and indicates some paths for future improvements of the model.

2. The model

In this section, we briefly recall the set-up of the oligopoly game proposed in Part I, to which we refer the reader for more details. For expository reasons, we consider below only the case of firms arranged in two disjointed R&D networks.

Let us consider $N \geq 2$ quantity setting firms producing an homogeneous good whose price is determined by the linear inverse demand function $p = a - bQ$, $a, b > 0$, Q being the total output in the market. The N firms are ex-ante partitioned into two groups (called *sub-network*) such that two firms of the same sub-network are neighbors (i.e. they are linked) if they establish a bilateral agreement for a full sharing of their R&D results. The two sub-networks, denoted s_1 and s_2 in the following, are formed by n_1 and n_2 firms, with $N = n_1 + n_2$. Following [9], each sub-network s_j is assumed to be symmetric of degree k_j, with $0 \leq k_j \leq n_j - 1$, i.e. every firm belonging to the same sub-network has the same number of collaborative links k_j. Fixing the structure of the two groups, each firm decide R&D efforts and quantities to produce. In addition, some R&D results may involuntarily spill over for free also to non-neighbors inside the same sub-network s_j (internal spillovers) as well as to firms of the other sub-network s_k with $k \neq j$ (external spillovers). Assuming that firm i bears a linear cost $c_i q_i$ for producing q_i, a representative firm in sub-network s_j has a marginal cost c_i of the form

$$c_i(s_j) = c - e_i - k_j e_{l_i} - \beta_j e_{l_{-i}}[(n_j - 1) - k_j] - \beta_{-j} \sum_{m \in s_k} e_m \tag{1}$$

where $c < a$ is the marginal cost without R&D efforts (equal for all firms), e_i represents R&D effort of firm i, $k_j e_{l_i}$ represents the total effort exerted by firms with whom i is linked in s_j, $\beta_j \in [0, 1)$ are related to internal spillovers with non-neighbors in network s_j, and regulate external spillovers, i.e. originating from non-neighbors out of s_j toward firm i. Following [6,9], all N firms are rivals in the market place, and they calculate their optimal outputs by solving individual profit maximization problems. The reduced-form profit for an oligopolist i in sub-network s_j can be written as

$$\pi_i(s_j) = \left\{ a - b \left[q_i(s_j) + \sum_{p \neq i} q_p \right] - c_i(s_j) \right\} q_i(s_j) - \gamma e_i^2(s_j)$$

where $q_i(s_j)$ and $e_i(s_j)$ are, respectively, the quantity and the R&D effort by agent i in sub-network s_j, and γe_i^2, $\gamma > 0$, is the cost of effort (see [6]). The optimal quantity of firm i in sub-network s_j is

$$q_i(s_j) = \frac{a - N c_i(s_j) + \sum_{p \neq i} c_p}{b(1 + N)} \tag{2}$$

with corresponding optimal profit

$$\pi_i(s_j) = \left[\frac{a - Nc_i(e_i) + \sum_{p \neq i} c_p}{\sqrt{b}(1+N)}\right]^2 - \gamma e_i^2 \tag{3}$$

Given this setting, each firm i tries to maximize its individual profit with respect to its own R&D efforts $e_i \in [0, c]$. We refer the reader to Part I for the issues of existence of a Nash equilibrium E^*.

In Part I, we argued that, due to the complex network structure of R&D collaborations and spillover externalities, it is unlikely that agents are able to play the Nash equilibrium strategy in one shot. Consequently, an adaptive system is proposed where firms adjust their efforts over time toward the 'optimal' strategy, following the direction of the local estimate of expected marginal profits, according to the so called "gradient dynamics"

$$e_j(t+1) = e_j(t) + \alpha_j(e_j)\frac{\partial \pi_j}{\partial e_j}; \quad j = 1, 2 \tag{4}$$

where $e_j(t)$ represents the R&D effort at time period t of a representative firm belonging to the sub-network s_j; $\alpha_j(e_j)$ are positive functions that represent speeds of adjustment. Nash equilibria are also equilibrium points for the dynamic process (4). If such an equilibrium is stable, then we can say that the adaptive agents are able to learn, in the long run, how they can choose R&D efforts in an optimal way. However, we have seen that these equilibria are not always stable under the gradient dynamics (4).

Assuming linear speeds of adjustment $\alpha_j(e_j) = \alpha_j e_j$, the dynamical system becomes

$$e_i(t+1) = e_i(t) + \frac{\alpha_i e_i(t)}{b(1+n_i+n_j)^2}[A_i + B_i e_j(t) + C_i e_i(t)], \quad i, j = 1, 2; i \neq j \tag{5}$$

where:

$$\begin{aligned}
A_i &= 2(a-c)[(n_i-k_i)(1-\beta_i) + \beta_i + n_j(1-\beta_{-j})] \\
B_i &= 2n_j[(1-\beta_i)(n_i-k_i) + \beta_i + n_j(1-\beta_{-j})] \cdot [-\beta_j(n_j-k_j-1) + \beta_{-i}(n_j+1) - k_j - 1] \\
C_i &= 2\{(-k_i + \beta_i(1+k_i-n_i) + N - \beta_{-j}n_j) \cdot (1 + k_i + n_j + k_i n_j - \beta_{-j}n_i n_j - \beta_i(1+k_i-n_i)(1+n_j)) - b\gamma(1+N)^2\}
\end{aligned} \tag{6}$$

with $A_i > 0$ for all economic meaningful parameters, $C_i < 0$ being equivalent to $(\partial^2\pi_i/\partial e_i^2) < 0$ so that the profit function $\pi_i(e_i)$ is strictly concave and the necessary condition for a maximum is also sufficient. The dynamical model (5) always admits three boundary equilibria and

$$O = (0, 0), \quad E_1 = \left(\frac{-A_1}{C_1}, 0\right), \quad E_2 = \left(0, \frac{-A_2}{C_2}\right), \tag{7}$$

located on the invariant coordinate axes, with nonzero coordinate strictly positive if and only if the corresponding profit function $\pi_i(e_i)$ is strictly concave, and a unique interior equilibrium

$$E^* = \left(\frac{A_2B_1 - A_1C_2}{C_1C_2 - B_1B_2}, \frac{A_1B_2 - A_2C_1}{C_1C_2 - B_1B_2}\right) \tag{8}$$

provided that $C_1C_2 - B_1B_2 \neq 0$. The main results from standard local stability analysis obtained in Part I are summed up in the following:

Proposition 1. *For the map (5):*

- $O = (0, 0)$ *is a repelling node;*
- E_i, $i = 1, 2$, *is attracting along the* e_i *axis as long as* $A_i < 2b(1+N)^2/\alpha_i$, *and, in the direction transverse to* e_i, E_i *is stable if condition*

$$-2 < \frac{\alpha_j(A_jC_i - B_jA_i)}{C_ib(1+N)^2} < 0$$

holds;

- *at* $A_jC_i = B_jA_i$ *a transcritical bifurcation occurs at which equilibria* E_i *and* E^* *merge;*
- *if* $C_1C_2 \geq B_1B_2$, *with* $C_i < 0$, $i = 1, 2$ *a necessary condition for the stability of* E^* *is*

$$4 + \frac{\alpha_1 C_1 e_1^* + \alpha_2 C_2 e_2^*}{b(1+N)^2} \geq 0 \tag{9}$$

In Part I we have also shown that for sufficiently high values of the parameters α_i or low values of k_i all the equilibria are unstable and periodic or chaotic dynamics can be obtained in the long run. Moreover, coexisting attractors with quite intermingled basins of attraction have been numerically observed, thus giving a strong path dependence.

3. Two opposite benchmark cases

In this section we show how the sub-network structure influences fixed points coordinates and their stability properties for two opposite benchmark cases. In the first one it is posited that all firms compete in the market and possibly invest in R&D, but no ties are established among them (*empty network*). Thus, individual effort has never the effect to reduce costs to competitors, neither in form of agreements nor in form of involuntary spillovers. The second benchmark is, in some sense, the opposite case, represented by two competing networks that are fully connected, i.e. each firm completely shares its R&D cost-reducing results within its network (*complete networks*).

3.1. Empty sub-networks

Let us assume that no ties are present: $k_i = 0$ as well as $\beta_i = \beta_{-i} = 0$, $i = 1, 2$. In this case the equilibria E_1, E_2 and E^* are given by

$$E_1 = (\overline{p}_1, 0) \quad \text{and} \quad E_2 = (0, \overline{p}_2), \quad \text{with } \overline{p}_i = \frac{(a-c)N}{b\gamma(1+N)^2 - (1+n_j)N} \tag{10}$$

and

$$E^* = (\overline{q}, \overline{q}) \quad \text{with } \overline{q} = \frac{(a-c)N}{b\gamma(1+N)^2 - N} \tag{11}$$

We observe that boundary equilibrium values $\overline{p}_i$ are increasing functions of n_j, i.e. the number of agents that compete in the market but do not invest in R&D. This has an immediate economic intuition: if it is reduced the number of firms that only compete in the marketplace without investing, then R&D investing firms can safely reduce their R&D efforts, as a result of a decrease in competition.

The non-zero coordinate of E_i is strictly positive provided that $\gamma > ((1+n_j)N/b(1+N)^2)$, corresponding to $C_i < 0$; on the other hand, the inner equilibrium E^* is in the positive quadrant if $\gamma > (N/b(1+N)^2)$. It is useful to observe that condition $\gamma \geq (1/b) > (N/b(1+N)) > ((1+n_i)N/b(1+N)^2)$ ensures that the boundary equilibrium E_i has positive coordinate and this in turn implies that also E^* has strictly positive coordinates. Moreover, by comparing the coordinates of E_i and E^*, it is straightforward to observe that R&D equilibria levels decrease as the number of firms investing in it decreases; again by (10) and (11), we deduce that R&D efforts at equilibrium are always lower when both sub-networks invest in R&D than when only one group invests. Stability analysis of equilibria can be carried out by particularizing the results recalled in Proposition 1 (see Part I for additional details).

Proposition 2. *Assume that* $\gamma \geq (1/b)$ *and* $k_i = \beta_i = \beta_{-i} = 0$, $i = 1, 2$. *Then the equilibrium* E_i *is stable along* e_i *axis as long as* $a - c < (b(1+N)^2/N\alpha_i)$ *and it is always unstable along the direction transverse to* e_i.

Proof. See Appendix A.

From an economic point of view this result is quite interesting: without network structure and spillovers, starting from an equilibrium where only n_i firms invest in R&D and n_j do not, if a representative firm from the second group starts R&D investments, even for very small amounts, then it will be better off by continuing these investments over time.

With respect to $E^* = (\overline{q}, \overline{q})$, we observe that $\overline{q} > 0$ as long as $b\gamma \neq (N/(1+N)^2)$ so that condition $\gamma \geq (1/b)$ ensures its existence and positiveness; moreover, with $C_1, C_2 < 0$ and $C_1C_2 > B_1B_2$ (see (A.2)), we can apply directly the results of Proposition 1 to obtain suitable intervals of stability for E^*. In particular, we observe that the stability condition (i) in (B.2) is always satisfied. In the case $\alpha_1 = \alpha_2 = \alpha > 0$, by applying the results in Part I, Section 3.2, conditions (ii) and (iii) in (B.2) hold, and so E^* is stable, as long as $a - c < (b(1+N)^2/\alpha N)$ (that coincides with the stability of E_i along e_i). At $a - c = (b(1+N)^2/\alpha N)$ condition (ii) in (B.2) holds as an equality so that the equilibrium looses stability through a flip bifurcation. No other bifurcations are possible in this case.

3.2. Complete sub-networks

The second benchmark case is obtained by assuming that both sub-networks are fully connected. As a consequence, all firms in the same sub-network fully share their R&D efforts, so it is unnecessary to consider internal spillovers. For the sake of comparison, we also disregard the presence of external spillovers, i.e. in this case $k_i = n_i - 1$; $\beta_{-i} = 0$, $i = 1, 2$.

The aggregate parameters are given by

$$A_i = 2(a-c)(1+n_j) > 0$$
$$B_i = -2n_j^2(1+n_j) < 0$$
$$C_i = 2n_i(1+n_j)^2 - 2\gamma b(1+N)^2$$

whence $C_i < 0$ holds provided that $\gamma > \widetilde{\gamma}_i = (n_i(1+n_j)^2/b(1+N)^2)$, also ensuring that the quantity

$$-\frac{A_i}{C_i} = \frac{(a-c)(1+n_j)}{b\gamma(1+N)^2 - n_i(1+n_j)^2} = \widetilde{p}_i, \tag{12}$$

the nonzero coordinate of the boundary equilibrium $\widetilde{E}_i$, is strictly positive.

By confronting $\widetilde{p}_i$ with $\overline{p}_i$ in (10), we have that $\widetilde{p}_i < \overline{p}_i$ when $\gamma > ((1+n_j)^2N/b(1+N)^2)$, whereas if $\widetilde{\gamma}_i < \gamma < (N(1+n_j)^2/b(1+N)^2)$ then $\widetilde{p}_i > \overline{p}_i$. Hence, if only one group (empty or complete network) invests in R&D and this is very expensive, then equilibrium investment is lower with a complete network than with an empty R&D network; the opposite holds when R&D efforts are cheap.

By Proposition 1, $\widetilde{E}_i$ is stable along the e_i axis for $b > ((a-c)(1+n_j)\alpha_i/(1+N)^2)$, so we conclude that under full connection, it is possible to observe convergence to a boundary equilibrium for an interval of b that is larger here than with empty networks. More interestingly and differently to the previous case, it is possible for $\widetilde{E}_i$ to be stable in the direction transverse to e_i. In fact by applying again Proposition 1, it is possible to show that stability of $\widetilde{E}_i$ transverse to e_i is obtained if $\widetilde{\gamma}_i < \gamma < (n_i(1+n_j)/b(1+N))$ and α_i is sufficiently low.

These results on transverse stability have an immediate economic translation: in the empty network case, a non-zero boundary equilibrium corresponds to a situation where only some individuals invests in R&D and others just produce. Since in this case this equilibrium is always transversally unstable, the latter firms would be better off by starting their own R&D investments. In the complete network case, a non-zero boundary equilibrium corresponds to a case where there is a group of fully linked firms while the rest of the firms are isolated, i.e. the N agents form a network with the *dominant group architecture* (see [8]). In this case, depending on R&D costs, fringe firms could be worst off by starting their own R&D investments. Thus, the existing dominant group can create a barrier that prevents other firms to start their own R&D network.[1] For an analysis of the inner equilibrium E^* see Appendix B.

3.3. Some consideration on private and social optimality

In the previous subsections, we showed that proper subsets of the parameters' space exist where the inner effort equilibrium E^* is positive and stable for the two main benchmark cases. For these cases, it is interesting to carry out

[1] This results is similar to the one stated in [9] (Proposition 6) stating that in an empty network it is convenient to form bilateral agreements whereas it is not the case in a complete network. However in that paper the focus is on link formation rather than decisions on effort exertion.

some consideration on welfare analysis at the positive equilibrium. For the Cournotian game without network structure, this detailed analysis is examined in [13]. In particular we would like to assess whether an high degree of collaboration activity of the networks is beneficial. Notice that the results for the case with a single network (e.g. $n_2 = 0$) corresponds exactly to proposition 7 and 8 in [9]. For sake of simplicity in this subsection we set $b = 1$.

In case of empty sub-networks, by making use of E^* given by (11), we can write equilibrium quantity and individual profits of a representative firm, with consumer surplus[2] respectively as

$$q^E = \frac{(a-c)\gamma(1+N)}{\gamma(1+N)^2 - N}; \quad \pi^E = \frac{(a-c)^2 g(\gamma(1+N)^2 - N^2)}{(\gamma(1+N)^2 - N)^2}; \quad C_S^E = \frac{(a-c)^2\gamma^2 N^2(1+N)^2}{2[\gamma(1+N)^2 - N]^2} \tag{13}$$

where the index E stays for “Empty”.

Correspondingly, for the case of complete sub-networks,[3] we obtain by (B.4) equilibrium quantity and individual profits of a representative firm with consumer surplus respectively as

$$q^C = \frac{(a-c)\gamma(1+2n)}{\gamma(1+2n)^2 - n(1+n)}; \tag{14}$$

$$\pi^C = \frac{(a-c)^2\gamma(\gamma(1+2n)^2 - (1+n)^2)}{(n+n^2 - \gamma(1+2n)^2)^2}; \tag{15}$$

$$C_S^C = \frac{2(a-c)^2\gamma^2 n^2(1+2n)^2}{[n+n^2 - \gamma(1+2n)^2]^2}$$

where the index C stays for “Complete”. Moreover, in the following we denote the total welfare by $W_T = N\pi + C_S = \sum_{i=1}^{N}\pi_i + (1/2)\left(\sum_{i=1}^{N} q_i\right)^2$.

By direct comparison of the previous quantities we can prove the following

Proposition 3. *Assume that $b = 1$, $n_1 = n_2 = n$ and no knowledge spillovers. Then the following relations hold:*

$$\pi^E < \pi^C; \quad C_S^E < C_S^C; \quad W_T^E < W_T^C.$$

Under these circumstances, it is preferable, both from a private and a social point of view, to have a competition between two complete networks than two empty ones. Three main factors concur to determine this result. First, greater cost reduction is achieved in the complete network by (1); second, by concavity of profit functions, individual efforts at equilibrium are greater in an empty network than in the complete network [see (11) and (B.4)], so that in the empty network firms spend more in R&D activities; third, efforts of unlinked firms are strategic substitutes (see Part I), so that, in empty networks, firms are penalized by the higher R&D effort by competitors; however, as previously shown, this effect is reduced as the degree increases and it vanishes completely in a complete sub-network. Stated differently, it is possible to show that efforts by same network non-neighbors are negative externalities (see [7]), which are fully internalized within a complete sub-network. With respect to consumer surplus and total welfare, the conclusion in Proposition 3 follows from the fact that equilibrium quantities are lower in the empty network than in the complete one [see (13) and (14)].

These results are also confirmed in our numerical simulations with random networks, as outlined below. Of course, Proposition 3 does not rule out that more efficient solutions can be achieved for intermediate levels of collaboration activity in the sub-networks. These topics, as well as the impact of knowledge spillovers, are explored in the next sections, where some typical numerical examples are considered.

---

[2] In this case, assuming linear demand, consumer surplus reduces to $(b/2)Q^2$

[3] For sake of simplicity here we only consider the case $n_1 = n_2 = n$, where agents in different sub-networks are homogeneous.

4. Symmetric networks: global analysis and numerical simulations

In this section, we present some numerical experiments on long-run levels of efforts, individual and collective profits and total welfare, as some of the underlying parameters of the model (5) are changed, namely the sub-networks degree, internal or external spillovers. The main example of this section is then revisited with randomly generated networks in the next part of this paper. For all examples presented, it has been also verified that the other relevant quantities (productions, prices and costs) are nonnegative. In order to avoid any ambiguity on the concept of the "solution" of the game, here we focus on cases where only convergence to steady states in the effort space is achieved, with quantities given in (2). However, as shown in Part I, other attractors of (5) are possible, such as cycles or chaotic attractors that characterize the long-run behavior of the model. But a detailed analysis of such disequilibrium cases and the influence of the structure of networks on them, are left to future works.

4.1. Varying the level of connectivity

In this part, we fix the degree of one sub-network while increasing the degree of the other one. The main finding is that efforts are not necessarily decreasing in networks's degree, and profits are not maximized for intermediate degrees, in contrast to the results with only one network in [9]. In fact, an increment in the degree of a sub-network s_j brings several counterbalancing effects on profits. From a local network point of view, it lowers marginal revenues and the magnitude of strategic complementarity of any neighbor, thus lowering marginal profits (see (5) in Part I); however, from a global network point of view, it reduces the non-neighbor's strategic substitutability, thus increasing marginal profits (see (6) and (7) in Part I).

In the leading example of this section, an overall beneficial effect to firms of a specific sub-network is granted as its degree is increased; this is a consequence of the cost reduction to firms of that sub-network and is clearly a particular global network effect, as firms outside that sub-network are penalized by the increased degree of non-neighbors in the sub-network. These results are also confirmed in the case of competition between two randomly generated networks, as shown in the next section.

A typical example is shown in Fig. 1. Here, we consider a case where the whole industry is constituted by two groups of ten firms ($n_1 = n_2 = 10$) and firms in the second group form a symmetric network of degree 5 ($k_2 = 5$); the demand function is characterized by parameters $a = 200$ and $b = 1$; unitary effort cost is $\gamma = 6$ and unitary production cost without R&D is $c = 80$. Moreover, spillovers are absent ($\beta_1 = \beta_2 = \beta_{-1} = \beta_{-2} = 0$), and we set equal speed of adjustment of the gradient process for all firms $\alpha_1 = \alpha_2 = 0.05$. In all simulations we took an initial condition (i.c.) $(e_1(0), e_2(0)) = (0.5, 0.5)$. With these parameters, we let the level of connectivity of the first network assume all possible values $k_1 \in [0, 9]$ (for graphical purposes we plotted k_1 as a continuous variable). First notice that the equilibrium effort level inside network 1 is non-monotonic in k_1 (see Fig. 1a). Clearly, the increment in the level of connectivity in the first network weakens the competitors of the second network, whose profit is progressively reduced (see Fig. 1d). In particular we observe that in this situation, firms in network 1 are better off when they form a complete network (see Fig. 1c), whereas an intermediate level of collaboration in both networks ($k_1 = k_2 = 5$) brings a minimum of total profits and social welfare (see Fig. 1d and e). This is in contrast to what happens when only a single network exists, as analyzed in [9], Proposition 8. With different sets of parameters, it is possible to exhibit examples showing that when the first network is fully connected the second network is completely out of the market, i.e. firms from the second network do not sell any good, and the system converges to the boundary equilibrium where efforts of network 1 are given by $\widetilde{p}_i$ in (12).

4.2. Varying the degree of knowledge spillovers

Let us consider a case of two homogeneous sub-networks and no spillovers, so that a representative firm in each sub-network exerts the same effort and gains the same profit of its rivals. Now we are interested in the effects of increasing internal and external spillovers between sub-networks.

To start with, consider the case depicted in Fig. 2 where parameters are given as $\alpha_1 = \alpha_2 = 0.05$; $n_1 = n_2 = 10$; $k_1 = k_2 = 5$; $\beta_2 = \beta_{-1} = \beta_{-2} = 0$; $a = 200$; $c = 80$; $\gamma = 6$; $b = 1$, and $\beta_1 \in [0, 1)$ and i.c. $(e_1(0), e_2(0)) = (0.5, 0.5)$. Obviously for $\beta_1 = 0$, the two networks are homogeneous. An increase in the internal spillovers within network 1, i.e. in parameter β_1, leads to a greater average cost reduction for firms in network 1, so that they become tougher

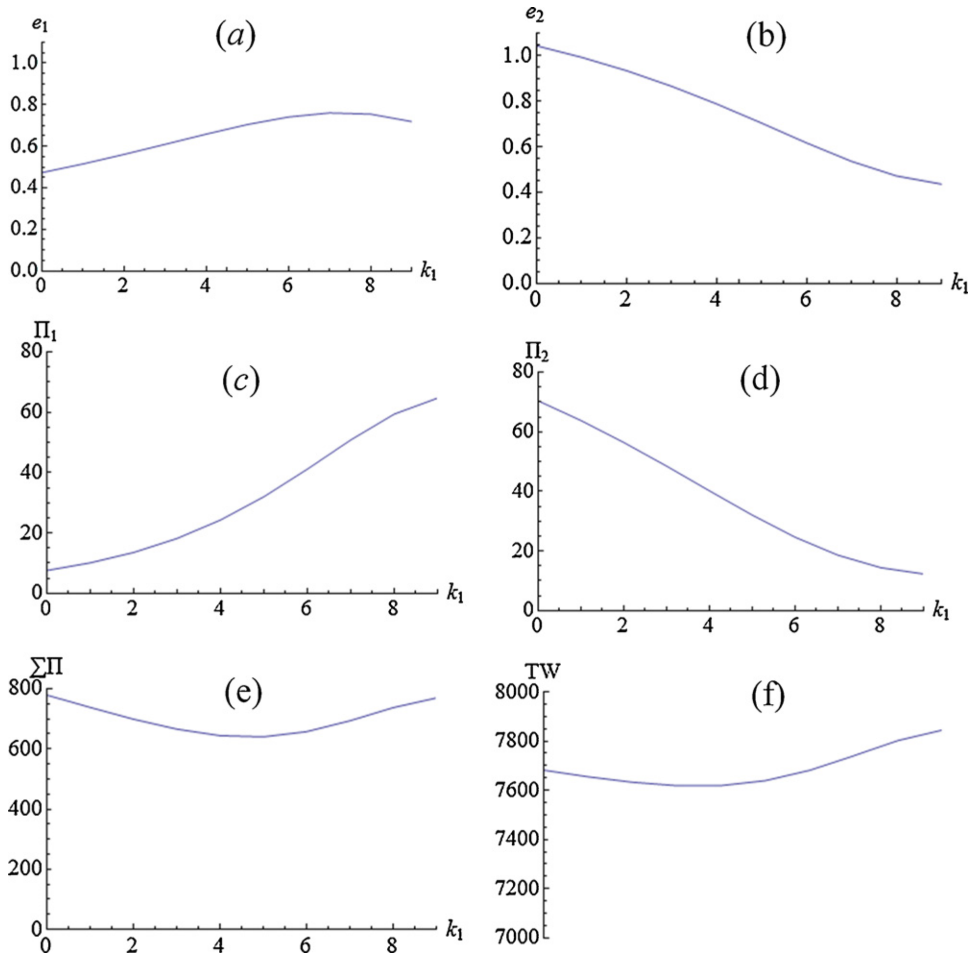

Fig. 1. For varying degrees of collaboration in network 1, $k_1 \in [0, 9] \cap \mathbb{Z}$: (a and b) asymptotic levels of efforts in both networks; (c and d) asymptotic levels of profits in both networks; (e) corresponding total industry profits; (f) corresponding total welfare. Parameters are given as $\alpha_1 = \alpha_2 = 0.05$; $n_1 = n_2 = 10$; $k_2 = 5$; $\beta_1 = \beta_2 = \beta_{-1} = \beta_{-2} = 0$; $a = 200$; $c = 80$; $\gamma = 6$; $b = 1$, i.c. $(e_1(0), e_2(0)) = (0.5, 0.5)$.

competitors against firms of networks 2. As a consequence, we observe an increment of efforts by a representative firm in network 1, as long as it serves to weaken competitors on the other network. Firms in network 1 enjoy a greater cost reduction than rivals and, consequently, an higher profit. On the other hand, R&D's investments of firms in network 2 induce low cost reduction, so they progressively reduce their efforts as β_1 is increased (see Fig. 2b). Observe that R&D investment inside network 1 reaches a maximum level exactly at a point $\overline{\beta}_1 \approx 0.586$, corresponding to the spillover level at which R&D efforts pass from strategic complements to strategic substitutes, as explained above. As a result, we observe an inflection point in the overall industry profit and total welfare, which are nonetheless maximized when spillovers are the highest possible (see Fig. 2e and f).

Now we consider the same case, but we increment the external spillovers β_{-2}, which regulate the fraction of R&D effort from network 1 that spills over for free to competitors in network 2. All parameters are given as in Fig. 2, but $\beta_1 = 0$ and $\beta_{-2} \in [0, 1)$. Clearly, the higher the parameter β_{-2} is, the more an unit of effort exerted in network 1 advantages firms in network 2. Consequently, firms in network 1 keep reducing their effort so that the advantage to competitors in network 2 lowers. As a consequence we observe that there exists an intermediate level of external spillover $\overline{\beta}_{-2}$ such that profits in network 1 and 2 are, respectively, minimized and maximized (see Fig. 3c and d) and total industry profit has a local but not global maximum at $\overline{\beta}_1$ (see Fig. 3e). The final outcome is thoroughly explainable within the framework we are dealing with. In fact, high levels of external spillovers reduce the advantage to invest in R&D for the firms from which spillovers originate. As a consequence, these firms reduce their investments and switch

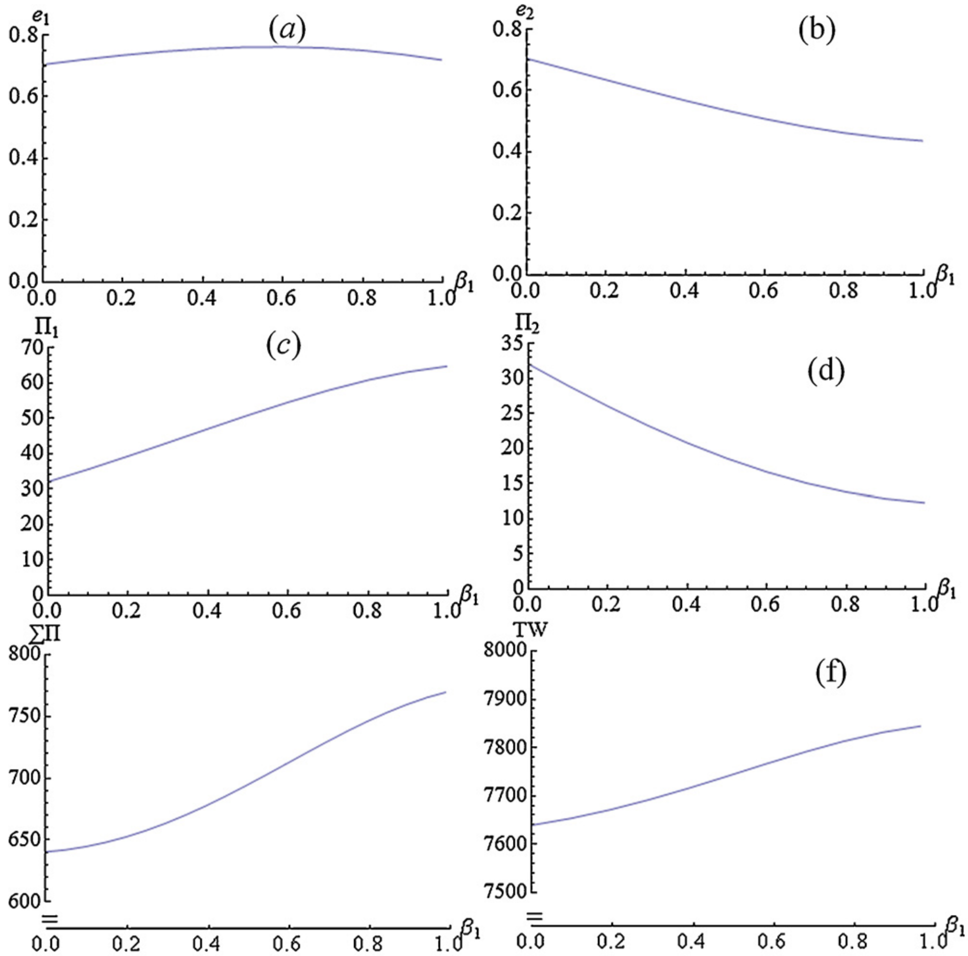

Fig. 2. For varying degrees of spillovers internal to network 1, $\beta_1 \in [0, 1)$: (a and b) asymptotic levels of efforts in both networks; (c and d) asymptotic levels of profits in both networks; (e) corresponding total industry profits; (f) corresponding total welfare. Parameters and i.c. are given as in Fig. 1, with $k_1 = k_2 = 5$.

to a kind of competition based on poor innovation, where higher profits go to firms that innovate less and produce more. It is possible to show that similar phenomena are also associated with intermediate levels of internal and external spillovers.

To end this section, we slightly modify our leading example, for exploring a case where network 2 has a greater connectivity level than network 1, but spillovers internal to network 1 are increased. All parameters are as in Fig. 2, except $\gamma = 5$ and $k_2 = 8$. Analogously to the case described in Fig. 1, with such an high degree of connectivity, firms in network 1 operate at very low levels, i.e. their R&D efforts and production is nearly zero. If the degree of internal spillovers β_1 within firms in network 1 is increased, we do not observe any consequence in the outcomes of the game, up to the level $\widetilde{\beta}_1 \approx 0.6765$; in fact, firms in network 1 do not invest in R&D as they can not compete with network 2 because of their few ties; on the other hand, the equilibrium level of investments in network 2, E_2, does not depend on β_1, as easily seen from (6) and (7).

At the level $\widetilde{\beta}_1$, we observe a jump of equilibrium effort levels of both networks (see Fig. 5a and b): firms of network 1 invest at the level E_1, that indeed depends on β_1, whereas firms in the other network cease to invest. More interestingly, the degree of internal spillovers determines which is the network that takes over the whole market, as shown in Fig. 4c and d, where again $\widetilde{\beta}_1$ represents the point of jump for profits of both representative players; see also Fig. 4e and f, for aggregate performances in this example.

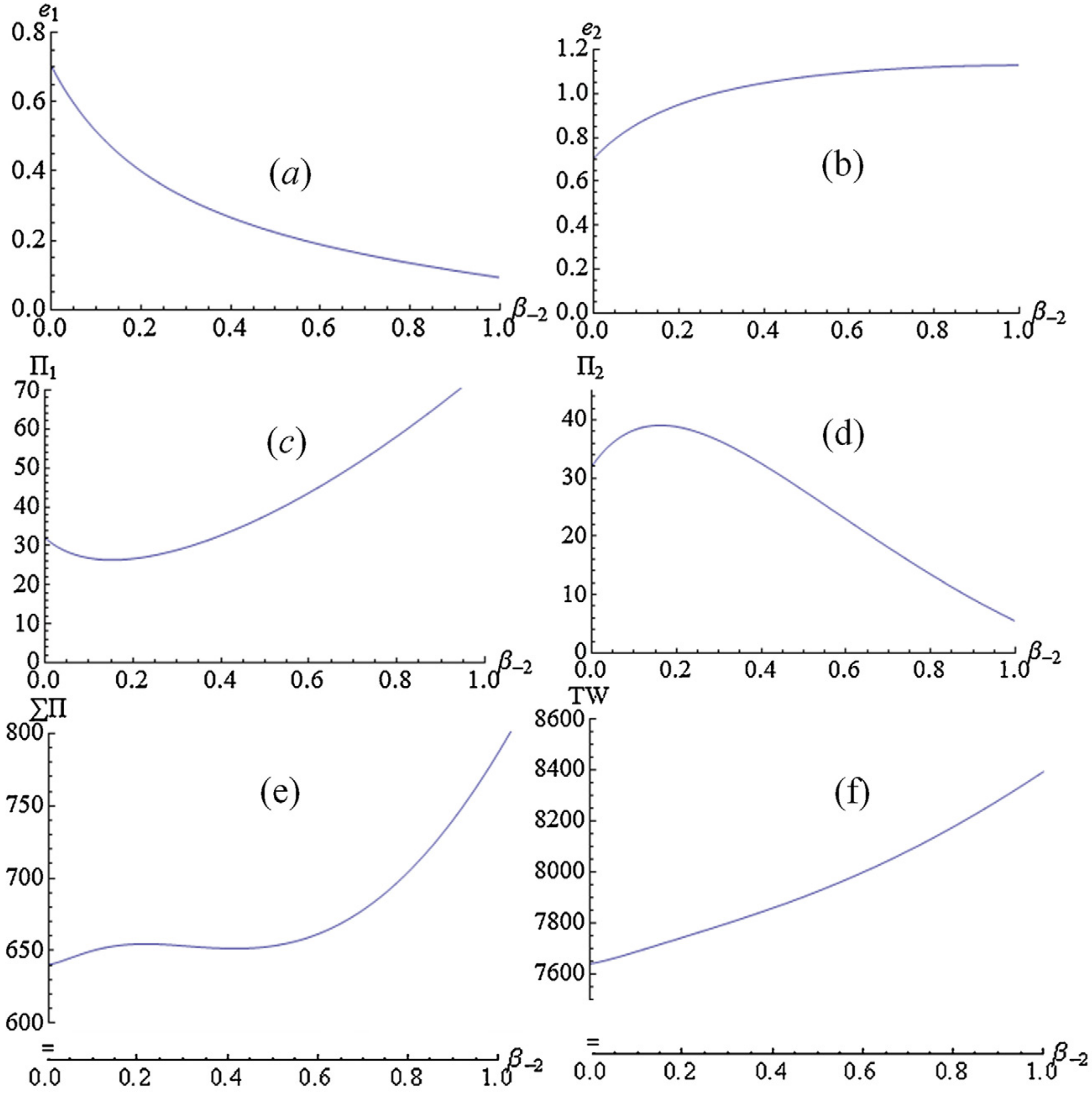

Fig. 3. For varying degrees of spillovers external to network 1, $\beta_{-2} \in [0, 1)$: (a and b) asymptotic levels of efforts in both networks; (c and d) asymptotic levels of profits in both networks; (e) corresponding total industry profits; (f) corresponding total welfare. Parameters and i.c. are given as in Fig. 2, with $\beta_1 = 0$.

From a technical point of view, with this set of parameters, the boundary equilibria E_1 and E_2, given by (7), are attracting fixed points, whereas E^* is a saddle, whose stable set delimits the boundary of the basins of attraction of the two steady states. When $\beta_1 < \widetilde{\beta}_1$, a trajectory that starts at the i.c. $(e_1(0), e_2(0)) = (0.5, 0.5)$ is attracted to E_2, as this i.c. belongs to the basin of attraction of E_2. However, the parameter β_1, even if does not influence the coordinate of the boundary fixed point E_2, it influences the stable set of the saddle point E^*, and consequently the basin of attraction of E_2. Thus, a trajectory at the same i.c. as before is now attracted to the boundary fixed point E_1. Nonetheless, E_2 is still an attractor also for level of spillovers above $\widetilde{\beta}_1$, but this increment of β_1 leads to a progressive shrinking of the basin

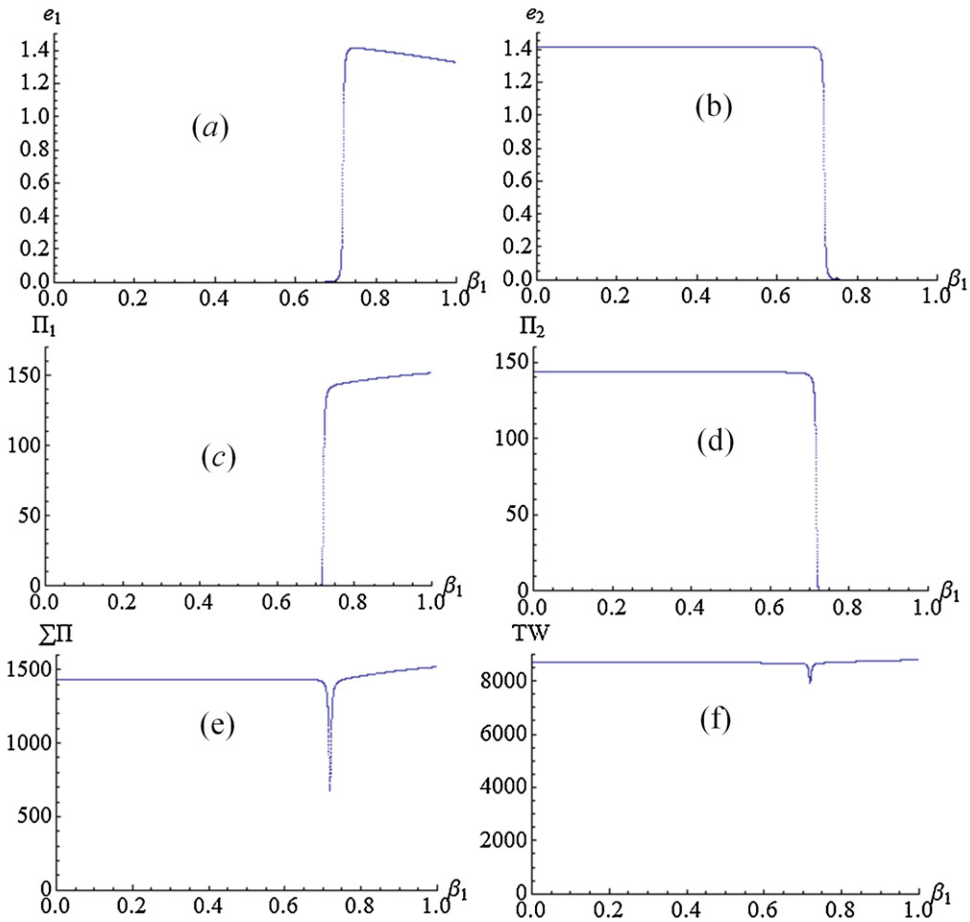

Fig. 4. For varying degrees of spillovers internal to network 1, $\beta_1 \in [0, 1)$: (a and b) asymptotic levels of efforts in both networks; (c and d) asymptotic levels of profits in both networks; (e) corresponding total industry profits; (f) corresponding total welfare. Parameters and i.c. are given as in Fig. 2, with $k_1 = 5$ and $k_2 = 8$.

of attraction of E_2. This kind of bifurcation, which leads to the bistability of the system, is a typical global one, as it cannot be detected by the eigenvalues of the Jacobian matrix at the fixed point.

5. Asymmetric networks: some numerical experiments

In this section, we relax some assumptions on network structure to perform a numerical treatment of the model in a more realistic setting thus confronting the main results with the ones previously obtained. In particular, we are interested in exploring the long run effort levels, and the corresponding profits, for a ‘large number’ of randomly generated couples of competing networks, i.e. when a competition between two random networks takes place.

The generation of random networks is done as follows. First of all, a given number of N firms is subdivided into two groups with $N = n_1 + n_2$. Within each group, a collaborative tie between any two of n_i agents is formed independently, with probabilities p and q in the first and in the second group respectively. The structure of each group is usually referred to as an Erdös-Rényi network and beliefs about neighbors’ degrees follow asymptotically a binomial distribution (see [7]).

Each sub-network is then represented by an adjacency matrix, i.e. for sub-network i a symmetric (agreements are bilateral) $n_i \times n_i$ matrix $G_i = \{g_{i,kh}\}$, where $g_{i,kh} = 1$ implies that a tie between firms k and h in sub-network i is present. We follow the convention of setting $g_{i,kk} = 1$.

The natural extension of (1), i.e. the cost function for firm m in network i reads

$$c_m = c - \mathbf{u}_{m,n_i}^T[G_i - \beta_i(U_{n_i} - G_i)]\mathbf{e}_i - \beta_{-i}\mathbf{u}_{1,n_j}^T U_{n_j}\mathbf{e}_j, \quad i = 1, 2, i \neq j \tag{16}$$

where:

- $\mathbf{e}_i^T = [e_1, e_2, \ldots, e_{n_i}]$ is the $1 \times n_i$ vector of efforts in network i ($\mathbf{e}_j$ is the corresponding vector of efforts in network j, with n_j firms)
- $\mathbf{u}_{i,n}$ is the i-th fundamental column vector of length n (with all elements equal to 0 but the i-th which is 1)
- U_n is an $n \times n$ matrix with all entries equal to 1.

The assessment of asymptotic levels of efforts and profits for a given couple of probabilities (p, q) is obtained by inspecting how these quantities change as a large number n of couples of networks is generated with equal probability, as described below.

5.1. Varying the probability of connections

Let us consider the case of two competing networks with, respectively, n_1 and n_2 firms. For assessing long-run average level of efforts and profits, we implemented the following Monte Carlo procedure:

1. Select a number of meshes S and define two vectors $\mathbf{p} = \{p_i\}_{i=1}^S$ and $\mathbf{q} = \{q_i\}_{i=1}^S$ representing the probability grids to establish bilateral links in the two networks;
2. Choose initial conditions $\mathbf{e}_1(0)$ and $\mathbf{e}_2(0)$, i.e. the initial efforts exerted by firms in both networks and specify all the parameters of the model;
3. Do $i = 1, S$
4. Do $j = 1, S$
5. Do $w = 1, n$

a) Generate two random networks, i.e. two adjacency matrices G_1 and G_2, where a link between any two agents of the same group is formed independently with probabilities p_i and q_j respectively;

b) Construct the industry profit functions with cost structure (16) and iterate the corresponding gradient dynamical system (4) with i.c. $\mathbf{e}_1(0)$ and $\mathbf{e}_2(0)$; delete the transient and memorize the effort trajectory for each firm in the industry;

6. End Do

c) Calculate average efforts, profits and their variances over the n trials;

7. End Do
8. End Do

In our numerical experiments we considered two groups of 10 firms each, i.e. $N = 20$ with $n_1 = n_2 = 10$; the probability space $[0, 1]^2$ is subdivided with equally spaced grid points with step 0.1, (i.e. $S = 11$), $n = 2500$ runs for each pair of probabilities p and q. For comparison purposes with the main symmetric example of the previous section, we considered the same parameters as for Fig. 1, i.e. $n_1 = n_2 = 10$; $\beta_1 = \beta_2 = \beta_{-1} = \beta_{-2} = 0$; $a = 200$; $c = 80$; $\gamma = 6$; $b = 1$, $\alpha_i = 0.05$ and i.c. $e_i(0) = 0.5$, $i = 1, \ldots, 20$. The main results can be summarized in Fig. 5, where probabilities p (to have a link in network 1) are in the horizontal axis and probabilities q (to have a link in network 2) are the small numbers on each plot. In Fig. 5a and b average R&D efforts in the first and second network respectively are depicted as the probabilities p of establishing a link in the first network is increased. Consider Fig. 5a. On average, the lower q is, the higher R&D efforts in the first networks are. Moreover, if q is low, these efforts are maximized for intermediate levels of p. This is in perfect analogy with the symmetric example shown in Fig. 1a. However, when q is high ($q > 0.7$ in Fig. 5a), the average effort in network 1 is strictly increasing with the probability p to make a tie. Analogously to Fig. 1b, efforts in the second network are decreasing when firms in network 1 have an higher probability to establish bilateral links. In Fig. 5c and d average profits are depicted, respectively, for firms in network 1 and 2. Firms in network 1 are better off when firms in the competing network 2 have a low probability to be connected (see Fig. 5c); on the other hand, firms in network 2 are better off when there is an high probability for them to have links and/or the competitors in network 1 have a low probability to establish links (see Fig. 5d). Again, it is interesting to compare Fig. 5c and d with Fig. 1c and d, to conclude that, on average, employing this structure of randomly generated networks leads qualitatively to results which are similar to the ones obtained in the simplified setting of this paper with symmetric network.

Now, we look at the total welfare, see Fig. 6 (the figure for total profits is qualitatively identical). It is immediate to observe that the competition between two networks where firms have zero probability to be connected is less efficient

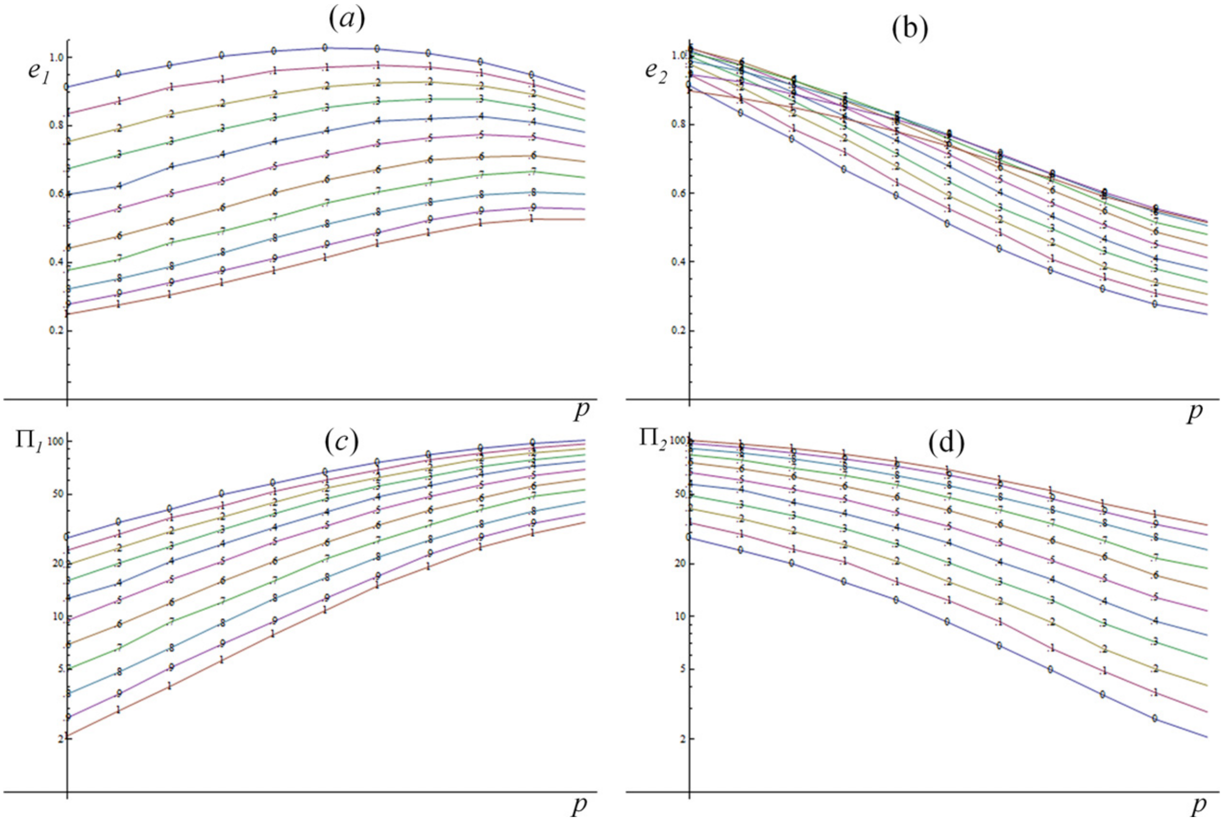

Fig. 5. For varying link's probability p in network 1: (a and b) asymptotic levels of efforts in network 1 and 2 respectively; (c and d) asymptotic levels of profits in networks 1 and 2 respectively; parameters are given as in Fig. 1. The small numbers on each graphic are link's probabilities q in network 2.

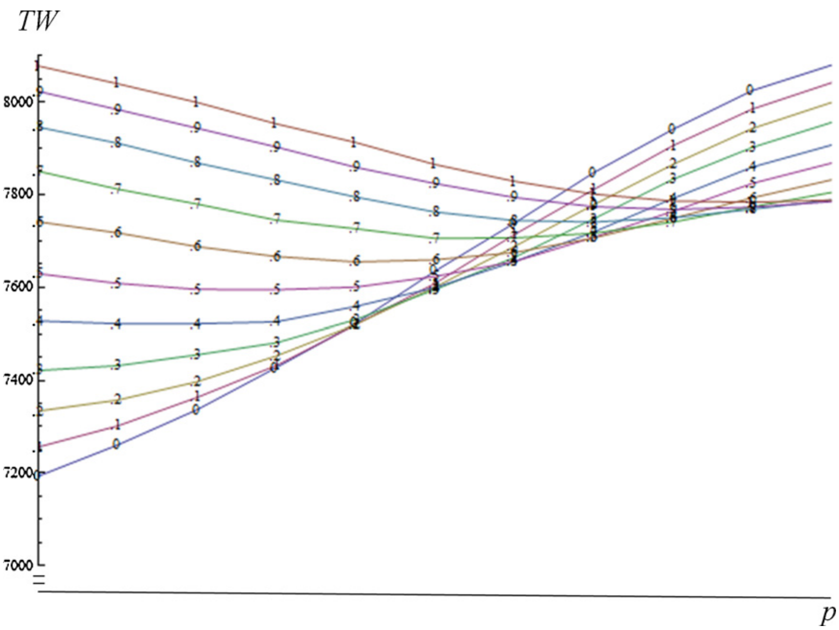

Fig. 6. Total welfare of the example in Fig. 5.

than competition between two networks where that probability is one, similarly to the result in the symmetric case of Proposition 3. However, total welfare is always maximized when only one network has the highest probability to be connected, whereas the other one has a low probability.

Essentially, all the results shown in Figs. 5 and 6 in terms of average efforts, profits and welfare are qualitatively identical to the ones in Fig. 1, where the simplified symmetric structure with fixed connectivity is posited (confront of the graphs in Fig. 1 with Figs. 5 and 6 and $q = 0.5$). The examples we carried out with varying spillover parameters confirm that the results are very similar if not identical to the ones we described earlier in the simplified symmetric model also when spillovers are present. However, we do not report them here, but we leave them to a future work focused on random networks.

6. Conclusions and further analysis

In this paper we have analytically and numerically analyzed some properties of a repeated two-stage game, proposed in [4] and referred to as Part I in this paper, which describes the competition between firms constituting Research Joint Ventures for sharing R&D results.

We first considered two relevant benchmark cases, namely the empty and the complete network cases, where the collaboration level is, respectively, absent and maximum. These benchmark cases allowed us to state that an empty network of R&D share agreements is never preferable to a structure with an high degree of collaboration, both from a private and social point of view.

By carrying out a typical numerical simulation, we showed the effects of introducing spillovers, both internal and external to a network. When two networks with equal level of connectivity compete, spillovers that are internal to a network can reduce progressively the market share of the other network. Furthermore, in cases where a network starts with the disadvantage of having less links than its competing network, internal spillovers can completely overturn the positions, with discontinuous changes in equilibrium investments levels and profits. This is due to the coexistence of different long run attractors, i.e. to coexisting fixed points in the space of efforts, each with its own basin of attraction, leading to a situation of *multistability*. With respect to external spillovers, when they are low, they provide substantial benefits to the network they are directed to, whereas above a threshold level, they turn out to be harmful for the receiving network. For all these cases we explored also the effects of these parameters on social indicators of performance.

In the last part of the paper, we relaxed the assumption on networks' symmetry, showing how the framework of the paper can be adapted to perform numerical analysis with asymmetric networks, which are randomly generated. We leave the detailed results to a future work, but we notice results very similar on average to the ones obtained with regular networks.

All numerical examples with regular networks have been carried out for cases where only convergence to an equilibrium level of efforts (with consequent optimal quantities) takes place, whereas examples where different attractors arise (as shown by some numerical simulations in Part I) will be more deeply characterized and interpreted in future works.

In the general case of different sub-networks we provided examples where some conclusions of [9], obtained in the case of only one collaboration network, no longer hold: in fact, in the case where only a network operates, as in [9], R&D efforts are not decreasing as the level of connectivity is increased, and profits in the network are indeed maximized when the level of connectivity is the highest, provided that other firms sell the (homogeneous) good in the marketplace. Indeed, when other firms sell the good, an higher level of connectivity inside the operating network can provide a winning tool to defeat outside competitors.

These results can be appraised in a stylized model with just two competing networks, and are confirmed in the case of competition between randomly generated networks, as reported in the last part of the paper.

The model and the analysis given in this paper can be extended in several ways. First of all, the problem of network formation and strategic stability with multi-network competition has to be addressed and analyzed. It is also interesting to perform simulations with asymmetric random networks and more general degree distributions. In a work focused on spillovers, it would be more reasonable to assume that internal spillovers are not constant, but their effects fade with network distance.

In addition, a remarkable improvement of the model can be obtained by assuming that knowledge gained through R&D efforts accumulates over time. Following [2], a first step in this direction can be found in [3], where an R&D

network game with knowledge accumulation is analyzed. However, both the cost reduction effect and the capacity to exploit spillovers (i.e. the “absorptive capacity”, see [5]) should be assumed to depend on the accumulated knowledge.

Finally, we stress that the framework we propose can be employed to formulate models with different objective functions to be maximized at the second stage, in the spirit of [6,10]. We leave these topics, as well as the comparison between the different models, to future works.

Acknowledgments

The authors thank an anonymous referee for valuable suggestions and remarks on the paper. The usual disclaimers apply. This work has been performed within the activity of the PRIN project “Local interactions and global dynamics in economics and finance: models and tools”, MIUR, Italy and in the framework of COST Action IS1104.

Appendix A. Proof of Proposition 2

The first part follows from the first part of Proposition 2 of Part I, where we considered the fact that the equilibrium E_i is located on the invariant axis e_i, along which the dynamics are described by the unidimensional map

$$e_i(t+1) = e_i(t) + \frac{\alpha_i e_i(t)}{b(1+N)^2}[A_i + C_i e_i(t)] \tag{A.1}$$

topological conjugate to the well known Myrberg quadratic map $q(x) = x^2 - c$ (see e.g. [11] or [12], chapter 2) through the linear homeomorphism $\tau(x) = (\alpha C_i/b(1+N)^2)x + (\alpha A_i/2b(1+N)^2) + (1/2)$, with $c = (1/4)(1 + (\alpha A_i/b(1+N)^2))^2 - (1/2)(1 + (\alpha A_i/b(1+N)^2))$. The first flip bifurcation for $q(x)$ at $c = (3/4)$ translate for (A.1) to the condition aforementioned.

We also remark that $b > a - c$ is sufficient for stability along e_i. We observe that in this case (6) reduce to

$$\begin{aligned} A_i &= 2(a-c)N > 0 \\ B_i &= -2n_j N < 0 \\ C_i &= 2(1+n_j)N - 2b\gamma(1+N)^2 < 0 \end{aligned} \tag{A.2}$$

As for stability along the direction transverse to e_i, we observe that $b\gamma > (N/(1+N))$ implies that $C_i < B_j$, and so, being $C_i < 0$ and $A_i = A_j > 0$, it is $((\alpha_j(A_j C_i - B_j A_i))/(C_i b(1+N)^2)) > 0$. By Proposition 1 we have that E_i is always unstable.

Appendix B. Inner equilibrium with complete sub-networks

The study of the local stability of the equilibria of model (5) starts, as usual, from the Jacobian matrix

$$J(e_1, e_2) = \begin{bmatrix} 1 + \frac{\alpha_1}{b(1+N)^2}(A_1 + B_1 e_2 + 2C_1 e_1) & \frac{\alpha_1 B_1 e_1}{b(1+N)^2} \\ \frac{\alpha_2 B_2 e_2}{b(1+N)^2} & 1 + \frac{\alpha_2}{b(1+N)^2}(A_2 + B_2 e_1 + 2C_2 e_2) \end{bmatrix} \tag{B.1}$$

computed at E^*, given by:

$$J(E^*) = \begin{bmatrix} 1 + \frac{\alpha_1 C_1 e_1^*}{b(1+N)^2} & \frac{\alpha_1 B_1 e_1^*}{b(1+N)^2} \\ \frac{\alpha_2 B_2 e_2^*}{b(1+N)^2} & 1 + \frac{\alpha_2 C_2 e_2^*}{b(1+N)^2} \end{bmatrix}$$

Necessary conditions for stability of E^* can be expressed by

$$\begin{cases} 1 - Tr + Det \geq 0 \\ 1 + Tr + Det \geq 0 \\ Det \leq 1 \end{cases} \tag{B.2}$$

where Tr and Det represent the trace and the determinant of $J(E^*)$ respectively.[4]

These conditions become

$$\begin{cases} \dfrac{\alpha_1\alpha_2 e_1^* e_2^*}{b^2(1+N)^4}(C_1C_2 - B_1B_2) \geq 0 & \text{(i)} \\ 4 + \dfrac{2\alpha_1 C_1 e_1^*}{b(1+N)^2} + \dfrac{2\alpha_2 C_2 e_2^*}{b(1+N)^2} + \dfrac{\alpha_1\alpha_2 e_1^* e_2^*}{b^2(1+N)^4}(C_1C_2 - B_1B_2) \geq 0 & \text{(ii)} \\ -\dfrac{\alpha_1 C_1 e_1^*}{b(1+N)^2} - \dfrac{\alpha_2 C_2 e_2^*}{b(1+N)^2} - \dfrac{\alpha_1\alpha_2 e_1^* e_2^*}{b^2(1+N)^4}(C_1C_2 - B_1B_2) \geq 0 & \text{(iii)} \end{cases} \tag{B.3}$$

When $n_i = n_j = n$, the Nash equilibrium reads

$$E^* = (\widetilde{e}^*, \widetilde{e}^*) \,\text{with}\, \widetilde{e}^* = \frac{(a-c)(1+n)}{b\gamma(1+2n)^2 - (1+n)n}. \tag{B.4}$$

For this simple formulation and $\alpha_1 = \alpha_2 = \alpha$, by applying the conditions for stability of E^* in the homogeneous case, we get that the Nash equilibrium E^* has strictly positive coordinates and it is stable as long as $\gamma > (n(1+n)/b(1+2n))$ (which holds true whenever $\gamma > \widetilde{\gamma}_i$) and $a - c < (b(1+2n)^2/\alpha(1+n))$, whose violation destabilizes E^* through a flip bifurcation. Similarly to stability of boundary equilibria, we observe that the interval of parameters $(a-c)$ ensuring the stability of the Nash equilibrium is larger than in the empty network case.

Without loss of generality, we now assume that $n_1 > n_2$. Both components of E^*, as given in (8), are strictly positive when

$$\widetilde{\gamma}_2 = \frac{n_2(1+n_1)^2}{b(1+N)^2} < \gamma < \frac{(1+n_1)n_2}{b(1+N)} \quad \text{or} \quad \gamma > \frac{n_1(1+n_2)}{b(1+N)} \tag{B.5}$$

Differently from the benchmark case of the empty network, when the first condition in (B.5) is verified, the Nash equilibrium E^* is unstable with strictly positive components of E^*. In fact, in this case it is $C_i < 0$ (strict concave profit functions) but $C_1C_2 < B_1B_2$ so that the second part of Proposition 3 applies.

When $\gamma > (n_1(1+n_2)/b(1+N))$, condition (B.5) holds, $C_i < 0$ and it is also $C_1C_2 > B_1B_2$. Conditions 1 in (B.3) is always verified and so a lost of stability of E^* for high values of γ, can be originated by flip or Neimark–Sacker bifurcations.[5]

References

[1] G. Bischi, F. Lamantia, Nonlinear duopoly games with positive cost externalities due to spillover effects, Chaos, Solitons & Fractals 13 (2002) 805–822.

[2] G. Bischi, F. Lamantia, A competition game with knowledge accumulation and spillovers, International Game Theory Review 6 (2004) 323–342.

[3] G. Bischi, F. Lamantia, Knowledge Accumulation in an R&D Network, in 'Nonlinear economic dynamics', Nova Science Publishers, New York, 2011.

[4] G. Bischi, F. Lamantia, A dynamic model of oligopoly with R&D externalities along networks. Part I, Mathematics and Computers in Simulation (2012), http://dx.doi.org/10.1016/j.matcom.2012.08.006

[5] G. Confessore, P. Mancuso, A dynamic model of R&D competition, Research in Economics 56 (2002) 365–380.

[6] C. D'Aspremont, A. Jacquemin, Cooperative and noncooperative R&D duopoly with spillovers, The American Economic Review 78 (5) (1988) 1133–1137.

[7] A. Galeotti, S. Goyal, M. Jackson, F. Vega-Redondo, L. Yariv, Network games, Review of Economic Studies 77 (2010) 218–244.

[4] Notice that these three conditions, all taken as strict inequalities, give sufficient conditions for stability of E^*.

[5] When $\gamma \in \left[\frac{(1+n_1)n_2}{b(1+N)}, \frac{n_1(1+n_2)}{b(1+N)}\right] / \overline{\gamma}$, only one coordinate of the equilibrium E^* is strictly positive, being $\overline{\gamma} > 0$ the only positive value for which $C_1C_2 = B_1B_2$, so that E^* is undefined, see (8).

[8] S. Goyal, Connections: An Introduction to the Economics of Networks, Princeton University Press, Princeton, 2007.
[9] S. Goyal, J. Moraga-Gonzales, R&D networks, RAND Journal of Economics 32-4 (2001) 686–707.
[10] M. Kamien, E. Muller, I. Zang, Cooperative joint ventures and R&D cartels, The American Economic Review 82 (1992) 1293–1306.
[11] C. Mira, Chaotic Dynamics. From the one-dimensional endomorphism to the two-dimensional diffeomorphism, World Scientific, Singapore, 1987.
[12] C. Mira, A. Barugola, J. Cathala, Chaotic dynamics in two-dimensional noninvertible maps, World Scientific, Singapore, 1996.
[13] L.D. Qiu, On the dynamic efficiency of bertrand equilibria, Journal of Economic Theory 75 (1997) 213–229.
[14] M. Spence, Cost reduction, competition and industry performance, Econometrica 52 (1984) 101–121.