Volume 389, No. 4, 15 February 2010
ISSN 0378-4371

PHYSICA A
STATISTICAL MECHANICS AND ITS APPLICATIONS

Recognized by the European Physical Society

Editors:
K.A. DAWSON
J.O. INDEKEU
H.E. STANLEY
C. TSALLIS

Available online at www.sciencedirect.com
ScienceDirect

http://www.elsevier.com/locate/physa

Physica A 389 (2010) 843–853

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Binary choices in small and large groups: A unified model

Gian-Italo Bischi [a,1], Ugo Merlone [b,*]

[a] DEMQ (Dipartimento di Economia e Metodi Quantitativi), Università di Urbino Carlo Bo, via Saffi n. 42, I-61029 Urbino, Italy

[b] Dipartimento di Statistica e Matematica Applicata "de Castro", Università di Torino, Corso Unione Sovietica 218/bis, I-10134 Torino, Italy

ARTICLE INFO

Article history:
Received 31 August 2009
Available online 14 October 2009

Keywords:
Sociophysics
Binary games
Social externalities
Discrete dynamical systems
Small groups
Large groups

ABSTRACT

Two different ways to model the diffusion of alternative choices within a population of individuals in the presence of social externalities are known in the literature. While Galam's model of rumors spreading considers a majority rule for interactions in several groups, Schelling considers individuals interacting in one large group, with payoff functions that describe how collective choices influence individual preferences. We incorporate these two approaches into a unified general discrete-time dynamic model for studying individual interactions in variously sized groups. We first illustrate how the two original models can be obtained as particular cases of the more general model we propose, then we show how several other situations can be analyzed. The model we propose goes beyond a theoretical exercise as it allows modeling situations which are relevant in economic and social systems. We consider also other aspects such as the propensity to switch choices and the behavioral momentum, and show how they may affect the dynamics of the whole population.

1. Introduction

In the recent literature several authors have proposed models to describe how collective behaviors arise from individual choices, assuming that these choices are influenced by the society in some way (see e.g. Refs. [1–3]). In particular, Ref. [4] provides an interesting analysis of opinion dynamics in a reaction–diffusion like model. This model has been extended along several directions and together with other models by the same author [5–8] has produced a great deal of research papers (for a survey see Ref. [9]).

A peculiarity of the system described in Ref. [4] is that the population of agents form a social system evolving as a whole. Systems characterized by such a trade-off between individual choices and collective behavior are ubiquitous and have been studied extensively in many contexts. Among the different contributions, the seminal work [10] stands out on its own as it provides a simple model, based on payoff functions which can qualitatively explain a wealth of every-day life situations such as wearing a helmet in a hockey game, boycotting a country or a product and traffic congestion. Indeed, the model proposed in Ref. [10], and in the successive generalizations, is general enough to include several games, such as the well known n-players prisoner's dilemma or the minority games [11]. Multiperson and dynamic extensions of prisoners' dilemma and battle of sexes games have been proposed in Refs. [12,13] by using suitable payoff functions derived from the corresponding payoff matrices. There these games have been analyzed as adaptive dynamic models and also by agent based simulations (see also Ref. [14]). Recently, Ref. [15] presented an explicit discrete-time dynamic model which is based on the qualitative properties described in Ref. [10] and simulates an adaptive adjustment process of repeated binary choices with externalities.

* Corresponding author. Tel.: +39 011 6705753.
E-mail addresses: gian.bischi@uniurb.it (G.-I. Bischi), merlone@econ.unito.it (U. Merlone).
1 Fax: +39 0722 305550.

0378-4371/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.physa.2009.10.010

The interest in applying formal methods for explaining social problems has several reasons, among the others we mention the possibility of predicting political events as described in Ref. [9]. This approach has produced several contributions by important authors such as Coleman [1], Granovetter and coworkers [16,17], Simon [18], and Schelling [19,10,2].

In this paper we propose an unified model which embeds both the model of rumors spreading [4] and a formalization of Schelling's binary choices model [15]. Our modelization allows us to extend the two original models in terms of the phenomena they can describe and also to interpret their mutual relations. In fact, the models we propose can be used to integrate the Galam's dynamics in a large group dynamics and also to fragment Schelling's binary choices in small groups. The distinction between large and small groups is important both theoretically (see Refs. [20,21]) and empirically (see Ref. [22] for an application to Group Decision Supporting Systems).

The paper is organized as follows. In Sections 2 and 3 respectively, Galam's model of rumors [4] and a formalization of Schelling's binary choices [10] are summarized. In Section 4 the general model we propose is explained. In Section 5 we show how the model we present is a general form which includes both Galam and Schelling models. In Section 6, we show how the model we propose can be used to study different kinds of social interactions which were not analyzed before. Some conclusions and further research areas are given in Section 9.

2. The Galam model

In his contribution [4] Galam proposes a model which explains the so called *Pentagon French hoax*,according to which no plane crashed on the Pentagon on September the 11th. He provides a model in which individuals try to choose an opinion (true or false) on this rumor on the basis of repeated discussions in social gatherings. At each of them, a small group of people get together and line up with a consensual opinion in which everyone agrees with the majority inside the group. This process is formalized in Ref. [4] as follows. Denote by a_i, $i = 1, 2, \ldots, L$, the probability to be sitting at a group of size i, with the obvious constraint

$$\sum_{i=1}^{L} a_i = 1. \tag{1}$$

As remarked in Ref. [4], including one-person groups makes the assumption "everyone gathering simultaneously" realistic. Given the social spaces, individuals distribute among them according to probabilities a_i at each social meeting.

Assume there are just two possible opinions, denoted by "$+$" and "$-$". Consider a N person population where at time t everyone is holding an opinion, i.e. $N_+(t)$ individuals are believing to opinion "$+$" and $N_-(t)$ persons are sharing the opinion "$-$", with $N_+(t) + N_-(t) = N$. Therefore, the probabilities to hold on "$+$" or "$-$" are, respectively

$$P_+(t) = \frac{N_+(t)}{N} \tag{2}$$

and

$$P_-(t) = 1 - P_+(t). \tag{3}$$

From this initial configuration, people start discussing the issue at the first social meeting. Each new cycle of multi-size discussions is marked by an unitary time increment. In a group of size k with j agents sharing opinion "$+$" and $(k-j)$ sharing opinion "$-$", all k members adopt opinion "$+$" if $j > k/2$; on the contrary, everybody adopts opinion "$-$" if $j < k/2$. In the symmetric case $j = k/2$ the outcome is determined assuming of a bias in favor of one of the two opinions. In Ref. [4] the following majority rule dynamics is proposed for a generic group of size k:

$$P_+^k(t+1) = \sum_{j=\left\lfloor \frac{k}{2}+1 \right\rfloor}^{k} C_j^k P_+(t)^j (1 - P_+(t))^{k-j} \tag{4}$$

where $C_j^k = \frac{k!}{(k-j)!j!}$ are binomial coefficients and $\left\lfloor \frac{k}{2}+1 \right\rfloor$ indicates the integer part of $\frac{k}{2}+1$. The careful reader can observe that the bias in favor of opinion "$-$" in case of a local doubt, is obtained considering as initial index of the sum $\left\lfloor \frac{k}{2}+1 \right\rfloor$. In fact, when k is even the sum starts at $k/2+1$, while when k is odd the sum starts considering the minimum majority.

Then, the overall updating process for all groups of size $k = 1, \ldots, L$, becomes

$$P_+(t+1) = \sum_{k=1}^{L} a_k P_+^k(t+1) \tag{5}$$

with $P_+^k(t+1)$ given by (4).

In the course of time, according to Galam's model, the same people will meet again and again randomly in the same cluster configuration of size groups. Fig. 1 is the original illustration of the process provided in Ref. [4]. At each new encounter they discuss locally the issue at stake and may change their mind according to above majority rule applied to each local group. In Ref. [4] the Eq. (5) is iterated to follow the time evolution of P_+.

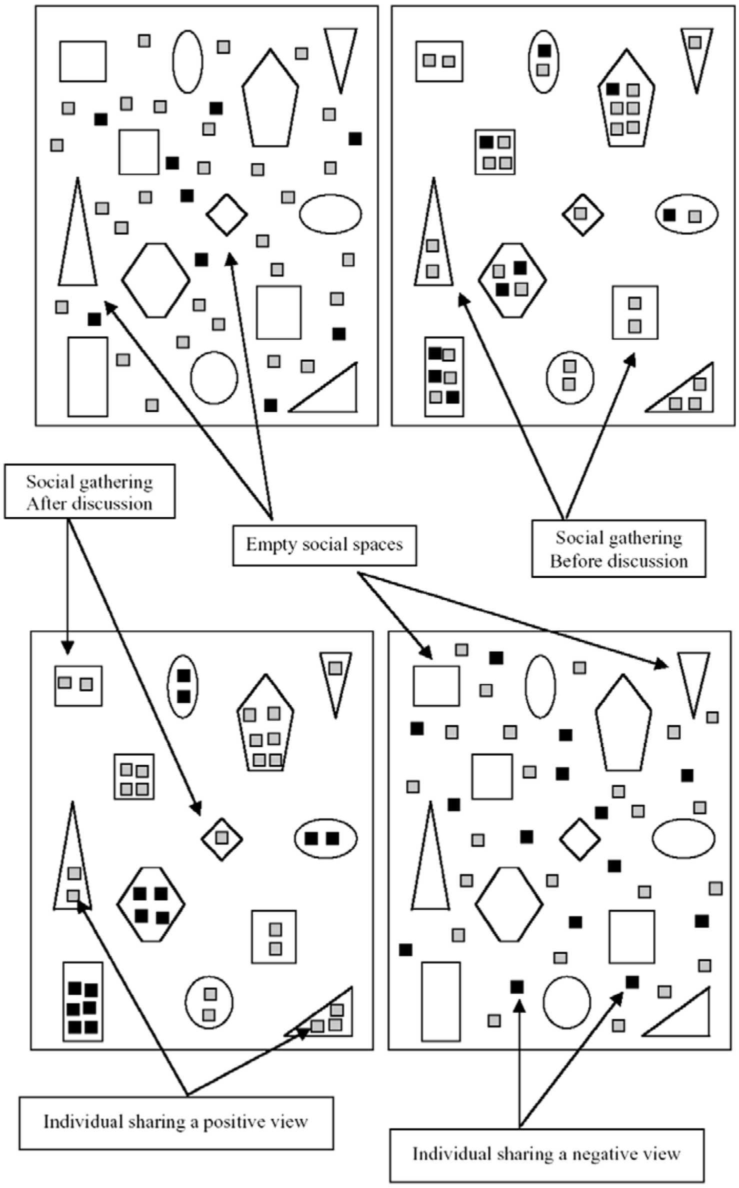

Fig. 1. A one step social gathering dynamics. Up left, people sharing the two opinions are moving around. Grey have + opinion while black have the other opinion. No discussion is occurring with 28 grey and 9 black. Upper right, people is partitioned in groups of various sizes from one to six and no one yet changes its mind. Below left, consensus has been reached within each group. As a result, they are now 23 grey and 14 black. Below right, people are again moving around with no discussion. The balance stays at 23/14 (the illustration has been taken from Ref. [4], page 576).

Finally, it is interesting to observe how, in this model, the bias driven by the tie effect may explain the propagation of "absurd" rumors from initial tiny minorities. An analysis of how group shared belief may favor one opinion against the other can be found in Ref. [7].

3. The Schelling model

In order to model how individual choices are influenced by social interactions (social externalities), Ref. [10] proposes a simple model where agents that face binary choices are assumed to interact *impersonally*, i.e., each agent's payoff depends only on the number of agents who choose one way or the other and not on their identities. The model he obtains provides a qualitative explanation of a wealth of every-day life situations, and is general enough to include several games, such as the well known *n*-players prisoner's dilemma and the minority games.

Following this approach, in Ref. [15] a discrete-time dynamic model is proposed where a population of players is assumed to be engaged in a game where they have to choose between two strategies, say *A* and *B* respectively.[2] They assume that

2 In Refs. [10,15] the two choices are denoted respectively *R* and *L*; here we modify the notation in order to avoid confusion with some of the symbols used in Ref. [4].

the set of players is normalized to the interval [0, 1] and denote by the real variable $x \in [0, 1]$ the fraction of players that choose strategy A. The payoffs are functions of x, say $A : [0, 1] \to \mathbb{R}$, $B : [0, 1] \to \mathbb{R}$, where $A(x)$ and $B(x)$ represent the payoff associated to strategies A and B respectively. Obviously, since binary choices are considered, when fraction x is playing A, then fraction $1 - x$ is playing B. As a consequence, $x = 0$ means that the whole population is playing B and $x = 1$ means that all the agents are playing A. The basic assumption modeling the dynamic adjustment is the following: *x will increase whenever $A(x) > B(x)$ whereas it will decrease when the opposite inequality holds*. This assumption, together with the constraint $x \in [0, 1]$, implies that equilibria are located either in the points $x = x^*$ such that $A(x^*) = B(x^*)$, or in $x = 0$ (provided that $A(0) < B(0)$) or in $x = 1$ (provided that $A(1) > B(1)$). In this process the agents update their binary choice at each time period $t = 0, 1, 2, \ldots$, and x_t represents the fraction of those playing strategy A at time period t. At time $(t + 1)$, x_t is assumed to become common knowledge, hence each agent is able to compute (or observe) payoffs $A(x_t)$ and $B(x_t)$. Finally, agents are homogeneous and myopic, that is, each of them is only interested to increase its own next period payoff. In this discrete-time model, at time t if x_t players are playing strategy A and $A(x_t) > B(x_t)$ then a fraction δ_A of the $(1 - x_t)$ agents that are playing B will switch to strategy A in the following turn; analogously, if $A(x_t) < B(x_t)$ then a fraction δ_B of the x_t players that are playing A will switch to strategy B. In other words, at any time period t, agents decide their action for period $t + 1$ comparing $A(x_t)$ and $B(x_t)$ according to

$$x_{t+1} = f(x_t) = \begin{cases} x_t + \delta_A g\left[\lambda\left(A(x_t) - B(x_t)\right)\right](1 - x_t) & \text{if } A(x_t) \geq B(x_t) \\ x_t - \delta_B g\left[\lambda\left(B(x_t) - A(x_t)\right)\right] x_t & \text{if } A(x_t) < B(x_t) \end{cases} \tag{6}$$

where $\delta_A, \delta_B \in [0, 1]$, $g : \mathbb{R}_+ \to [0, 1]$ is a continuous and increasing function such that $g(0) = 0$ and $\lim_{z\to\infty} g(z) = 1$, λ is a positive real number. The function g modulates how the fraction of switching agents depends on the difference between the previous turn payoffs; the parameters δ_A and δ_B represent the propensity of agents to switch to A and B respectively. When $\delta_A = \delta_B$, there are no differences in the propensity to switch to either strategies. On the contrary, $\delta_A \neq \delta_B$ represents a form of bias: given any payoff difference $|A(x) - B(x)| > 0$, $\delta_A > \delta_B$ implies that when $A(x) > B(x)$ switching from choice B to choice A is favored over switching switches from A to B when $A(x) < B(x)$. Finally, the parameter λ represents the switching intensity (or speed of reaction) of agents as a consequence of the difference between payoffs. In other words, small values of λ imply more inertia, i.e. anchoring attitude, of the actors involved, while, on the contrary, larger values of λ can be interpreted in terms of impulsiveness. In fact, according to the Clinical Psychology literature [23], impulsiveness can be separated in different components such as acting on the spur of the moment and lack of planning. In order to investigate the effects of impulsivity, Ref. [24] examines what happens when the parameter λ increases, up to the limiting case obtained as $\lambda \to +\infty$. This is equivalent to consider $g(\cdot) = 1$, i.e. the switching rate only depends on the sign of the difference between payoffs, no matter how much they differ. In this case the dynamical system assumes the form of the following discontinuous map

$$x_{t+1} = f_\infty(x_t) = \begin{cases} \delta_A + (1 - \delta_A)x_t & \text{if } B(x_t) < A(x_t) \\ x_t & \text{if } B(x_t) = A(x_t) \\ (1 - \delta_B)x_t & \text{if } B(x_t) > A(x_t). \end{cases} \tag{7}$$

4. A general model with different switching propensities

In this section we propose a dynamic model that merges several aspects of the two models described in Sections 2 and 3. It includes these two models as particular cases, moreover it allows us to describe other situations which cannot be studied by any of the models described above.

Firstly, in order to obtain a common notation for both models, we assume that the opinion "+" (believing the truth in Ref. [4]) corresponds to the A choice in the Schelling's model, and analogously the opinion "−" corresponds to the B choice. We assume that in any group of size k we have two payoff functions $A_k(j)$ and $B_k(j)$ where $j \in 0, 1, \ldots, k$ is the number of people in the group choosing A, or believing the truth. We assume that individuals gather in groups at time t and, after the interaction, leave the group with an updated preference which depends on the outcome of the interaction, in the sense that in each group a fraction of the agents with lower payoff will switch to the choice that gives higher payoff. To formalize this dynamic adjustment let us consider, at each time t, a size k group where $0 \leq j_t \leq k$ agents are choosing A. Then, after the social gathering, i.e. at time $t + 1$, the number of individuals choosing A will be

$$j_{t+1} = h_k(j_t, \delta_A, \delta_B) = \begin{cases} j_t + \lfloor \delta_A (k - j_t) \rfloor = \lfloor k\delta_A + (1 - \delta_A) j_t \rfloor & \text{if } A_k(j_t) > B_k(j_t) \\ j_t & \text{if } A_k(j_t) = B_k(j_t) \\ j_t - \lfloor \delta_B j_t \rfloor = \lfloor j_t (1 - \delta_B) \rfloor & \text{if } A_k(j_t) < B_k(j_t) \end{cases} \tag{8}$$

where $\delta_A, \delta_B \in [0, 1]$ represent the probability according to which agents may switch to A and B respectively, according to the greater payoff observed. Furthermore, since in groups the number of agents is integer, it makes little sense to consider fraction of agents who are switching choices, therefore we have introduced integer parts in the right hand side of (8) to consider integer number of agents. In formulation (8) we consider the integer part and this indicates the behavioral momentum of agents, that is, how difficult is for agents to switch choices when facing inferior payoffs. Behavioral momentum is a well known psychological construct which has been examined for example in Ref. [25]. In Sections 7 and 8 we will compare the effects of considering different rounding methods and their relations to bias. Notice that in this model the switching mechanism has no bias in the sense of Ref. [4]. In order to obtain a bias towards B it is sufficient to consider, for example,

$$j_{t+1} = h_{k,B}(j_t, \delta_A, \delta_B) = \begin{cases} \lfloor k\delta_A + (1-\delta_A) j_t \rfloor & \text{if } A_k(j_t) > B_k(j_t) \\ \lfloor j_t (1-\delta_B) \rfloor & \text{if } A_k(j_t) \le B_k(j_t). \end{cases} \tag{9}$$

By symmetry it is immediate to obtain a switching function biased towards A. It is worth to stress that, in this formulation, such a bias -which is related to how local uncertainty $A_k(j_t) = B_k(j_t)$ is resolved- is combined to the bias related to the asymmetric switching propensity $\delta_A \neq \delta_B$, as described in Section 3.

Following Ref. [4], it is now possible to obtain the dynamics of the probability of choosing A:

$$P_A(t+1) = \sum_{k=1}^{L} a_k \sum_{j=0}^{k} C_j^k P_A(t)^j (1-P_A(t))^{k-j} \frac{h_k(j, \delta_A, \delta_B)}{k} \tag{10}$$

where the last term is the relative number of agents choosing option A in a group of size k.

In such formalization of the general model we assumed that individuals are impulsive in the sense described in Ref. [24]. However, it possible to model inertia in the agents' reaction, as in Ref. [15], by introducing a modulating function g as described in Section 3.

5. Some particular cases: Galam's modeling rumors and Schelling's binary choices

As mentioned in the introduction, the model we have proposed is a generalization of both the Galam model [4] and formalization [15] of Schelling binary choices model [10].

We now show how it is possible to derive, from (10), the Galam's probability dynamics described in Eq. (5). Consider $h_{k,B}(j_t, \delta_A, \delta_B)$ and fix $\delta_A = \delta_B = 1$. With these choices, in order to obtain the majority rule it is sufficient to consider

$$A_k(j) = j; \qquad B_k(j) = k - j.$$

With this notation the assumption of a majority rule dynamics with a bias in favor of choice B means

$$h_{k,B}(j_t, 1, 1) = \begin{cases} k & \text{if } j_t > \frac{k}{2} \\ 0 & \text{if } j_t \le \frac{k}{2} \end{cases}$$

and Eq. (10) becomes

$$\begin{aligned} P_A(t+1) &= \sum_{k=1}^{L} a_k \sum_{j=0}^{k} C_j^k P_A(t)^j (1-P_A(t))^{k-j} \frac{h_{k,B}(j_t, 1, 1)}{k} \\ &= \sum_{k=1}^{L} a_k \left[\sum_{0 \le j < \lfloor \frac{k}{2}+1 \rfloor} C_j^k P_A(t)^j (1-P_A(t))^{k-j} \frac{0}{k} + \sum_{j=\lfloor \frac{k}{2}+1 \rfloor}^{k} C_j^k P_A(t)^j (1-P_A(t))^{k-j} \frac{k}{k} \right] \\ &= \sum_{k=1}^{L} a_k \sum_{j=\lfloor \frac{k}{2}+1 \rfloor}^{k} C_j^k P_A(t)^j (1-P_A(t))^{k-j}. \end{aligned}$$

We now show how Schelling's model of binary choices, as formalized in Ref. [15], can be obtained as a particular case of the general model we have proposed.

As in Ref. [15], a single group consisting of a continuum of agents is considered. It is then necessary to assume $a_k = 0$ for all values of k, and to introduce an infinitely large group such that $a_\infty = 1$ together with the appropriate payoff functions $A_\infty(\cdot)$ and $B_\infty(\cdot)$. With these assumptions the population can be assumed unitary ($k = 1$) and j_t becomes the fraction of population choosing A. Since there is only one single group, Eq. (10) reduces to

$$j_{t+1} = \begin{cases} \delta_A + (1-\delta_A) j_t & \text{if } A_\infty(j_t) > B_\infty(j_t) \\ j_t & \text{if } A_\infty(j_t) = B_\infty(j_t) \\ j_t (1-\delta_B) & \text{if } A_\infty(j_t) < B_\infty(j_t) \end{cases}$$

which is the same as the dynamic process formalized in Eq. (7).

Simply changing notation, $A_\infty(\cdot) := A(\cdot)$ and $B_\infty(\cdot) := B(\cdot)$, we obtain the binary choices model considered in Ref. [15] and studied in Ref. [24].

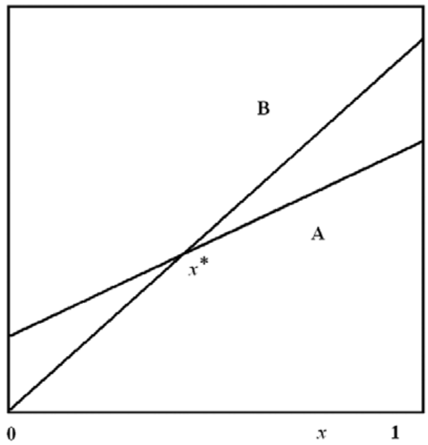

Fig. 2. A graphical representation of possible payoffs for the example on traffic congestion after a blizzard given in Ref. [10].

6. Integrating small groups and fragmenting large ones

By using the model we propose, it is possible to simulate different situations besides the original ones proposed in Refs. [4,10]. Among the different possibilities here we consider only two of them. The first one considers the case of a single large group. This situation is important from the theoretical point of view as it represents the case in which all the groups merge into one. Furthermore it illustrates how the size of the group and the payoff affect the dynamics. In fact, while in the case of the majority rule [4], it takes just one iteration for the population to reach unanimity on the majority's opinion, and the killing point for this model coincides with the floor (i.e. the integer part) of $N/2$, when considering different payoffs the dynamics may become quite interesting. This latter case will be illustrated in Section 8.

The second example consists of payoff functions with — using Schelling's terminology, see Ref. [10] — both contingent internality and contingent externality. This is illustrated in Fig. 2; in this case it is possible to observe that an A choice benefits those who choose B, a B choice those who choose A. Among the vivid real life examples provided in Ref. [10], we quote[3] one situation related to traffic congestion after a blizzard: "…let A be staying home and B using the car right after a blizzard. The radio announcer gives dire warnings and urges everybody to stay home. Many do, and those who drive are pleasantly surprised by how empty the roads are; if the others had known, they would surely have driven. If they had, they would all be at the lower left extremity of the B curve." ([10], page 405). This kind of games has been examined in Ref. [26] as compounded dispersion game. Now assume that in each group the payoff functions assume the same structure, then an interesting dynamics would be determined. A first possible interpretation of the revisited example could be that of individuals making their driving decisions in different neighborhoods. Fig. 3 represents a step of the social gathering dynamics in this case; this is similar to the one provided by Galam and reported in Fig. 1.

7. The effect of propensity to switch choices

One of the interesting aspects of both Galam's and Schelling's contribution is that, in their models, the collective behavior is based on the individual choices of the single agents. As mentioned in Section 4, given the necessity to maintain an integer number of agents during the dynamic process, the fractions of agents switching choices must be rounded. Obviously, this applies when the switching propensity of agents is not integer, that is either $0 < \delta_A < 1$ or $0 < \delta_B < 1$ or both. We observe that switching propensities model two aspects in agents' decision. First, when the two propensities are identical, they model behavioral momentum. The lower the propensity is, the less agents are likely to switch choices when facing inferior payoff. For a discussion about cases in which decision-makers continue to allocate resources to courses of action that have resulted in disappointing outcomes, the reader may refer to Ref. [25]. Second, different switching propensities model bias in choices. Bias in favor of one opinion is explicitly considered in Ref. [4]. In our general model, formula (9) shows how another bias can be introduced when the two choices give the same payoff. Different propensities values extend bias when the choices give different payoffs.

Furthermore, non-integer switching propensities can have consequences on how non-integer numbers of agents who are switching choices in each group are rounded. We analyze and compare the dynamics in majority games when different rounding functions are assumed. In particular, we consider *floor*, where $\lfloor x \rfloor$ defines the largest integer $n \leq x$; *ceiling*, where $\lceil x \rceil$ defines the smallest integer $n \geq x$. Finally we consider *nearest integer*, where $\lfloor x \rceil$, defines the integer closest to x; to avoid ambiguity we adopt the convention according which half-integers are always rounded to even numbers, see Ref. [27]. The results are reported in Tables 1–3, the example we consider is the same as the one presented by Galam in Ref. [4].

In Fig. 4 we illustrate the dynamics of choice A in terms of fraction of population with the different rounding choices and compare them to the original dynamics considered in Ref. [4].

3 Recall that we use respectively A and B instead of R and L.

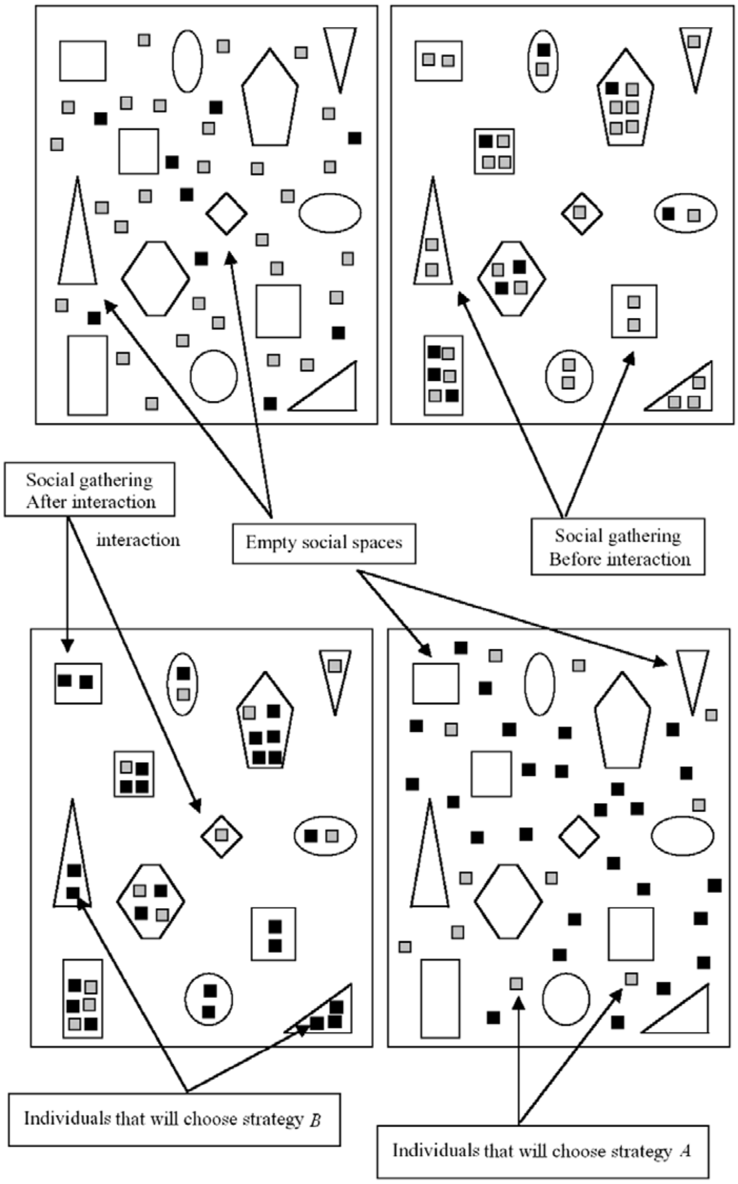

Fig. 3. The analogous of Galam [4] social gathering dynamics when considering Schelling [10] payoffs with both contingent internality and contingent externality, and no bias. In this case individuals belonging to social spaces with ties do not switch strategy. At the end of the step the balance is 11/26.

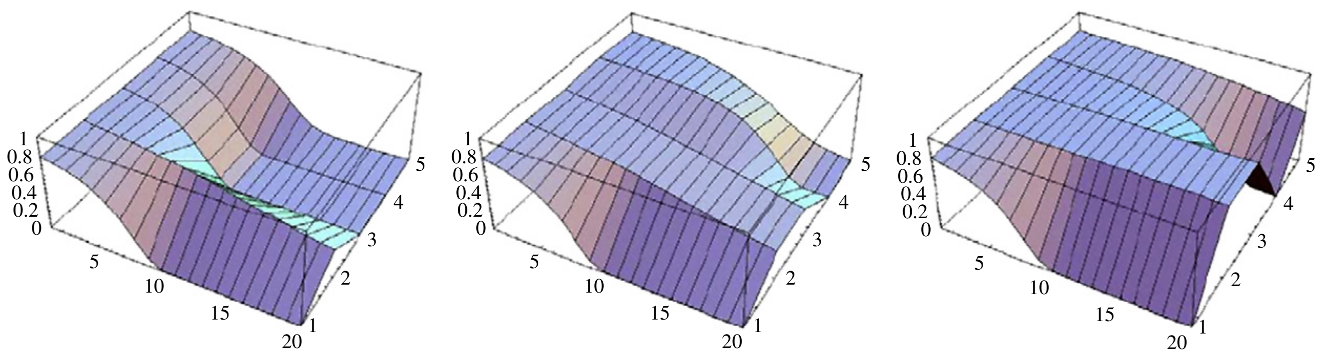

Fig. 4. Comparison of different rounding functions for $\delta_A = \delta_B = .75$; $\delta_A = \delta_B = .5$; $\delta_A = \delta_B = 25$; 1 = Galam, 2 = floor, 3 = nearest integer, 4 = ceiling, 5 = no rounding.

Table 1
Values of $P_A(t)$ to test the effects of different rounding functions with propensity to switch choices $\delta_A = \delta_b = .75$.

	Galam	floor	nearest integer	ceil	no rounding
1	0.785067	0.787200	0.785067	0.785067	0.788800
2	0.764484	0.773169	0.764484	0.764484	0.774449
3	0.735544	0.757790	0.735544	0.735544	0.755835
4	0.693956	0.740946	0.693956	0.693956	0.731368
5	0.632981	0.722525	0.632981	0.632981	0.698758
6	0.542763	0.702428	0.542763	0.542763	0.654740
7	0.412600	0.680583	0.412600	0.412600	0.594880
8	0.244830	0.656954	0.244830	0.244830	0.513937
9	0.086113	0.631559	0.086113	0.086113	0.408164
10	0.010258	0.604486	0.010258	0.010258	0.281823
11	0.000141	0.575906	0.000141	0.000141	0.156340
12	0.000000	0.546079	0.000000	0.000000	0.064990
13	0.000000	0.515358	0.000000	0.000000	0.020595
14	0.000000	0.484167	0.000000	0.000000	0.005577
15	0.000000	0.452980	0.000000	0.000000	0.001425
16	0.000000	0.422280	0.000000	0.000000	0.000358
17	0.000000	0.392522	0.000000	0.000000	0.000090
18	0.000000	0.364093	0.000000	0.000000	0.000022
19	0.000000	0.337290	0.000000	0.000000	0.000006
20	0.000000	0.312308	0.000000	0.000000	0.000001
21	0.000000	0.289245	0.000000	0.000000	0.000000
22	0.000000	0.268113	0.000000	0.000000	0.000000
⋮					
100	0.000000	0.026178	0.000000	0.000000	0.000000

Table 2
Values of $P_A(t)$ to test the effects of different rounding functions with propensity to switch choices $\delta_A = \delta_b = .50$.

	Galam	floor	nearest integer	ceil	no rounding
1	0.785067	0.787200	0.787200	0.797867	0.792533
2	0.764484	0.773169	0.773169	0.795155	0.783678
3	0.735544	0.757790	0.757790	0.791696	0.773115
4	0.693956	0.740946	0.740946	0.787266	0.760433
5	0.632981	0.722525	0.722525	0.781563	0.745107
6	0.542763	0.702428	0.702428	0.774175	0.726452
7	0.412600	0.680583	0.680583	0.764527	0.703588
8	0.244830	0.656954	0.656954	0.751810	0.675388
9	0.086113	0.631559	0.631559	0.734852	0.640450
10	0.010258	0.604486	0.604486	0.711934	0.597119
11	0.000141	0.575906	0.575906	0.680496	0.543664
12	0.000000	0.546079	0.546079	0.636711	0.478761
13	0.000000	0.515358	0.515358	0.575019	0.402499
14	0.000000	0.484167	0.484167	0.488146	0.317866
15	0.000000	0.452980	0.452980	0.369695	0.231893
16	0.000000	0.422280	0.422280	0.224387	0.154507
17	0.000000	0.392522	0.392522	0.087274	0.094113
18	0.000000	0.364093	0.364093	0.013714	0.053200
19	0.000000	0.337290	0.337290	0.000344	0.028533
20	0.000000	0.312308	0.312308	0.000000	0.014817
21	0.000000	0.289245	0.289245	0.000000	0.007556
22	0.000000	0.268113	0.268113	0.000000	0.003816
⋮					
100	0.000000	0.026178	0.026178	0.000000	0.000000

8. The dynamics of the dispersion game

The different groups and the switching propensity have interesting effects when considering dispersion games. In Ref. [15] the dynamics of binary choices for a continuum of agents meeting in a single group has been examined. In this section we show some examples of the dynamics when agents are allowed to meet in finite size groups. Obviously, in this case, when considering non-trivial switching propensities the different rounding may have stark consequences; the reason is that the internal representation of numbers may cause *floor* and *ceiling* functions to behave differently from what would have been the result if real numbers were considered. In order to improve the reliability of our simulation we have all simulations have been performed using long double precision variables in C++, i.e., floating-point variables with 80 bits of precision. Furthermore some of the trajectories have been replicated using Microsoft Office Excel and the results were quite similar.

Table 3
Values of $P_A(t)$ to test the effects of different rounding functions with propensity to switch choices $\delta_A = \delta_b = .25$.

	Galam	floor	nearest integer	ceil	no rounding
1	0.785067	0.800000	0.800000	0.797867	0.796267
2	0.764484	0.800000	0.800000	0.795155	0.792189
3	0.735544	0.800000	0.800000	0.791696	0.787729
4	0.693956	0.800000	0.800000	0.787266	0.782842
5	0.632981	0.800000	0.800000	0.781563	0.777478
6	0.542763	0.800000	0.800000	0.774175	0.771580
7	0.412600	0.800000	0.800000	0.764527	0.765082
8	0.244830	0.800000	0.800000	0.751810	0.757910
9	0.086113	0.800000	0.800000	0.734852	0.749977
10	0.010258	0.800000	0.800000	0.711934	0.741185
11	0.000141	0.800000	0.800000	0.680496	0.731422
12	0.000000	0.800000	0.800000	0.636711	0.720558
13	0.000000	0.800000	0.800000	0.575019	0.708446
14	0.000000	0.800000	0.800000	0.488146	0.694920
15	0.000000	0.800000	0.800000	0.369695	0.679790
16	0.000000	0.800000	0.800000	0.224387	0.662850
17	0.000000	0.800000	0.800000	0.087274	0.643873
18	0.000000	0.800000	0.800000	0.013714	0.622616
19	0.000000	0.800000	0.800000	0.000344	0.598837
20	0.000000	0.800000	0.800000	0.000000	0.572305
21	0.000000	0.800000	0.800000	0.000000	0.542828
22	0.000000	0.800000	0.800000	0.000000	0.510294
⋮					
100	0.000000	0.800000	0.800000	0.000000	0.000000

In the following we present only some results in order to show how the propensities to switch may affect the dynamics. Assume that payoff functions are

$$A_k(j) := j$$
$$B_k(j_{t:}) := k - j$$

with switching propensities $\delta_A = .3$ and $\delta_B = .9$. Further assume that, when payoff are identical, agents have a bias toward A, that is those who have played B will switch choice with propensity δ_A. As mentioned in Section 7 the different switching propensities model important aspects in agent choices.

The probabilities to be sitting at the different sizes groups are defined as follows:

$$a_2 = 0.1, \qquad a_3 = 0.1, \qquad a_k = 0.8, \quad \text{where } k \text{ is fixed as any value greater than 3}$$

all other groups have probability 0. We analyze the dynamics of choices as k varies in the range $[4, 276]$; for each value of k, starting from the initial condition $x_0 = 0.80$, after discarding the first 500 steps x_t is computed and plotted on the vertical line $x = k$. The result is depicted in Fig. 5. We can observe that when the size of the third group belongs to the set $\{4, 5, \ldots, 19, 21\}$ there exist a fixed point. When k belongs to the set $\{20, 22, 23, \ldots, 63, 65\}$ there is a 2-period cycle. The period doubles when k belongs to the set $\{64, 66, \ldots, 89, 91\}$ and so on until the trajectory becomes chaotic. Finally when k is strictly greater than 207 the population steady dynamics is a 3-period cycle. In Section 7 we examined how the different rounding functions affect the dynamics in the majority game; it is interesting to perform a similar analysis with the dispersion game. When considering either ceil or no rounding the results are quite similar to the previous case, therefore they are not represented. By contrast the trajectories are quite different when considering the floor rounding, they are depicted in Fig. 6. The differences in the dynamics are to be interpreted not as numerical artifacts rather as the result of the different biases implicitly modeled by the rounding functions.

Finally we illustrate the effect of group size when this game is played in a single size k group, i.e., $a_k = 1$ and $a_j = 0$ when $j \neq k$. The different dynamics for k in the range $[4, 276]$, are illustrated in Fig. 7.

9. Conclusions

In this paper we have proposed a model which constitutes a generalization and a synthesis of two well known models of social interaction. This model aims to understand how choices within small groups affect those in the population. While the model we consider is general enough to include the original ones — namely Galam's model of rumors spreading [4] and a formalization of Schelling's binary choices [15] — it can be further used to analyze other multi-player social dilemmas. Among the many possible, all the situations that are considered in Ref. [10] for a single population can be extended to small groups by using the generalization proposed in this paper. In particular, we have provided two different models which naturally arise when combining the main aspects of those presented in Refs. [4,10]. For these models we have given a preliminary analysis and we have related them to some real life situations.

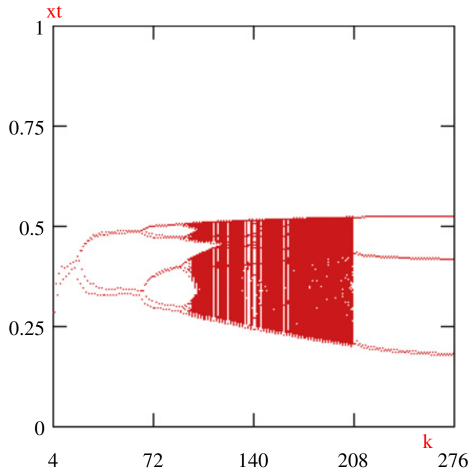

Fig. 5. Trajectories for the dispersion game when using the *nearest integer* rounding.

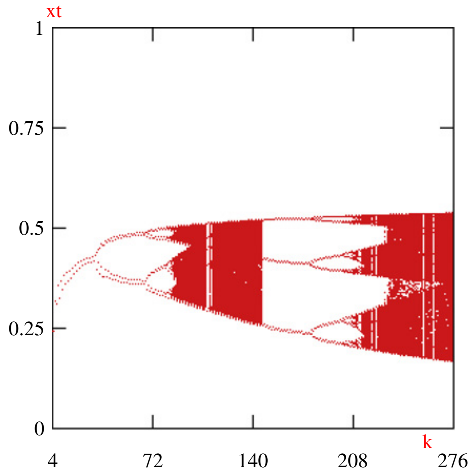

Fig. 6. Trajectories for the dispersion game when using the *floor* rounding.

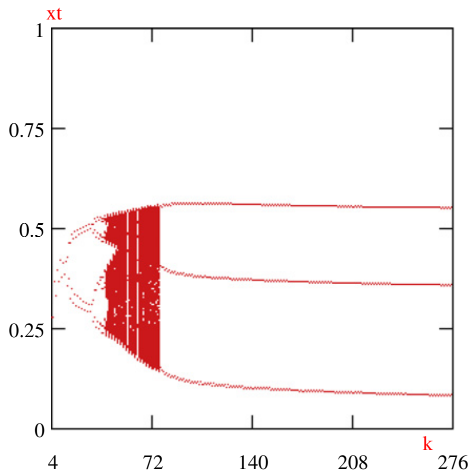

Fig. 7. Trajectories for the dispersion game played in a *k* size group when using the *nearest integer* rounding.

Although in large groups a fraction of unitary population is considered and therefore numerical rounding poses little concern, when considering small groups this is different since each single individual may become pivotal. Therefore, we considered different rounding functions, interpreted them in terms of behavioral momentum and compared them. In fact, when considering different propensities in switching choices, we showed how the different rounding functions may affect the dynamics. Our analysis was conducted on the majority game studied in Ref. [4], but can be extended to other games as well. The results show how the small groups dynamics – in terms of rounding function – may become crucial for the whole group dynamics.

Other aspects of this model can be investigated in further research, for example the analysis of the role of impulsiveness in small groups. Finally, it would be interesting to provide an analysis on how some social parameters, such as agents' impulsiveness and population size, affect the dynamics. In particular to investigate what makes a group large and how its dynamic behavior differs from that of a small one.

References

[1] J. Coleman, The Mathematics of Collective Action, Aldine, Chicago, IL, 1973.
[2] T.C. Schelling, Micromotives and Macrobehavior, W.W. Norton, New York, NY, 1978.
[3] H.P. Young, Individual Strategy and Social Structure, Princeton University Press, Princeton, NJ, 1998.
[4] S. Galam, Modelling rumors: The no plane pentagon French Hoax case, Physica A 320 (2003) 571–580.
[5] S. Galam, Majority rule, hierarchical structures and democratic totalitarism: A statistical approach, Journal of Mathematical Psychology 30 (1986) 426–434.
[6] S. Galam, B. Chopard, A. Masselot, M. Droz, Competing species dynamics, The European Physical Journal B 4 (1998) 529–531.
[7] S. Galam, Heterogeneous beliefs, segregation, and extremism in the making of public opinions, Physical Review E 71 (2005) 046123-1-5.
[8] S. Galam, Local dynamics vs. social mechanisms: A unifying frame, Europhysics Letters 70 (2005) 705–711.
[9] S. Galam, Sociophysics: A review of Galam models, International Journal of Modern Physics C 19 (2008) 409–440.
[10] T.C. Schelling, Hockey helmets, concealed weapons, and daylight saving, Journal of Conflict Resolution 17 (3) (1973) 381–428.
[11] D. Challet, Y.C. Zhang, Emergence of cooperation and organization in an evolutionary game, Physica A (1997) 246–407.
[12] J. Zhao, M.N. Szilagyi, F. Szidarovszky, A continuous model of *n*-person prisoners' dilemma, Game Theory and Applications 12 (2007) 192–228.
[13] J. Zhao, M.N. Szilagyi, F. Szidarovszky, An *n*-person battle of sexes, Physica A 387 (2008) 3669–3677.
[14] J. Zhao, M.N. Szilagyi, F. Szidarovszky, *n*-person battle of sexes — A simulation study, Physica A 387 (2008) 3678–3688.
[15] G.I. Bischi, U. Merlone, Global dynamics in binary choice models with social influence, Journal of Mathematical Sociology 33 (4) (2009) 277–302.
[16] M. Granovetter, Threshold models of collective behavior, The American Journal of Sociology 83 (6) (1978) 1420–1443.
[17] M. Granovetter, R. Soong, Threshold models of diffusion and collective behavior, Journal of Mathematical Sociology 9 (1978) 165–179.
[18] H.A. Simon, A formal theory of interaction in social groups, American Sociological Review 17 (1952) 202–212.
[19] T.C. Schelling, Dynamic models of segregation, Journal of Mathematical Sociology 1 (1971) 143–186.
[20] E.J. Miller, A.K. Rice, Systems of Organisations, Tavistock, London, UK, 1967.
[21] J.E. Mcgrath, Time, interaction and performance (TIP): A theory of groups, Small Group Research 22 (2) (1991) 147–174.
[22] G. DeSanctis, R.B. Gallupe, A foundation for the study of group decision support systems, Management Science 33 (5) (1987) 589–609.
[23] J.H. Patton, M.S. Stanford, E.S. Barratt, Factor structure of the Barratt impulsiveness scale, Journal of Clinical Psychology 51 (1995) 768–774.
[24] G.I. Bischi, L. Gardini, U. Merlone, Inpulsivity in binary choices and the emergence of periodicity, Discrete Dynamics in Nature and Society 2009 (2009) doi:10.1155/2009/407913. Article ID 407913, 22 pages.
[25] S.M. Goltz, Can't stop on a dime, Journal of Organizational Behavior Management 19 (1) (1999) 37–63.
[26] A. Namatame, Adaptation and Evolution in Collective Systems, World Scientific, Singapore, 2006.
[27] E.W. Weisstein, Nearest Integer Function, retrieved May 13, 2009, from MathWorld–A Wolfram Web Resource at http://mathworld.wolfram.com/NearestIntegerFunction.html.