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Chapter 13

AN ADAPTIVE DYNAMIC MODEL OF SEGREGATION

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Abstract

Starting from a seminal paper of Thomas Schelling (1969) we formalize a two-dimensional discrete time dynamical system to study segregation. The simple adaptive mechanism we propose may lead to the segregation of two different populations whose members are characterized by a limited tolerance about the presence of individuals of the other group. We provide a global analysis of the model, based on a computer-assisted interplay of analytical, numerical and geometrical methods. This allows us to emphasize the role of the parameters that represent the distribution of tolerance within the populations and their inertia in moving in or out of the system considered, as well as the role of constraints imposed. When several attractors are present, each with its own basin of attraction, the adaptive dynamics can act as a path-dependent selection device, i.e., the collective behavior that prevails in the long run depends on the starting conditions and on the historical accidents occurring along the trajectories. The study shows how simple adaptive rules, repeatedly applied over time, can be used to analyze the evolutive paths leading to the emergence of different collective behaviors in the long run, i.e., the trade-off between myopic individual behavior and the emergence of social structures.

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13.1. Introduction

In a celebrated seminal paper [1] (see also the expanded version, [2]), Schelling proposed two models for the description of residential separation of a population formed by two kinds of inhabitants, differing, e.g., for racial or religious or cultural features. The separation into exclusive districts is explained by Schelling in terms of individual preferences, or *tolerance* — following Schelling's terminology — on coexistence with neighbors of the opposite kind. Schelling analyzed the effects of local decisions on global behavior, and showed that even if agents have mild preferences for same-type neighbors, in the long run, the system can evolve towards a complete separation, even if this is not the outcome preferred by the individuals. In his papers [1] and [2], Schelling refers to *blacks* and *whites* that have to decide if they want to stay in a given city district or leave it, and denotes the phenomenon of separation as *segregation*, to stress the dramatic problem of formation of ghettos. However, as stressed by other authors (see e.g. [3]), individuals tend to categorize themselves in several ways; this has consequences on several situations, for example when they have to decide on joining a given club, or entering an organization, or a political party or an academic group. Indeed, Schelling himself begins his paper with the sentence "People get separated along many lines and in many ways. There is segregation by sex, age, income, language, religion, color, taste ... and the accidents of historical location"([2, p. 143]).

The first model proposed by Schelling is a typical agent-based simulation model. According to [4] it can be considered as a migration model, i.e., a cellular automata where actors are not confined to a particular cell. By contrast, the second one is formulated in terms of a two-dimensional dynamical system, and, even if no explicit expression is given, a qualitative graphical dynamical analysis is proposed to show that several equilibria coexist. Moreover, no information on basins of attraction is given, due to the lack of an explicit analytic formulation of the dynamical system.

While the first model has inspired a flourishing stream of literature, and many researchers have developed extensions, refinements, computer and graphical implementations of Schelling's agent-based simulation model, the second approach, based on the qualitative theory of nonlinear dynamical systems, has been rather neglected.

In this chapter we try to fill this gap, and propose an explicit analytic formulation of the model described in [1] and [2]. This explicit formulation will allow us to emphasize the effects of the variations of the parameters that represent individual behavior of the members of the two populations. Moreover, when the system has several different attractors, an analysis of the extension and the shapes of their basins of attraction is possible. This will allow us to give a more accurate description of the role of initial conditions on the system evolution, and of the influence of their changes as consequences of small variations, as those induced by particular laws or policies. In fact, as already stressed by Schelling, "In some cases, small incentives, almost imperceptible differentials, can lead to strikingly polarized results"([2, p. 146]). Finally, we shall study the effects of constraints on local and global stability analysis.

The structure of the chapter is the following. In section 13.2 we present the formal dynamic model and the assumptions on which it is based. In section 13.3 we describe all the possible equilibrium points and in section 13.4 we examine, through numerical explorations, the structure of the basins and the kind of dynamic evolutions generated by the model, as well as the influence of the main parameters and constraints in the case of linear distribution of tolerance. In section 13.5 we introduce a distribution of tolerance different from the linear one, and stress the effects of this new assumption on the dynamic behaviors of the model. Section 13.6 concludes the chapter and gives some directions for future extensions of the proposed model.

13.2. The Model

Following [2], we assume that individuals are partitioned in two classes C_1 and C_2 (say "color 1" and "color 2") of respective numerosity N_1 and N_2, and postulate that the individuals of each group care about the color of the people in the district they live in or — according to the different interpretations attached to the modeled system — in the association they belong to, or political party and so on.

Any individual of color i, $i = 1,2$, can observe the ratio of the two types at any moment, and can decide to move in (out) depending on its (dis)satisfaction with the observed proportion of opposite color agents to its own color. As in [2], this can be expressed through the definition of a *distribution of Tolerance*, $R_i = R_i(x_i)$ for each population i. This distribution represents a cumulative density function giving the maximum ratio R_i of individuals of population C_j to individuals of population C_i which is tolerated by a fraction x_i/N_i of population C_i. As suggested by Schelling (see [1] and [2]), the simplest assumption is a linear cumulative distribution (see Figure 13.1a):

$$R_i = \tau_i \left(1 - \frac{x_i}{N_i}\right) \qquad i = 1,2, \qquad (13.1)$$

where τ_i is a parameter giving the *maximum ratio of tolerance* of individuals of color C_i.

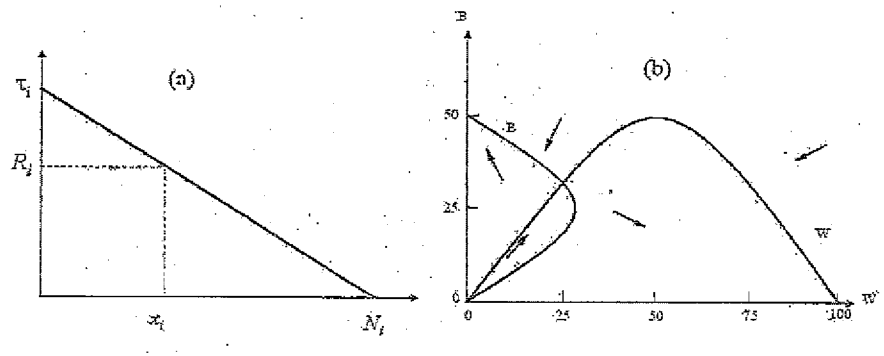

Figure 13.1. (a) Linear distribution of tolerance, $0 < x_i < N_i$ individual of color C_i can tolerate at most a ratio $R_i = x_j/x_i$ of individuals of different color C_j, where nobody can tolerate a ratio τ_i or more of different individuals. All can tolerate 0 different individuals. (b) The two functions $x_iR_i(x_i)$, $i = 1,2$ considered in [2].

Following the qualitative arguments of Schelling (see [1] and [2]), let $x_i(t)$ be the number of individuals of color C_i present in the system at time period $t = 0,1,2....$ They can tolerate at most $x_i(t)R_i(x_i(t))$ individuals[1] of color C_j, $j \neq i$. If this number is larger than the current number $x_j(t)$ of individuals of color C_j in the system, then we assume that, at time t, some individuals of color C_i will enter the system, and consequently x_i will increase in the next time period $t+1$; vice versa, if the opposite inequality holds, i.e. $x_j(t)$ is greater than $x_i(t)R_i(x_i(t))$, then some individuals of color C_i will leave the system, and x_i will decrease (see the arrows in Figure 13.1b, reproduced from [2]). In order to obtain an explicit dynamic model, we assume that this difference times γ_i (the *speed of adjustment*), gives the relative variation of class C_i individuals. Formally:

$$\frac{x_i(t+1)-x_i(t)}{x_i(t)} = \gamma_i\left[x_i(t)R_i(x_i(t)) - x_j(t)\right], \tag{13.2}$$

where we adopt a discrete time scale in order to allow a comparison with the agent based simulation model proposed by [2]. Notice that low values of the speed of reaction γ_i denote inertia, or patience, whereas high values represent strong reactivity and fast decisions.

Of course, we also have to consider the natural constraints $0 \leq x_i(t) \leq N_i$ for each $t \geq 0$. However, in some situations, it may be interesting to introduce even stronger restrictions on the upper limits for the number of individuals of a given color that are allowed to enter the system: say $0 \leq x_i(t) \leq K_i$, with $K_i \leq N_i$, as possible exogenous controls imposed by an authority in order to regulate the system. Putting together all these assumptions we obtain the following piecewise differentiable dynamical system

$$x_1(t+1) = \begin{cases} 0 & \text{if } F_1(x_1(t),x_2(t)) \leq 0 \\ K_1 & \text{if } F_1(x_1(t),x_2(t)) \geq K_1 \\ F_1(x_1(t),x_2(t)) & \text{otherwise} \end{cases}$$

$$x_2(t+1) = \begin{cases} 0 & \text{if } F_2(x_1(t),x_2(t)) \leq 0 \\ K_1 & \text{if } F_2(x_1(t),x_2(t)) \geq K_2 \\ F_2(x_1(t),x_2(t)) & \text{otherwise} \end{cases} \tag{13.3}$$

where

$$F_1(x_1(t),x_2(t)) = x_1(t)\left[1+\gamma_1\left(x_1(t)R_1(x_1(t)) - x_2(t)\right)\right]$$
$$F_2(x_1(t),x_2(t)) = x_2(t)\left[1+\gamma_2\left(x_2(t)R_2(x_2(t)) - x_1(t)\right)\right] \tag{13.4}$$

The presence of the constraints gives rise to a piecewise differentiable dynamical system, i.e., the phase space of the dynamical system can be divided into disjoint regions where the dynamical system is smooth, whereas it is not differentiable along the boundaries that separate these regions. In the presence of piecewise smooth dynamical systems the adjustment process may reveal the occurrence of so-called *border-collision bifurcations*, which are related to the crossing of invariant sets through the borders of non differentiability. These bifurcations may cause sudden stability switches and/or the appearance/disappearance of periodic cycles or chaotic attractors (see [5, 6]). Moreover, (13.3) is a noninvertible (or

[1] If R_i is the maximum tolerated ratio of C_j to C_i individuals, then x_iR_i represents the absolute number of C_j individuals tolerated by C_i ones.

many-to-one) map, that is, if we solve the system (13.3) with respect to the variables $x_1(t)$ and $x_2(t)$ in terms of $x_1(t+1)$ and $x_2(t+1)$ we obtain several solutions, i.e. distinct points of the plane exist that are mapped by (13.3) into the same point. This property may have important consequences on the global dynamic behaviour of the model, see, e.g., [7, 8]. In this chapter we will not deal with such advanced global dynamic properties of (13.3), however the reader should be aware that these features are underlying many of the numerical results that will be illustrated in the following.

13.3. Equilibrium Points

The equilibrium points of the model (13.3), expressed by the steady state conditions $x_i(t+1) = x_i(t)$, $i = 1,2$, include the trivial equilibrium $E_0 = (0,0)$, the "segregation" (or "ghetto's") equilibria $E_1 = (K_1, 0)$ and $E_2 = (0, K_2)$, and some "coexistence "equilibria $E = (x_1^*, x_2^*)$ where $x_i^* > 0$, $i = 1,2$, which are either the solutions (if any) of

$$\begin{cases} x_2^* = x_1^* R_1(x_1^*) \\ x_1^* = x_2^* R_2(x_2^*) \\ x_1^* < K_1; x_2^* < K_2 \end{cases}$$

or $x_1^* = K_1$ and x_2^* solution (if any) of $x_2^* R_2(x_2^*) = K_1$, or $x_2^* = K_2$ and x_1^* solution (if any) of $x_1^* R_1(x_1^*) = K_2$, or, finally, $x_1^* = K_1$ and $x_2^* = K_2$ provided that $K_2R_2(K_2) > K_1$ and $K_1R_1(K_1) > K_2$.

Some situations, obtained with two linear tolerance functions (13.1) are illustrated in Figure 13.2.

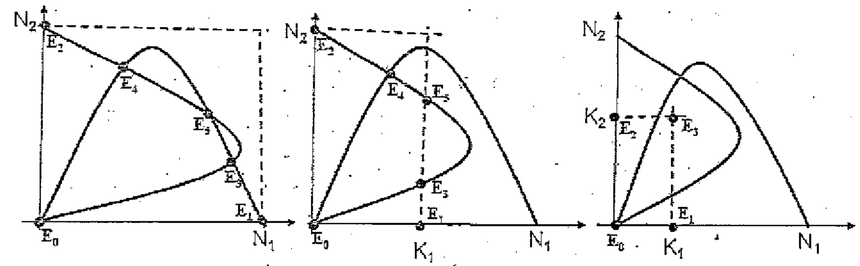

Figure 13.2. Coexisting equilibria (black circles) assuming linear tolerance distributions and different values of the parameters τ_i, $i = 1,2$, and constraints K_i, $i = 1,2$. (a) $K_1 = N_1$, $K_2 = N_2$ and curves $x_iR_i(x_i)$, $i = 1,2$, intersecting in three points; (b) $K_1 < N_1$ and $K_2 = N_2$; (c) $K_1 < K_2R_2(K_2)$ and $K_2 < K_1R_1(K_1)$.

Given the existence of so many equilibrium points, the first question to be investigated is their stability and, when several stable equilibrium points are present, the delimitation of their basins of attraction. A deeper analytic study of existence and stability of the equilibria is out of the goals of the present chapter, and will be given elsewhere. In the next sections, in order to investigate these questions, as well as the influence of the parameters of the model, we present the results of some numerical explorations obtained with different combinations of parameters.

13.4. Global Dynamics and Basins of Attraction: Numerical Explorations

To start our numerical explorations, we consider the model (13.3) with linear tolerance distributions (13.1) and two populations of the same size that, without loss of generality, we consider normalized to $N_1 = N_2 = 1$. We do not consider further limitations (i.e. $K_1 = K_2 = 1$) and set the maximum tolerances to $\tau_1 = 3$ and $\tau_2 = 3.5$ and the speeds of reaction to $\gamma_1 = 0.5$ and $\gamma_2 = 0.3$ for population of color C_1 and C_2 respectively. A simple numerical computation of the eigenvalues of the Jacobian matrix of the model (13.3) at the equilibrium points shows that the trivial equilibrium E_0 is a non-hyperbolic unstable equilibrium, whereas the two "segregation" equilibria $E_1 = (1, 0)$, characterized by all color C_1 individuals, and $E_2 = (0, 1)$, characterized by all color C_2 individuals, are stable. The unique "coexistence" equilibrium E_3, is located at the positive intersection of the two parabolas $x_1 = \tau_2 x_2 (1 - x_2/N_2)$ and $x_2 = \tau_1 x_1 (1 - x_1/N_1)$, and is a saddle point. Its stable set constitutes the boundary that separates the two basins of attraction $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$ represented in Figure 13.3a by the light grey and dark grey regions respectively. It is worth to notice that the basin of E_2 is more extended. This is related to the fact that $\tau_2 > \tau_1$: since the less tolerant population has a higher attitude to leave the system thus giving an easier path towards a "ghetto" completely occupied by the other one. Notice also that, in the situation illustrated in Figure 13.3a, the less tolerant population C_1 also has a higher speed of reaction, being $\gamma_1 > \gamma_2$, this emphasizes even more the segregation phenomenon. However, by increasing the tolerance of population C_1 from $\tau_1 = 3$ to $\tau_1 = 3.8$, due to the contact and crossing of the two parabolas, a saddle-node bifurcation occurs, and both a stable node E_5 and a saddle point E_4 are created (Figure 13.3b). The boundary of the basin of the stable "coexistence" equilibrium E_5, represented by the white region in Figure 13.3b, is now delimitated by the stable sets of two saddles — the already existing one E_3 and the newborn E_4. The condition of tangency between the two parabolas, as well as the exact coordinates of the equilibria located at their intersections, can be analytically computed by the Cardano's formulas for a third degree algebraic equation. Their expressions are quite cumbersome and not useful for our purposes; therefore they will be omitted. What is interesting for our analysis is the fact that by decreasing the difference between the parameters of tolerance of the two populations a stable equilibrium is created. There, the coexistence (or integration) of the two populations is possible, provided that the initial condition is taken inside the basin $\mathcal{B}(E_5)$, i.e. if the system starts with a sufficiently balanced initial mixture of the two populations. By contrast, starting from an initially unbalanced situation, i.e. a marked prevalence of one of the two populations so that the initial condition belongs to one of the two grey regions, the evolution is different. In this case, the endogenous long run evolution of the system will enhance the initial difference and finally move towards a completely segregated situation; we can observe convergence to a segregation equilibrium, either E_1 or E_2, according to the initial bias.

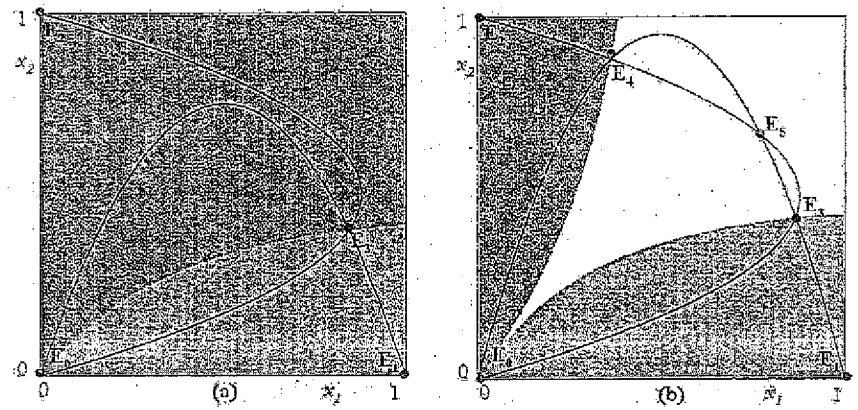

Figure 13.3. Equilibria and basins of attraction: the light grey regions represent the basin of E_1, the dark grey regions represent the basin of E_2; the white region represents the basin of the coexistence equilibrium E_5, for the following sets of parameters: (a) $N_1 = N_2 = 1$, $\gamma_1 = 0.5$, $\gamma_2 = 0.3$, $\tau_1 = 3$, $\tau_2 = 3.5$, $K_1 = 1$, $K_2 = 1$; (b) Same values as in (a) except $\tau_1 = 3.8$.

In the dynamic situations described above, the eigenvalues of the Jacobian matrix computed in the equilibrium points are real and positive, and only monotonic motions are observed. However, situations of oscillatory convergence towards to the coexistence equilibrium E_5 can be easily observed by increasing the speeds of reactions γ_1 and/or γ_2. In fact, high reactivity may imply overshooting effects, which are quite common in the presence of emotional human decisions. As stressed by [9, ch. 3, p. 86] "The phenomenon of overshooting is a familiar one at the level of individual" and consequently "Numerous social phenomena display cyclical behavior". The coexistence equilibrium E_5 may even be destabilized via a flip (or period doubling) bifurcation as the speeds of reaction increase. This is illustrated in Figure 13.4a, with parameters $\tau_1 = 3.3$, $\tau_2 = 3.5$ and high speeds of reaction $\gamma_1 = \gamma_2 = 1.2$; there E_5 has become a saddle point after a flip bifurcation at which a stable periodic cycle of period two — represented by the periodic points p_1 and p_2 in Figure 13.4a — is created. In this case, coexistence can be obtained, even if the coexistence equilibrium is unstable, being characterized by oscillations in the number of the individuals of the two integrated populations. If either speed of reaction increases further, the first flip bifurcation is followed by other ones, and the usual period-doubling cascade leads to the creation of chaotic attractors around E_5. Another remarkable global dynamic property that can be seen in Figure 13.4a is the existence of non connected portions of the basins $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$ located in the portion of the phase plane where both the populations are almost entirely included in the system considered. From the point of view of the mathematical properties of discrete dynamical systems, this occurrence can only be observed when the dynamic model is obtained by the iteration of a noninvertible (or many-to-one) map (see, e.g., [7, 8]). Indeed, the points of the non connected portion of $\mathcal{B}(E_1)$ are mapped into the bigger portion of the basin around E_1 (also called immediate basin of E_1) in one step, due to the folding properties of a noninvertible map. In other words, the points of the non connected portion are rank-1 preimages of points located inside the immediate basin. In terms of the adaptive behavior considered in the model, this property represents an extreme form of overshooting, as it is due to an over-reaction of many members of a population due to the excessive presence of the other population. This massive migration of members of population C_2 out of the system leads to the final dominance of the population C_1 even if the starting situation was characterized by a slight prevalence of C_2. This is due to an excessive reactivity of the C_2 individuals to the strong presence of C_1 individuals, even if the C_2 population initially represents the majority. Similar arguments explain the presence of the non connected portion of $\mathcal{B}(E_2)$ in the upper right portion of the state space.

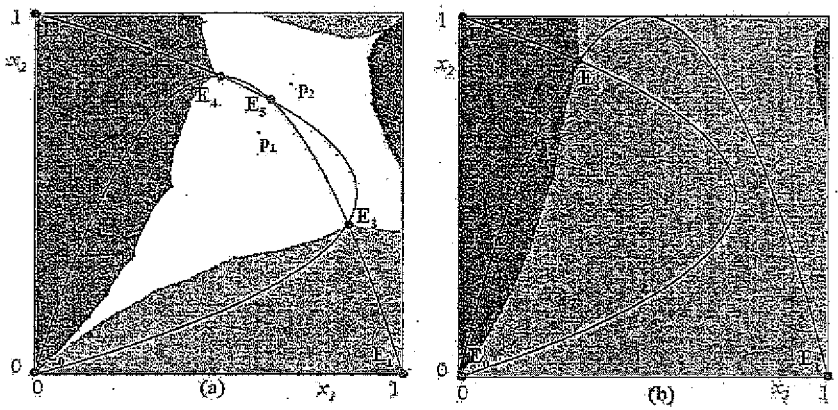

Figure 13.4. Equilibria and basins of attraction for the following sets of parameters: (a) $N_1 = N_2 = 1$, $\gamma_1 = \gamma_2 = 1.2$, $\tau_1 = 3.3$, $\tau_2 = 3.5$, $K_1 = 1$, $K_2 = 1$; (b) Same values as in (a) except $\tau_1 = 4$ and $\tau_2 = 3$. The meaning of the colors is the same as in Figure 13.3.

If the difference between the tolerances of the two populations is slightly increased, any possibility of stable coexistence, both stationary and oscillatory, is ruled out, as shown in Figure 13.4b. This is obtained with the same parameters as Figure 13.4a, but the two parameters of tolerance are now fixed at $\tau_1 = 4$ and $\tau_2 = 3$; in this case all the attractors interior to the state space disappear, and only the two segregation equilibria E_1 and E_2 are stable. Furthermore, the basin of E_1 is much more extended because of the higher tolerance that characterizes the population of color C_1. However, due to the high value of speeds of reaction, a nonconnected portion of $\mathcal{B}(E_2)$ persists in the upper-right portion of the phase space, because of the excessive reactivity of C_1 population with respect to the presence of C_2 individuals, even if C_1 population represents the majority inside the system.

We now consider what happens if one population is more numerous, $N_i > N_j$, i.e. color C_j represents a "minority" population. For example, the situation shown in Figure 13.5a is obtained with $N_1 = 1$ and $N_2 = 0.5$, identical reactivity ($\gamma_1 = \gamma_2 = 1$) and tolerance ($\tau_1 = \tau_2 = 3$) for both populations. If there are no further restrictions (i.e. $K_i = N_i$, $i = 1, 2$) then only the segregation equilibria are stable, with a much stronger probability of convergence to the equilibrium E_1, being the basin $\mathcal{B}(E_1)$ more extended than $\mathcal{B}(E_2)$. The possibility of stable coexistence is obtained by increasing the tolerance of the minority population, as in Figure 13.5b where $\tau_1 = 2$ and $\tau_2 = 8$. In this case, a stable coexistence equilibrium exists, with a quite extended basin of attraction. This can be stated by saying that a stable coexistence between a majority and a minority can be obtained only if the minority is more tolerant.

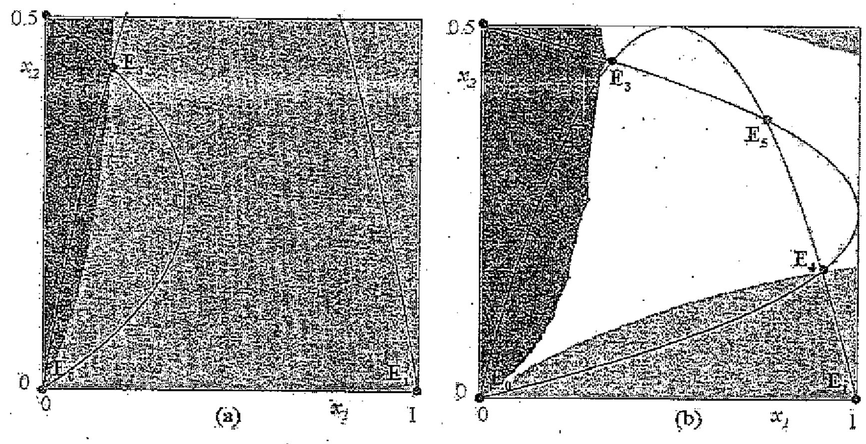

Figure 13.5. Equilibria and basins of attraction for the following sets of parameters: (a) $N_1 = 1$, $N_2 = 0.5$, $\gamma_1 = \gamma_2 = 1$, $\tau_1 = \tau_2 = 3$, $K_1 = 1$, $K_2 = 0.5$; (b) Same values as in (a) except $\tau_1 = 2$ and $\tau_2 = 8$. The meaning of the colors is the same as in the previous pictures.

Let us finally consider the effects of introducing constraints on the maximum number of individuals of one or both the populations allowed to enter the system, i.e. $K_1 < N_1$ and/or $K_2 < N_2$. As stressed in the previous section, this may be considered as a kind of exogenous control in order to force the system to converge to a programmed equilibrium. The effect of such a kind of control is here investigated by a numerical exploration starting from a given combination of parameters without any imposed constraints, for example $N_1 = N_2 = 1$, $\gamma_1 = \gamma_2 = 1$, $\tau_1 = 4$, $\tau_2 = 2$, $K_1 = K_2 = 1$ as in Figure 13.6a, and then gradually reducing K_1 respectively to $K_1 = 0.6$ (Figure 13.6b), $K_1 = 0.4$ (Figure 13.6c), and $K_1 = 0.2$ (Figure 13.6d). It can be noticed that as K_1 decreases, stable coexistence is possible only at intermediate levels (see Figure 13.6c). By contrast, for high values of K_1 only segregation is possible — with the segregation equilibrium E_1 prevailing (figures 13.6a,b). Finally, at low levels of K_1 only segregation can be obtained with equilibrium E_2 that dominates. This confirms that an exogenously regulated maximum number of allowed entrance can be a useful way to control the system. The introduction of constraints is also interesting from a mathematical point of view, because the bifurcations at which equilibrium points are created or destroyed are not standard bifurcations of differentiable dynamical systems, but can be characterized as border collision bifurcations, i.e. bifurcations related to the contacts between attractors and curves where the dynamical system is not differentiable, see, e.g., [5, 6].

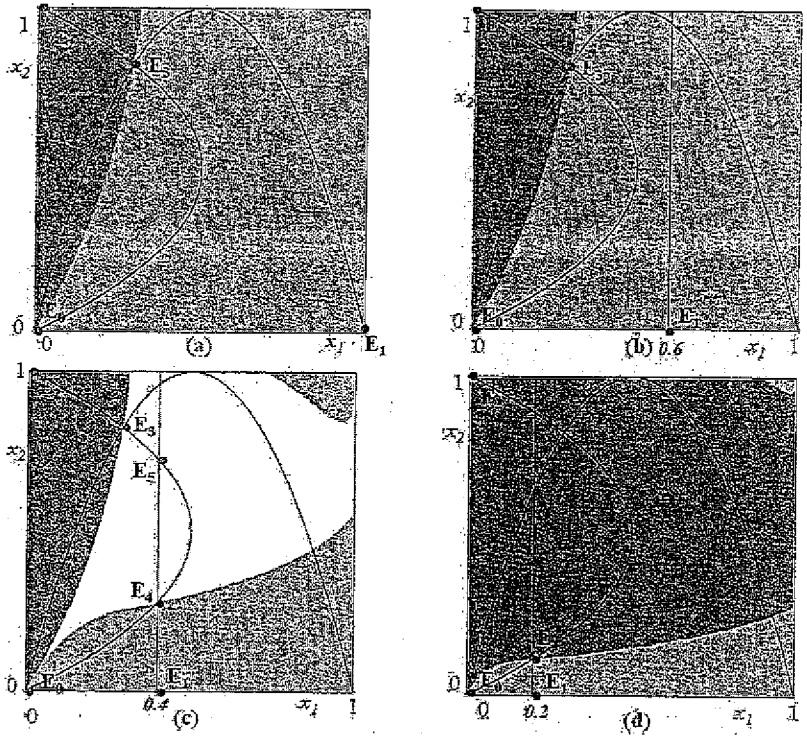

Figure 13.6. Effects of constraints. Equilibria and basins of attraction for the following sets of parameters: (a) $N_1 = N_2 = 1$, $\gamma_1 = \gamma_2 = 1$, $\tau_1 = 4$, $\tau_2 = 2$, $K_1 = K_2 = 1$; (b) Same values as in (a) except $K_1 = 0.6$; (c) Same values as in (a) except $K_1 = 0.4$; (d) Same values as in (a) except $K_1 = 0.2$. The meaning of the colors is the same as in the previous pictures.

13.5. An Alternative Tolerance Distribution

In this section we introduce a different distribution of tolerance for population C_2, given by

$$R_2(x_2) = \frac{\tau_2 (N_2 - x_2)}{x_2} \tag{13.5}$$

and characterized by the property that a positive fraction of the population C_2 that tolerates any ratio of different colored individuals always exists (see Figure 13.7a). The corresponding function $x_2 R_2(x_2) = \tau_2 (N_2 - x_2)$ is a straight line, as represented in Figure 13.7b

together with the function $x_1 R_1(x_1)$ associated with linear distribution of tolerance (13.1). The fact that the function $x_1 = x_2 R_2(x_2)$ does not vanish for $x_2 = 0$, i.e. has a positive intercept, is different from the Schelling original assumption (13.1) which implies that both curves cross at the point $(0, 0)$. This new assumption is supported by empirical data, as shown by [10], for black populations living in American cities (Figure 13.7c). We now wonder what are the effects induced by the introduction of one distribution of tolerance of the form (13.5) in the model (13.3).

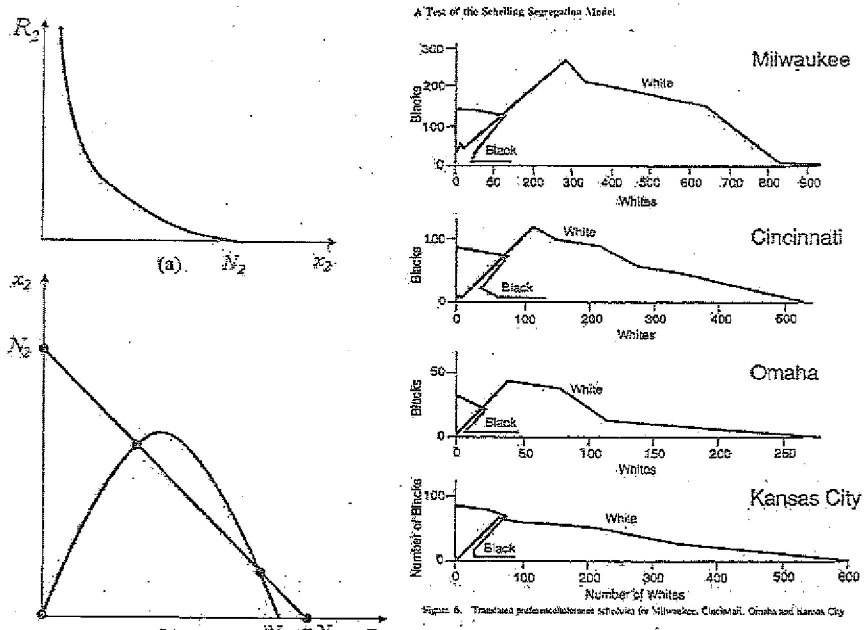

Figure 13.7. (a) The hyperbolic distribution of tolerance for population of color C_2; (b) the functions $x_1 R_1(x_1)$ assuming linear tolerance distribution for C_1 and $x_2 R_2(x_2)$ assuming the hyperbolic distribution for C_2; (c) empirical curves $x_i R_i(x_i)$ taken from [10].

In this case at most two coexistence equilibria can be obtained; they can be easily computed as the intersections between the straight line $x_1 = \tau_2 (N_2 - x_2)$ and the parabola $x_2 = \tau_1 x_1 (1 - x_1/N_1)$ (see Figure 13.7b). This gives rise to new dynamic situations that allow us to get interesting real life interpretations. Indeed, starting from the dynamic situation shown in Figure 13.8a, obtained with distributions of tolerance R_1 in the form (13.1) and R_2 in the form (13.5) respectively and parameters $N_1 = 1$, $N_2 = 0.8$, $\gamma_1 = 0.4$, $\gamma_2 = 0.5$, $\tau_1 = 2$, $\tau_2 = 1$, $K_1 = 1$, $K_2 = 0.8$, the usual bi-stability situation, with the two segregation equilibria E_1 and E_2, is obtained. However, as the two tolerance parameters increase, a new situation is obtained. This is illustrated in Figure 13.8b where $\tau_1 = 3$, $\tau_2 = 2$ and the other parameters are the same as in Figure 13.8a. Here a new positive coexistence equilibrium exists and is stable; it is created via a transcritical (or "stability exchange") bifurcation at the merging with the segregation equilibrium E_1 that, as a consequence, becomes unstable. This is different from the previous cases: here, only the segregation equilibrium E_2 is stable, whereas the other one, E_1, is unstable. This means that in such a situation the population of color C_2 will never disappear: it may either become the only one (if the system converges to E_2) or coexist with population of color C_1 (if the system converges to E_4) but it cannot disappear. This is possible since, due to assumption (13.5), some individuals of C_2 population can tolerate any ratio x_1/x_2, hence they never leave the system. Consequently, the equilibrium with $x_2^* = 0$ is not stable, i.e. it cannot prevail in the long run.

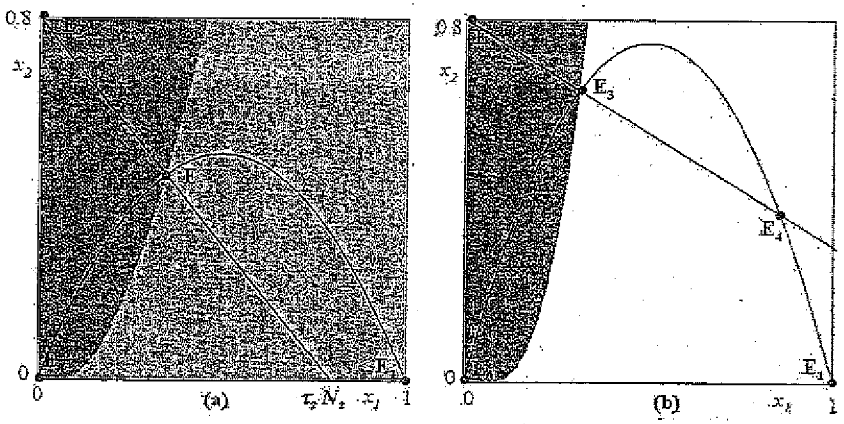

Figure 13.8. Equilibria and basins of attraction for the following sets of parameters: (a) $N_1 = 1, N_2 = 0.8, \gamma_1 = 0.4, \gamma_2 = 0.5, \tau_1 = 2, \tau_2 = 1, K_1 = 1, K_2 = 0.8$; (b) Same values as in (a) except $\tau_1 = 3$ and $\tau_2 = 2$. The meaning of the colors is the same as in the previous pictures.

13.6. Conclusion

We have proposed a simple two-dimensional discrete-time dynamical system to model the long run outcomes emerging from repeated individual choices to enter or not a given system (a city district, or an association, or a political party), taken by agents belonging to two different populations, according to their tolerance about the presence of agents of the other population.

The explicit dynamic model we have proposed, even if it is based on a simple adaptive mechanism, allowed us to get some useful information about the effects of some parameters that characterize the two populations, such as their maximum tolerance or their reactivity (or their inertia) in taking decisions as a consequence of the observed proportions of individuals belonging to different populations.

Adaptive dynamic models constitute useful equilibrium selection devices when several equilibria (or other kind of invariant sets) are present. Indeed, the explicit adaptive dynamic adjustment, we have proposed in this chapter, allowed us to investigate some global dynamical properties, in particular the structure of the basins of attraction when several attractors coexist, thus giving indications on the long-run outcome of path-dependent evolutions through which different emerging collective behaviors can emerge from repeated (or step by step) myopic individual decisions.

By properly tuning the values of the parameters, we observed two different kinds of results. Some were both intuitive and expected; others, more interestingly, were somehow counterintuitive. Both kinds of results are important: the former confirm the proper setup of the model; the latter illustrate the overshooting effects due to impulsive (or emotional) behavior of the agents. The presence of overshooting should not be seen as an artificial effect or a distortion of reality due to discrete time scale we have considered. Instead, as stressed in [9], overshooting and over-reaction arise quite naturally in social systems, due to emotional attitude, impulsivity or lack of information.

Such phenomena are mathematically expressed in terms of oscillatory behavior, and extreme forms are also related with the peculiar properties of noninvertible iterated maps, whose geometric action consists in folding and pleating the state space, so that distinct points are mapped into the same point. This folding action is also caused by the presence of imposed constraints, see, e.g., [11].

The approach followed in this chapter is mainly numerical, based on computer assisted explorations of dynamic scenarios obtained by using different combination of parameters, properly chosen in order to illustrate the main dynamic features of the model and their real life interpretations. However, the discrete dynamical system proposed, being represented by a two-dimensional piecewise differentiable and noninvertible iterated map, with several equilibria as well as more complicated attractors, is worth to be investigated more deeply by geometric and analytic methods. Because of the analytic difficulties, as the map is represented by a high degree polynomial and is piecewise-differentiable, due to the presence of constraints, such a study goes far beyond the goals of the present chapter, and will be approached in a future work.

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