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Learning the demand function in a repeated Cournot oligopoly game

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In this article, single product Cournot oligopolies are considered, where the demand and cost functions are linear. While cost functions are completely known by all firms, they only partially know the demand function, as they misspecify the slope. At any stage of the repeated oligopoly game firms update the slope of their subjective demand functions on the basis of the discrepancy they observe between the expected price, computed according to believed demand, and the price they actually observe. This adjustment process has a unique steady state, where any subjective demand function coincides with the true demand function. If such steady state is stable, then the true slope of the demand function can be learned by all oligopolists, even if they start from misspecified initial guesses. Sufficient conditions for the stability of the steady state are given for n-firms oligopolies. In the particular case of a duopoly, an exact delimitation of the stability region in the parameters' space is given, and with the help of numerical simulations, the size and the shape of the basins of attraction is analysed, as well as the kinds of attracting sets that characterise the long-run dynamics of the learning process when the steady state is unstable.

Keywords: Oligopoly game; Heterogeneity; Dynamical systems; Stability; Bifurcations

What we have to learn to do, we learn by doing
Aristotle, Nocomacheam Ethics, Book II

1. Introduction

Oligopoly theory is one of the most frequently discussed areas in the literature of mathematical economics. This field dates back to Cournot (1838), and since then many researchers have devoted their efforts to the different variants of the Cournot model. The main results on the existence and stability of equilibria in Cournot oligopoly games are summarised in Okuguchi (1976), Okuguchi and Szidarovszky (1999), where the most relevant references are also given.

A common assumption in game-theoretic models is that each player has a perfect knowledge of the payoffs of the game and is able to correctly forecast the choices of the other players. However, in the literature on oligopoly games, these assumptions of full rationality have been weakened in several ways. First of all, firms have been considered unable to predict their competitors' decisions. So, at each repetition of the game, they are assumed to form expectations about the next period decisions of the competitors, and they base their own decisions upon such expectations. This problem was already present in the model proposed by Cournot (1838), where he assumed that players use the outputs observed in the current period as expected outputs for the next period (the so called naive expectations). Other authors proposed adaptive expectations as a more general forecasting rule (see, e.g., Okuguchi 1976, Okuguchi and Szidarovszky 1999, Szidarovszky and Okuguchi 1988, Bischi and Kopel 2001).

However, firms' information sets may be incomplete on several accounts. As stressed by Kirman (1975), firms are, in general, imperfectly aware of their environment, so they may have an imperfect knowledge of the payoffs (i.e., the expected profits) of the oligopoly

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game they are playing. For example, players may misspecify the estimates of the demand function. However, under suitable assumptions regarding the information sets available to the firms, they may realise that their view of the economic system is imperfect, since the economic data observed will not always coincide with their predictions. This will give rise to an adjustment process, that is, firms will update their believed demand functions according to the observed data when the game is played repeatedly. In other words, firms try to learn the game they are playing. This learning process was introduced and analysed in Szidarovszky (2003), where local and global stability analysis was performed in continuous time scale.

Following this stream of thought, in this article we propose a n-firm single product Cournot oligopoly where the demand and cost functions are linear. However, while cost functions are completely known by all firms, they only partially know the demand function, as they misspecify the slope. In other words, even if firms have a correct understanding of the structure of the economic system of which they are part, they have an incorrect wrong estimate of some parameters. Moreover, we assume heterogeneity in the estimates, i.e., different firms may have different beliefs about the slope of the demand function. At each repetition of the game, by using its believed demand function each firm computes its own profit maximising production, as well as those of its competitors. The sum of all these production quantities gives the total supply expected by the single firm and this allows it to compute its expected next period price. While firms will never learn how much the actual total supply is, they do observe that the expected price does not coincide with the actual market price. This will make players aware that the demand function they are using may be misspecified. In light of these observations, a reasonable response for the firms would then be to update the slope of the believed demand, through an adaptive mechanism, at each stage of the repeated game.

In other words, when players perceive that their game (subjective game) is different from the real (objective) one they correct their subjective estimates of the demand function in order to learn the true game they are playing. The question is: can boundedly rational players who myopically play the game repeatedly learn about the real (objective) elements of the game starting from perceived (subjective) misspecified elements?

For the oligopoly game proposed in this article, we show that the learning process always has a unique steady state, corresponding to the situation where all the believed demand functions coincide with the true market demand. If the adjustment process converges at such unique steady state, then we say that all the firms *learn* the true demand, even if they started from misspecified (and different) initial guesses about the slope of the demand function. We give conditions for the stability of the steady state, that is, we identify the sets of parameters which ensure the convergence of the adjustment process.

However, the adjustment process proposed is not always convergent, and we examine some bifurcations that lead to instability of the steady state, and what kind of disequilibrium dynamics may occur when the learning process does not converge. These results show that, in general, excessive reactivity of the firms, as well as an excessive cost heterogeneity, increases the likelihood of failing convergence at the steady state. The first statement is not surprising, as stability in adaptive systems is often lost due to over-reaction by the agents. The second statement may deserve some economic reflections. A more detailed analysis of the duopoly case will show that some counterintuitive effects can be observed if the reactivity of firms is gradually increased in the presence of strong heterogeneity. Moreover, the analysis of the particular case of n identical firms shows that if the number of firms is increased, the likelihood of convergence also increases.

It is worth taking a brief look at some related models in the literature. The effect of imperfect knowledge of the market demand function has been examined in Cyert and DeGroot (1971, 1973) by using Bayesian methodology in duopolies. Kirman (1975, 1983) examined differentiated products and linear demand functions, what the firms try to estimate, and investigated the convergence of the resulting process. Gates *et al.* (1982) introduced an economically guided learning process. All previously discussed learning schemes used either least squares learning or Bayesian updating of the unknown parameter, or considered the outputs of the competitors as random variables and introduced a similar learning process for estimating it. In the model introduced in this article we assume that the firms know their own cost functions and those of the competitors. It is a realistic assumption in real economies, since firms are usually aware of each other's technology. Firms can also observe the market price without the output of the competitors. Price observations indicate the values of the price function at unknown total production levels. Therefore, these observations cannot be considered as independent observations from the same statistical family. For this reason we did not select Bayesian, least squares, or other statistics-based optimal learning methods. Okuguchi and Szidarovszky (1990) discussed the asymptotic properties of dynamic oligopolies with perceived marginal costs. Léonard and Nishimura (1999) examined discrete dynamic duopolies and illustrated how the asymptotic properties of the steady states change as the result of the incorrect assessment of the demand function, the misspecification due to a multiplicative scale factor. Chiarella and Szidarovszky (2001) analysed the continuous time-scale version of Léonard–Nishimura model and investigated how the asymptotic properties are further altered by time delays in obtaining and implementing information on the output of the rivals. They show that (under concavity assumptions) the resulting dynamical system converges towards a steady state, which in general differs from the full information equilibrium. This model has been further generalised in Chiarella and Szidarovszky (2003), in which firms may also misspecify the shape of the demand function and not only its scale as assumed in the original model of Léonard and Nishimura. Léonard and Nishimura's discrete time model has also been extended in Bischi *et al*. (2003b), where a duopoly model is proposed for situations in which the players lack knowledge of the market demand, and cost externalities between the firms are allowed. So, differently from the Léonard and Nishimura model, the assumption of decreasing reaction functions is relaxed. This implies that the presence of demand misspecification à la Léonard and Nishimura may imply that new steady states are created, when (one or both) players over- or underestimate the demand. However, in these papers no learning occurs on the subjective (misspecified) demand function. Indeed, in Léonard and Nishimura (1999) and the above quoted extensions, it is assumed that the information set available during the repeated game is not sufficient to reveal to the players that they are using an incorrect demand function. In fact, these works are based on the assumption that firms do not know the cost functions of their competitors and therefore are not able to compute or estimate the output decisions of their competitors. This implies that they are not able to estimate the whole quantity sold in the market. So, the price they observe does not provide sufficient information to them that they are using a misspecified demand function.

In the model proposed in this article, the observed discrepancy not only allows the players to be aware that they are using an incorrect demand function, but also give them the possibility to correct, adaptively, the believed demand.

Adaptive learning ideas go back several decades, and first were used to form predictions about the simultaneous outputs of the competitors (Fisher 1961). Okuguchi (1969, 1970) has examined the effect of adaptive expectations on the stability of dynamic oligopolies. A comprehensive summary of early results are given in Okuguchi (1976).

This article is organised as follows. In section 2, the dynamic learning model is introduced. The existence and uniqueness of the steady state is examined in section 3, and the local stability analysis is given in section 4. Sections 5 and 6 examine particular cases where a global analysis of the model is possible, because the dimension of the model is reduced. In fact, section 5 deals with the one-dimensional model obtained under the assumption of n identical players, and section 6 analyses the two-dimensional model that simulates the case of duopoly with heterogeneous players. Some concluding remarks follow in section 7.

2. The Cournot oligopoly with learning on demand

We consider an n-firm single product oligopoly, without product differentiation. Let $C_k(x_k)=\alpha_k x_k+\beta_k$, with $\alpha_k > 0$ and $\beta_k \geq 0$, be the *cost function* of firm k, where x_k is its output. We assume that all cost functions are known by all firms, i.e., each firm knows not only its own cost function, but also the cost functions of the competitors (i.e., the production technologies that they adopt, their salaries etc.). Let $Q=\sum_{k=1}^{n} x_k$ be the *total production* of the good considered. We assume that the price p that prevails in the market is determined by the *linear inverse demand (or price) function*

$$p=f(Q)=B-AQ \tag{1}$$

where the positive parameter B indicates the price when the production of the good is zero, and the positive parameter A represents the slope of the inverse demand function.

We assume that this demand function is not fully known by the firms. More precisely, we assume that the firms know that the demand function is linear and decreasing, but they have only a misspecified estimate of the slope. In other words, firms are assumed to know the value of B/A, i.e., the total output level that makes the price zero, but each firm selects a *subjective "reference" price function* $p=u_k f(Q)=u_k(B-AQ)$, $u_k>0$, that is, a function with a slope that in general is different from the one of the true price function (1). As we shall see below, the information set of the firms allows them to realize that the demand functions they are using may be misspecified, and they guess that the true price function is a constant multiple of the subjective reference price function. In other words, starting from a subjective guess given by $p=u_k(B-AQ)$, at each time t firm k will use the price function

$$f_k(t)=\frac{u_k}{\varepsilon_k(t)}(B-AQ) \tag{2}$$

where $\varepsilon_k(t)$ is a *firm-selected scale factor* that they will adjust over time, on the basis of the observed data, in order to obtain a better approximation of the true demand. More precisely, as we shall describe in more detail below, at each time period each firm will adjust its believed scale factor ε_k on the basis of the discrepancy between their computation of the expected price and the realised price they observe in the market.

We now describe, step by step, how this adaptive learning process is developed. First, we will describe the output selection and price expectation of a particular firm k that behaves, at each repetition of the game, as a Cournot oligopolist, i.e., a profit maximising quantity setting producer. The believed (or expected) profit† of firm k is

$$\pi_k^e = x_k \frac{u_k}{\varepsilon_k}(B - Ax_k - AQ_{-k}) - (\alpha_k x_k + \beta_k) \qquad (3)$$

where $Q_{-k} = \sum_{l \neq k} x_l$ is the aggregated output of the competitors of firm k, so that $Q = x_k + Q_{-k}$. Assuming interior optimum, the first-order condition for the maximization of the expected profit gives

$$\frac{u_k}{\varepsilon_k}(B - Ax_k - AQ_{-k}) - x_k \frac{u_k}{\varepsilon_k} A - \alpha_k = 0$$

that is

$$x_k = \frac{u_k B - \alpha_k \varepsilon_k}{u_k A} - Q. \qquad (4)$$

Firm k also believes that the optimal choice of any other firm l is given, similarly, as

$$x_l = \frac{u_k B - \alpha_l \varepsilon_k}{u_k A} - Q \qquad (5)$$

i.e., firm k also computes the expected x_l by using its own believed demand function (2). Of course, this computation is possible only if firm k knows the cost function of any other firm l. Therefore, firm k computes its believed equilibrium output of the industry, say $Q^{(k)}$, by adding equations (4) and (5):

$$Q^{(k)} = \frac{nB}{A} - \frac{\varepsilon_k}{u_k A} \sum_i \alpha_i - nQ^{(k)}$$

so that

$$Q^{(k)} = \frac{nB}{(n+1)A} - \frac{\varepsilon_k}{(n+1)u_k A} \sum_i \alpha_i. \qquad (6)$$

Based on this belief, firm k computes its expected equilibrium price by using its own believed price function (2):

$$\begin{aligned} p^{(k)} &= f_k(Q^{(k)}) = \frac{u_k}{\varepsilon_k}\left(B - AQ^{(k)}\right) \\ &= \frac{u_k}{\varepsilon_k}\left(B - \frac{nB}{n+1} + \frac{\varepsilon_k}{(n+1)u_k} \sum_i \alpha_i\right) \\ &= \frac{u_k B + \varepsilon_k \sum_i \alpha_i}{(n+1)\varepsilon_k} \end{aligned} \qquad (7)$$

and, according to (4), it selects its output as

$$\begin{aligned} x_k &= \frac{u_k B - \alpha_k \varepsilon_k}{u_k A} - Q^{(k)} \\ &= \frac{Bu_k - (n+1)\alpha_k \varepsilon_k + \varepsilon_k \sum_i \alpha_i}{(n+1)u_k A}. \end{aligned} \qquad (8)$$

In reality, each firm reasons in the same way, so the actual outputs of each firm k is given by equation (8). By adding all the x_k, computed according to (8), we can obtain the actual total output of the industry:

$$\begin{aligned} Q = \sum_{k=1}^{n} x_k &= \frac{nB}{(n+1)A} - \frac{1}{A} \sum_i \frac{\alpha_i \varepsilon_i}{u_i} \\ &+ \frac{1}{(n+1)A}\left(\sum_i \alpha_i\right)\left(\sum_j \frac{\varepsilon_j}{u_j}\right). \end{aligned} \qquad (9)$$

So, the equilibrium price that prevails in the market is:

$$\begin{aligned} p &= f(Q) = B - AQ \\ &= \frac{B}{n+1} - \frac{1}{n+1}\left(\sum_i \alpha_i\right)\left(\sum_j \frac{\varepsilon_j}{u_j}\right) + \sum_i \frac{\alpha_i \varepsilon_i}{u_i}. \end{aligned} \qquad (10)$$

Of course, a discrepancy between the expected price (7) and the realised price (10) is observed by the firms. This discrepancy shows the firms that their computation of the expected price is based on a misspecified description of the economic environment they are modeling, and it provides them with a method of improving their estimate of the slope of the demand function. The learning process works in the following way. At each time period, each firm computes its subjective expected price (7), and receives the realised equilibrium price (10). The difference $\Delta_k = p^{(k)} - p$ can be used by firm k to adjust its price assessment by altering the value of the multiplier ε_k. If $\Delta_k > 0$, then the expected price is too high, so firm k wants to decrease its

†Believed means that it is computed by firm k according to the believed demand function (2).

price assessment by increasing the value of ε_k; if $\Delta_k < 0$, then the expected price is too low, so firm k wants to increase its price estimate by decreasing the value of ε_k; if $\Delta_k = 0$, then there is no need to change the price estimate. This adjustment process, in discrete time scale, can be conveniently modeled by assuming that, at each time period, each firm k computes its new ε_k value by adding a positive multiple of Δ_k to the previous ε_k value:

$$\varepsilon_k(t+1) = \varepsilon_k(t) + K_k \Delta_k . \tag{11}$$

This gives rise to the following system of n nonlinear difference equations:

$$\varepsilon_k(t+1) = \varepsilon_k(t) + K_k \left[\frac{B}{n+1} \left(\frac{u_k}{\varepsilon_k(t)} - 1 \right) + \frac{\sum_i \alpha_i}{n+1} + \sum_l \frac{\varepsilon_l(t)}{(n+1)u_l} \left(\sum_i \alpha_i - (n+1)\alpha_l \right) \right] \tag{12}$$

for $k = 1, 2, \ldots, n$, where the parameter K_k denotes the "adjustment speed" of firm k: smaller values of K_k imply an higher inertia of firm k in revising the scale factor ε_k, i.e., a stronger anchoring attitude. This discrete-time n-dimensional dynamical system expresses the adjustment process by which the n firms update, at any repetition of the oligopoly game, the scale factors ε_k, $k = 1, \ldots, n$ of their believed demand functions. We are interested in studying the long-run behaviour of this dynamical system, in order to see if the time evolution of the scale factors is such that the firms are able to learn, during the repeated game, the true demand function. Indeed, in the next section we shall prove that this occurs whenever the long-run evolution of the dynamical system (12) converges to a steady state.

3. Existence and uniqueness of the steady state

In this section, we prove the theorem that essentially states that the dynamical system (12) indeed represents a learning process, since the system is in a steady state if and only if all the believed demand functions f_k coincide with the true demand function f:

Theorem 1: *The discrete time n-dimensional non-linear dynamical system* (12) *has a unique steady state given by* $\bar{\varepsilon}_k = u_k$ *for* $k = 1, 2, \ldots, n$.

Before giving the proof of Theorem 1, let us remark that the unique steady state of (12) corresponds to the perfect knowledge of the demand, because for $\varepsilon_k = u_k$ we get $f_k(Q) = f(Q)$ for all k.

Proof: Let $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_n$ be a steady state, i.e., the expression inside the square bracket in each of the difference equations in (12) vanishes. Notice that in the bracketed expression only the first term depends on k, hence if all such expressions vanish, then

$$\frac{u_1}{\bar{\varepsilon}_1} = \frac{u_2}{\bar{\varepsilon}_2} = \cdots = \frac{u_n}{\bar{\varepsilon}_n} . \tag{13}$$

Let v denote this common value. Of course, being both u_k and $\bar{\varepsilon}_k$ positive, $v > 0$. The bracketed expression can be rewritten as

$$\frac{B}{n+1}(v-1) + \frac{\sum_i \alpha_i}{n+1} + \sum_l \frac{\sum_i \alpha_i - (n+1)\alpha_l}{(n+1)v} = \frac{B}{n+1}(v-1) + \frac{\sum_i \alpha_i}{n+1}\left(1 - \frac{1}{v}\right).$$

If $v > 1$, then both terms are positive, if $0 < v < 1$, then both terms are negative, and if $v = 1$, then both terms are equal zero. Hence, this expression is zero if and only if $v = 1$, that is, $\bar{\varepsilon}_k = u_k$. □

Theorem 1 states the existence and uniqueness of the steady state, but it doesn't say anything about its stability. However, we need the asymptotic stability of the steady state in order to guarantee that the firms have the possibility to learn the true demand by repeatedly playing the game, i.e., as the long-run outcome of an endogenous adjustment process.

4. Local stability analysis

In order to study the local stability of the unique steady state, we will use the standard linearization method, based on the localization of the eigenvalues of the Jacobian matrix computed at the fixed point of (12). We notice, first, that the derivative of Δ_k with respect to ε_k is:

$$D_{\epsilon_k}(\Delta_k) = \frac{-Bu_k}{(n+1)\varepsilon_k^2} + \frac{\sum_i \alpha_i - (n+1)\alpha_k}{(n+1)u_k}$$

and, for $l \neq k$, the derivative of Δ_k with respect to ε_l is:

$$D_{\epsilon_l}(\Delta_k) = \frac{\sum_i \alpha_i - (n+1)\alpha_l}{(n+1)u_l}.$$

Therefore at the steady state, where $u_k/\varepsilon_k = 1$, the Jacobian of the dynamical system (12) has the special form

$$\underline{J} = \underline{I} + \underline{K}\left[\begin{pmatrix} -A_1 & & & \\ & -A_2 & & \\ & & \ddots & \\ & & & -A_n \end{pmatrix} + \begin{pmatrix} B_1 & B_2 & \cdots & B_n \\ B_1 & B_2 & \cdots & B_n \\ \vdots & \vdots & \ddots & \vdots \\ B_1 & B_2 & \dots & B_n \end{pmatrix}\right] \tag{14}$$

where $\underline{I}$ is the $n \times n$ identity matrix, $\underline{K} = \text{diag}(K_1, K_2, \dots, K_n)$, and for $k = 1, 2, \dots, n$,

$$A_k = \frac{B}{(n+1)u_k}, \qquad B_k = \frac{\sum_i \alpha_i - (n+1)\alpha_k}{(n+1)u_k}.$$

By introducing the notation $d_i = 1 - K_i A_i$ $(1 \le i \le n)$, $\underline{D} = \text{diag}(d_1, d_2, \dots, d_n)$,

$$\underline{k} = \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_n \end{pmatrix} \quad \text{and} \quad \underline{b}^T = (B_1, B_2, \dots, B_n)$$

it is easy to see that

$$\underline{J} = \underline{D} + \underline{k}\,\underline{b}^T \tag{15}$$

so the characteristic polynomial of the Jacobian is the following:

$$\begin{aligned} \varphi(\lambda) &= \det(\underline{D} + \underline{k}\,\underline{b}^T - \lambda\underline{I}) \\ &= \det(\underline{D} - \lambda\underline{I}) \det(\underline{I} + (\underline{D} - \lambda\underline{I})^{-1}\underline{k}\,\underline{b}^T) \\ &= \prod_{i=1}^{n} (d_i - \lambda)\left[1 + \underline{b}^T(\underline{D} - \lambda\underline{I})^{-1}\underline{k}\right] \end{aligned}$$

where we used the fact that for any n-element column vectors $\underline{u}$ and $\underline{v}$, $\det(\underline{I} + \underline{u}\,\underline{v}^T) = 1 + \underline{v}^T\underline{u}$. Therefore

$$\varphi(\lambda) = \prod_{i=1}^{n}(1 - K_iA_i - \lambda)\left[1 + \sum_{i=1}^{n}\frac{K_iB_i}{1 - K_iA_i - \lambda}\right]. \tag{16}$$

For the sake of simplicity let $\delta_1 < \delta_2 < \cdots < \delta_s$ denote the different $1 - K_iA_i$ values with multiplicities $m_1, m_2, \dots, m_s$. Let furthermore $I_l = \{i | 1 - K_iA_i = \delta_l\}$. Then

$$\varphi(\lambda) = \prod_{l=1}^{s}(\delta_l - \lambda)^{m_l}\left[1 + \sum_{l=1}^{s}\frac{\gamma_l}{\delta_l - \lambda}\right] \tag{17}$$

with $\gamma_l = \sum_{i \in I_l} K_iB_i$. The main result of this section is expressed by the following theorem, that gives sufficient conditions for the local asymptotic stability of the unique equilibrium:

Theorem 2: *Assume that for all l,*

(a) $|\delta_1| < 1$;
(b) $\gamma_1 \le 0$;
(c) $\sum_{l=1}^{s} \frac{\gamma_l}{\delta_l + 1} > -1$.

Then the unique steady state of system (12) *is locally asymptotically stable.*

Proof: Under assumption (a), all roots of the first part of $\varphi(\lambda)$ are inside the unit circle. In order to complete the proof, we will show that all roots of equation

$$\sum_{l=1}^{s}\frac{\gamma_l}{\delta_l - \lambda} = -1 \tag{18}$$

are also inside the unit circle. If $h(\lambda)$ denotes the left-hand side of equation (18), then it is easy to see that

$$\lim_{\lambda \to \pm\infty} h(\lambda) = 0, \quad \lim_{\lambda \to \delta_l + 0} h(\lambda) = +\infty, \quad \lim_{\lambda \to \delta_l - 0} h(\lambda) = -\infty$$

and

$$h'(\lambda) = \sum_{l=1}^{s}\frac{\gamma_l}{(\delta_l - \lambda)^2} < 0. \tag{19}$$

The graph of $h(\lambda)$ is shown in figure 1. Notice first that equation (18) is equivalent to a polynomial equation of degree s, so there are s real or complex roots. From the graph of function $h(\lambda)$ it is clear that there is a root before δ_1, and one root between each pair δ_i, δ_{i+1} $(i = 1, 2, \dots, s-1)$. Therefore all roots are real and inside the unit circle if $h(-1) > -1$. Assumption (c) is equivalent to this condition. □

Assumption (a) can be rewritten as $-1 < 1 - K_lA_l < 1$, that is

$$\frac{K_l}{u_l} < \frac{2(n+1)}{B}, \qquad l = 1, \dots, n \tag{20}$$

So, assumption (a) holds if the speeds of adjustment K_1 are sufficiently small with respect to the subjective

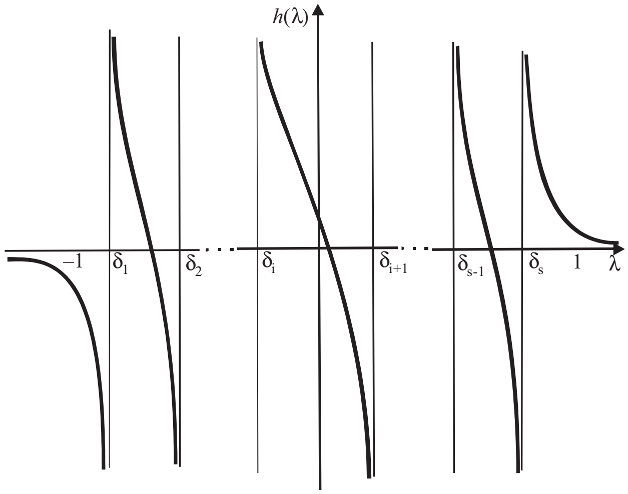

Figure 1. Graphical representation of the function $h(\lambda)$ in the proof of Theorem 2.

"reference" misspecified slope. This means that if firm l selected a demand function that underestimates the price (i.e., $u_l < 1$) then a smaller speed of adjustment (i.e., higher inertia) is requested to have stability. This means that a firm who updates his estimates by small steps has good chances to learn the true demand function. Assumption (a) is more easily satisfied for a large value n of firms. This result suggests that, when the number of firms present in the same market is high, there is a lot of information available about the market and thus it is easier to learn the demand. Finally, assumption (a) is satisfied for small values of the parameter B. This shows that the described adaptive process is suitable for a small market. All these considerations can be summed up in the following statement: the adaptive process (11) is suitable for a market with many, small, cautious firms.

Assumption (b) holds necessarily if

$$\sum_i \alpha_i < (n+1)\alpha_l \tag{21}$$

for all l. If the marginal costs, α_i, are almost the same, then this inequality is satisfied. This means that for each firm it's easier to learn the true demand if firms are similar to each others.

Assumption (c) can be rewritten as

$$\sum_{l=1}^{s} \frac{\sum_{i\in I_l} K_i B_i}{2-K_l A_l} = \sum_{k=1}^{n} \frac{K_k B_k}{2-K_k A_k} > -1 \tag{22}$$

which holds if all B_k values are sufficiently small in absolute value.

The sufficient conditions for the local asymptotic stability of the unique steady state, as given in Theorem 2, ensure that for sufficiently small speeds of adjustment K_i, sufficiently small values of B_k, and sufficiently homogeneous marginal costs, the adaptive adjustment described in section 2 will lead the firms to learn the true demand function in the long run, after several repetitions of the Cournot oligopoly game, even if some (or all) the firms start from subjective misspecified beliefs about the demand function.

However, the results of this section concern only the local stability of the equilibrium, i.e., the sufficient conditions of Theorem 2 ensure the convergence of the learning process provided that the initial scale factors selected by the firms, $\varepsilon_k(0)$, are sufficiently close to the respective "reference" slope u_k. So this local analysis, based on the linearization of the model (12) around the steady state, leaves several open questions to solve. First of all, what are necessary conditions for local stability, such that an exact delimitation of the stability region in the space of the parameters can be obtained. Then, the study of what kinds of bifurcations occur when the boundaries of such stability region is crossed, i.e., how does the steady state lose stability and what kind of disequilibrium asymptotic dynamics of (12) should be expected when the steady state is unstable. Finally, what are the extension and the shape of the basin of attraction of the steady state (when it is stable) or of other attractors when the steady state is unstable. Answering these questions is not generally easy for the highly nonlinear n-dimensional dynamical system (12), so we shall try to gain some insight into these problems by considering some simple situations: an oligopoly with n identical players starting from the same initial guesses on the scale factors, a duopoly with two heterogeneous players that start from arbitrary initial guesses on the scale factors $\varepsilon_k(0)$.

5. Oligopoly with *N* identical players starting from identical initial guesses on scale factors

Let us now consider the case of n homogeneous players, i.e., players that are characterised by identical parameters

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha;\; u_1 = u_2 = \cdots = u_n = u;$$
$$K_1 = K_2 = \cdots = K_n = K. \tag{23}$$

Then equations (12) are the same for each k, and starting from an homogeneous initial condition

$$\varepsilon_i(0) = \varepsilon_0 \quad \forall i$$

we have $\varepsilon_i(t)=\varepsilon(t)$ for each $t\geq 0$ and $i=1,2,\dots,n$. This corresponds with the obvious statement that, in a deterministic framework, identical players, starting from identical initial conditions, behave identically for each time. These trajectories, characterised by $\varepsilon_i(t)=\varepsilon(t)$ for each $t\geq 0$ and $i=1,2,\dots,n$, are called 'synchronised trajectories', and are governed by the following one-dimensional difference equation

$$\varepsilon(t+1)=g(\varepsilon(t))=\left(1-\frac{Kn\alpha}{(n+1)u}\right)\varepsilon(t)+\frac{BKu}{n+1}\frac{1}{\varepsilon(t)}+\frac{K(n\alpha-B)}{n+1}. \tag{24}$$

5.1 *Stability*

The one-dimensional dynamical system (24) can be considered as the model of a *representative player* whose dynamics summarise the common behaviour of the n synchronised firms (Bischi *et al.* 1999, Bischi and Gardini 2000). This reduction may be quite misleading (on this point see also Bischi *et al.* 1999, Kopel *et al.* 2000) if suitable hypothesis are not done. However, the study of this simplified one-dimensional model may give us some insight into the dynamic properties of the general n-dimensional model (12). The following result holds:

Theorem 3: *A necessary and sufficient condition for the local asymptotic stability of the unique steady state* $\bar{\varepsilon}=u$ *of* (24) *is*

$$\frac{K(n\alpha+B)}{u(n+1)}<2. \tag{25}$$

If the aggregate parameter at the left-hand side of (25) *is increased through value* 2 *then the fixed point loses its stability via a flip* (*or period doubling*) *bifurcation, at which a stable cycle of period* 2 *is created around it.*

Proof: The derivative of the map g is

$$Dg(\varepsilon)=1-\frac{Kn\alpha}{(n+1)u}-\frac{BKu}{n+1}\frac{1}{\varepsilon^2} \tag{26}$$

and, computed at the unique steady state $\bar{\varepsilon}=u$, gives

$$Dg(u)=1-\frac{K(n\alpha+B)}{(n+1)u}. \tag{27}$$

As $Dg(u)<1$ for all values of the parameters, the only stability condition becomes $Df(u)>-1$, i.e., the condition (25). Moreover, when the aggregate parameter at the left hand side of (25) is increased through the value 2, $Dg(u)$ crosses through the value -1, thus giving a flip (or period doubling) bifurcation, at which a stable cycle of period 2 is created. □

The stability condition (25) confirms the stabilising role of small values of K/u. The role of the number of firms is not so trivial, because the left-hand side of the stability condition (25) is a decreasing function of n if $\alpha<B$, and an increasing function of n if $\alpha>B$, so that in the former case a higher number of identical firms helps the learning process, whereas in the latter case an higher number of identical firms may prevent the learning process.

5.2 *Instability and complex dynamics*

In order to understand the global dynamic properties of the one-dimensional map (24), the graph of $g(\varepsilon)$ has to be examined for $\varepsilon>0$. It is a hyperbola with a vertical asymptote at $\varepsilon=0^+$, whereas and for $\varepsilon\to+\infty$ it approaches the asymptote of equation

$$y=\left(1-\frac{Kn\alpha}{(n+1)u}\right)\varepsilon+\frac{K(n\alpha-B)}{n+1}. \tag{28}$$

So, if $Kn\alpha/(u(n+1))>1$ then the map g is decreasing, and for $\varepsilon\to+\infty$ it tends to $-\infty$ along the negatively sloped line (28). In this case, any positive trajectory converges to the steady state if the stability condition (25) is satisfied, whereas if (25) does not hold a stable cycle of period 2 may be the unique attractor. Instead, if $Kn\alpha/(u(n+1))<1$ then the map g is unimodal (see figure 2(a)): it decreases for $\varepsilon<\varepsilon_{\min}$, where

$$\varepsilon_{\min}=u\sqrt{\frac{BK}{(n+1)u-Kn\alpha}},$$

and it increases for $\varepsilon>\varepsilon_{\min}$. As $\varepsilon\to+\infty$ it approaches the positively sloped line (28). This case may give rise to more complex dynamic properties: in fact, in this case the first period doubling bifurcation, at which the steady state loses stability, is followed by other period doublings and, in general, by the well-known period doubling cascade, that constitutes the typical route to chaotic behaviours for smooth unimodal maps. So, complex dynamics, that include periodic cycles of any

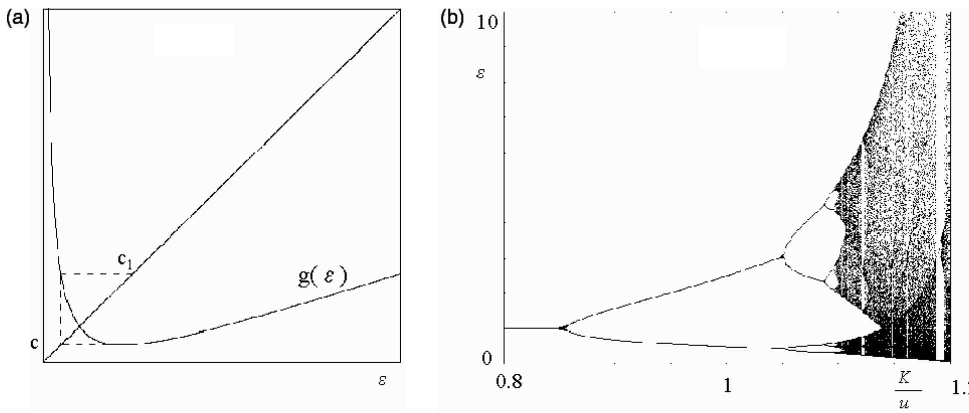

Figure 2. (a) Graphical representation of the map g with the set of parameters $B = 5$, $K = 1$, $\alpha = 1$, $n = 2$, $u = 1$. As far as the minimum value $c = f(x_{\min}) > 0$, the asymptotic dynamics are trapped inside the interval $[c, c_1]$, where $c_1 = f(c)$ is the rank-1 image of the minimum point. (b) Bifurcation diagram for the one-dimensional map g with the same parameters $B = 5$, $\alpha = 1$, $n = 2$ and K/u as a bifurcation parameter in the range $[0.8, 1.2]$.

period and chaotic motion, can be obtained if the map is unimodal and the fixed point is unstable, i.e.,

$$\frac{2(n+1)u}{n\alpha + B} < K < \frac{(n+1)u}{n\alpha} \tag{29}$$

This range is nonempty provided that $B > n\alpha$, i.e., the maximum price is greater than the aggregated marginal cost. For example, if we consider the set of parameters $n = 2$, $B = 5$, $\alpha = 1$, $u = 1$, the range (29) is $6/7 < K < 3/2$. This is confirmed by a numerical computation of the bifurcation diagram shown in figure 2(b). As far as the minimum value $c = g(\varepsilon_{\min}) > 0$, the asymptotic dynamics are trapped inside the interval $[c, c_1]$, where $c_1 = g(c)$ is the rank-1 image of the minimum point (see figure 2(a)). For increasing values of the adjustment coefficient K, as shown in the bifurcation diagram of figure 2, the minimum value c decreases until it reaches the value $c = 0$ (for the set of parameters used to obtain the bifurcation diagram of figure 2(b), this occurs at $K \simeq 1.3$). This is the *final bifurcation*, after which the generic trajectory involves negative values.

It is worth stressing that the same kind of bifurcation diagram as the one shown in figure 2(b) can be obtained by increasing the parameter B or by increasing the parameter α, of by decreasing u.

6. The case of duopoly with heterogeneous players

In this section, we consider the case of two heterogeneous firms, i.e., a duopoly system, and we give a detailed study of the region of stability in the space of the parameters of the learning process. In the case $n = 2$, the dynamic model (12) becomes two-dimensional, the learning process is given by the iteration of the two-dimensional map T: $(\varepsilon_1(t), \varepsilon_2(t)) \to (\varepsilon_1(t+1), \varepsilon_2(t+1))$ defined by:

$$\begin{aligned}
\varepsilon_1(t+1) &= \varepsilon_1(t) - K_1\left[\frac{B}{3}\left(1 - \frac{u_1}{\varepsilon_1(t)}\right) + \frac{2\alpha_1 - \alpha_2}{3u_1}\varepsilon_1(t)\right. \\
&\qquad \left. + \frac{2\alpha_2 - \alpha_1}{3u_2}\varepsilon_2(t) - \frac{\alpha_1 + \alpha_2}{3}\right] \\
\varepsilon_2(t+1) &= \varepsilon_2(t) - K_2\left[\frac{B}{3}\left(1 - \frac{u_2}{\varepsilon_2(t)}\right) + \frac{2\alpha_1 - \alpha_2}{3u_1}\varepsilon_1(t)\right. \\
&\qquad \left. + \frac{2\alpha_2 - \alpha_1}{3u_2}\varepsilon_2(t) - \frac{\alpha_1 + \alpha_2}{3}\right].
\end{aligned} \tag{30}$$

6.1 *Stability*

In order to study the stability of the unique positive steady state $\bar{\varepsilon} = (\bar{\varepsilon}_1, \bar{\varepsilon}_2) = (u_1, u_2)$, we consider the Jacobian matrix (14) with $n = 2$ computed at the equilibrium:

$$DT(u_1, u_2) = \begin{bmatrix} 1 - \dfrac{K_1}{3u_1}(B + 2\alpha_1 - \alpha_2) & -K_1\dfrac{2\alpha_2 - \alpha_1}{3u_2} \\ -K_2\dfrac{2\alpha_1 - \alpha_2}{3u_1} & 1 - \dfrac{K_2}{3u_2}(B + 2\alpha_2 - \alpha_1) \end{bmatrix}. \tag{31}$$

The analysis of local stability is obtained by the standard linearization procedure, that is by the study of the eigenvalues, solutions of the characteristic equation $P(z) = z^2 - \mathrm{Tr} \cdot z + \mathrm{Det} = 0$, where Tr and Det are, respectively, the Trace and the Determinant of the

Jacobian matrix (31). A sufficient condition for the stability is expressed by the following system of inequalities

$$P(1) = 1 - \mathrm{Tr} + \mathrm{Det} > 0;$$
$$P(-1) = 1 + \mathrm{Tr} + \mathrm{Det} > 0; \quad \mathrm{Det} - 1 < 0 \tag{32}$$

that give necessary and sufficient conditions for the two eigenvalues be inside the unit circle of the complex plane (see, e.g., Medio and Lines 2001, p. 52, or any standard book on discrete dynamical systems). The first condition is always satisfied, hence the stability conditions reduce to $P(-1) > 0$ and $\mathrm{Det} - 1 < 0$, that, after some algebraic manipulations and a division by $u_1 u_2$ become, respectively

$$B(\alpha_1 + \alpha_2 + B)\frac{K_1}{u_1}\frac{K_2}{u_2} - 6(2\alpha_1 - \alpha_2 + B)\frac{K_1}{u_1} - 6(2\alpha_2 - \alpha_1 + B)\frac{K_2}{u_2} + 36 > 0 \tag{33}$$

and

$$B(\alpha_1 + \alpha_2 + B)\frac{K_1}{u_1}\frac{K_2}{u_2} - 3(2\alpha_1 - \alpha_2 + B)\frac{K_1}{u_1} - 3(2\alpha_2 - \alpha_1 + B)\frac{K_2}{u_2} < 0. \tag{34}$$

These two inequalities define a *region of stability* (we may call it *learning region*) in the space of the parameters of the model (30). Moreover, the conditions (33) and (34) taken as equalities, i.e., the equations $P(-1) = 0$ and $\mathrm{Det} = 1$, define bifurcations hypersurfaces. This means that when one or more parameters are varied so that the equilibrium $\bar{\varepsilon}$ becomes unstable, if the stability loss is due to a change of sign $P(-1)$, i.e., of the left-hand side of (33), then a flip (or period doubling) bifurcation occurs, whereas if the stability loss is due to a change of sign $\mathrm{Det} - 1$, i.e., of the left-hand side of (34), then a Neimark–Hopf bifurcation occurs.

It is useful to represent the learning region projected in the two-dimensional plane $((K_1/u_1), (K_2/u_2))$, where the bifurcation curves that bound the region of stability are equilateral hyperbolas (see figure 3, where F denotes the positive branch of the hyperbola $P(-1) = 0$, H denotes the positive branch of the hyperbola $\mathrm{Det} = 1$, and the shaded area represents the learning region). If

$$\frac{\alpha_1}{2} < \alpha_2 < 2\alpha_1 \tag{35}$$

then the two hyperbolas do not intersect, and the learning region is bounded only by the flip bifurcation curve (figure 3(a)), whereas if

$$2\alpha_2 < \alpha_1 < 2\alpha_2 + B \quad \text{or} \quad 2\alpha_1 < \alpha_2 < 2\alpha_1 + B \tag{36}$$

then the two hyperbolas intersect in the positive orthant of the plane $((K_1/u_1), (K_2/u_2))$, so that the learning region is bounded by an arc of the Neimark-Hopf bifurcation curve and by two arcs of the flip bifurcation curve† (figure 3(b)).

If the parameters K_1/u_1 and/or K_2/u_2 are varied, so that they cross the boundary of the stability region along the portion of curve F, then the equilibrium point $\bar{\varepsilon}$ changes from a stable node to a saddle point via a supercritical flip bifurcation‡. This means that, just after the stability loss of $\bar{\varepsilon}$, the long run evolution of the trajectories of (30) is characterised by the convergence to a periodic cycle of period 2. So, although the learning process is adopted during the repeated game, the players will never learn the true demand function. They will keep on underestimating/overestimating it, as the subjective scale factors continue to oscillate.

When the cost parameters α_1 and α_2 are not too different, according to (35), i.e., in the case of moderate heterogeneity in costs, the steady state $\bar{\varepsilon}$ can lose stability only via a period doubling bifurcation. This is particularly true if players are identical, according to the analysis in section 5.

Let us now consider what happens if the parameters K_1/u_1 and K_2/u_2 are varied, so that they cross the boundary of the learning region along the portion of curve H. In this case, the equilibrium $\bar{\varepsilon}$ changes from a stable focus to an unstable focus via a supercritical Neimark–Hopf bifurcation§. This means that the long-run evolution of the trajectories of (30) converges

†For $\alpha_1 = 2\alpha_2$ the curve F degenerates into the pair of straight lines $K_1/u_1 = 6/(3\alpha_2 + B)$ and $K_2/u_2 = 6/B$. For $\alpha_2 = 2\alpha_1$ the curve F degenerates into the pair of straight lines $K_1/u_1 = 6/B$ and $K_2/u_2 = 6/(3\alpha_1 + B)$.

‡A rigorous proof of the supercritical nature of the flip bifurcation requires a centre manifold reduction and the evaluation of higher order derivatives, up to the third order (see, e.g., Guckenheimer and Holmes 1983). This is rather tedious in a two-dimensional map, and we prefer to rely on numerical evidence as a stable 2-cycle close to the saddle $\bar{\varepsilon}$ is numerically detected whenever the parameters cross the bifurcation curve F.

§Also in this case, a rigorous proof of the supercritical nature of the Neimark–Hopf bifurcation requires a centre manifold reduction and the evaluation of higher order derivatives, up to the third order (see, e.g., Guckenheimer and Holmes 1983). This is rather tedious in a two-dimensional map, and we prefer to rely on numerical evidence as a stable orbit surrounding the unstable focus $\bar{\varepsilon}$ is numerically detected whenever the parameters cross the bifurcation curve H.

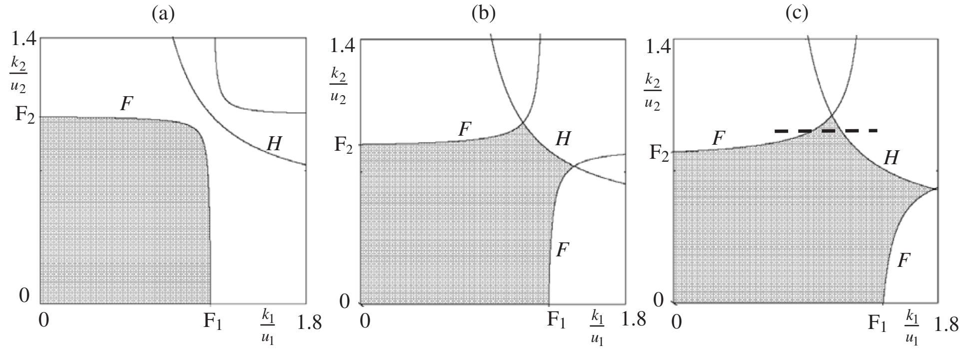

Figure 3. Learning region projected in the two-dimensional plane $(K_1/u_1, K_2/u_2)$. The shaded area represents the learning region, the bifurcation curves that bound it are equilateral hyperbolas: F denotes the positive branches of the Flip (period doubling) bifurcation curve, H the positive branch of the Neimark-Hopf bifurcation curve. (a) With the set of parameters $B=5$, $\alpha_1=0.5$, $\alpha_2=0.8$, the condition $\alpha_1/2 < \alpha_2 < 2\alpha_1$ holds, then the two hyperbolas do not intersect, and the learning region is bounded only by the Flip bifurcation curve; (b) With the set of parameters $B=5$, $\alpha_1=0.5$, $\alpha_2=1.3$, the two hyperbolas intersect in the positive orthant, so the learning region is bounded by an arc of the Neimark-Hopf bifurcation curve and by two arcs of the flip bifurcation curve; (c) With the set of parameters $B=5$, $\alpha_1=0.3$, $\alpha_2=1.4$, the arc of the curve H included in the boundary of the stability region becomes more extended.

to a quasi-periodic motion around the steady state. Again, this implies that, on the basis of the adjustment process adopted, players will never learn the true demand function, as they will continue cyclically to over/under estimate prices. This kind of route to instability can only occur if the two players are sufficiently heterogeneous with respect to cost parameters, according to (36). Moreover, if the difference between the cost parameters is increased, then the arc of the curve H included in the boundary of the stability region becomes more extended (see figure 3(c)).

In summary, on the basis of the results on the stability region in the plane $(K_1/u_1, K_2/u_2)$, we can say that the adjustment process can converge to the true demand function provided that both the ratios K_1/u_1 and K_2/u_2 are sufficiently small. This means that, given u_1 and u_2, the speeds of adjustment K_1 and K_2 cannot be too great, i.e., a sufficiently small degree of inertia is necessary, in order to ensure convergence of the adaptive learning process to the true demand. Indeed, increasing one or both speeds of adjustment may cause overshooting, characterised by oscillations of the scale factors that never settle to the true demand function. This result confirms the observation made in section 4, assumption (a), and the underlying idea that this learning process is suitable for cautious firms.

We may also consider a different point of view. Given fixed values of the speeds of adjustment K_1 and K_2 (or, equivalently, given levels of inertia of the players in revising their scale factors), subjective price overestimation (i.e., high values of u_1 and u_2) favours the convergence of the learning process to the true demand, and if we decrease u_1 or u_2 with fixed values of K_i the steady state of the learning process may loose stability through one of the bifurcations described above. This suggests that when firms are very sensitive it's better for them to overestimate the price.

However, when (36) holds, so that the stability region has a shape like the one shown in figure 3(b) or 3(c), it is interesting to notice that some bifurcation paths exist such that an increase of one or both the parameters K_i/u_i may have both a stabilizing and a destabilizing effect. This occurs, for example, on the bifurcation path indicated by the dashed line of figure 3(c). Along the first portion of this path an increase of K_1/u_1 and/or K_2/u_2 has a stabilizing effect, as $\bar{\varepsilon}$ from unstable becomes stable via a backward flip (or period halving) bifurcation, and if we continue to increase K_1/u_1 and/or K_2/u_2 along the same path we get a destabilising effect because $\bar{\varepsilon}$ loses stability via a supercritical Neimark–Hopf bifurcation.

Of course, this "double effect" can only occur in the presence of a considerable degree of heterogeneity in costs because, as remarked above, the portion of the boundary of the learning region formed by the Neimark–Hopf bifurcation curve becomes smaller and smaller (until it disappears) as the heterogeneity in marginal costs is reduced.

It is also worth noticing that, in any case, the stability region shrinks as, ceteris paribus, the parameter B increases, i.e., the maximum price is higher. In fact, the intersections F_1 and F_2 of the curve F with the coordinate axes of the parameter plane $(K_1/u_1, K_2/u_2)$ are given by $F_1=(6/(2\alpha_1-\alpha_2+B), 0)$ and $F_2=(6/(2\alpha_2-\alpha_1+B), 0)$. So, the convergence of the learning process to the true demand is more difficult if the inverse demand function is shifted towards higher maximum prices. In other words, this learning process is

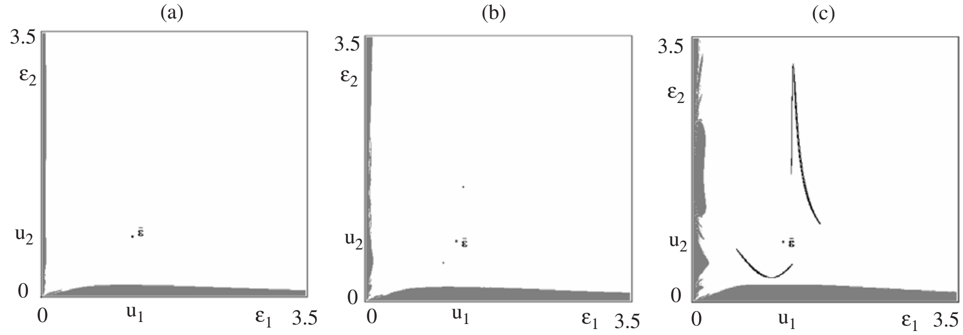

Figure 4. The white region represents the set of points that generate feasible trajectories (i.e. trajectories entirely included inside the positive orthant) and converging to the steady state, or another attractor around it, whereas the grey region represents the set of points that generate unfeasible trajectories. (a) With $B=5$, $\alpha_1=0.5$, $\alpha_2=0.8$ (like in figure 3b) and $u_1=1.2$, $u_2=0.8$, $K_1=0.6$, $K_2=0.7$, the parameters are inside the stability region and the steady state is a stable node; (b) With $B=5$, $\alpha_1=0.5$, $\alpha_2=0.8$, $u_1=1.2$, $u_2=0.8$, $K_1=0.6$, $K_2=0.9$, the parameters are outside the region of stability (close to the boundary F, see figure 3a), the steady state is a saddle point, and the only attractor is a stable cycle of period 2, represented by the two small dots; (c) With $B=5$, $\alpha_1=0.5$, $\alpha_2=0.8$, $u_1=1.2$, $u_2=0.8$, $K_1=0.6$, $K_2=1.05$, a chaotic attractor exist.

better adapted for small markets. This confirms the results on local stability for the n-dimensional model given in section 4, assumption (a).

6.2 *Instability and complex dynamics*

The stability analysis given above is only based on local stability and local bifurcations of the unique steady state. With the help of some numerical simulations we can explore what happens when the parameters are moved far from the boundaries of the stability region, and we can obtain some indications about the extension and the shape on the basis of attraction of the steady state or of the more complex attractors that replace it when the parameters are out of the learning region.

Let us consider, first, the values of the parameters $B=5$, $\alpha_1=0.5$ and $\alpha_2=0.8$ that give the learning region of figure 3(a). In this case, when the parameters are inside the stability region the steady state is a stable node, like in figure 4(a) obtained with $u_1=1.2$, $u_2=0.8$, $K_1=0.6$, $K_2=0.7$, so that $K_1/u_1=0.5$ and $K_2/u_2=0.875$. In this case, there are two real eigenvalues, one positive and one negative. This means that any trajectory of (30) starting close to the steady state $\bar{\varepsilon}$ converges to it through oscillations of decreasing amplitude. In figure 4 the white region represents the set of points that generate feasible trajectories (i.e., trajectories entirely included inside the positive orthant) and converging to the steady state, whereas the grey region represents the set of points that generate unfeasible trajectories (i.e., trajectories involving negative values). Figure 4(b) is obtained with a higher value of the speed of adjustment K_2, namely $K_2=0.9$, so that the ratio $K_2/u_2=1.125$ is outside the region of stability (close to the boundary, see figure 3(a)). In this case, as expected on the basis of the local stability analysis, the steady state is a saddle point, because a period doubling bifurcation created a stable cycle of period 2, represented by the two small dots in figure 4(b). This means that none of the two firms learns the demand and they keep on underestimating and overestimating it. As K_1/u_1 and/or K_2/u_2 are further moved away from the stability region, the periodic points move away from the unstable steady state, i.e., the amplitude of the oscillations increase. Moreover, other local bifurcations may occur, at which also the cycle of period 2 loses stability and more complex attractors may appear. For example, the two-cycle may flip bifurcate to give rise to a stable cycle of period *4*, and so on, until chaotic attractors appear after the well-known period-doubling cascade (see figure 4(c), obtained with $K_2=1.05$ and the other parameters like in the previous figures, so that $K_2/u_2=1.3125$).

We now consider the case of a greater difference between the cost parameters α_1 and α_2, so that the condition (36) is satisfied and, consequently, the stability region is also bounded by a portion of the curve H where a Neimark–Hopf bifurcation occurs. Indeed, by using $B=5$, $\alpha_1=0.5$ and $\alpha_2=1.3$, like in figure 3(b), we consider a set of parameters inside the stability region, namely $u_1=1.2$, $u_2=0.8$, $K_1=1.2$, $K_2=0.64$, so that the equilibrium, shown in figure 5(a) with its feasible basin of attraction, is a stable focus (complex conjugate eigenvalues of modulus less than 1). As expected, if we increase K_1 and K_2, so that $(K_1/u_1, K_2/u_2)$ crosses the boundary H of the stability region, a supercritical Neimark–Hopf bifurcation occurs, at which the steady state is transformed into an unstable focus, and an attracting closed invariant curve is created around it (see figure 5(b), obtained with $K_1/u_1=1.3$ and $K_2/u_2=0.9$). As the parameters K_1 and K_2 are further increased, so

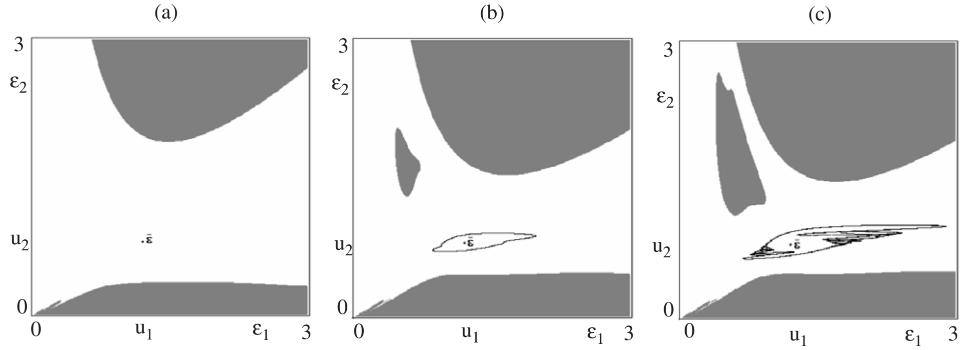

Figure 5. The condition $\alpha_1/2 < \alpha_2 < 2\alpha_1$ is not satisfied and, consequently, the stability region is also bounded by a portion of the curve H where a Neimark–Hopf bifurcation occurs. (a) Using $B=5$, $\alpha_1=0.5$, $\alpha_2=1.3$ (like in figure 3b) and $u_1=1.2$, $u_2=0.8$, $K_1=1.2$, $K_2=0.64$, the parameters are inside the stability region and the steady state is a stable focus; (b) $B=5$, $\alpha_1=0.5$, $\alpha_2=1.3$, $u_1=1.2$, $u_2=0.8$, $K_1=1.56$, $K_2=0.72$, so that $(K_1/u_1, K_2/u_2)$ is outside of the learning region, close to the boundary H. The steady state is an unstable focus, and an attracting closed invariant curve exists around it; (c) With $B=5$, $\alpha_1=0.5$, $\alpha_2=1.3, u_1=1.2$, $u_2=0.8$, $K_1=1.63$, $K_2=0.75$, the attractor is a chaotic ring around the unstable focus $\bar{\varepsilon}$.

that K_1/u_1 and K_2/u_2 are further moved away from the stability region, the size of the attracting closed orbit around the steady state increases, and consequently the long-run oscillations of the scale factors $\varepsilon_i(t)$ will increase their amplitude, according to the Neimark–Hopf bifurcation theorem. The closed invariant curve may change its shape and be replaced by a more complex attractor, such as a chaotic ring (see figure 5(c), obtained with $K_1=1.63$ and $K_2=0.75$).

In both numerical explorations, if the parameters are moved away from the boundaries of the learning region, more and more complex attractors can be observed that characterise the nonlearning dynamics. These attractors become larger, i.e., the long run dynamic of the learning process is characterised by greater and lesser regular oscillations until a contact between the boundaries of the attractor and the boundary of the feasible region occurs. This contact represents a global bifurcation (called *final bifurcation* in Mira *et al*. 1996 and Abraham *et al*. 1997, or *boundary crisis* in Grebogi *et al*. 1983) that marks the disappearance of the attractor, i.e., after the contact the generic trajectory is unfeasible. We do not analyse in greater detail these dynamic properties of the model, as in this context we are mainly interested in studying the conditions for the occurrence of learning dynamics. For this purpose, we explore the qualitative changes of long-run behaviour of the learning process when the heterogeneity condition (36) holds. We want to see what happens when K_1/u_1 and/or K_2/u_2 are gradually increased in such a way that we obtain two bifurcations, which cause a transition between two different instability situations separated by a "window" of stability. This particular sequence will be illustrated by numerical examples obtained following the bifurcation path, in the space of parameters, represented by the dashed line in figure 3(c). Increasing the value of the K_1 parameter the learning process goes through, first, a period halving (or backward flip) bifurcation and then a supercritical Neimark–Hopf bifurcation. In figure 6(a) the learning process approaches, in the long run, a stable cycle of period 2. In figure 6(b), obtained after increasing K_1, the learning process converges to the true demand (i.e., the parameters are inside the learning region). In figure 6(c), obtained after a further increase of the speed of adjustment K_1, the steady state is again unstable and the long-run dynamics of the learning process are characterised by quasi-periodic oscillations along a stable close invariant orbit around the unstable steady-state.

To conclude this section, it is worth to make some remarks about the extension and the shape of the feasible region, represented by the white area in the figures 4–6. Of course, when the parameters are inside the learning region, the extension of the feasible region provides important information about the robustness of the learning process. In fact, a knowledge of the feasible basin of the steady state $\bar{\varepsilon}$ gives an answer to the fundamental question: how far from true demand can the guesses of the players be in regard to their subjective scale factors, in order to guarantee the success of the learning process? First of all, it can be noticed that the maximum "distance" of a single subjective scale factor is not important, as the distance of all the scale factors must be considered. One firm may start its learning process from an initial guess on ε_i very close to the corresponding u_i, even from $\varepsilon_i=\bar{\varepsilon}_i=u_i$, and the endogenous dynamics of the global learning process may prevent it to use the true demand in the long run, due to the initial errors of its competitors. This remark may sound quite trivial for an interconnected nonlinear dynamical system, but we think that it is worth stressing. The idea is that a firm with the right guess thinks that his

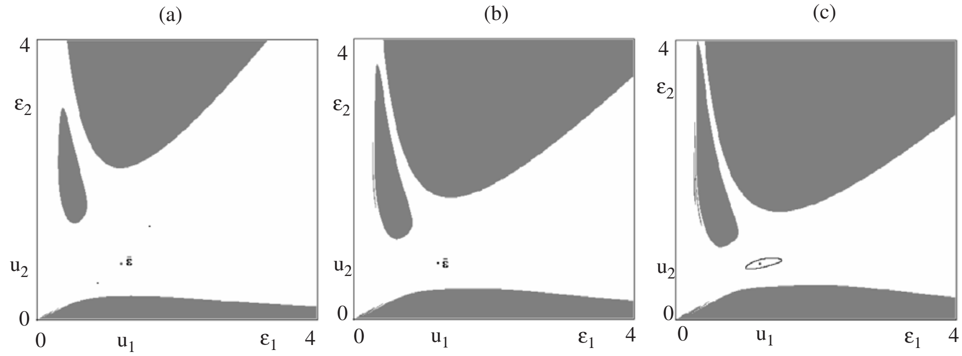

Figure 6. Qualitative changes of long-run behaviour of the learning process when the heterogeneity condition (36) holds. Effects of increasing values of K_1/u_1 and/or K_2/u_2 when we follow the bifurcation path, in the space of parameters, represented by the dashed line in figure 3c. (a) With $B=5$, $\alpha_1=0.3$, $\alpha_2=1.4$, $u_1=1.2$, $u_2=0.8$, $K_1=0.84$, $K_2=0.7$, the learning process approaches, in the long run, a stable cycle of period 2; (b) With the same set of parameters, but increasing the value of K_1 ($K_1=1.2$), the learning process converges to the true demand (i.e. the parameters are inside the learning region); (c) This figure is obtained after a further increase of the speed of adjustment K_1 ($K_1=1.44$). The steady state is again unstable and the long-run dynamics of the learning process are characterized by quasi-periodic oscillations along a stable close invariant orbit around the unstable steady-state.

competitor is going to use his same demand function, which is a mistake. However, based on the market price information the competitor will select a wrong demand function at the beginning of the second time period. Depending on the initial incorrect guess, the trajectory may converge to the steady state or may not.

As well, the boundaries of the feasible region may be quite complicated, as it can be seen in the figures 4–6, where some irregular boundaries can be seen (figure 4(c)) as well as the presence of "islands" (or "holes") of the unfeasible region nested inside the feasible basin of the attractor (figures 5 and 6). A study of this kind of complexity requires an analysis of the global dynamic properties of the map (30). In particular, the creation of complicated topological structures of the feasible region, such as the presence of islands of the unfeasible set nested inside the feasible region, is related to the fact that the two dimensional map (30) is noninvertible (see, e.g., Mira *et al.* 1996, for the general theory and methods for the study on noninvertible maps, see also Bischi *et al.* 2000, Bischi and Kopel 2001, Bischi *et al.* 2003a, for some recent applications). We do not go deeper into this question, as we prefer to develop these aspects of global dynamic properties elsewhere. We just stress that the two kinds of complexity, one related to a more and more complex structure of the attractors that characterise the nonlearning dynamics and one related to complex topological structures of the feasible set, are not generally related.

However, from our numerical simulations we observed that increasing values of K_1/u_1 and/or K_2/u_2, as well as increasing values of B, may cause both the exit of the parameters from the learning region and a reduction of the size of the feasible region. The effects of the cost parameters α_1 and α_2 on the shape of the feasible region do not seem to be so strong.

7. Conclusions

In this article we have analysed a single-product Cournot oligopoly where firms have incomplete knowledge of the demand function. This implies that they compute both their own and their competitors' "optimal" production choices, by using a misspecified slope of the demand function. Under the assumption that each firm knows all the cost functions of the firms of the oligopoly system, a dynamic adaptive process of belief revision is proposed, based on the observed discrepancy between forecasted and observed prices. The adjustment process has been expressed in the form of a nonlinear n-dimensional discrete dynamical system, where n is the number of firms, such that its unique steady state corresponds to a situations where all the subjective believed demand functions are equal and coincide with the true demand function. So, the stability of the steady state means that true demand can be learned by all the oligopolists even if they start from misspecified (and different) initial guesses in the slope of the demand function. This means that, even if firms behave as myopic players that always follow the same adaptive adjustment mechanism, under suitable conditions they can "learn" the true demand in the sense that in the long-run they become fully informed about the game they are playing. However, the adjustment process may not converge, i.e., the myopic firms continue to misspecify the demand. We examined some bifurcations that lead to instability of the steady state, and what kind of disequilibrium dynamics may occur when the adjustment process does not converge.

Following Oechssler and Schipper (2002), the study we have proposed in this article concerns the problem of *learning about the game*, i.e., players, who have incomplete knowledge about some elements of the game, such as the payoff functions, try to learn about such imperfectly known elements while playing the game repeatedly. This kind of learning differs from the one which is more frequently addressed in the literature, related to *learning how to play a game*, where players try to complete their imperfect knowledge on the behaviour of the opponents (see, e.g., Fudenberg and Levine 1988, and references therein).

Our results concerning the conditions for the local stability of the steady state have allowed us to understand the influence of the parameters of the model on the convergence of the learning process. We showed that, in general, excessive reactivity of the firms as well as an excessive cost heterogeneity increase the likelihood of failing convergence to the steady state. The first statement is not surprising, as in adaptive systems stability is often lost due to over-reaction of the agents. However, our studies reveal that when firms are very reactive the likelihood of learning the true demand increases if they initially overestimate the price.

The second statement, concerning the role of firms' heterogeneity, may deserve some economic reflections. Loosely speaking, the intuition behind such statement is that it is easier for a firm to predict the choices of the competitors if they are similar.

Another parameter that has an important role in the effectiveness of the learning process is the highest price, which people are willing to pay when quantity tends to zero, denoted by B in the model. In fact, the learning process works better if B is decreased.

Moreover, the analysis of the particular case of n identical firms showed that the learning process is helped if the number of players is increased. The intuition behind this may be expressed by the idea that if more firms are present in the market, then more information goes through it, so it's easier for the firms to collect data to learn the true demand function.

We can summarise all this by saying that the proposed adaptive learning process is more suitable for a small market with many similar cautious firms. These conclusions have been drawn on the basis of the sufficient conditions for the n-dimensional dynamical system that describes the learning mechanism for a n-firms oligopoly. However, the more detailed analysis of the duopoly case showed that some counterintuitive effects can be observed if the reactivity of firms is gradually increased in the presence of strong heterogeneity. In fact, a two-dimensional dynamical system is obtained in the case of duopoly, and this allowed us to get a more detailed study of the stability region in the space of the parameters and of the local bifurcations that cause a loss of stability of the steady state. This analysis has shown that when the two firms have significantly different marginal costs, some peculiar paths in the space of the parameters exist such that a gradual increase of the reactivity of the firms may lead from situations of instability to stability and then to instability again, through the occurrence of two different local bifurcations. In other words, even if generally an increase in the reactivity of firms has a destabilising role (as it may prevent the convergence of the learning process) under certain circumstances, starting from a situation of non convergence, an increase of one or both the speeds of adjustment may first have a stabilising effect and then destablise again (i.e., two successive bifurcations occur which cause a transition between two different instability situations separated by a "window" of stability). This "double effect" can only occur in the presence of a considerable degree of heterogeneity in costs.

With the help of numerical simulations, the size and the shape of the basins of attraction, as well as the kinds of attracting sets that characterise the nonequilibrium dynamics of the learning process, are described. This allowed us to gain some insight about the kinds of long-run behaviour of the learning process when the stability conditions are not fulfilled, i.e., firms do not learn the true demand even after infinite repetitions of the game.

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