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An extension of the Antoci–Dei–Galeotti evolutionary model for environment protection through financial instruments

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ABSTRACT

This work moves from a recent paper by Antoci et al. (2009) [1] where a dynamic model is proposed to describe an innovative method for improving environmental quality based on the exchange of financial activities, promoted by a Public Administration, between firms and tourists in a given region. We extend their analysis in two directions: we first perform a global analysis of the basins of attraction to check the stability extents of the coexisting stable attractors of the model, and we show that some undesirable and sub-optimal stable equilibria always exist, whose basins may be quite intermingled with those of the optimal equilibrium; then we introduce a structural change of the model by assuming that the Public Administration, besides its action as an intermediary between visitors and polluting firms, also performs a direct action for the pollution control. We show how the cost of this direct action of the Public Administration can be balanced by proper taxes and we prove that undesired equilibria can be ruled out by a suitable balance of financial instruments and direct actions of Public Administration for environmental remediation.

1. Introduction

In a recent paper, Antoci et al. [1] (quoted as *ADG* henceforth) propose an evolutionary game, to model an innovative system for environment protection based on the exchange of financial activities, promoted by a Public Administration (*PA* henceforth). In a tourist region (R) two financial options are supposed to be issued which work like contracts between the *PA* and, respectively, visitors and firms operating in R, and can be regarded as (cash-or-nothing) environmental call (*EC*) and environmental put (*EP*) options. A visitor who wishes to spend a period of time in the region R can choose between the payment of a given tax (or ticket) or purchasing an *EC* sold by the *PA* at a given price, which will be refunded in the case of a low environmental quality, a sort of satisfied-or-reimbursed contract. On the other side, the *PA* offers to a potentially polluting firm operating in the region R the choice between the payment of a fixed tax or subscribing an *EP* issued by the *PA*. This financial activity is a contract, which binds the firm to adopt a new environmentally friendly technology, thus bearing a supplementary cost, which can be covered with a financial aid of *PA* only if the environmental quality target is reached (the same used for deciding the refunding or not of visitors subscribing to the *EC*).

In a continuous-time dynamic setting, at each time potentially polluting firms have to choose between two strategies: adopting an environmentally friendly technology and subscribing to the *EP*, or using cheaper but polluting technology and paying a fixed tax, while each visitor can choose purchasing reimbursable *EC* options or just paying for a ticket. Given the observed pollution level Q, if such a level is above a threshold $\overline{Q}$ (poor environment quality) then visitors are refunded by *PA* and firms receive no financial aid, whereas if $Q < \overline{Q}$, then firms receive financial support to refund the cost they afforded for adopting non-polluting technology and visitors receive nothing (but they can enjoy the fine environmental quality). Of

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course, the *PA* has only the role of an intermediary, without direct costs, as it just has the role of organizing the trading of environmental options and checking the pollution level.

In *ADG* the short-run time evolution of the fractions of visitors and firms adopting the different strategies is modelled by using replicator dynamics, which states that the fraction of agents (firms or visitors) that make a given choice increases as long as its expected payoff is greater than the one associated with the opposite choice. This short-run profit-maximizing behaviour may trigger a virtuous self-sustained long-run evolution leading to a desirable Nash equilibrium, where all firms adopt non-polluting technologies and, consequently the environmental quality is so good that no visitors are attracted by the purchasing environmental call options. However, as often occurs in evolutionary games, this is not the only possible long-run evolution, because other attractors may exist corresponding to "perverse" evolutions towards a "bad" equilibrium where no firms adopt environmentally friendly technologies and, consequently, characterized by permanently high levels of pollution. Sometimes other situations may be obtained as well, that can be defined as intermediate (or sub-optimal) equilibria. This coexistence of several attractors is always associated with uncertainty, due to the fact that the final outcome is path dependent, with different outcomes obtained as the consequence of (even small) changes of the initial conditions of the evolutionary game, according to the structure of the basins of attraction.

We stress that some dynamic scenarios can be evidenced such that even if the basin of the "bad" equilibrium has a small extension, it may be so intermingled with the basin of the "good" one that the evolutionary stability of the latter is quite low, i.e. the desired equilibrium may be very vulnerable.

The aim of this paper is to show that if in the *ADG* model we introduce the possibility of a direct action of the *PA* for the pollution control (for example by interventions for environmental damages remediation) then the attractivity of the desired equilibrium can be enhanced, and some perverse and sub-optimal long-run evolutions can be eliminated by a proper combination of the role of *PA* as regulator of the market of environmental options and its direct intervention. Through a global dynamic study of attractors and their basins we show that even a moderate direct action of the *PA* for environmental protection can help to induce a time evolution leading to the desirable Nash equilibrium, by a process fuelled both by the mechanism of environmental options regulated by the *PA* and its direct intervention.

Of course, the direct environmental remediation performed by the Public Administration has a cost, but we shall prove that it can always be covered by properly tuning the ticket paid by visitors that choose to pay the fixed tax, so that a "balanced budget" of the Public Administration can always be reached.

The plan of the work is as follows. In Section 2 we describe the *ADG* model and recall some of their results. In Section 3 we numerically explore the structures of the basins that can be obtained with some particular constellations of parameters, and we show that the structure of the boundaries that separate the basins of attraction of the coexisting equilibria may be quite involved, so that in some situations it is difficult to forecast the long-run outcome of the evolutionary mechanism proposed, as already stressed in a remark given by *ADG*. In Section 4 we propose the modified model which includes a direct action of *PA* for environment remediation, besides the role as regulator of the environmental options market, and we study the effects, on the local and global dynamic properties of the model, of this modification. Section 5 concludes.

2. The Antoci–Dei–Galeotti model

In this section we briefly describe the *ADG* model and recall some of its properties. For more details the reader is referred to [1] and the references therein. Following [1], we consider a tourist region R and we assume that, at each time t, each visitor has to choose (ex ante) between strategy $V1$ (buying the *EC* from the *PA* at a price $\widetilde{p}$) and $V2$ (paying a fixed entrance ticket at a price $\overline{p}$), whereas a firm has to choose between strategy $F1$ (subscribing to the *EP* offered by the *PA* and paying a cost c_{NP} for the adoption of an innovative non-polluting technology) and $F2$ (paying to the *PA* a fixed amount $\overline{q}$ as an environmental tax associated with a lower cost c_P for using a standard polluting technology).

The number of visitors and that of firms are assumed to be constant over time, and $x(t) \in [0, 1]$ denotes the fraction of visitors adopting choice $V1$ at time t, $y(t) \in [0, 1]$ denotes the fraction of firms choosing $F1$ at time t. Of course, the complementary fractions choose the opposite strategy, i.e. a binary game is assumed.

Following *ADG*, the quality of the environment in region R at each time t is characterized by a given measure of pollution level, denoted by $Q(t)$, which can be compared with a threshold level $\overline{Q}$ used to distinguish a good from a bad environment at time t, according to $Q(t) < \overline{Q}$ or $Q(t) > \overline{Q}$ respectively. Let $\theta(y)$ be the probability that $Q > \overline{Q}$, assumed to be a decreasing function of the number of firms that choose to adopt non-polluting technologies. *ADG* assume

$$\theta(y) = 1 - y, \tag{1}$$

i.e. a linear dependence such that $\theta(1) = 0$, that is the environmental quality target is surely reached if all firms choose strategy $F1$, i.e. all adopt non-polluting technologies, whereas if no firms adopt such technologies, i.e. all choose strategy $F2$, then $\theta(0) = 1$, that is, pollution is above the threshold level and the environmental quality target is not reached for sure.

According to *ADG*, the expected payoffs associated with the four strategies are

$$EV_1(x, y) = -\widetilde{p}(x) + \widetilde{r}(x, y)\,\theta(y)$$
$$EV_2(x, y) = -\overline{p}$$

where $\overline{p} > 0$ is the price of the ticket for visitors choosing option $V2$, $\widetilde{p}(x) > 0$ is the price of the *EC* purchased by visitors choosing option $V1$, $\widetilde{r_v}(x, y)$ is the reimbursement that the *PA* gives to the visitors that owe the *EC* when $Q \geq \overline{Q}$. In *ADG* the following linear functions are proposed:

$$\widetilde{p}(x) = \overline{p} + \alpha + \beta x; \qquad \widetilde{r}(x, y) = \gamma - \delta x - \varepsilon y$$

where $\alpha \geq 0$ and $\beta > 0$, $\gamma > 0$, $\varepsilon > 0$, $\delta \gtreqless 0$ are parameters controlled by the *PA*: α is the minimum difference between the price of the *EC* and the fixed ticket, obtained for $x = 0$; β indicates the positive correlation between the price of the *EC* and the number of visitors asking to buy them, i.e. *EC* demand; the positive coefficient ε indicates that the amount of money refunded to visitors in the case of high pollution is negatively related to the number of non-polluting firms, i.e. the same number as influences the environmental quality. The sign of the coefficient δ is undetermined because two opposite effects coexist: from one side if the number of visitors asking for the *EC* increases, then the *PA* collects more money to be used for reimbursement; on the other side if more visitors hold the *EC* then more must be refunded when $Q > \overline{Q}$.

Analogously, the expected payoffs for firms are

$$EF_1(x, y) = -c_{NP} + \widetilde{r}_F(x, y)(1 - \theta(y))$$
$$EF_2(x, y) = -c_P - \overline{q}$$

where $\overline{q}$ is the environmental tax that polluting firms must pay, $c_P > 0$ and $c_{NP} > 0$ have been defined above, and

$$\widetilde{r}_F(x, y) = \lambda + \mu x + \nu y,$$

where $\lambda \geq 0$, $\mu > 0$ and $\nu \gtreqless 0$, are coefficients fixed by the *PA*; μ indicates a positive correlation between the number of visitors that purchase the *EC* and the amount of financial aids given by the *PA* to firms adopting non-polluting technologies whenever $Q < \overline{Q}$, whereas the undetermined sign of ν depends on contrasting effects related to the number of firms to be refunded by the *PA* in the case of $Q < \overline{Q}$ and the fact that increasing y means more probability of reaching the desired environmental quality.

All in all, if these assumptions are inserted into the dynamic equations that express the "replicator dynamics" (see [2–4]):

$$\dot{x} = x\left(EV_1 - \overline{EV}\right) = x(1 - x)(EV_1 - EV_2) \tag{2}$$
$$\dot{y} = y\left(EF_1 - \overline{EF}\right) = y(1 - y)(EF_1 - EF_2)$$

where the expression for average payoffs $\overline{EV} = xEV_1 + (1 - x)EV_2$; $\overline{EF} = yEF_1 + (1 - y)EF_2$ has been used, one gets the dynamic equations of the *ADG* model:

$$\dot{x} = x(1 - x)\left[\gamma - \alpha - (\beta + \delta)x - (\varepsilon + \gamma)y + \delta xy + \varepsilon y^2\right] \tag{3}$$
$$\dot{y} = y(1 - y)\left[-(c_{NP} - c_P - \overline{q}) + \lambda y + \mu xy + \nu y^2\right].$$

The following further conditions on the parameters are given in *ADG* to avoid trivial dynamic behaviours of the model.

(a) $EV_1(1, 0) - EV_2(1, 0) > 0$, that is, if $y = 0$ then purchasing the *EC* must be convenient for visitors even if $x = 1$; this condition becomes

$$\gamma - \alpha - \beta - \delta > 0. \tag{4}$$

(b) Even if $\delta < 0$, a quota of the money obtained by the *PA* from *EC* must return to subscribers if $Q > \overline{Q}$, i.e.

$$\beta + \delta > 0. \tag{5}$$

(c) The non-polluting technologies (strategy $F1$) are more costly than fixed taxes and polluting technologies (strategy $F2$)

$$c_{NP} - c_P - \overline{q} > 0. \tag{6}$$

(d) If environmental quality is reached, i.e. $Q < \overline{Q}$, then the adoption of non-polluting technologies must be convenient (thanks to the financial aid of *PA*) even if $x = 0$, i.e.

$$\lambda + \nu - (c_{NP} - c_P - \overline{q}) > 0. \tag{7}$$

Under these conditions, the following result is given in *ADG*:

Proposition 1. *Six equilibrium points exist along the boundary of the phase space* $[0, 1]^2$*, given by the four vertices* $0 = (0, 0)$, $G = (0, 1)$, $I = (1, 1)$, $B = (1, 0)$*, as well as two further equilibria* $S_0 = (0, y_4)$ *and* $S_1 = (1, y_3)$*. G and B are stable nodes and the other ones are saddle points. Moreover, an interior equilibrium may exist,* $P_0 = (x_0, y_0) \in (0, 1)^2$*, that can be an unstable node, a saddle, an unstable focus or a stable focus. In the latter case an unstable closed invariant curve may exist which bounds the basin of attraction of* P_0.

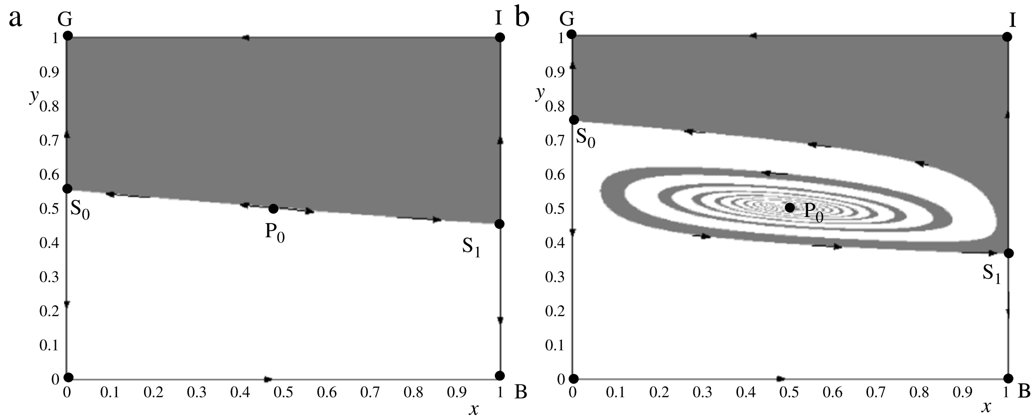

Fig. 1. Equilibrium points and basins for the model (3). (a) With $\alpha = 0$, $\beta = 8.3$, $\gamma = 91$, $\delta = 1.4$, $\varepsilon = 164$, $\lambda = 2.5$, $\mu = 1$, $\nu = 2$, $c_{NP} = 12$, $c_P = 8$, $\overline{q} = 2$ and $\rho = 0.4$, the same set of parameters in [1], example 3. (b) With the same parameters as in (a) except $\rho = 0.7$, so that the interior fixed point P_0 is an unstable focus instead of an unstable node, $C_{NP} = 13$ so that the boundary saddle point S_0 is shifted upwards, and $\mu = 5$ so that the boundary saddle point S_1 is moved downwards, in order to make the spiralling structure of the basins more visible.

3. Multistability and basins of attraction

As shown by *ADG*, and recalled in the previous section, multistability, that is, the coexistence of several attractors, is a feature of the *ADG* evolutionary model. In particular, the "good" equilibrium $G = (0, 1)$, where all firms adopt non-polluting technologies and consequently no visitors are motivated to purchase the *EC* options, always coexists with a "bad" equilibrium $B = (1, 0)$, where no firms adopt non-polluting technologies. This can be considered as a "poverty trap", because it can be seen as the outcome of a "perverse evolution" leading to a situation completely different from the desired one. In *ADG* a short remark is given stating that different structures of the boundaries of the basins of attraction can be obtained. In this section we numerically show some typical structures of the basins of attraction, both in the case of two coexisting stable equilibria, G and B, and in the case of a third one, the inner stable equilibrium P_0.

Following a suggestion given in *ADG*, in order to investigate some bifurcation patterns leading to different structures of the basins, we multiply all the coefficients of the first dynamic equation by a scaling parameter $\rho \in [0, 1]$. In fact, if we consider the Jacobian $J = \left[J_{ij}\right]$ of the model (3) computed at P_0, the multiplication of the first dynamic equation by ρ leaves unchanged the sign of the determinant, as the new one is given by $\rho^2 \det J$, whereas the new trace becomes $\rho J_{11} + J_{22}$, so a change of its sign, leading to instability of P_0 for decreasing values of ρ, may occur at $\rho = \overline{\rho} = -\frac{J_{22}}{J_{11}}$ provided that $\overline{\rho} \in (0, 1)$. For example, starting from the same set of parameters as are proposed in *ADG*, example 3, namely $\alpha = 0$, $\beta = 8.3$, $\gamma = 91$, $\delta = 1.4$, $\varepsilon = 164$, $\lambda = 2.5$, $\mu = 1$, $\nu = 2$, $c_{NP} = 12$, $c_P = 8$, $\overline{q} = 2$, we have $\overline{\rho} = 0.5556$ (as computed in *ADG* as well). In Fig. 1(a), obtained with $\rho = 0.4$, the interior equilibrium P_0 is an unstable node, and the boundary that separates the two basins is determined by the union of the stable sets of the two saddle points along the vertical boundaries, joining at P_0. This is the simplest situation, as shown in Fig. 1(a) where the grey region represents the basin of attraction of the "good" equilibrium $G = (0, 1)$ and the white region is the basin of the "bad" equilibrium $B = (1, 0)$.

Fig. 1(b), obtained after changing of the two parameters $\mu = 5$, $c_{NP} = 13$, and with $\rho = 0.7$, shows a more involved structure of the boundaries that separate the two basins. In fact, with this value of ρ the interior fixed point P_0 is an unstable focus instead of an unstable node, the increase of C_{NP} causes an upward shift of the boundary saddle point S_0 and due to the increase of μ the other boundary saddle point S_1 is moved downwards. These two opposite displacements of the boundary saddles are forced in order to make the spiralling structure of the basins more visible, so that the reader can more easily appreciate the uncertainty induced by the involved structure of the basins' boundaries. The effects on the boundary equilibria of these two parameters can be easily deduced from their expressions

$$S_0 = \left(0, \frac{-\lambda + \sqrt{\lambda^2 + 4v\left(C_{NP} - C_P - \overline{q}\right)}}{2v}\right)$$

and

$$S_1 = \left(1, \frac{-(\lambda + \mu) + \sqrt{(\lambda + \mu)^2 + 4v\left(C_{NP} - C_P - \overline{q}\right)}}{2v}\right)$$

from which it is straightforward to see that increasing C_{NP} both S_0 and S_1 shift upward along the vertical lines $x = 0$ and $x = 1$ respectively, whereas if we increase the parameter μ only S_1 moves downward along the line $x = 1$. Of course, decreasing values of ρ will further accentuate the spiral shape of the basins' boundaries.

Instead, if ρ is increased starting from the set of parameters of Fig. 1(b), the interior equilibrium undergoes a subcritical Hopf bifurcation at $\rho = \overline{\rho} = 0.\overline{7}$, at which it becomes a stable focus surrounded by an unstable closed invariant curve which

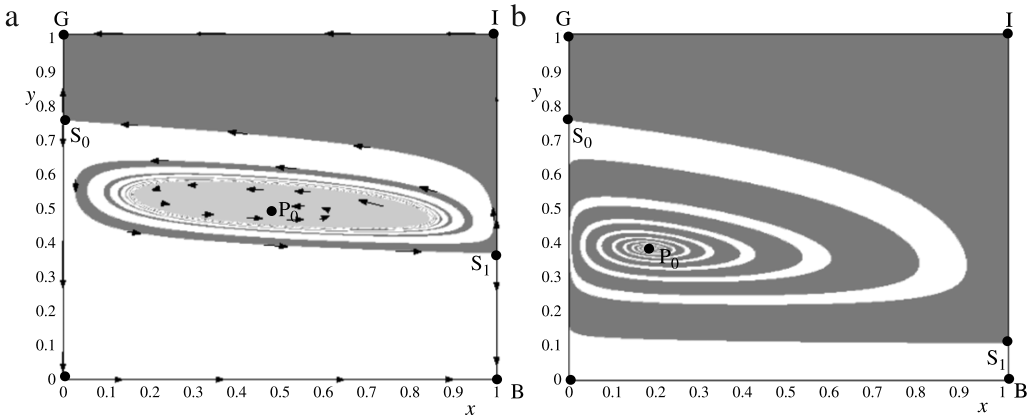

Fig. 2. Equilibrium points and basins for the model (3) (a) with the same set of parameters as in Fig. 1(b) and $\rho = 0.9 > \overline{\rho} = 0.\overline{7}$, and (b) with the same set of parameters as in (a) except $\alpha = 16$, that causes the loss of stability of P_0, and $\mu = 25$, that causes a downward shift of S_1 which is now quite close to the "bad" equilibrium B.

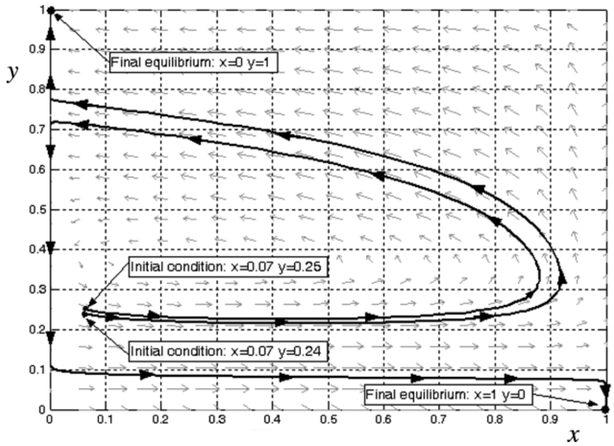

Fig. 3. With the same set of parameters as in Fig. 2(b), two typical trajectories that, even starting from two initial conditions which are quite close, evolve towards the good and the bad equilibrium respectively.

constitutes the boundary of the basin of P_0. This is the situation shown in Fig. 2(a), obtained with the same set of parameters as Fig. 1(b) and $\rho = 0.9$, where the basin of P_0 is represented by the light grey region. In this case the stable sets of the two saddle points located on the vertical boundaries, that still form the boundary that separates the basins of G and B, wing around the unstable invariant curve from outside.

It is worth noticing that even if the *PA* can control the parameters so that the inner "sub-optimal" equilibrium is not stable and the basin of the "bad" equilibrium $B = (1, 0)$ shrinks, according to Proposition 1 it remains stable for each set of the parameters. Moreover, even if the extension of its basin is reduced, such a basin may be quite intermingled with the one of the "good" equilibrium $G = (0, 1)$ and consequently its presence remains quite disturbing. An example is shown in Fig. 2(b), obtained with the same set of parameters as in Fig. 2(a) except $\alpha = 16$, that causes the loss of stability of P_0, and $\mu = 25$, that causes a downward shift of S_1 which is now quite close to the "bad" equilibrium B. It is rather evident that even if we have a smaller extension of the basin of the bad equilibrium, the good one is still very vulnerable because points of the white basin are quite close to it.

Fig. 3, obtained with the same set of parameters as in Fig. 2(b), exhibits two typical trajectories that, even starting from two initial conditions which are quite close, evolve towards the good and the bad equilibrium respectively. This kind of dynamic evolution can be observed in all the cases shown in the previous pictures, and may be quite misleading because the first portions of the two trajectories are almost identical, both characterized by an increase of the fraction of visitors choosing the strategy $V1$, i.e. purchasing the *EC* options. Then the two trajectories suddenly turn towards opposite reactions of firms.

4. The effects of a direct environmental protection by the Public Administration

As argued in the previous sections, the financial market of environmental put and call options offered by the Public Administration to firms and visitors respectively, as described in the *ADG* model, may trigger a virtuous and self-sustained endogenous evolution leading to a final situation where all firms adopt non-polluting technologies and, consequently, the environmental quality becomes so good that no visitors are attracted by the *EC* options. However, this is not the only possible evolution in the long run, because for any set of parameters there is space for "perverse" evolutions towards a stable "bad" equilibrium, characterized by no firms adopting environmentally friendly technologies and, consequently, high levels of pollution. Moreover, other situations may be obtained where the system evolves towards intermediate (say sub-optimal) situations, leading to high degrees of uncertainty due to the fact that the final outcome is path dependent, with different outcomes obtained as the consequence of (even small) changes of the initial conditions of the evolutionary game.

It is worth stressing that in the *ADG* model the Public Administration only has an indirect role of intermediary between firms and visitors, just to set up the financial market of put and call environmental options with proper prices, as well as deciding alternative fixed taxes for both. In this section we assume that the Public Administration, besides its action as an intermediary between visitors and polluting firms for trading environmental financial options, also performs a direct action to control pollution, for example by (even moderate) interventions for environmental damages remediation. As we shall see, such direct action, properly combined with the regulation of financial market of environmental options described above, may enhance the attractivity of the desired equilibrium and reduce, or even eliminate, perverse and sub-optimal long-run evolutions. Of course, the direct environmental remediation performed by the *PA* has a cost, but we shall prove that it can always be covered by properly fixing the price $\overline{p}$ of the ticket paid by visitors, so that a "balanced budget" for the Public Administration can always be obtained.

In order to model the direct action of the Public Administration let us modify the probability $\theta(y)$ as follows:

$$\theta\left(y\right) = 1 - a - by$$

were $a \in [0, 1]$ represents the effect of direct action of Public Administration on the pollution level and $b \in [0, 1]$ modulates the effect on the environmental quality of the fraction of firms choosing non-polluting technologies. With this modification we have $\theta\left(0\right) = 1 - a$, i.e. if no firms adopt environmentally friendly techniques the probability that $Q \geq \overline{Q}$ can decrease only thanks to the direct action of the Public Administration. On the other side $\theta\left(1\right) = 1-a-b$; hence the usual assumption $\theta\left(1\right) = 0$, i.e. if all firms adopt non-polluting techniques then the environmental quality goal is surely reached, implies that $a + b = 1$, i.e. $a = 1 - b$. So, the probability that $Q \geq \overline{Q}$ becomes

$$\theta\left(y\right) = b\left(1 - y\right) \tag{8}$$

where a small value of b means a high direct intervention of the Public Administration, whereas for $b = 1$ the same expression for the *ADG* model, without any direct action of the Public Administration, is obtained. This means that (8) is a generalization of (1).

In the following we show how this change is reflected in the dynamic properties of the evolutionary model, in particular in the stability of the steady states and the structure of the basins.

Of course, the model proposed in this section is quite similar to the one of *ADG*, and reported in the previous section, so in what follows we only stress the differences between the two models as the new parameter b varies. All the other parameters have the same meaning as in the previous section. With (8), the expected payoffs become

$$EF_1\left(x, y\right) = -c_{\mathrm{NP}} + \left(\lambda + \mu x + \nu y\right)\left(1 - b + by\right)$$

whereas EF_2 does not change. Analogously, we get

$$EV_1\left(x, y\right) = -\overline{p} - \alpha - \beta x + \left(\gamma - \delta x - \varepsilon y\right)\left(b - by\right)$$

and EV_2 remains the same. So, the complete dynamic model becomes

$$\begin{aligned} \dot{x} &= x\left(1 - x\right)\left[-\alpha - \beta x + \left(\gamma - \delta x - \varepsilon y\right)\left(b - by\right)\right] := x\left(1 - x\right)F(x, y) \\ \dot{y} &= y(1 - y)\left[-\left(c_{\mathrm{NP}} - c_P - \overline{q}\right) + \left(\lambda + \mu x + \nu y\right)\left(1 - b + by\right)\right] := y(1 - y)G(x, y). \end{aligned} \tag{9}$$

Some conditions on the parameters are modified, due to the presence of the parameter b. The condition (4) becomes

$$b\gamma - \alpha - \beta - b\delta > 0 \tag{10}$$

which states that if $y = 0$, i.e. all firms are polluting, then the visitors purchasing the *EC* will be refunded even if $x = 1$. It should be noted that the amount of money refunded to visitors is no longer $\gamma - \alpha - \beta - \delta$, like in the *ADG* model, but is smaller, because the *PA* uses some money to pay for its direct action to repair pollution damages, a cost for the *PA* which is higher for smaller values of b. Notice that (10) can be written as

$$b > \frac{\alpha + \beta}{\gamma - \delta}, \tag{11}$$

i.e. the price of the *EC*, given by $\alpha + \beta$, cannot be greater that its expected value, $b - by$ (which becomes equal to b when $y = 0$), times $(\gamma - \delta)$; otherwise no rational visitor will decide to buy the *EC*.

Conditions (5)–(7) remain the same, and the following new condition is imposed:

$$\lambda\left(1 - b\right) - \left(c_{\mathrm{NP}} - c_P - \overline{q}\right) < 0 \tag{12}$$

which states that the cost expected by a non-polluting firm (given by the cost of non-polluting technologies minus the expected financial support given by the *PA* if the environmental quality target is reached) must be greater than the cost of a polluting firm when $y = 0$ and $x = 1$, i.e. all other firms are polluting and all visitors prefer to pay for the ticket. This condition is imposed in order to rule out opportunistic behaviours.

As we shall see, the (undetermined) sign of the expression

$$L(b) = (\lambda + \mu)(1 - b) - \left(c_{\text{NP}} - c_p - \overline{q}\right), \tag{13}$$

will be important in the study of the stability of the "bad" equilibrium. Its meaning is the following. If all visitors decide to purchase the *EC*, then the *PA* gets money that can be used both for the financial support of non-polluting firms and for direct intervention to repair environmental damages. From condition (6), $L(1) < 0$. However, for decreasing values of b, i.e. if the *PA* decides to increase its direct intervention, of course taking into account (11), $L(b)$ can change its sign and, as we shall see, this has an important role in forcing the "bad" equilibrium to lose its stability, so that "perverse" trajectories no longer occur. This is proved in the next section.

4.1. Existence and stability of equilibrium points

Besides the four equilibrium points located at the vertices of the phase space $[0, 1]^2$, the existence of other equilibria of the model (9) can be obtained through a study of the nullclines $F(x, y) = 0$ and $G(x, y) = 0$. Indeed, boundary equilibria are located at the intersections of $F(x, y) = 0$ with the horizontal edges $y = 0$, $y = 1$ (with $0 < x < 1$) or at the intersections between $G(x, y) = 0$ and the vertical edges $x = 0$, $x = 1$ (with $0 < y < 1$). Moreover, the intersections between the two nullclines inside $(0, 1)^2$ define interior equilibria.

The nullcline $F(x, y) = 0$ is given by the following function:

$$x = \varphi(y) = \frac{(\gamma - \varepsilon y)(b - by) - \alpha}{\beta + \delta(b - by)} \tag{14}$$

and the nullcline $G(x, y) = 0$ can be defined as

$$x = \psi(y) = \frac{c_{\text{NP}} - c_P - \overline{q}}{\mu(1 - b + by)} - \frac{\lambda + \nu y}{\mu}. \tag{15}$$

Along the horizontal edges $y = 0$ and $y = 1$ there are no equilibria, because $\varphi(1) = \frac{-\alpha}{\beta} < 0$, and $\varphi(0) = \frac{\gamma b - \alpha}{\beta + \delta b} > 1$ due to condition (10). Instead, along the vertical edges $x = 0$ and $x = 1$ two boundary equilibria may exist. In fact, from $\psi(y) = 0$ we get

$$\nu b y^2 + [\lambda b + \nu(1 - b)]\, y - [A - \lambda(1 - b)] = 0$$

where $A = c_{\text{NP}} - c_P - \overline{q}$, and hence $[A - \lambda(1 - b)] > 0$ for (12). The real and positive solution

$$y_3 = \frac{-(\lambda b + \nu(1 - b)) + \sqrt{(\lambda b + \nu(1 - b))^2 + 4\nu b\,(A - \lambda(1 - b))}}{2\nu b},$$

gives an equilibrium $S_0 = (0, y_3)$ provided that $y_3 \leq 1$, i.e. $A \leq \lambda + \nu$, which is always true for (7).

Analogously, $\psi(y) = 1$ gives

$$\nu b y^2 + [(\lambda + \mu)b + \nu(1 - b)]\, y + L(b) = 0$$

where $L(b)$ is given in (13). If $L(b) < 0$ then

$$y_4 = \frac{-((\lambda + \mu)b + \nu(1 - b)) + \sqrt{((\lambda + \mu)b + \nu(1 - b))^2 - 4\nu b L(b)}}{2\nu b}$$

is real and positive, and $y_4 = 0$ for $L(b) = 0$. Moreover, if $y_4 \leq 1$, i.e. $-L(b) \leq \nu + (\lambda + \mu)\, b$, then an equilibrium $S_1 = (1, y_4)$ exists along the vertical edge $x = 1$.

The following result concerns the stability of the boundary equilibria, and should be compared with Proposition 1.

Proposition 2. *For the dynamical system (9) with parameters satisfying the conditions (5)–(7), (10) and (12), $0 = (0, 0)$ and $I = (1, 1)$ are saddle points, with the stable set along the vertical edge and the unstable set along the horizontal edge; $S_0 = (0, y_4)$ and $S_1 = (1, y_3)$, when they exist, are saddle points with unstable sets along the vertical edges.*

The equilibrium point $G = (0, 1)$ is a stable node.

The equilibrium point $B = (1, 0)$ is a stable node if $b_T < b < 1$, where

$$b_T = 1 - \frac{c_{\text{NP}} - c_P - \overline{q}}{\lambda + \mu}$$

and B is a saddle point for

$$b < b_T.$$

If $b = b_T$ then $y_3 = 0$, i.e. $S_1 \equiv B$, and the two equilibria undergo a transcritical (or stability exchange) bifurcation.

Remark. The bifurcation condition $b = b_T$ corresponds to $L(b) = 0$, and hence the stability loss of the "bad" equilibrium B corresponds to a change of sign of $L(b)$ from negative to positive.

Proof. The Jacobian matrix of (9) computed at the equilibrium $O = (0, 0)$ becomes

$$J(0,0) = \begin{pmatrix} \gamma b - \alpha & 0 \\ 0 & -\left(c_{np} + p_p - c_p - \overline{q}\right) + \lambda(1-b) \end{pmatrix}.$$

The first eigenvalue $\gamma b - \alpha > \beta + b\gamma > 0$ for (10) and the second eigenvalue $-\left(c_{np} + p_p - c_p - \overline{q}\right) + \lambda(1-b) < 0$ for (12). Then O is a saddle point.

At the equilibrium point $I = (1, 1)$ the Jacobian matrix is

$$J(1,1) = \begin{pmatrix} \alpha + \beta & 0 \\ 0 & (c_{NP} - c_P - \overline{q}) - (\lambda + \mu + \nu) \end{pmatrix}.$$

It does not depend on b, and as in the *ADG* model it is a saddle point due to condition (7).

Also at $G = (0, 1)$ the Jacobian does not depend on b, being

$$J(0,1) = \begin{pmatrix} -\alpha & 0 \\ 0 & (c_{NP} - c_P - \overline{q}) - (\lambda + \nu) \end{pmatrix}.$$

As in the *ADG* model, both of the eigenvalues are always negative for (7), and hence G is a stable node.

Finally, from

$$J(1,0) = \begin{pmatrix} \alpha + \beta + b(\delta - \gamma) & 0 \\ 0 & (c_{NP} - c_P - \overline{q}) - (\lambda + \mu)(1-b) \end{pmatrix}$$

it follows that the eigenvalue $\alpha + \beta + b(\delta - \gamma) < 0$ because of condition (10), whereas the second eigenvalue, associated with the eigenvector along the invariant line $y = 1$, is negative for $b > b_T$, and positive for $b < b_T$. It is easy to check that for $b = b_T$ we have $\psi(0) = 1$, i.e. $S_1 \equiv B$. This concludes the proof. □

4.2. Global dynamic properties and "environmental policy implications"

The results of the previous section show that a suitable direct action of the *PA* for pollution control, associated with its indirect action as promoter of the environmental financial assets, can eliminate the presence of the "bad" attractor through a local bifurcation that transforms it into an unstable equilibrium. In this section we show, by some numerical simulations, that decreasing values of the parameter b may induce important changes in the global dynamic scenarios as well.

For example, Fig. 4(a), obtained with the set of parameters $\alpha = 0.2$, $\beta = 0.4$, $\gamma = 5.5$, $\delta = 0.3$, $\varepsilon = 0.1$, $c_{NP} = 11.3$, $c_P = 8$, $\overline{q} = 2$, $\lambda = 1$, $\mu = 0.8$, $\nu = 0.31$ and $b = 0.3$, shows a situation where the structure of the basins of attraction is such that the "good" equilibrium $G = (0, 1)$, whose basin is represented by the grey region, is very vulnerable because some portions of the white basin of the "bad" equilibrium $B = (1, 0)$ are very close to it. However, a computation of the bifurcation value $b_T = 1 - \frac{c_{NP} - c_P - \overline{q}}{\lambda + \mu} = 1 - \frac{0.3}{1.8} = 0.2\overline{7}$ suggests that a slight decrease of b, i.e. a slight increase of the direct action of the *PA* for pollution abatement, will eliminate this problem. This can be seen by the numerical computation of the basins shown in Fig. 4(b), obtained with $b = 0.27$, where the good equilibrium $G = (0, 1)$ is the unique global attractor, i.e. its basin covers the whole region $(0, 1)^2$.

The beneficial effects of decreasing values of b can also be appreciated starting from a situation with three attractors, like the one shown in Fig. 2(a). Such a situation can be seen as a benchmark case, for the model considered in this paper, obtained with $b = 1$, i.e. no direct intervention of the *PA*. If b is decreased, the basin of attraction of the interior sub-optimal equilibrium, bounded by a closed invariant repelling curve, shrinks (see Fig. 5(a), obtained with the same set of parameters as Fig. 2(b) and $b = 0.7$) until it disappears after a subcritical Hopf bifurcation occurs, at which the interior equilibrium becomes an unstable focus (see Fig. 5(b), obtained for $b = 0.65$). It is quite evident that the dynamic situation is now characterized by a higher degree of evolutionary stability of the "good" equilibrium. Moreover, if b is further decreased until the bifurcation value $b_T = 1 - \frac{c_{NP} - c_P - \overline{q}}{\lambda + \mu} = 1 - \frac{3}{7.5} = 0.6$ is crossed, then $G = (0, 1)$ remains the unique global attractor, and a situation similar to the one shown in Fig. 4(b) is obtained.

A final remark is necessary in order to state the economic sustainability of a direct action included in the *PA* environmental policy. In fact, a direct pollution abatement has a cost, so a source of income that covers the expenses for refunding non-polluting firms as well as to finance the direct action for pollution abatement must be found. Indeed, a balanced budget can always be obtained by the *PA* by properly tuning the ticket $\overline{p}$ that visitors are asked to pay. In fact, $\overline{p}$ does not appear at all in the dynamic Eq. (9), and hence it does not have any influence on the dynamics of the model even if it increases the income of the *PA*. So, it can be used as an exogenous parameter to control the budget balance.

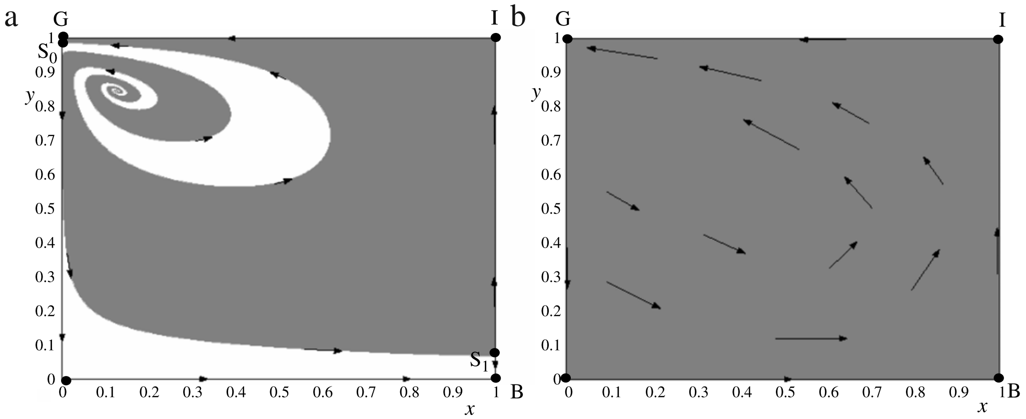

Fig. 4. Equilibrium points and basins of the model (9). (a) With parameters $\alpha = 0.2$, $\beta = 0.4$, $\gamma = 5.5$, $\delta = 0.3$, $\varepsilon = 0.1$, $c_{NP} = 11.3$, $c_P = 8$, $\bar{q} = 2$, $\lambda = 1$, $\mu = 0.8$, $\nu = 0.31$ and $b = 0.3$. (b) With $b = 0.27$ and all the other parameters at the same values as in (a).

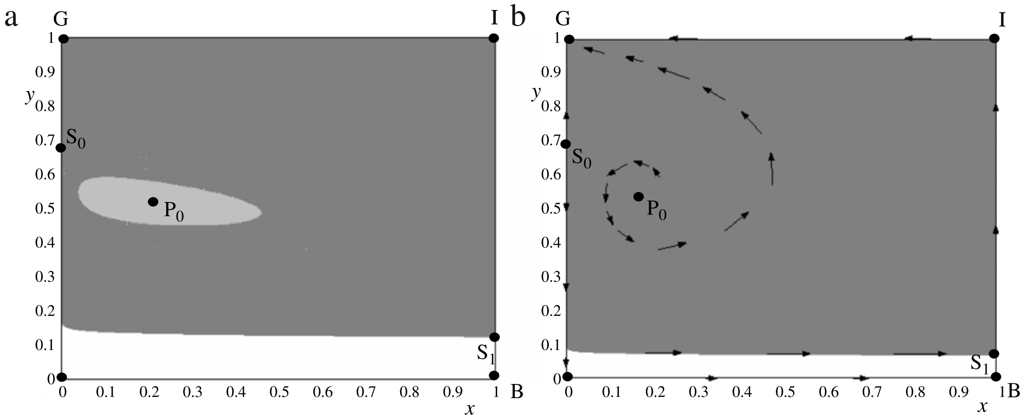

Fig. 5. Equilibrium points and basins of the model (9). (a) With the same set of parameters as in Fig. 2(b) and $b = 0.7$. (b) With $b = 0.65$.

5. Conclusions

We have proposed a modification of the evolutionary model proposed by Antoci et al. [1] to describe the time evolution of a system where a Public Administration tries to reconcile the presence of polluting firms and visitors in a tourist region by adopting financial instruments to induce environmental maintenance. We have introduced in the model an additional parameter that describes the possibility of a moderate direct intervention of the Public Administration in the pollution control, added to its role as intermediary in the market of environmental financial options. We have shown that by properly tuning this parameter the presence of bad asymptotic evolutions leading to a high pollution trap, always present in the original model, can be eliminated through a transcritical bifurcation. Moreover, by a numerical analysis of the global dynamical behaviour of the modified model, we have shown some beneficial effects on the structure of the basins of attraction as well. The model analysed in this paper can also be applied to different situations, not only for tourists but also for citizens living in an industrial town, where the Municipality may add to its usual direct action against pollution a role as intermediary for some kinds of financial options in order to set up a virtuous trade-off between the way citizens pay taxes and financial support to firms that adopt less polluting technologies. As shown in our model, a suitable trade-off between the direct action of the Public Administration and the adoption of financial instruments is necessary to help the virtuous dynamic process leading to the desired environment quality goals.

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