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Research paper

A dynamic model of adaptive consumers with endogenous unimodal preferences[☆]

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ABSTRACT

We consider a discrete-time dynamic model to describe the repeated choices of adaptive consumers that at each time period adjust their request for a given good according to the discrepancies between the observed price and the fair price perceived on the basis of their utility function. Moreover, consumers' preferences are endogenously modified on the basis of past consumption experience. The model considered is derived from the one proposed in D'Orlando and Rodano (2006), with a different assumption about the way the utility function changes according to the past consumption. In fact, in D'Orlando and Rodano (2006) the consumption preferences increase whenever past consumption increases, whereas in the model proposed in this paper a saturation effect is introduced, so that the same assumption holds for low and moderate past consumption, whereas current consumption decreases if the quantity consumed in the previous time period was too high. This leads to a unimodal preference function instead of an increasing one, which implies that the two-dimensional map, whose iteration represents the time evolution of the consumers' choices, is transformed from an invertible map to a noninvertible one. Hence different global dynamic properties are obtained that influence the structure of the attractors and basins of attraction. These global dynamic features, described by the method of critical curves, interact with the property that the map is also characterized by the presence of a denominator that can vanish, giving rise to different kinds of singularities denoted as focal points and prefocal curves in Bischi et al. (1999), Bischi et al. (2003), Bischi et al. (2005), that strongly influence the structure of the basins of attraction. We describe the structure of the basins of attraction, and the contact bifurcations that change the qualitative properties of their global structure, by using geometric and numerical methods guided by the combined study of critical and prefocal curves.

1. Introduction

In this paper we re-consider a dynamic model proposed in [1] to describe the iterated choices of boundedly rational consumers that at each time period update their consumption choices on the basis of the observed discrepancy between

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expected price (expressed as the relative utility gain due to good consumption) and the current one. Moreover, the utility function is updated according to the consumption choice of the previous period, i.e. the consumers are assumed to learn from past consumption. This evolutionary approach to consumers' behavior departs from the one prevailing in the traditional economic theory based on the assumption that consumers are *rational agents*, that is, they decide their consumption of a given good according to the maximization of a given utility function, that depends on the quantities of goods consumed, subject to budget constraints. This requires that consumers have a complete knowledge of their personal utility function, are able to fully exploit all the available information on the economic parameters involved (e.g. prices and budget) and have computational skillness required to solve the optimization problem. However, many authors have recently argued that these assumptions are too strong, and economic models should take into account limited human ability to solve maximization problems as well as the uncertainties that force real people to adapt to circumstances (see e.g. [2]). This leads to the weaker concept of *bounded rationality*, which assumes that agents make repeated choices following a trial and error (or adaptive) method, and at each time they correct the previous choices on the basis of their observations. Sometimes this repeated adaptive process converges to an equilibrium point that corresponds to the same choice of a rational agent, i.e. the agents learn to behave rationally in the long run. This possibility may be seen as an "evolutionary explanation" of the assumption of rational behavior. In this case we say that boundedly rational agents are able to learn from their past experience and become rational in the long run. In the seminal paper [3] such an "evolutionary approach" is described to explain how non fully rational economic agents (firms in that case) follow a "Darwinian" evolution, characterized by adaptive trial and error methods, that may lead them to converge to a rational behavior (i.e. they learn to be rational) in the long run.

Of course, if several stable equilibria coexist, a problem of equilibrium selection arises and the adaptive mechanism proposed becomes a device for the selection of which equilibrium will prevail in the long run. In such a situation the role of initial conditions is crucial, the process becomes path dependent, and the delimitation of the basins of attraction must be considered in the study of the dynamical system. This requires a global analysis of the dynamical system, a problem often approached by a combination of analytical, geometric and numerical methods. In particular, for the model of evolutive consumer proposed in [1], this study of the structure of the basins and their global bifurcations is given in [4], see also [5] and [6].

The time evolution of the adaptive choices of the boundedly rational consumer considered in these papers is represented by the iteration of a nonlinear invertible two-dimensional map, whose steady states represent possible alternative choices of rational consumers. According to [1], all these papers assume that consumers increasingly prefer a good consumed in the past due to habits or skillness gained, i.e. the learning mechanism connecting current to past consumption of the good considered is based on an increasing function (more consumption in the past implies more even now). By contrast, in this paper we assume a non monotonic relation, so that the same assumption holds for low and moderate past consumption, whereas current consumption decreases if the quantity consumed in the previous time period was too high, a typical consumption saturation effect (see [7,8] and [9]). This leads to a unimodal (or one-hump) preference function instead of an increasing one, which implies that the two-dimensional map, whose iteration represents the time evolution of the consumer's choices, is transformed from an invertible map to a noninvertible one. This implies different global dynamic properties of the map that influence the structure of the attractors as well as of the basins of attraction. It is now well known that the global dynamic properties and bifurcations of a noninvertible map of the plane can be usefully described by the method of critical curves (see e.g. [10]). Moreover, as already stressed in [4,5] and [6], the map considered is also characterized by the presence of a denominator that can vanish, giving rise to different kinds of singularities denoted as focal points and prefocal curves in [11–13], whose presence strongly influences the structure of the basins of attraction. Roughly speaking, a *prefocal curve* is a set of points which are mapped (or "focalized"), by the inverse function (if the map is invertible) or by at least one of the inverses (if the map is noninvertible), into a single point, called *focal point,* where a component of the map assumes the form 0/0. In this paper we consider the interaction between critical and prefocal curves, i.e. between the properties of noninvertible maps and those of maps with a vanishing denominator, as a consequence of the assumption of non monotonic function expressing learning of preferences from past consumption experiences.

The paper is organized as follows. In Section 2 a description of the economic dynamic model is given with a general preference function as well as some particular assumptions about it. In Section 3 the existence of equilibrium points is studied and how their local stability conditions are influenced by the shape of the preference function. In Section 4 some definitions related to the basins are given together with a deep analysis of global geometric properties of the dynamic model considered. These definitions and results are applied to the study of the global bifurcations of the basins of the economic dynamic model considered in this paper through some numerical simulations. Section 5 concludes and outlines further studies.

2. The model

Following [1], let us consider the utility function $U(x, y)$ where x is the quantity of a given good and y represents the aggregated quantity of all the other goods that the consumer can buy. The utility function, that loosely speaking represents the satisfaction obtained by the consumer as a consequence of the consumption of the quantities of goods considered, is a smooth function of its arguments. If p is the unitary price of the good considered and the price of all other goods is conventionally taken as a reference unitary price, the budget constraint becomes:

$$px + y = m \tag{1}$$

where m is the amount of money that a consumer can use to buy goods. The rational choice of the consumer is the solution (x^*, y^*) of the problem of maximization of U under the budget constraint (1). If we exclude corner solutions, the rational solution is identified by:

$$MRS = \frac{\partial U/\partial x}{\partial U/\partial y} = p \quad ; \quad px + y = m \tag{2}$$

where MRS is the Marginal Rate of Substitution, that represents the relative gain of utility caused by consumption of a unit of good x with respect to the utility gain for consumption of complementary goods y. Under optimality conditions, in the standard economic theory this ratio represents the fair price . However, under the assumption of bounded rationality, in [1] the consumers are considered unable to compute the solutions of this problem, and they follow a discrete time adjustment process:

$$x_{t+1} = x_t + \mu \left[MRS\left(x_t\right) - p\right] \tag{3}$$

where $\mu > 0$ represents the speed of adjustment. This adaptive process is based on the assumption that at any time period t the quantity x_{t+1} that the consumer decides to buy in the next period, x_{t+1}, is obtained as a correction of the quantity chosen in the current period, x_t, according to the discrepancy between the given price p and the experienced relative utility gain $MRS(x_t)$. It is straightforward to notice that a steady state $x_{t+1} = x_t$ of this process is a rational choice, i.e. a solution of (2).

A utility function which is commonly used in this context, also adopted in [1], is:

$$U(x, y) = x^{\alpha} y^{1-\alpha} \tag{4}$$

known as the Cobb–Douglas function, where the real parameter $\alpha \in [0, 1]$ measures the preference (marginal utility) for good x (if we multiply the quantity x by a factor $k > 0$, then the utility is multiplied by a factor k^{α}). From (4) we get:

$$MRS(x) = \frac{\alpha}{1-\alpha}\frac{y}{x} = \frac{\alpha}{1-\alpha}\frac{m - px}{x} \tag{5}$$

and the adjustment process (3) becomes:

$$x_{t+1} = x_t + \frac{\mu}{1-\alpha}\left(\frac{m\alpha}{x_t} - p\right). \tag{6}$$

In [1] it is also assumed that the consumer's preferences are influenced by past choices, i.e. the preference parameter is endogenized to become a dynamic variable depending on past consumption:

$$\alpha_{t+1} = g\left(x_t\right) \tag{7}$$

and the dynamic model becomes two-dimensional, the evolutive process being represented by the iteration of the two-dimensional map $(x_{t+1}, \alpha_{t+1}) = T(x_t, \alpha_t)$, where:

$$T : \begin{cases} x_{t+1} = x_t + \frac{\mu}{1-\alpha_t}\left(\frac{m\alpha_t}{x_t} - p\right) \\ \alpha_{t+1} = g\left(x_t\right) \end{cases} . \tag{8}$$

Several different assumptions can be made about the function $g(x)$. In [1] the authors assume that $g(x)$ is a continuous and increasing function, i.e. a consumer prefers to consume increasingly more a good according to the quantity consumed in the previous period, due to acquired consumption habits or skills attributable to past consumption. In particular, they propose the following sigmoidal function:

$$g(x) = \frac{1}{k_1 + k_2 \cdot k_3^x} \tag{9}$$

with $0 < k_3 < 1$, $k_2 > 0$ and $k_1 > 1$ in order to ensure $\alpha < 1$ being the range of this function $\alpha \in \left(\frac{1}{k_1+k_2}, \frac{1}{k_1}\right)$. However, as the same authors suggest, different assumptions can be made about the relation between past consumption and current preferences of a given good. For example, we may assume that if past consumption is too high then the consumer becomes tired of that good, or a saturation effect occurs due to a decreased necessity to buy that good or lack of space to store it, etc. However, up to now these different assumptions have not been explored in the literature. So, in this paper we assume that the function $g(x)$ increases only for low values of x and it decreases for higher values of the past consumption. In other words, we assume a continuous unimodal function. As a particular example, to be compared with (9), we propose:

$$g(x) = \frac{1}{k}x^2 e^{-hx} + l \tag{10}$$

with $h > 0$ and $kh^2 > \frac{4}{(1-l)e^2}$ in order to ensure $\alpha < 1$, as the range of (10) is $\alpha \in \left(l, l + \frac{4}{4e^2kh^2}\right)$. The parameter $l > 0$ has been introduced to mimic the shape of the sigmoid function (9)$0 \leq x < \frac{2}{h}$ (the increasing portion of (10)).

In the following we study the existence and stability of steady states of the dynamic process described above, and then, in the cases of coexistence of stable equilibria, we shall focus our attention on the study of their basins of attraction, based on a global view of the dynamical system.

3. Existence of rational equilibria

The steady states of the adaptive model described in the previous section are the fixed points of the map (8) obtained by setting $x_{t+1} = x_t$ and $\alpha_{t+1} = \alpha_t$, i.e. the solutions of the system:

$$\begin{cases} \alpha = \dfrac{p}{m}x \\ \alpha = g(x) \end{cases} . \tag{11}$$

The solutions of this system can be graphically represented as the intersections between a line through the origin, of angular coefficient p/m, and the curve that represents the relation between current preferences and past consumption (see Fig. 1).

It is worth noticing that from the first equation in (11) and (5) we get $MRS = p$, hence any steady state of the model is a rational equilibrium, so convergence to a stable equilibrium means that the choices of a boundedly rational (adaptive) consumer coincide in the long run with the choice of a rational consumer, i.e. such rational behavior can be learned through the trial and error adaptive process.

Concerning the existence of equilibrium points, under the assumption of sigmoidal preference function (9) in [4] has been proved that up to 3 positive fixed points, say $0 < x_1^* < x_2^* \leq x_3^*$ with respective $\alpha_i^* = g(x_i^*)$ can be obtained, one always exists and a further couple can be created through a fold (or saddle node) bifurcation as $\frac{p}{m}$ decreases or increases (see the left panel of Fig. 1). A very similar situation holds if we consider the unimodal function (10), with the only difference that an equilibrium (the one characterized by higher consumption) may belong to the decreasing branch of the function $g(x)$ if the equilibrium consumption is greater than $2/h$. This may imply some differences concerning the stability and local bifurcations of such equilibrium.[2]

Unfortunately, general conditions for the existence of three equilibrium points, related to conditions of tangency between the line and the curve, cannot be expressed analytically, except for the particular case $l = 0$. In this case we can prove the following proposition:

Proposition 1. *The map (8) with preference function (10) if $l = 0$ is characterized by two equilibria with consumptions x_1^* and x_2^* such that $0 < x_1^* \leq \frac{1}{h} \leq x_2^*$ provided that $\frac{kp}{m} \leq \frac{1}{eh}$, and $x_1^* = x_2^* = \frac{1}{h}$ for $\frac{p}{m} = \frac{1}{ekh}$.*

Proof. From the equilibrium condition (11) we get:

$$\frac{1}{k}x^2e^{-hx} = \frac{p}{m}x$$

which is solved by the value $x = 0$, that cannot be an equilibrium value because the map (8) is not defined there. Besides that, by dividing both terms by x we obtain:

$$\frac{1}{k}xe^{-hx} = \frac{p}{m}$$

that can be rewritten as:

$$xe^{-hx} = \frac{kp}{m}$$

where the function at the left hand side is unimodal with maximum at $x^* = \frac{1}{h}$ and maximum value $\frac{1}{eh}$. So, real positive solutions $0 < x_1^* \leq \frac{1}{h} \leq x_2^*$ exist for $\frac{kp}{m} \leq \frac{1}{eh}$, and $x_1^* = x_2^* = \frac{1}{h}$ for $\frac{p}{m} = \frac{1}{ekh}$. □

Concerning the local stability of an equilibrium point $E = \left(x^*, \frac{p}{m}x^*\right)$, let us consider the standard linearization procedure based on the Jacobian matrix of (8):

$$J(x, \alpha) = \begin{bmatrix} 1 - \frac{\mu m\alpha}{x^2(1-\alpha)} & \mu\frac{m-px}{x(1-\alpha)^2} \\ g'(x) & 0 \end{bmatrix} \tag{12}$$

computed at the equilibrium:

$$J(x^*, \frac{p}{m}x^*) = \begin{bmatrix} 1 - \frac{\mu mp}{x^*(m-px^*)} & \frac{\mu m^2}{x^*(m-px^*)} \\ g'(x^*) & 0 \end{bmatrix} \tag{13}$$

[2] In the literature it is called saturation (or bliss) point (see [7–9]).

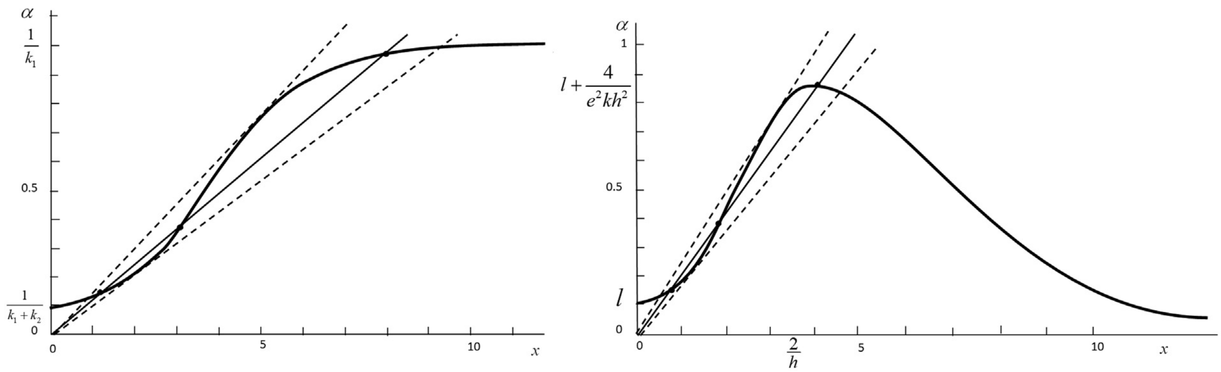

Fig. 1. Left. Sigmoidal preference curve (9) proposed in [1] with $k_1 = 1$, $k_2 = 45$, $k_3 = 0.4$. The case of three equilibrium points is shown (solid line) as well as the two cases of tangent bifurcations (dashed lines). Right. The same for the unimodal preference curve (10) proposed in this paper with $k = 2$, $h = 0.6$ and $l = 0.1$.

where $m - px^* = m(1 - \alpha^*) > 0$. The trace and the determinant of (13), $Tr^* = 1 - \frac{\mu mp}{x^*(m-px^*)}$ and $Det^* = -\frac{\mu m^2 g'(x^*)}{x^*(m-px^*)}$, allow us to state that ${Tr^*}^2 - 4Det^* > 0$ if $g'(x^*) > 0$, i.e. the eigenvalues are always real if the preference curve is increasing at the equilibrium. From the Schur stability conditions (see e.g. [14–16]):

$$1 - Tr^* + Det^* = \frac{\mu m}{m - px^*}\left(p - mg'(x^*)\right) > 0$$

$$1 + Tr^* + Det^* = \frac{1}{x^*(m - px^*)}\left(2x^*(m - px^*) - \mu mp - \mu m^2 g'(x^*)\right) > 0$$

$$1 - Det^* = 1 + \frac{\mu m^2 g'(x^*)}{x^*(m - px^*)} > 0$$

we can state that:

A saddle–node (or fold) bifurcations at which a couple of fixed points is created can only occur when $g'(x^*) > 0$, i.e. x^* belongs to the increasing portion of the preference curve, and the bifurcation is characterized by the tangency condition $g'(x^*) = \frac{p}{m}$;

A Neimark–Sacker bifurcation, at which the equilibrium loses stability and a closed invariant curve is created around it, can only occur if x^* belongs to the decreasing portion of the preference curve, and the bifurcation is characterized by the condition $g'(x^*) = -\frac{x^*(m-px^*)}{\mu m^2}$.

4. Global properties of the map, basins of attraction and numerical simulations

We first recall some basic definitions and properties[3] of the basins of attraction for a discrete dynamical system defined by the iteration of a two-dimensional map $T : (x_t, \alpha_t) \to (x_{t+1}, \alpha_{t+1})$. The point $(x_{t+1}, \alpha_{t+1}) \in \mathbb{R}^2$ is called a rank-1 image of the point (x_t, α_t) under T, and (x_t, α_t) is called a rank-1 preimage of (x_{t+1}, α_{t+1}). A set $A \subset \mathbb{R}^2$ is *trapping* if it is mapped into itself, $T(A) \subseteq A$, i.e. if $(x, \alpha) \in A$ then also $T(x, \alpha) \in A$. A trapping set is *invariant* if it is mapped onto itself: $T(A) = A$, i.e. all the points of A are images of points of A. A closed invariant set A is an attractor if it is *asymptotically stable*, i.e. if a neighborhood U of A exists such that $T(U) \subseteq U$ and $T^t(x, \alpha) \to A$ as $t \to +\infty$ for each $(x, \alpha) \in U$.

The *Basin of an attractor* A is the set of all points that generate trajectories converging to A:

$$\mathcal{B}(A) = \left\{(x, \alpha) | T^t(x, \alpha) \to A \text{ as } t \to +\infty\right\}. \tag{14}$$

Starting from the definition of stability, let U be a neighborhood of an attractor A whose points converge to A. Of course $U \subseteq \mathcal{B}(A)$, but also the points of the phase space which are mapped inside U after a finite number of iterations belong to $\mathcal{B}(A)$. Hence, the *basin* of A is given by the open set $\mathcal{B}(A) = \bigcup_{n\geq 0} T^{-n}(U)$, where $T^0(x, \alpha) = (x, \alpha)$ and $T^{-n}(x, \alpha)$ represents the set of rank-n preimages of (x, α), i.e. the set of points that are mapped into (x, α) after n iterations of the map T. The basin $\mathcal{B}(A)$ is trapping under T and invariant under T^{-1}, i.e.:

$$T^{-1}(\mathcal{B}(A)) = \mathcal{B}(A), \; T(\mathcal{B}(A)) \subseteq \mathcal{B}(A).$$

The boundary $\partial\mathcal{B}(A)$ behaves as a repelling set for the points near it, since it acts as a watershed for the trajectories of the map T. Points belonging to $\partial\mathcal{B}(A)$ are mapped into $\partial\mathcal{B}(A)$ both under forward and backward iteration of T. More exactly:

$$T^{-1}(\partial\mathcal{B}(A)) = \partial\mathcal{B}(A), T(\partial\mathcal{B}(A)) \subseteq \partial\mathcal{B}(A).$$

[3] For a more detailed and rigorous treatment, see e.g. [10].

We remark that $T^{-1}(\partial\mathcal{B}(A)) = \partial\mathcal{B}(A)$ implies that if a curve segment belongs to $\partial\mathcal{B}(A)$ then also all its preimages must belong to $\partial\mathcal{B}(A)$. In particular, $\partial\mathcal{B}(A)$ includes the stable set of any fixed point (or cycle) of T belonging to $\partial\mathcal{B}(A)$. So, in order to study the structure of the boundaries of a basin, the properties of the inverse (or inverses if a map is noninvertible, see e.g. [10]) must be considered.

For the model (8) considered in this paper, if the function $g(x)$ is invertible, then also the two-dimensional map (8) is invertible. In fact, in this case from $\alpha_{t+1} = g(x_t)$ a unique preimage x_t exists, given by $x_t = g^{-1}(\alpha_{t+1})$. After inserting such x_t in the right hand side of the first component and solving with respect to α_t (a first degree algebraic equation) the following solution is obtained:

$$\alpha_t = \frac{(x_{t+1} - g^{-1}(\alpha_{t+1}))g^{-1}(\alpha_{t+1}) + \mu p g^{-1}(\alpha_{t+1}) - \gamma\left(x_r - g^{-1}(\alpha_{t+1})\right)}{\mu m + (x_{t+1} - g^{-1}(\alpha_{t+1}))g^{-1}(\alpha_{t+1})}.$$

For example, for the increasing sigmoid function (9) we get:

$$x_t = g^{-1}(\alpha_{t+1}) = \frac{\ln(1 - k_1\alpha_{t+1}) - \ln(k_2\alpha_{t+1})}{\ln k_3}.$$

On the other hand, if a unimodal preference map $g(x)$ is considered, then the two dimensional map T is a noninvertible $Z_0 - Z_2$ map, as explained in the next subsection. For example, with the function (10), given $\alpha_{t+1} \in \left(l, l + \frac{4}{e^2kh^2}\right)$ two distinct preimages are $0 < x_{t,1} \le \frac{2}{h} \le x_{t,2}$, and if $\alpha_{t+1} = l + \frac{4}{e^2kh^2}$ then $x_{t,1} = x_{t,2} = \frac{2}{h}$; instead, if $\alpha_{t+1} = l$ then $x_{t,1} = 0$ and $x_{t,2} \to \infty$.

4.1. Critical curves

The global properties of a noninvertible map can be studied by using the critical curves LC (from the French "Ligne Critique") defined as the locus of points having two, or more, coincident rank-1 preimages, located in a set denoted by LC_{-1}. Arcs of LC separate regions of the plane characterized by different numbers of rank-1 preimages, say region Z_k, whose points have k distinct preimages, from region Z_{k+2}, as pairs of real preimages appear or disappear crossing through LC. Accordingly, such boundaries are characterized by the presence of two coincident (merging) preimages. LC is the two-dimensional generalization of the notion of critical value (local minimum or maximum value) of a one-dimensional map, and LC_{-1} is the generalization of the notion of critical point (local extremum point). Analogously to the case of differentiable one-dimensional maps, where the derivative necessarily vanishes at the local extremum points, for a two-dimensional differentiable map LC_{-1} belongs to the set of points in which the Jacobian determinant vanishes, i.e. $LC_{-1} \subseteq \left\{(x,\alpha) \in \mathbb{R}^2 | \det J = 0\right\}$, and LC is obtained as the image of LC_{-1}, i.e., $LC = T(LC_{-1})$.

The Jacobian determinant of the map (8) is $\det J(x,\alpha) = -\mu\frac{m-px}{x(1-\alpha)^2}g'(x)$, hence it vanishes along the lines $x = \frac{m}{p}$ and $x = \bar{x}$ whenever $\bar{x}$ exists such that $g'(\bar{x}) = 0$. If the unimodal preference function (10) is considered, then $g'(x) = \frac{1}{k}xe^{-hx}(2 - hx)$, hence $\bar{x} = \frac{2}{h}$. The image of the line $x = \frac{2}{h}$ is a critical curve:

$$LC = T\left(x = \frac{2}{h}, \alpha\right) = \left(\frac{2}{h} + \frac{\mu}{2}\left(\frac{mh\alpha - 2p}{1-\alpha}\right), l + \frac{4}{e^2kh^2}\right)$$

i.e. the line $\alpha = l + \frac{4}{e^2kh^2}$ separates the region $Z_2 = \left\{(x,\alpha) \in \mathbb{R}^2 | \alpha < l + \frac{4}{e^2kh^2}\right\}$, whose points have two rank-1 preimages, from the complementary region Z_0, whose points have no preimages.

Instead, the image of the line $x = \frac{m}{p}$ is a single point:

$$T(x = \frac{m}{p}, \alpha) = Q^{-1} = \left(\frac{m}{p} - \mu p, \frac{m^2}{kp^2}e^{-h\frac{m}{p}} + l\right)$$

i.e. the whole line is "focalized" by T into the point Q_{-1}. By using the terminology introduced in [11], we can say that the line $x = \frac{m}{p}$ is a prefocal line of T^{-1}, as explained in the next subsection. As we shall see, a consequence of this property is that if a chaotic attractor crosses the line $x = \frac{m}{p}$, then it must include a "knot".

4.2. Definition of focal point and prefocal set

Let us consider a two-dimensional map with at least one of the components that contains a denominator which can vanish. This implies that the map is not defined in the whole plane. For example, in the map (8) the first component has a denominator $D(x,\alpha) = x(\alpha - 1)$ that vanishes along the lines $x = 0$ and $\alpha = 1$, on which the map is not defined. Let us denote this as the *set of nondefinition* of the map T:

$$\delta_s = \{(x,\alpha) \in \mathbb{R}^2 | D(x,\alpha) = 0\}. \tag{15}$$

Now let us consider a bounded and smooth simple arc γ transverse to δ_s. In general, the image $T(\gamma)$ is made up of two disjoint unbounded arcs, but a different situation may occur if the point where γ intersects δ_s is such that not only the denominator but also the numerator vanishes in it, as it occurs in the point $Q = (0,0)$ for the map (8). In this case the curve $T(\gamma)$ may be bounded, and the following definition of *focal point* and *prefocal curve* can be given (see [11]):

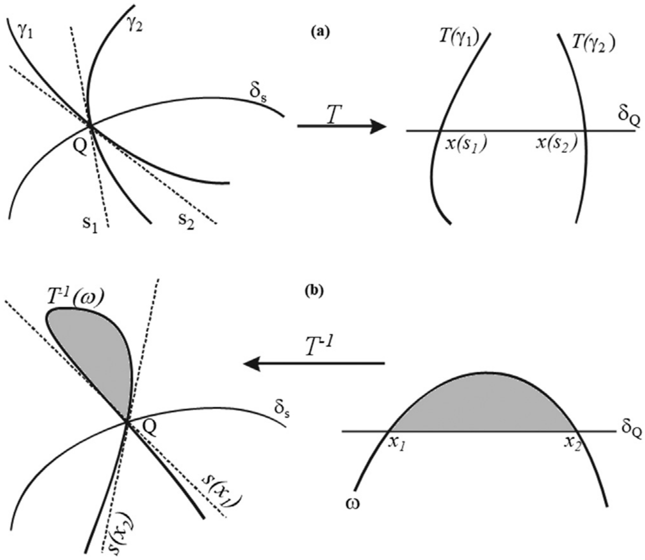

Fig. 2. Upper panel. Arcs through a focal point Q with different slopes are mapped into arcs crossing through δ_Q in different points. Lower panel. A preimage of an arc crossing through the prefocal line δ_Q into distinct points is a loop with double point in the focal point Q.

Definition. *A point Q is a focal point for the map T if at least one component of T takes the form 0/0 in Q and there exist smooth simple arcs γ through Q such that their image $T(\gamma)$ is finite. The set of all the finite images of Q computed along different arcs γ through Q is the prefocal set δ_Q.*

Indeed, let us assume that the first component of the map has the form $\frac{N(x,\alpha)}{D(x,\alpha)}$. The point $Q = (0, 0)$ is a simple focal point, i.e. a simple root of the algebraic system:

$$N(x, y) = 0, \qquad D(x, y) = 0.$$

We recall that Q is simple if $\overline{N}_x\overline{D}_y - \overline{N}_y\overline{D}_x \neq 0$, where $\overline{N}_x = \frac{\partial N}{\partial x}(Q)$ and analogously for the other partial derivatives. In this case the prefocal line $\alpha = g(0)$ where $g(0)$ is the preference function computed in the focal point. Following [11], a one-to-one correspondence is defined between the point $(x, \alpha(0))$, in which $T(\gamma)$ crosses δ_Q, and the slope s of γ in Q, given by:

$$s \to (x(s), g(0)), \qquad \text{with} \qquad x(s) = (\overline{N_x} + s\overline{N_y})/(\overline{D_x} + s\overline{D_y}) \tag{16}$$

and

$$(x, \alpha(0)) \to s(x) \quad \text{with} \quad s(x) = (\overline{D_x}x - \overline{N_x}) \;/\; (\overline{N_y} - \overline{D_y}x). \tag{17}$$

These relations can be obtained by using a method either based on a series expansion of the functions $N(x, y)$ and $D(x, y)$ in a neighborhood of $Q = (x_0, y_0)$, or by considering the Jacobian determinant of the inverse map T^{-1} (or one of the inverses if the map is noninvertible). In fact, from the definition of the prefocal curve, it follows that the Jacobian det $\left(J^{-1}\right)$ must necessarily vanish in the points of δ_Q. Indeed, if the map T^{-1} is defined in δ_Q, then all the points of the line δ_Q are mapped by T^{-1} into the focal point Q. This means that T^{-1} is not locally invertible in the points of δ_Q, being a many-to-one map, and this implies that its Jacobian cannot be different from zero in the points of δ_Q. So, roughly speaking, a *prefocal curve* is a set of points for which at least one inverse exists which maps (or "focalizes") the whole set into a single point, called *focal point* or, more concisely, that $T^{-1}(\delta_Q) = Q$. From the relations (16), (17) it follows that different arcs γ_j, passing through a focal point Q with different slopes s_j, are mapped by T into bounded arcs $T(\gamma_j)$ crossing δ_Q in different points $\left(x(s_j), g(0)\right)$, and interesting properties are obtained if the inverse of T (or the inverses, if T is a noninvertible map) is (are) applied to a curve that crosses a prefocal curve. Let δ_Q be a prefocal curve whose corresponding focal point is Q. Then each point sufficiently close to δ_Q has its rank-1 preimage in a neighborhood of the focal point Q, and if an arc ω crosses δ_Q in two distinct points, say $(x_1, g(0))$ and $(x_2, g(0))$ then its preimage $T^{-1}(\omega)$ must include a loop with double point in Q, as shown in the qualitative picture in Fig. 2.

In the map (8), the first component can be written as:

$$x_{t+1} = \frac{N(x_t, \alpha_t)}{D(x_t, \alpha_t)} = \frac{(1-\alpha_t)x_t^2 - \mu p x_t + \mu m \alpha_t}{x_t(1-\alpha_t)}$$

and it becomes 0/0 in $Q=(0,0)$ and $R=\left(\frac{m}{p},1\right)$. These are both focal points, with corresponding prefocal curves:

$$\delta_Q=\{(x,\alpha)\,|\alpha=l\}$$

and

$$\delta_R=\left\{(x,\alpha)\,|\alpha=\frac{m^2}{kp^2}e^{-\frac{hm}{p}}+l\right\}.$$

The one-to-one relations (16) between slope s (through the focal point) and position $x(s)$ along the corresponding prefocal line are given by:

$$x(s)=\mu ms-\mu p$$

for the focal point Q, and:

$$x(s)=\frac{\left(m^2-\mu mp^2+m\right)s-\mu p^3}{pms}$$

for the focal point R. The existence of such biunivocal relations proves that, indeed, both Q and R are focal points.

The presence of these focal points and corresponding prefocal curves has important effects on the geometrical and dynamical properties of the dynamical system considered. In fact, a contact of an arc ω with a prefocal curve gives rise to important qualitative changes in the shape of the preimages $T_j^{-1}(\omega)$, and when ω is an arc belonging to a basin boundary $\partial\mathcal{B}$, the qualitative modifications of the preimages $T_j^{-1}(\omega)$ of ω, due to a tangential contact of ω with a prefocal curve can be particularly important for the global structure of the basin boundary. In fact, as $\partial\mathcal{B}$ is backward invariant, i.e. $T^{-1}(\partial\mathcal{B})=\partial\mathcal{B}$, if ω is an arc belonging to $\partial\mathcal{B}$, then all its preimages of any rank must belong to $\partial\mathcal{B}$. This implies that if a portion ω of $\partial\mathcal{B}$ crosses a prefocal curve in two points, then the basin boundary must include loops, denoted as "lobes", somewhere along the basin boundary. As we shall see in the next section, this occurrence, together with the contacts and intersections of basin boundaries with critical curves LC, constitute the basic mechanisms leading to the involved structures of the basins of attraction.

4.3. Numerical explorations of basins of attraction and their global bifurcations

In this section we show some numerical explorations of attractors and basins of attraction of the dynamic model (8) with preference function (10), and explain their geometric structures, as well as their qualitative changes (i.e. global bifurcations) as some parameters are varied, on the basis of the global properties and singularities of the map defined in the previous section.

The dynamic situation shown in the left panel of Fig. 3 is obtained with parameters $m=10$, $p=2.5$, $\mu=0.98$ and $k=0.95$, $h=1$ and $l=0.01$ in the preference function. For this parameters' constellation there are three equilibrium points, characterized by rational consumptions $x_1^*=0.05$, $x_2^*=0.26$ and $x_3^*=2.26$ (with corresponding equilibrium preference values $\alpha_1^*\simeq 0.0125$, $\alpha_2^*\simeq 0.064866$ and $\alpha_3^*\simeq 0.571$, respectively). The two equilibria with lower consumption are unstable (saddle point and unstable node respectively) whereas the one with higher consumption, located in the decreasing portion of the preference curve being $x_3^*>2/h$, is a stable focus, denoted by E in the picture, with eigenvalues $\lambda_{1,2}=-0.75\pm 0.35i$ (hence with modulus $|\lambda_i|=0.83$). The basin of attraction of this stable rational equilibrium is represented by the white region, whereas the gray region represents the set of initial conditions that generate unfeasible trajectories because they involve negative values of consumption x. Indeed, the initial conditions with small values of x and α (gray region in the bottom-left) as well as the initial conditions with high values of x and α generate negative values of x according to the model (8). It can be noticed that the boundary of the gray region with small values of x and α, crossing in two points the prefocal line δ_Q of equation $\alpha=l$, also includes a small lobe issuing from the focal point $Q=(0,0)$. A second preimage of it is visible in the upper right portion of the picture, but it generates no further gray preimages because it is in Z_0. However, we can notice that a portion of this gray region representing the set of unfeasible initial conditions, is quite close to the critical line LC. Indeed, if some parameter is varied so that LC moves upwards, then a portion of such gray region will enter the zone Z_2 and new preimages will be created belonging to the basin of unfeasible initial conditions. This is shown in the right panel of Fig. 3, obtained with a slightly decreased value of the parameter k. This parameter change causes a contact between the basin's boundary and LC and then a portion of the gray basin enters Z_2, as indicated by the arrow in Fig. 3, obtained with $k=0.9$. This portion, denoted by H_0, has two preimages located around LC_{-1} (more exactly two preimages joining along LC_{-1}) indicated by H_{-1} in the picture. Following the terminology introduced by [10], we say that this is a hole (or lake) of the gray basin nested inside the white basin of the rational equilibrium E, or equivalently we may say that the white basin from simply connected set has been transformed into a multiply connected set (or connected with holes). This occurrence is not possible when dealing with invertible maps. So, we may say that this kind of qualitative change (or global bifurcation) of the basin of the rational equilibrium is caused by the modification of the preference function from increasing to non monotonic.

No other preimages exist because $H_{-1}\in Z_0$. However, if k is further decreased then LC is further shifted and a portion of H_{-1} enters Z_2 after a contact with LC, as shown in the left panel of Fig. 4. This gives rise to the creation of further holes.

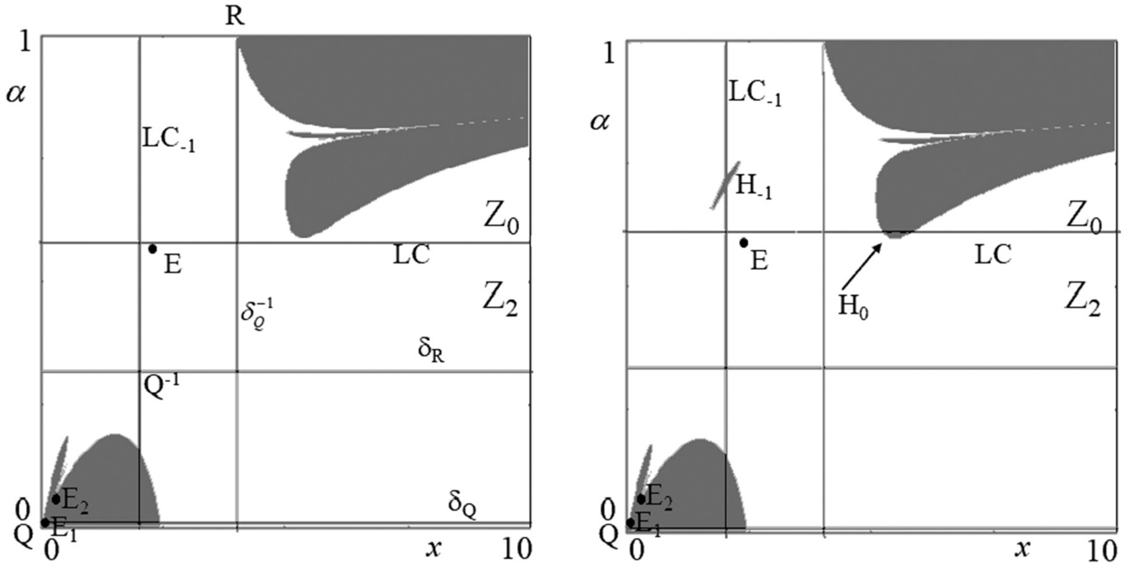

Fig. 3. Left. Phase portrait of the model (8) with preference function (10) and parameters $m = 10$, $p = 2.5$, $\mu = 0.98$, $k = 0.95$, $h = 1$, $l = 0.01$. Right. $k = 0.9$.

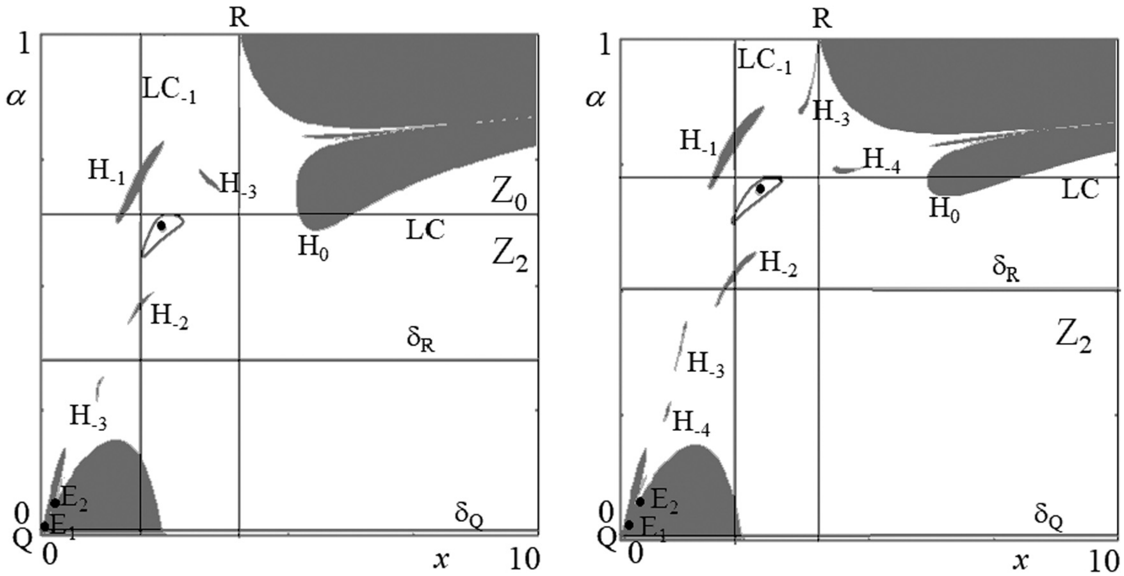

Fig. 4. Parameters used in the simulations are the same as Fig. 3 with $k = 0.86$ (left panel) and with $k = 1$, $h = 0.87$ (right panel).

Moreover, $H_{-2} \in Z_2$ and this generates other holes, some of them again in Z_2 and so on. These holes are too small to be seen in the picture, however zooming it would show an arborescent sequence of holes. Incidentally, the parameter's modification introduced also caused a stability loss of the rational equilibrium E through a supercritical Neimark–Sacker bifurcation. Of course, for the set of parameters used to get the picture in the left panel of Fig. 4, the two complex conjugate eigenvalues of the Jacobian matrix computed at the equilibrium E moved out of the unit circle of the complex plane and their modulus became $|\lambda_1| = 1.07$. So, for this parameters' constellation the adaptive process does not converge to a rational equilibrium, but continues to move around it along a stable closed invariant curve. This is another effect of the unimodal preference function because, as stressed in the previous section, rational equilibria cannot have complex eigenvalues if an increasing preference function is considered.

Up to now no role has been played by the focal point R. However, if a basin boundary crosses the corresponding prefocal curve δ_R then a lobe belonging to that basin will arise from R. This can be easily obtained by moving δ_R upwards, as shown in the right panel of Fig. 4, obtained for $h = 0.87$ (and $k = 1$ in order to avoid other qualitative changes). Now the non connected portion of the basin $H_{-2} \in Z_2$ crosses δ_R hence one of its two preimages H_{-3} has the shape of a lobe

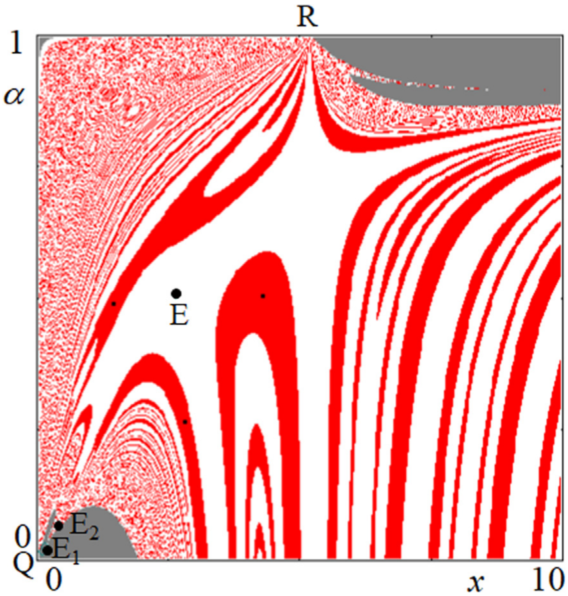

Fig. 5. Phase portrait of the model (8) with preference function (10) and parameters $m = 13$, $p = 2.5$, $\mu = 0.77$, $k = 1$, $h = 1$, $l = 0.01$.

issuing from R. This lobe has no further preimages (being entirely included into Z_0) whereas the other hole is $H_{-3} \in Z_2$ and generates a sequence of holes (two are visible in the picture).

The role of the two focal points Q and R is much more evident in Fig. 5, obtained with the set of parameters $m = 13$, $p = 2.5$, $\mu = 0.77$, $k = 1$, $h = 1$, $l = 0.01$. In this case a stable cycle of period 3 (indicated by three black dots inside the red basin of attraction) coexists with the stable rational equilibrium E (whose basin is again represented by the white region).

In the previous section we noticed that the whole line $x = \frac{m}{p}$ is mapped by T into the single point $Q^{-1} = \left(\frac{m}{p} - \mu p, \frac{m^2}{kp^2} e^{-h\frac{m}{p}} + l\right)$. As explained in [11], this implies that whenever a chaotic attractor crosses the line $x = \frac{m}{p}$, then it must include what is called a "knot". This occurrence is shown in Fig. 6, obtained for a set of parameters so that the attracting invariant curve created around the rational equilibrium E via the Neimark–Sacker bifurcation has been transformed into an annular (and rather involved) chaotic attractor. Such attractor intersects the line $x = \frac{m}{p}$, denoted as δ_Q^{-1} in Fig. 6, and all the intersections with this line are shrunk into the knot Q^{-1}.

5. Conclusions

In this paper a dynamic adaptive process which describes the repeated choices of a boundedly rational consumer has been considered. A first version of the economic model was proposed in [1] and its dynamical properties have been studied in [4,5] and [6]. In this paper a substantial modification has been introduced, concerning the way the consumers' preferences are endogenously determined by the past consumption: instead of an increasing relation, i.e. higher consumption in a given period implies higher consumption in the next period as well, a unimodal relation has been proposed, i.e. an excessive consumption in a given period implies less consumption in the next one, i.e. an adverse reaction or saturation effect. This implies that the discrete-time two-dimensional map, whose iteration describes the time evolution of the economic process, becomes noninvertible from invertible. This mathematical feature has important consequences on the structure of the basins of attraction, and this is particularly meaningful in this economic model because stable equilibria represent rational choices, i.e. the same consumption decisions that a rational consumer takes by solving an optimization problem under the assumption of full information.

After a study of existence and local stability of equilibrium points, the main global dynamic properties of the dynamical system have been analyzed, both related with noninvertibility of the map and the ones arising from the existence of a vanishing denominator. The first property was studied by the analytical determination of the critical curves, following [10], the second one by the analytical study of focal points and related prefocal curves, following [11,12] and [13]. The economic model considered in this paper allowed us to study, through a trade-off of analytical, geometric and numerical methods, the interactions and reciprocal influences among these different kinds of singularities.

Related to the economic consequences of the assumption of preferences saturation introduced in the model proposed in [1], we detected that non monotonic preferences may cause the loss of stability through a Neimark–Sacker bifurcation,

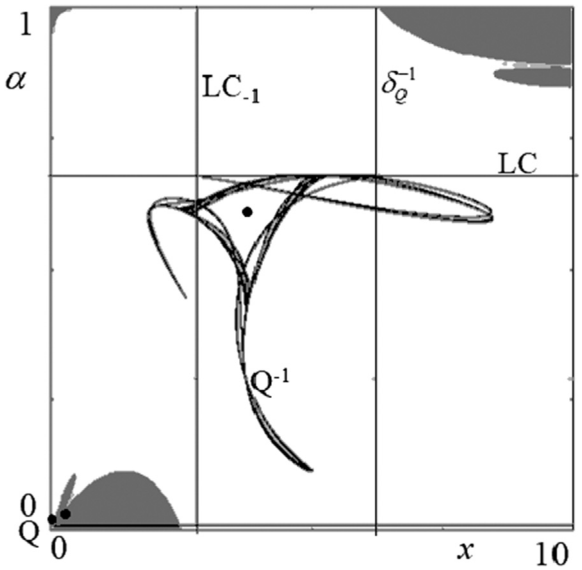

Fig. 6. Phase portrait of the model (8) with preference function (10) and parameters $m = 10$, $p = 2$, $\mu = 1$, $k = 1$, $h = 0.9$, $l = 0.01$.

an occurrence that is not possible with increasing preference functions. This kind of bifurcation is related with complex eigenvalues of the Jacobian matrix calculated at the equilibrium and imply that preferences and consumption oscillate, alternating periods with high preference and consumption that are the consequence of a decrease in both due to the saturation effect. When the consumption decreases the saturation effect disappears and the preference for the good rises again, and so on. Moreover, the creation of complex structures of the basins of attraction typical of noninvertible maps, such as non connected and multiply connected basins, can be observed as a consequence of saturation effects.

This work can be extended in several directions. We can combine the saturation effect with other behavioral features of consumers, such as the imitative behavior and the presence of a reference level of consumption [17]. It is also possible to conduct a more in depth investigation into the main features of the basins of attractions of coexisting attractors, in order to check the robustness of the results with respect to initial consumption habits and starting preferences. All the extensions can also be compared with the original D'Orlando and Rodano model [1].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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