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Adaptive and evolutionary mathematical models in economics

Gian Italo Bischi
Dipartimento di Economia, Società, Politica, Università di Urbino "Carlo Bo",

Abstract

This paper provides a short overview of dynamic models, denoted as adaptive or evolutive, recently adopted in economics and social sciences to describe systems with heterogeneous and boundedly rational agents. Some simple mathematical formalizations are shown, and a comparison between this kind of emerging literature and the orthodox main stream paradigm based on the assumption of fully rational and fully informed agents is discussed.

Sintesi

In questo lavoro viene offerta una breve rassegna di modelli dinamici adattivi ed evolutivi recentemente utilizzati in economia e nelle scienze sociali per la descrizione di situazioni in cui operano agenti eterogenei e limitatamente razionali. Vengono anche mostrate delle semplici formalizzazioni matematiche e discussi alcuni elementi di confronto fra questo tipo di letteratura emergente e i modelli ortodossi, finora dominanti, caratterizzati dall’ipotesi di agenti pienamente razionali e informati.

1. Introduction

During the last decades, economic modelling has been witnessing a paradigm shift in methodology, and the recent economic and financial crisis has strengthened this trend. Indeed, despite its notable achievements, the standard approach based on the paradigm of the rational and representative agent (endowed with unlimited computational ability and perfect information) as well as the underlying assumption of efficient markets, fails to explain many important features of economic systems, and has been criticized on a number of grounds (see e.g. Kirman 1992, Simon 1997, Hommes, 2013). At the same time, a growing interest has emerged in alternative approaches to economic agents’ decision making, which allow for factors such as *bounded rationality* and *heterogeneity* of agents, social *interaction*, and *learning*, where agents' behaviour is governed by simpler "rules of thumb" (or "heuristics") or "trial and error" or even “imitation” mechanisms. Of course, starting from these assumptions of bounded rationality, the modelled agents are generally not able to choose what is optimal, but they can at most obtain, using a famous expression of Simon (1997) what is ‘good enough’for them, thus replacing the concept of *optimal behavior* with the (apparently lower) concept of *satisficing behaviour* (see Simon, 1955, 1956).
The result of this approach may seem, at a first sight, a quite unsatisfactory and dismissive (in the sense of understating, reductive) representation of economic agents facing a decision. However, the significance of this approach is much more interesting and meaningful if it is used at each step of a repeated decision process, i.e. iterated over time. In fact, under some circumstances, the repetition of boundedly rational decisions based on *trial and error*, or *imitation of the better*, or *comparison*

between expected and realized results, that we shall denote by the general term "*adaptive*" in the following, may be a much more realistic (and even more rational) behaviour with respect to a rigid optimizing attitude, based on fixed rules assumed as fundamental axioms of rational behavior. Indeed, the latter attitude (typical of mainstream economic modelling where economic agents are assumed to behave as if they already know the laws that govern the evolution of the system where they operate, like a physicist who knows the law of motion of simple physical devices) may be quite misleading (even dangerous) when applied in the presence of incomplete information about the system where the economic agents operate, or about other agents' degree of rationality, or if the system evolution is characterized by intrinsic uncertainty, as it occurs when the time evolution is governed by nonlinear laws allowing for chaotic behavior, with the associated phenomenon of sensitivity to arbitrarily small perturbations (a quite common situation in economics and social sciences). In fact, the adaptive agents are allowed, by definition, to adjust their repeated actions according to the information collected as the system evolves, and a repeated (step-by-step) comparison of the expected results of their decisions with the observed ones allows them to adapt to circumstances.

Moreover, an adaptive system, even if it is governed by local (or myopic) decision rules of boundedly rational and heterogeneous agents, may converge *in the long run* to a *rational equilibrium*, i.e. the same equilibrium forecasted (and instantaneously reached) under the assumption of full rationality and full information of all economic agents. This may be seen as an *evolutionary interpretation* of a rational equilibrium, and some authors say that in this case the boundedly rational agents are able to learn, in the long run, what rational agents already know under very pretentious rationality assumptions (see e.g. Fudenberg and Levine, 1998). However, it may happen that under different starting conditions or as a consequence of exogenous perturbations, the same adaptive process leads to non-rational equilibria as well, i.e. equilibrium situations which are different from the ones forecasted under the assumption of full rationality, as well as to dynamic attractors characterized by endless asymptotic fluctuations that never settle to a steady state. The coexistence of several attracting sets, each with its own basin of attraction, gives rise to path dependence, irreversibility, hysteresis and other nonlinear and complex phenomena commonly observed in real systems in economics, finance and social sciences, as well as in laboratory experiments. Indeed, an interesting stream of the empirical literature and experimental economics provides support to this view (Menkhoff and Taylor, 2007, Hommes, 2013).

From the description given above, it is evident that adaptive systems can be mathematically modelled in the form of discrete dynamical systems, i.e. systems of ordinary differential equations or difference equations; the qualitative theory of nonlinear dynamical systems, with the related concepts of stability, bifurcations, attractors and basins of attraction, is a major tool for the analysis of their long run (or asymptotic) properties. Not only in economics and social sciences, but also in physics, biology and chemical sciences, such models are a privileged instrument for the description of systems that change over time, often described as "nonlinear evolving systems", and their long-run aggregate outcomes can be interpreted as "emerging properties", sometimes difficult to be forecasted on the basis of the local (or step-by-step) laws of motion.

In the next section we give some simple examples of such mathematical formulation of adaptive systems taken from the recent literature on learning as well as evolutionary games. Section 3 concludes with possible extensions of these methods.

2. Some basic mathematical formulations of adaptive processes.

In this section we give some formal expressions of adaptive models in the simple form of deterministic discrete time dynamical systems, just to fix some of the general ideas described in the introduction. The overview proposed is quite partial and it is seen from a personal viewpoint. As an exemplary case we consider oligopoly models (see e.g. Bischi et al., 2010); however, similar

mathematical approaches and considerations can be applied to other kinds of economic and social systems where interacting agents behave according to step-by-step adaptive decision processes.
Let us consider an economic system characterized by the numerical values of n dynamic variables, i.e. a state vector $\boldsymbol{x}(t)=[(x_1(\mathrm{t}), x_2(\mathrm{t}), \ldots, x_n(\mathrm{t})]$. The standard form of the local law of motion, in the description of physical or biological systems in discrete time, is given by

$$x_i(t+1) = f_i(\boldsymbol{x}(t), \boldsymbol{\mu}), \qquad i=1,...,n; \qquad (1)$$

where $\boldsymbol{\mu}=[\mu_1, ..., \mu_K]$ represents a set of k parameters. Given the initial condition $\boldsymbol{x(0)}$, which represents the state of the system at a given time, assumed to be known, the local law of motion allows one to obtain, inductively, or step-by-step, the whole forward evolution of the system, $\mathbf{x}(t)$, for each $t\geq 0$. This difference equation expresses the usual causality principle stating that in a deterministic environment the future state is a consequence of the actual state through a given rule which determines how the forces acting on the system regulate such causal relation. However, the evolution of economic and social systems, based on decisions of humans, must involve some assumptions on the degree of rationality involved in the decision processes. Moreover, the decisions taken at a given time are often influenced by agents' expectations about future scenarios, and this leads to some strong modifications of the classical causality principle of dynamical systems. In fact, the classical paradigm "The actual state of a system evidently arises from the state at a previous time..." in economics and sociology is often modified into "The actual state of a system is influenced by agents' expectations about its future state...". More formally:

$$x_i(t+1) = f_i({}_i\boldsymbol{x}^e(t+1), \boldsymbol{\mu}) \qquad (2)$$

where ${}_i\boldsymbol{x}^e(t+1)$ represents the state vector expected to hold at time t+1 by agent i according to information available at time t. The mainstream assumption of full rationality of economic agents, that in a deterministic systems implies perfect foresight, i.e. ${}_i\boldsymbol{x}^e(t+1)=\boldsymbol{x}(t+1)$ for each i and for each t, states that each agent is assumed to be able to predict the future states of the system, i.e. she is assumed to know (and solve) the equations governing the system and that all other agents involved are rational as well. Under this strong assumption, the dynamic system reduces to a set of equations whose solutions, if any, are called rational expectations equilibria. For example, in oligopoly models the dynamic variables are the quantities produced by firms (in the formulation à la Cournot) and the functions f_i are the best reply (or reaction) functions computed as a result of profit maximization, i.e. $f_i = R_i = argmax\ \pi_i$ with respect to x_i. In this case, under the assumption of rational expectations, the solutions are located at the intersections of the reaction functions and are known as Nash equilibria, and are assumed to be computed by all the players immediately (in one time step).
Weaker assumptions of rationality have long been proposed in the literature. For example, the seminal work of Cournot (1838) is based on the assumption of naive expectations ${}_i x_j^e(t+1) = x_j(t)$, $j\neq i$, i.e. each player is assumed to expect that the other players will produce in the next period the same output observed in the current period. Of course, such expectations reveal to be systematically wrong, however, under this assumption of bounded rationality the model (2) assumes the form of a standard discrete-time dynamical system (1), and its equilibrium points are Nash equilibria as well, obtained by imposing the condition $x_i(t+1)=x_i(t)$. So, in the case of convergence to an equilibrium, the Cournot model with naive expectations can be seen as an evolutionary outcome of the Nash equilibrium, but obtained - in the long run - through repeated decisions of boundedly rational players. Moreover, if several Nash equilibria exist, i.e. the reactions functions intersect in several points, then the Cournot adaptive process, with naïve players, can be seen as an equilibrium selection device.

The same arguments apply to the following adaptive mechanism, where after the computation of the best reply (or in general an optimal choice), agents prefer to choose a convex combination (i.e. a weighted average) of the computed best reply and the choice takes in the previous one, with a form of inertia, or anchoring attitude:

$$x_i(t+1) = (1-\lambda_i)\, x_i(t) + \lambda_i f_i(\mathbf{x}(t), \boldsymbol{\mu}) \quad i=1,\ldots n \qquad (3)$$

where the parameter $\lambda_i \in [0,1]$ gives a measure of inertia, because for $\lambda_i=1$ it gives the best reply (no inertia), for $\lambda_i=0$ it just gives the same choice of the previous period (thus ignoring the computed one, i.e. complete inertia) and intermediate values of λ_i represent intermediate degrees of inertia of agent *i*. It can be noticed that different values of λ_i , $i=1,\ldots n.$ can be used to represent heterogeneous agents characterized by different levels of inertia (that can be interpreted as more or less prudent behaviours). By imposing the usual steady state condition $x_i(t+1)=x_i(t)$, it is straightforward to see that also the adjustment process (3) has the same equilibrium points as the model (2) with perfect foresight and the naïve model (2) without inertia, so it can be seen as a different equilibrium selection device. However, periodic or quasi-periodic or chaotic attractors can be different, hence non rational attracting sets may exist, and they may coexist with stable rational equilibria, each with its own basin of attraction. This means that the same adaptive process may give rise to both time evolutions converging to rational equilibria and non-rational attractors, according to the initial conditions of the system. Moreover, an exogenous perturbation may transform a trajectory along which the system will learn rational behavior into a time evolution along which the agents never learn to behave rationally (see Barucci, Bischi and Gardini, 1999). Another interesting adaptive process can be obtained by assuming adaptive expectations

$${}_i x_j^e(t+1) = {}_i x_j^e(t) + \alpha_i \left[x_j(t) - {}_i x_j^e(t) \right] = (1-\alpha_i)\, {}_i x_j^e(t) + \alpha_i x_j(t) \qquad (4)$$

where, at each time step, the expected value is computed by introducing a correction according to the discrepancy between the previous expectation and the previous observed realization of the forecasted variable. By plugging (4) into (2) interesting leaning processes can be obtained, whose equilibria are, again, the same as the rational ones. Several applications to oligopoly models can be found in the book by Bischi et al. (2010); see also Bischi and Kopel (2001), whereas Bischi and Marimon (2001) show an application of adaptive expectations to monetary models for inflation control.

Many other rules for expectations' formations can be considered. For example

$${}_i x_k^e(t+1) = {}_i x_k^e(t) + \alpha_i \left[x_k(t) - x_i^* \right] \qquad (5)$$

describes agent *i* believing that a given dynamic variable x_k will move at each time step towards a given "fundamental reference value" with speed of adjustment $\alpha_i \geq 0$.

Some agents may behave, instead, as trend followers

$${}_i x_k^e(t+1) = {}_i x_k^e(t) + \gamma_i \left[x_k(t) - x_k(t-1) \right] \qquad (6)$$

thus expecting a further increase if the observed variables increased in the last two periods, a further decrease if it decreased. Both expectations (5) and (6) have been often applied in the description of adaptive financial markets where the population of agents operating in a given market are subdivided into two subgroups according to the rule (5) or (6) adopted to form expected prices, called fundamentalists and chartists respectively (see e.g. Chiarella et al., 2001-2011, Dieci et al., 2010, Tramontana et al., 2010, Hommes, 2011)

Higher order expectation methods are obtained by considering *k* states observed in the past, i.e. ${}_i x_k^e(t+1) = \psi({}_i x_k^e(t-i), x_k(t-i)$, *i=1,...,k,* for example weighted averages of *k* values of the past (k memory)

$$ {}_i x_k^e(t+1) = \sum_{i=1}^{k} w_i x_k(t-i), \quad \text{with} \quad \sum_{i=1}^{k} w_i = 1 \qquad (7) $$

or even averages of all values observed since a given starting time period (increasing memory)

$$ {}_i x_k^e(t+1) = \sum_{i=0}^{t} w_i x_k(i) \qquad (8) $$

with exponentially fading weights (see e.g. Bischi and Naimzada, 1997).
Other adaptive models can be obtained by assuming boundedly rational and heterogeneous agents use misspecified assumptions about the system where they are making their choices, e.g. they compute their "optimal" choices by using a wrong demand function, or assuming wrong cost functions for their competitors, or other wrong assumptions about some parameters that characterise the environment where they operate (e.g. Bischi et al., 2004). Adaptive mechanisms may be introduced through the correction, step-by-step, of the wrong assumptions, according to the discrepancies between some economic indicators observed and those forecasted according to their wrong assumptions. So, by repeated corrections, the adaptive process may lead them to progressively correct the misspecified parameters and converge to the true values in the long run. Examples of such learning mechanisms can be seen in Bischi et al. (2007, 2008) or in the book by Bischi et al. (2010).
A different, and frequently used adaptive rule, is obtained by assuming that each agent *i*, *i=1,...,n,* wishes to get the maximum value of a given index of performance, expressed as a function $\pi_i(\mathbf{x}(t)) = \pi_i((x_1(t), x_2(t), \ldots, x_n(t))$, but she does not have a complete knowledge of the function π_i, or it does not have the computational ability to solve the optimization problem. In this case one can assume that agents only have a local knowledge of the partial derivative, i.e. the marginal value, of π_i with respect to their own decision variable. This information may be obtained at each time step through economic or social experiments - see e.g. Arrow and Hurwicz (1960), Bischi and Naimzada (2000) ; the agent uses this information to gradually change her decisions in the direction of increasing performance according to the following adaptive mechanism (also known as gradient dynamics)

$$ x_i(t+1) = x_i(t) + \alpha_i(x_i) \frac{\partial \pi_i}{\partial x_i} $$

where $\alpha_i > 0$ is the speed of adjustment and measures agent i reactivity to signals of increasing performance.
We conclude this section with a short description of another class of important and very promising adaptive processes, obtained by using the formalism of evolutionary games. In this case, a populations of N agents is considered, divided into K subgroups each adopting a different strategy (e.g. different kinds of expectation, or different kinds of optimization procedures, or taking decisions under different information sets, etc.) so that the subgroups represent heterogeneous behavioural rules. Let $n_i(t)$ be the number of agents adopting strategy (or rule) *i* at time *t* and let $r_i(t) = n_i(t)/N$ be the respective fractions, with $\sum_{i=1}^{K} r_i(t) = 1$ for each $t \geq 0$. The same notations can be used to describe a single agent that at each time chooses among K possible strategies and $r_i(t)$ is interpreted as the probability associated to strategy *i*.

An evolutionary mechanism is introduced if the fractions (or probabilities) $r_i(t)$ are considered as endogenously driven dynamic variables, whose time evolution is determined by the values of an index of performance that gives a measure of the success (or fitness, or payoff) of the strategy adopted, say $\pi_i(t) = \pi_i(\boldsymbol{x}(t)) = \pi_i((x_1(t), x_2(t), \ldots, x_n(t))$, where the dynamic variables $x_i(t)$ have been used to denote some measures associated to the strategies adopted.
Following the spirit of evolutionary games, the adaptive mechanism, that at each time step determines how the shares of the population adopting different strategies are updated, is based on the principle that the fraction of agents playing a strategy that, with respect to the other strategies, earns higher payoffs, will increase in the next period. In other words, at the end of each time period each agent is assumed to compare the payoff obtained with the average payoff of the population of agents, and decides to switch her strategy choice if she can adopt a different one giving higher performances. This may be interpreted by saying that each agent observes the performance, in terms of payoff obtained in the current period, of a randomly chosen agent among those that used a different behavioural rule, and decides to imitate her if her payoff reveals to be higher. The simplest model proposed in the literature which gives an evolutionary pressure in favor of groups obtaining the highest payoffs is the one denoted as *replicator dynamics* (Taylor and Jonker, 1978, see also Vega-Redondo, 1996, ch.3, Hofbauer and Sigmund, 1988, ch.7, Weibull, 1995, ch.3). The discrete time replicator dynamics can be written as

$$r_i(t+1) = r_i(t)\frac{\pi_i(t)}{\overline{\pi}} \qquad (9)$$

where $\overline{\pi}(t) = \sum_{i=1}^{K} r_i(t)\pi_i(t)$ represents the average payoff observed at time t. So, (9) states that $r_i(t+1)$ will be greater than $r_i(t)$ if $\pi_i(t) > \overline{\pi}(t)$, so that the population share related to the better performing strategy at time period t increases in the next period. The intuition is simple: behavioural rules with higher performances will expand in relative importance, and those with lower performance will contract (see Hofbauer and Sigmund 1988, p. 135).
Most evolutionary economics models consist of giving the variables that represent the different strategy adopted by agents different economic meaning, for example different kinds of expectations adopted, such as rational expectations versus naïve or adaptive expectations, and the index of performance is defined in terms of discrepancy between expected and realized values, or profits gained (see e.g. Brock and Hommes, 2007, 2008, Hommes 2013), or market competition (see e.g. the papers by Chiarella et al., Dieci et al. for several applications to financial markets with heterogeneous agents, such as chartists and fundamentalists, or bulls and bears, adopting different strategies in forecasting price trends), or differential profit rate driven selection mechanisms (Droste et al., 2002, Bischi et al., 2004, 2013). Applications have also concerned the problem of exploiting common resources (Sethi and Somanathan, 1996, Bischi et al. 2004, 2013).
Replicator dynamics is not the only evolutionary selection mechanism, many other have been proposed in the literature - see e.g. Brock and Hommes (1997, 1998), Droste et al., 2002, Chiarella et al. (2011) - for many different switching mechanisms and their applications.
Other generalizations are given by models with m populations (e.g. m nations or m industrial districts) with different number of individuals, say $N_1, \ldots, N_m$, each with a given number of strategies (or behavioural rules) available $K_1, \ldots, K_m$. For example, in Bischi, Dawid and Kopel (2003) the case of two populations of firms (two industrial districts) each with two strategies available (invest in the industry or in financial markets) is considered, with a form of "switching by imitation of the more profitable strategy". Other examples of switching by imitation can be found in Hofbauer and Sigmund (1988), Bischi et al. (2006).
All these models are ultimately expressed in the form of nonlinear discrete dynamical systems of dimension one, two or three. Several examples also exist of adaptive and evolutionary models

expressed in continuous time, i.e. in the mathematical form of system of first order differential equations. The choice between these two different representations of time is not trivial, as it is plain that time is a continuous variable, however changes in economic systems often occur at disctrete time intervals, driven by decisions that cannot be continuously revised.

An intermediate choice is given by hybrid systems, characterized by the interaction between dynamic variables evolving in continuous time and variables that evolve according to a decision-driven time scale, typically discrete. Methods for the study of such systems can be found in the very recent mathematical, physics and cybernetics literature, and have recently been used in economic modeling by Bischi et al. 2013. Further useful analytical tools are the systems with time delay (or memory effects), represented by integrodifferential equations or mixed (differential-difference) equations. These are often studied in the physics of elastic materials and in mathematical biology (ecological modeling, see e.g. Cushing 1977) and have been recently applied in oligopoly modelling, where learning and knowledge accumulation are taken into account, by Matsumoto and Szidarovszky (2010).

Of course, the use of nonlinear dynamics in economics has been well-established since the early contributions of the 1940s and 1950s of the last century, and even in more recent neoclassical models, the theory of nonlinear systems has yielded important results on the 'indeterminacy' and bifurcations of stationary competitive equilibria. However, its use is often restricted to the analysis of the local behavior of the systems, obtained through the well established method of linearization around equilibrium points. As a consequence, this approach is unsuitable to investigate the global effects of nonlinearity, i.e. emergent phenomena. Such effects are often remarkable, although they cannot be detected from the behavior of the system in the vicinity of the stationary equilibria.

Recent mathematical developments focus on global dynamic properties and global bifurcations of evolving systems, through suitable and often innovative geometric, analytic and numerical methods. The modelling strategy is largely inspired by the heterogeneous-agent approach to economics (Hommes, 2013) and, more generally, by Agent-Based models (Tesfatsion and Judd, 2006) and seeks to derive tractable nonlinear dynamic models as stylized representations of real-world complex interactions. Also, such models can be used as artificial "laboratories" to conduct policy experiments and assess the qualitative and quantitative impact of various kinds of interventions (e.g., different measures for financial market stabilization) and the possibility to manage complex economic environments. This approach takes advantage of a flexible set of assumptions regarding agent behaviors and interactions, which may be particularly important during economic and financial turmoils. Some results of this innovative approach to policy design are presented in Westerhoff and Dieci (2006), Dawid and Neugart (2011).

Recently developed mathematical tools are also provided by the theory of piecewise-linear and discontinuous dynamical systems (originally developed for applications in electrical engineering), particularly suitable to model realistic situations where exogenous breaks and constraints play a role, characterized by abrupt changes in the behavior of economic variables or policies, associated with events such as financial crises. Important economic applications have been recently proposed in Gardini et al. (2008), Tramontana et al. (2010).

Further specific tools include the theory of global bifurcations, in particular homoclinic bifurcations and contact bifurcations in general, included those involving the method of critical sets for noninvertible maps (see Agliari and Vachadze, 2011 for a recent application in economics). Other studies have addressed the numerical and graphical investigation of the basins of attraction, in case of multistability (see Bischi and Kopel, 2001, Bischi et al., 2003, Dieci and Gallegati, 2011, Dal Forno and Merlone 2010); the numerical simulation of nonlinear dynamical systems with noise, which helps relate the distributional characteristics and time series properties of economic and financial variables to the behavioral rules of economic agents (Chiarella et al., 2013).

3. Concluding remarks

In this paper we have proposed a general overview of some dynamic models which describe adaptive (myopic, or step-by-step) iterated processes used to model boundedly rational economic agents that try to reach, in the long run, a satisficing level of performance. While mainstream models adopted in economic studies consider unboundedly clever agents, and assume that they instantaneously achieve mutual consistency, as in competitive equilibrium or Nash equilibrium, by contrast adaptive (or evolutionary) economic modeling emphasizes the "trial and error" (or heuristic) adaptation processes followed by economic agents which are assumed to be more similar to those acting in real world systems (Boulding, 1991). In particular, the framework of evolutionary games - i.e. populations of economic agents that can choose among different behavioural rules associated with an adaptive switching mechanism used to increase the fraction of agents using the more performing rule, according to a given index of performance (or payoff function) - has received increasing attention in the recent literature and has considerable unrealized potential for modeling substantive economic and social issues. However, even if the literature based on bounded rationality, random matching, adaptive evolutionary approaches, learning and emerging asymptotic patterns has an increasing impact, both quantitative and qualitative, economic applications of adaptive models and evolutionary game theory remain few and isolated, while the dominant approach in applied economics is still the one of orthodox models with rational agents and standard game theory with fully informed players.

Adaptive models in general, and those based on evolutionary games in particular, offer a richer set of empirical predictions such as built-in selection criteria among multiple equilibria or more complex attractors, predictions of relatively rapid or slow convergence (or non-convergence), path dependence and role of starting conditions of the economic systems described. The specification of initial conditions is an additional requirement of these models; in addition to the orthodox specification of static objective functions (or payoff functions in game theoretic settings) to be optimized, adaptive and evolutionary game models require specification of initial conditions (or a slice of history) and also require detailed specification of adjustment (e.g. learning) dynamics.

Moreover, the time horizon considered in the long run emerging properties should be critically considered in applied dynamic modeling, because the mathematical concept of time-limit may imply a "cosmologically" long run in an economic system. Moreover, short or medium term transient behaviours should be considered as well, considering, in other words, the ways a given long-run equilibrium can be reached (slow or fast, oscillatory or monotonic, with path dependence or not).

The extension of natural selection arguments and mechanisms from biology to economic and social systems now plays an important role. It is a matter of fact that many recent economic models where economic agents are assumed to be boundedly rational and heterogeneous are often represented as myopic, interacting, adaptive and evolving systems, leading to long-run emerging structures which are difficult to be forecasted on the basis of local interactions among agents (Dosi, 1991, Dosi and Nelson, 1994). Moreover, from the game theoretic standpoint, the issue of selection through adaptive iterated mechanisms among different feasible strategies is also connected to the social dilemmas and evolution of conventions, e.g. selfishness versus cooperation in the iterated prisoner's dilemma (where cooperation is a Pareto superior outcome whereas selfish strategies are dominant, hence constitute Nash equilibria), or in different kinds of coordination games, with interesting applications to environmental issues and the economics of common resources (see Ostrom, 2000, Peyton Young, 1993, 1998). These questions become much more interesting when considering how cooperation or coordination evolve in an iterated evolutionary game with an indefinite time horizon (Hogson, 1993, Hogson and Knudsen, 2006, 2010).

As argued in section 2, a comprehensive analysis of the properties of adaptive and evolutionary models requires an interdisciplinary approach, based on the theory of nonlinear dynamical systems, optimal control, game theory, numerical analysis. Such studies can have an impact on policy analysis and design, too. The standard neoclassical approach to policy analysis is based on strong theoretical requirements of equilibrium and rationality and assumes that the economy reacts "optimally" to exogenous interventions. From this perspective, economic policy tends to be regarded as a "science" rooted in sound theoretical principles, although general criticism has been raised concerning its scientific foundations (see Aghion and Howitt, 2007). These models represent interesting "laboratories" to evaluate the impact of economic policies under alternative scenarios and more flexible assumptions about agents' behavior and reactions.

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