Multistability and role of noninvertibility in a discrete-time business cycle model

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Summary. We study a discrete-time business cycle model in income and capital, a Kaldor-type model, in order to discuss the problem of coexistence of attractors and the related problem of basins of attraction. The model considered is particularly useful for pedagogical purposes because economically meaningful ranges of parameters exist such that an attractor characterized by oscillatory motion (which may be periodic, quasi-periodic or chaotic) coexists with two stable steady states, and consequently the choice of the initial conditions is crucial to decide if economic fluctuations are obtained or not in the long run. Moreover, the map whose iteration gives the time evolution of the system may be invertible or noninvertible according to the parameter constellations considered. These features allow us to compare the different behaviors of the model in these two regimes to stress the role of noninvertibility in the global dynamical properties, due to the geometric action of folding the phase space. In particular, we describe the creation of non-connected basins, and we show that the two regions of the phase space separated by a closed invariant curve are not invariant. Such properties have no analogue neither in continuous time models nor in discrete time models described by invertible maps.

Key words: Business Cycle, Discrete Dynamical Systems, Noninvertible maps, Basins, Global Bifurcations.

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1 Introduction

Business cycle theory concerns the description and explanation of the observed ups and downs of main macroeconomic variables. After the early attempts to provide informal and non-mathematical explanations, based on verbal arguments and empirical observations, the business cycle theory has been mainly considered a problem of mathematical economics after the works by Samuelson [28] and Hicks [16]. So, a business cycle model is now considered a dynamic model, usually formulated in the framework of the theory of dynamical systems, whose mathematical structure allows for fluctuations in major macroeconomic variables. A broad, and purely formal, classification of business cycle models distinguishes between linear and nonlinear, continuous and discrete time models.

Linear models, which include the classical linear multiplier-accelerator models (see, for instance, [28], [29], [16]) are usually expressed by two-dimensional linear dynamical systems, characterized by damped oscillations converging to the unique steady state, with the superposition of random variables whose presence prevents convergence and causes the occurrence of persistent irregular oscillations. So, in these models the economy is endogenously stable and fluctuations emerge as a consequence of exogenous, possibly non-economic, disturbances.

Instead, nonlinear models, such as those proposed by Kaldor[2] [18] and Goodwin [12] allow for endogenously generated persistent oscillations, i.e. stable fluctuations which are driven by deterministic processes entirely related to internal economic mechanisms embodied in the structure of the model (see [22] or [9] for historical overviews).

The majority of the models proposed in the period between 1940 and 1960, both linear and nonlinear, were formulated in continuous time, because the theory of two-dimensional ordinary differential equations was very developed in that period, especially in connection with nonlinear oscillators in physics. However, after the eighties, several authors proposed discrete time models, following the general interest related to the discovery that chaotic dynamics can be easily obtained even with low dimensional discrete-time models. Of course, the choice between these two kinds of time concepts is very important also from the point of view of the economic meaning of the model and the interpretation of the

[2]The model propose by Kaldor in 1940 ([18]) was not properly a dynamic model, but several reformulations of the Kaldor model have been successively provided by using the language and the formalism of the theory of *dynamical systems*, see [6], [19], [8] just to quote a few.

results, and many discussions on this topic appeared in the literature (see e.g. [10], [9], [17], [23]).

In this paper we restrict our considerations to two-dimensional discrete-time business cycle models. As it is well known, a discrete dynamical system is essentially obtained by the repeated application (iteration) of a map defined in the space of the dynamical variables (state space). In what follows, we argue that a further distinction is worth to be emphasized in the framework of discrete-time business cycle models, according to the invertibility or noninvertibility of the map whose iteration gives the discrete time evolution. We recall that a map is invertible if it maps distinct points into distinct points, whereas whenever distinct points which are mapped into the same point exist, then we say it is a *noninvertible map* (NIM henceforth). Hence, the geometric action of a NIM can be expressed by saying that it "folds and pleats" the phase space, so that distinct points can be mapped into the same point (see e.g. [25] or [1] for recent studies of the properties of NIMs, [11], [4], [3], [7] and the monograph [26] for recent applications in economics). This introduces some peculiar dynamic properties when a business cycle model is represented by a discrete dynamical system obtained by the iteration of a NIM. In particular, we are interested in the creation of non connected basins and the non invariance of the regions separated by a closed invariant curve of the phase plane. Such properties have no analogue neither in continuous time models nor in discrete time models represented by invertible maps, and are very important in the global study of dynamic models characterized by *multistability*, i.e. by the presence of several coexisting attractors.

Indeed, even if the simplest two-dimensional nonlinear business cycle models analyzed in the literature, and often proposed as classroom exercises, are basically characterized by the presence of an unstable equilibrium around which a unique cyclic attractor exists, a nonlinear dynamical system often has several attractors, each with its own basin of attraction. The different attractors give different possible long-run behaviors according to the starting condition of the economy. Particularly interesting, in the context of business cycle models, are the situations of coexistence of stable steady states and stable oscillations, so that the occurrence of economic fluctuations is obtained only if the economy starts from initial conditions belonging to the basin of a cyclic attractor.

In a situation of multistability a system can exhibit one kind of behavior for a very long time, so that an observer may guess that the possible endogenous asymptotic dynamics of the system is unique, yet a completely different kind of behavior may arise if an exogenous perturbation moves the state of the system inside the basin of another coexisting attractor.

This naturally leads to the problem of the delimitation of the basins of attraction, a problem which is strongly influenced by the property of noninvertibility. In fact, as emphasized in the specialized literature (see e.g. [24], [25]), and as we shall argue in the present paper, noninvertible maps may lead to the creation of basins with complex structures, such as non connected and multiply connected sets.

In order to discuss the questions addressed above in a concrete and pedagogical case, in this paper we consider a classical business cycle model of Kaldor-type, proposed in [27] whose general properties have been described in [2]. This model is particularly suitable in order to analyze the issues raised above because:

i) economically meaningful ranges of parameters exist such that attractors characterized by oscillatory motion (which may be periodic, quasi-periodic or chaotic) coexist with two stable steady states;

ii) the map whose iteration gives the discrete time evolution may be invertible or noninvertible according to the parameter constellation considered, so that we can follow paths in the parameters space which give a gradual transition from invertibility to noninvertibility, thus allowing speculations on the qualitative changes associated to such a transition.

The paper is organized as follows. In Section 2 the discrete-time Kaldor-like model analyzed in this paper is described, the standard local stability analysis of the steady states is given, together with the study of the local bifurcation curves in the space of the parameters. In Section 3 we give the conditions under which the map representing the model is a NIM, and we define the *critical curves.* The main results of the paper are given in Section 4, where some global bifurcations are studied which are responsible for the creation of limit cycles and give rise to situations of coexistence of three attractors, and we follow some particular paths in the parameters' space in order to emphasize the effects of noninvertibility on the creation of the cyclic attractors and the structure of their basins. A short discussion of the mathematical results and their economic implications is given in section 5.

2 The dynamic model, the steady states and their local stability

The model proposed by Kaldor in [18] is one of the earliest and simplest nonlinear models of business cycles. Despite it is so simple and rather dated, it continues to generate a considerable amount of economic, pedagogical and methodological interest, both from the point of view of the economist and of the applied dynamicist (see e.g. [9]). Several reformulations of the Kaldor model have been given in the literature by using the language and the formalism of the theory of *dynamical systems*, both in continuous time (e.g. [6], [19], [13]) and in discrete time (see e.g. [8], [15], [20], [21]). We shall consider the following discrete-time version of the Kaldor model, which has the same structure of the above mentioned discrete time models (see also [2]):

$$\begin{aligned} Y_{t+1} &= Y_t + \alpha(I_t - S_t) \\ K_{t+1} &= (1-\delta)K_t + I_t \end{aligned} \tag{1}$$

where the dynamic variables Y_t and K_t represent, respectively, the income (or output) level and the capital stock in period t. The parameter α ($\alpha > 0$) represents a *speed of adjustment*, measuring the firm's reaction to the *excess demand*, which is equivalent, in a macroeconomic environment, to the difference between the investment demand (I_t) and savings (S_t). A small value of α means a prudent firms reaction, which can be explained with a high degree of risk aversion or with a relevant monopoly degree. Conversely, a high value of α means rash firms reactions due to a risk propensity or to competitivity pressures, which can cause a coordination failure. Finally, the parameter δ ($0 < \delta < 1$) represents the *capital stock's depreciation rate.*

As usual in a Keynesian framework, savings are assumed to be proportional to the current level of income,

$$S_t = \sigma Y_t$$

where the coefficient σ , $0 < \sigma < 1$, represents the *propensity to save.* While in many versions of the Kaldor model the savings function is assumed to be nonlinear, we prefer a linear specification, both for its analytical simplicity and for its sounder microfoundation. Moreover, in our case this assumption does not affect the nonlinearity of the model, which is related to the nonlinearity of the investment function, given below.

In our model, again in the logic of Keynesian setups, the normal level of income is exogenously assumed in the firms expectations. We indicate this normal level of income with the parameter μ ($\mu > 0$). Therefore, since the expected income $Y_t^e = \mu$, then $\sigma\mu$ represents the normal level of savings. As usual, the investment demand is assumed to be an increasing and sigmoid-shaped function of income. Without loss of generality, in the following we shall consider the form proposed in Rodano [27]

$$I_t = \sigma\mu + \gamma\left(\frac{\sigma\mu}{\delta} - K_t\right) + \arctan(Y_t - \mu) \tag{2}$$

where $\sigma\mu/\delta$ is the normal level of capital stock. In the equation (2) two short run investment components are considered: the first one is proportional to the difference between normal capital stock and current stock, according to a coefficient γ ($\gamma > 0$), usually explained with the presence of adjustment costs; the second one is an increasing, but not linear, function of the difference between current income and its normal level. This second component of the short run investment function is a convenient specification of the sigmoid-shaped relationship between investment and income proposed by Kaldor. We remark that this analytic specification does not compromise the generality of the results.

By substituting the expressions of I_t and S_t into the dynamic model (1), we obtain that the time evolution of income and capital is obtained by the iteration of a two-dimensional nonlinear map $T : (Y_t, K_t) \to (Y_{t+1}, K_{t+1})$ given by:

$$T: \begin{cases} Y' = Y + \alpha\left[\sigma\mu + \gamma\left(\frac{\sigma\mu}{\delta} - K\right) + \arctan(Y - \mu) - \sigma Y\right] \\ K' = (1-\delta)K + \sigma\mu + \gamma\left(\frac{\sigma\mu}{\delta} - K\right) + \arctan(Y - \mu) \end{cases}, \tag{3}$$

where the symbol $'$ denotes the unit time advancement operator.

The equilibrium points (or steady states) of the model (1) are the fixed points of the map T, solutions of the algebraic system:

$$\begin{cases} \sigma\mu + \gamma\left(\frac{\sigma\mu}{\delta} - K\right) + \arctan(Y - \mu) - \sigma Y = 0 \\ \sigma\mu + \gamma\left(\frac{\sigma\mu}{\delta} - K\right) + \arctan(Y - \mu) - \delta K = 0 \end{cases},$$

obtained by setting $Y' = Y$ and $K' = K$ in (3). This system can be rewritten as:

$$\begin{cases} K = \frac{\sigma}{\delta}Y \\ \sigma\left(1 + \frac{\gamma}{\delta}\right)(Y - \mu) = \arctan(Y - \mu) \end{cases}. \tag{4}$$

The first equation of (4) says that the fixed points belong to the line of equation $K = \frac{\sigma}{\delta}Y$ and from the second equation we have that the equilibrium values of Y can be obtained as intersections between the line of equation $z = \sigma\left(1+\frac{\gamma}{\delta}\right)(Y-\mu)$ and the sigmoid-shaped graph of the function $z = \arctan(Y-\mu)$. Such intersections may be one or three according to the value of the aggregate parameter $\sigma(1+\gamma/\delta)$: if $\sigma(1+\gamma/\delta) \geq 1$ then the exogenously given equilibrium $P = (\mu, \mu\frac{\sigma}{\delta})$ is the unique steady state, whereas in the complementary case $\sigma(1+\gamma/\delta) < 1$ two further steady states exist, say R and Q, located in symmetric positions with respect to the point P, given by $R = (Y_R, \frac{\sigma}{\delta}Y_R)$ and $Q = (Y_Q, \frac{\sigma}{\delta}Y_Q)$, with $Y_Q = 2\mu - Y_R$, $Y_R < \mu$ being the smallest real solution of the second equation in (4), which can be computed by any numerical method for finding the real roots of an equation. It is trivial to realize that the steady states are independent of the adjustment parameter α.

As shown in [2], the map T is symmetric with respect to the exogenous steady state $P = (\mu, \mu\frac{\sigma}{\delta})$. This means that symmetric points are mapped into symmetric points (with respect to P) and implies that a cycle of T is either symmetric with respect to P or admits a symmetric cycle.

In the following of this section we briefly recall the results on the local stability of the fixed point $P = (\mu, \mu\frac{\sigma}{\delta})$ already given in [2]. As usual, the analysis of the local stability of a fixed point is obtained through the localization, in the complex plane, of the eigenvalues of the Jacobian matrix evaluated at the fixed point, and their dependence on the parameters of the model. In the following we consider the parameters δ and γ as fixed, and we study the stability regions, and the local bifurcation curves, in the space of the parameters α, σ, with $\alpha > 0$ and $0 < \sigma < 1$. In order to simplify the mathematical treatment, we assume that the parameter γ belongs to the range $0 < \gamma < 2-\delta$, a condition which is satisfied in economically feasible situations, being usually $\gamma < 1$. The results of the standard analysis of the eigenvalues, given in [2], are summarized in the following proposition (see also Fig. *1*)

Proposition 1

(i) If $\sigma \geq \sigma_p$, with

$$\sigma_p = \frac{\delta}{\delta+\gamma} \tag{5}$$

then the point $P = (\mu, \mu\frac{\sigma}{\delta})$ is the unique fixed point of the map T, and if $\sigma < \sigma_p$ then two further fixed points exist, symmetric with respect to the point P.

(ii) If $\gamma < 2-\delta$, the point P is locally asymptotically stable if the parameters α and σ

belong to the region $ABCD$ of the plane (α,σ), with vertexes $A=\left(0,\frac{\delta}{(\delta+\gamma)}\right)$, $B=(0,1)$, $C=\left(\frac{\delta+\gamma}{\gamma},1\right)$, $D=\left(\frac{(\delta+\gamma)^2}{\gamma},\frac{\delta}{\delta+\gamma}\right)$, *where the sides AD and CD belong to the line $\sigma=\sigma_p$ and to the hyperbola of equation*[3]

$$\sigma=\sigma_{hP}(\alpha)=\frac{1}{1-\delta-\gamma}\left(1-\delta-\frac{\gamma+\delta}{\alpha}\right) \tag{6}$$

respectively.

(iii) If the point (α,σ) exits the stability region $ABCD$ by crossing the side AD, then a supercritical pitchfork bifurcation occurs at which the fixed point P becomes a saddle point and two stable nodes are created near it.

(iv) If the point (α,σ) exits the stability region $ABCD$ by crossing the side CD, then a supercritical Hopf bifurcation occurs at which the fixed point P is transformed from a stable focus into an unstable focus and an attracting closed invariant curve is created around it on which the dynamics may be periodic or quasi-periodic.

This proposition, concerning the usual local analysis based on the linear approximation of dynamical system near a steady state, seems to imply that for values of the parameters below the line $\sigma=\sigma_p$, where three equilibria exist, situations of bi-stability (without oscillations) are obtained, whereas self-sustained oscillatory behaviors seem to appear only for sufficiently high values of the propensity to save, i.e. above the line $\sigma=\sigma_p$, and for increasing values of the adjustment parameter α, i.e. when the curve CD of Fig.*1* is crossed. Indeed, in the rich literature on dynamical systems which represent Kaldor-like business cycle models, this is the stream followed by many authors: both in discrete time and in continuous time stable oscillations along limit cycles, generated via Hopf bifurcations, are considered for sufficiently high values of α and σ. However, as argued in [2], small values of the propensity to save σ are more realistic, hence it makes sense to wonder if oscillatory dynamics can be obtained in the region of the parameter space where three equilibria exist.

******** FIG. 1 APPROXIMATELY HERE ********

[3] *In the particular case $\gamma+\delta=1$ the side CD belongs to the vertical line $\alpha=1/\gamma$.*

3 Invertibility conditions

For particular values of the parameters, the map T is a noninvertible map of the plane. This means that while starting from some initial values for income and capital stock, say (Y_0, K_0), the iteration of (3) uniquely defines the trajectory $(Y_t\, , K_t) = T^t(Y_0, K_0)$, $t = 1, 2, ...$, the backward iteration of (3) is not uniquely defined. In fact, distinct points of the plane may have the same image, that is, equivalently, a point (Y', K') of the plane may have distinct rank-1 preimages. The conditions under which the map T given in (3) is noninvertible are discussed in the Appendix, together with some basic definitions, a minimal vocabulary of the theory of noninvertible maps of the plane and some basic facts about the *critical curves*.

In the Appendix it is shown that the map T is a noninvertible map in the region of the (α, σ) parameters space given by $0 < m < 1$, where

$$m = (\delta + \gamma - 1)\frac{(1 - \alpha\sigma)}{\alpha(1 - \delta)} \ .$$

Following the *critical curves* theory developed by Gumowski and Mira (see [14], [25]), the Appendix shows that for $0 < m < 1$ this map is of the so-called type $Z_1 - Z_3 - Z_1$ which means that a point (Y', K') of the phase-plane may have one or three distinct rank-*1* preimages: the region of points with three rank-*1* preimages is the strip contained between the two lines of equation

$$LC_a : \quad K = \frac{(\delta + \gamma - 1)}{\alpha\gamma}Y + \frac{1 - \delta}{\gamma} \left[q_1 + \sigma\mu\left(1 + \frac{\gamma}{\delta}\right)\right] \ ; \tag{7}$$

$$LC_b : \quad K = \frac{(\delta + \gamma - 1)}{\alpha\gamma}Y + \frac{1 - \delta}{\gamma} \left[q_2 + \sigma\mu\left(1 + \frac{\gamma}{\delta}\right)\right] \ , \tag{8}$$

where:

$$q_1 = -\arctan\left(\sqrt{\frac{1}{m} - 1}\right) - m\left(\mu - \sqrt{\frac{1}{m} - 1}\right);$$

$$q_2 = \arctan\left(\sqrt{\frac{1}{m} - 1}\right) - m\left(\mu + \sqrt{\frac{1}{m} - 1}\right) \ ,$$

while the points outside this strip have only one rank-1 preimage. Thus the set $LC = LC_a \cup LC_b$ (critical curve of rank-1) is the locus of points with two merging rank-1 preimages and the locus of such merging preimages, denoted by LC_{-1} (critical curve of rank-0) is also made up of two lines, $LC_{-1} = LC_{-1,a} \cup LC_{-1,b}$ where:

$$LC_{-1,a}: \quad Y = \mu - \sqrt{\frac{1}{m} - 1}\ ; \tag{9}$$

$$LC_{-1,b}: \quad Y = \mu + \sqrt{\frac{1}{m} - 1}\ , \tag{10}$$

with $LC_a = T(LC_{-1,a})$ and $LC_b = T(LC_{-1,b})$.

In order to compare the bifurcation curves in the parameter plane (α, σ), as shown in Fig. *1*, with the ranges of invertibility or non invertibility of the map T it is useful to draw the parameter region, in the same (α, σ) plane, in which the noninvertibility condition $0 < m < 1$ is fulfilled (NI region). In the complementary region of the (α, σ) plane the map is invertible. We notice that:

a) for $\gamma + \delta < 1$, taking into account that $m > 0$ corresponds to

$$\sigma > \frac{1}{\alpha}\ ,$$

while $m < 1$ corresponds to

$$\sigma < \frac{1-\delta}{1-\gamma-\delta} + \frac{1}{\alpha}\ ,$$

we have that $0 < m < 1$ holds between two branches of hyperbolae but considering only the interesting range, $0 < \sigma < 1$, we get the dark grey NI region shown in Fig. *1*;

b) for $\gamma + \delta > 1$, we have that $m > 0$ holds when

$$\sigma < \frac{1}{\alpha}\ ,$$

and $m < 1$ holds when

$$\sigma > \frac{1}{\alpha} - \frac{1-\delta}{\gamma+\delta-1}\ ,$$

so that condition $0 < m < 1$ holds between two branches of hyperbola.

In the following we shall limit our analysis to the case $\gamma + \delta < 1$, since it corresponds to ranges of the parameters which are rather realistic from the point of view of their economic meaning.

The fact that the map T is noninvertible may have remarkable consequences on the global dynamic properties of the business cycle model. In particular, in this paper we are interested in the structure of the basins and the problems of forward and backward invariance of subsets of the phase plane.

We recall that a set $A \subset \mathbb{R}^n$ is *trapping* if it is mapped into itself, $T(A) \subseteq A$, i.e. $\forall x \in A\ T(x) \in A$. A trapping set is *invariant* if it is mapped onto itself: $T(A) = A$, i.e. all the points of A are images of points of A. A closed invariant set A is an *attractor* if it is *asymptotically stable,* i.e. if a neighborhood U of A exists such that $T(U) \subseteq U$ and $T^n(x) \to A$ as $n \to +\infty$ for each $x \in U$.

The *Basin* of an attractor A is the set of all points that generate trajectories converging to A

$$\mathcal{B}(A) = \left\{ x \mid T^t(x) \to A \quad \text{as } t \to +\infty \right\} .$$

Geometrically, the action of a noninvertible map T can be expressed by saying that it "folds and pleats" the plane, so that the two distinct points p_1 and p_2 are mapped into the same point p. This is equivalently stated by saying that several inverses are defined in p, and these inverses "unfold" the plane.

The backward iteration of a noninvertible map *repeatedly unfolds* the phase space, and this implies that the basins may be non-connected, i.e. formed by several disjoint portions. This can be intuitively understood on the basis of the following arguments. Let A be an attractor for the iterated map T. This means that a neighborhood $U(A)$ of A exists whose points converge to A. Of course $U(A) \subseteq \mathcal{B}(A)$, but also the points of the phase space which are mapped inside U after a finite number of iterations belong to $\mathcal{B}(A)$, so that the total basin of A (or more briefly the basin of A) is given by

$$\mathcal{B}(A) = \bigcup_{n=0}^{\infty} T^{-n}(U(A))$$

where $T^{-1}(x)$ represents the set of the rank-1 preimages of x (i.e. the points mapped to x by T), $T^{-n}(x)$ represent the set of the rank-n preimages of x (i.e. the points mapped to x after n repeated applications of T), and T^0 is the identity map. In the case of a noninvertible map, the phase-plane may be thought of as the result of the overlapping of different sheets, with a different local inverse map defined on each sheet (*Riemann foliation*), so that it can be subdivided into regions Z_k ($k = 0, 1, 2, ...$) whose points have the same number k of rank-1 preimages. Thus, the total basin of A may be non connected because if $U(A)$, or its preimages, belong to regions Z_k with $k > 1$ distinct rank-1 preimages, the action of the distinct inverses defined in different sheets of the *Riemann foliation* may give preimages of $U(A)$ which are disjoint from $U(A)$ and far from it, due to the *unfolding* of the phase-plane under the action of the several distinct inverses.

4 Global dynamics with three coexisting equilibria

In this section we explore the global dynamic behaviors of the model when the values of the parameters are out of the region of stability of the exogenously given equilibrium P and in the set of parameters such that the three equilibria R, P, and Q exist. In all our numerical explorations we will assume $\mu = 200$, $\delta = 0.2$, $\gamma = 0.6$, and we will follow, in the (α, σ) parameter plane, particular routes characterized by low values of the propensity to save σ $(0.09 < \sigma < 0.2)$.

In [2] it is shown that for low values of the propensity to save σ situations of bi-stability can be obtained, with the exogenous steady state P unstable and two stable equilibria R and Q, each with its own basin of attraction, and that for increasing values of the propensity to save (but, in any case, in the range of low values) global bifurcations may occur at which a stable limit cycle appears which surrounds all the steady states. At the global bifurcation at which the stable limit cycle is created, a remarkable change occurs in the basins of attraction: before the bifurcation the basins of the two stable steady states share the whole state space of the dynamical system, whereas after the bifurcation these two basins suddenly become very small, and the majority of the initial conditions generate time evolutions converging to the limit cycle, that is, oscillatory behaviors dominate the long run dynamics of the system.

This transition from bi-stability to coexistence of the two stable steady states with an attractor characterized by oscillatory motion may occur through different kinds of bifurcation sequences, according to different ranges of the speed of adjustment α and to the invertibility or noninvertibility of the map T. One of these bifurcation-routes, observed for sufficiently low values of α and related to an homoclinic bifurcation of the saddle P, is described in [2]. More generally, for parameters constellations such that T is an invertible map, the appearance of a stable limit cycle is related to the fact that the stable set of P, which separates the basins of Q and R, becomes more and more involved as the propensity to save σ increases, leading to an increasing complexity in the structure of the basins of the stable steady states (see Figs. *2a, b, c*). We remark that in such cases, like the one considered in [2], the parameters are in the region where the map T is invertible, so that all the basins are connected sets, separated by the stable set of the saddle point P.

******** FIG. 2 APPROXIMATELY HERE ********

This situation of coexistence of two stable steady states with an attractor characterized by oscillatory motion (periodic, quasi-periodic, or chaotic) can also be found for higher values of the speed of adjustment α, when the map is noninvertible (remember that when $(\gamma + \delta) < 1$, as is the case in our examples, T is noninvertible for $\alpha > 1/\sigma$, as shown in Fig. *1*). In order to discuss the role of noninvertibility of the map, in the presence of multistability, let us consider higher values of α and follow a particular path in the (α, σ) parameter plane, obtained with $\sigma = 0.105$ and α in the region of noninvertibility. The dynamic scenario observed for $\alpha = 10.5$ (Fig. *3a*) is apparently similar to the one observed in Fig. *2c*, but opening a wider window on the phase-plane (Fig. *3b*) we can see that outside the closed invariant curve Γ regions of points exist, which belong to the basins of Q and R. These regions cannot exist when the map T is invertible, i.e. their appearance is due to the transition to the noninvertibility regime. As remarked in Section 3, the backward iteration of a two-dimensional noninvertible map "unfolds" the phase-plane, and this implies that the total basin of an attractor may be non connected. In other words, the "folding" action of the forward iteration of the noninvertible map makes it possible that points outside the closed invariant curve Γ are mapped inside it. The region *outside* Γ is no longer *forward invariant*, as it was in the regime of invertibility and, consequently, trajectories starting outside Γ exist which converge to a stationary state.

******** FIG. 3 APPROXIMATELY HERE ********

The noninvertibility property of T also enables us to explain other important qualitative changes of the non connected basins of Q and R, when the parameter α is further increased (the mechanism causing these basin-bifurcations is explained, for instance, in [24], [25], [1]). As already remarked, in the case of noninvertible maps, the phase-plane may be thought of as the result of the overlapping of different sheets, with a different local inverse map defined on each sheet (*Riemann foliation*), so that it can be subdivided into regions Z_k ($k = 0, 1, 2, ...$) whose points have the same number k of rank-1 preimages. The map T describing our model is of the so-called $Z_1 - Z_3 - Z_1$ type (see the Appendix) and the critical lines LC_a and LC_b separate the region Z_3 of points with three different rank-1 preimages from the region Z_1 of points with only one preimage (see Fig. *3b*). By further increasing α, the region denoted by H in Fig. *3b* approaches the boundary of the region Z_3 and crosses it (see Fig. *4a*, obtained with $\alpha = 11.5$). The portion of H which has crossed LC_a, say $H_0 = H \cap Z_3$, has three rank-1 preimages, and two of them (say $H_{-1,1}$ and $H_{-1,2}$) are located on opposite sides of $LC_{-1,a}$ and constitute the region denoted by $H_{-1} = H_{-1,1} \cup H_{-1,2}$, connected through a segment of $LC_{-1,a}$ which is the set of merging preimages of the portion of LC inside H. Of course the same qualitative changes occur to the basin of the steady state Q due to the symmetry of the map. In Fig. *4b*, obtained with $\alpha = 12$, the region H has now completely crossed over the curve LC, and therefore its preimages $H_{-1,1}$ and $H_{-1,2}$ now are separate regions.

******** FIG. 4 APPROXIMATELY HERE ********

So, noninvertibility is responsible for global bifurcations which create more and more non connected components of the basins of the steady states. The resulting dynamic scenario is rather counter-intuitive from an economic point of view. As an example, in the situation of Fig. *4a* assume that the economy is at the stationary state R: we can see that the system is stable with respect to small perturbations, but if a greater shock is considered then the economy enters a different dynamic regime, characterized by persistent endogenous fluctuations. What is surprising is that this switching of regime may not occur for very large perturbations, as one moving the system to the state x, in the region denoted by H_{-1}: in fact in this case the system again rapidly converges to the steady state R (our previous considerations about the appearance of the "islands" $H_{-1,1}$ and $H_{-1,2}$ allow us to conclude that a point inside the region H_{-1} is mapped inside the *immediate basin*[4] of R in only two iterations).

Besides the loss of the *forward invariance* of the region *outside* Γ, noninvertibility may also cause the loss of the forward invariance of the area *enclosed* by the curve: this change occurs when the limit cycle, increasing in size as the parameter α increases, crosses the lines $LC_{-1,a}$ and $LC_{-1,b}$.

[4] The widest connected component of $\mathcal{B}(R)$ including the fixed point.

In fact, as far as the attracting invariant closed curve Γ does not intersect LC_{-1} it can be thought of as entirely contained in one sheet of the Riemann foliation. This means that a neighborhood $U(\Gamma)$ of Γ exists such that not only $T(U) \subset U$ (since Γ is attracting) but a unique inverse exists, say T_1^{-1}, such that $T_1^{-1} : T(U) \to U$. This implies that the curve Γ, as well as the area of the phase plane enclosed by Γ, say $a(\Gamma)$, is both forward invariant (under T) and backward invariant (under T_1^{-1}). In fact, even if $\Gamma \in Z_k$ with $k > 1$, so that other (extra) rank-1 preimages of Γ exist, they do not intersect Γ.

The situation changes when Γ grows up until it has a contact with the two branches $LC_{-1,a}$ and $LC_{-1,b}$ of the set of merging preimages LC_{-1}, and then intersects them, as shown in Figs. *3* and *4*. We now describe the consequences of the contact between Γ and $LC_{-1,a}$. Of course, due to the symmetry property of the map T, the same description applies to the symmetric contact between Γ and $LC_{-1,b}$.

Let A_0 and B_0 be the two points of intersection between Γ and $LC_{-1,a}$, see Fig. *5a*, and let R_1 and R_2 the two regions, separated by $LC_{-1,a}$, where there are, respectively, the ranges of the two inverses T_1^{-1} and T_2^{-1}. Then the points $A_1 = T(A_0)$ and $B_1 = T(B_0)$, which must belong both to Γ and to $LC_a = T(LC_{-1,a})$, are points of tangential contact between Γ and LC_a. In fact, the arc $A_0B_0 \in \Gamma \cap R_2$ must be mapped by T in the arc $A_1B_1 = T(A_0B_0)$, entirely included in the region Z_3 on one side of LC_a (see also the enlargement *5b*). If we look at the preimages, we realize that now there is not a unique inverse under which Γ is backward invariant. In fact, now $T_1^{-1}(\Gamma)$ also includes arcs inside Γ, like the arc $A_0B_0^{(1)} \in R_1$, whereas $A_0B_0^{(2)} \in \Gamma \cap R_2$ is given by $T_2^{-1}(A_1B_1)$.

We can say that the region h_1 between the arc A_1B_1 of Γ and LC_a is "unfolded" by the action of the two inverses T_1^{-1} and T_2^{-1} in two distinct preimages, located in the regions R_1 and R_2 respectively, represented in Fig. *5a* by the two portions $h_0^1 = T_1^{-1}(h_1)$ and $h_0^2 = T_2^{-1}(h_1)$ of $a(\Gamma)$ bounded by the two arcs $A_0B_0^{(1)}$ and $A_0B_0^{(2)}$ inside and along Γ respectively. In other words, the two portions h_0^1 and h_0^2 of $a(\Gamma)$ are folded by T along LC_a to cover the area h_1, which is outside Γ. This implies that the area $a(\Gamma)$ bounded by Γ is no longer forward invariant, since some points inside Γ are mapped outside it (like the points belonging to h_0^1 and h_0^2). This phenomenon of forward invariance of a closed curve, together with noninvariance of the area inside it, is specific to noninvertible maps, that is, it cannot be observed in invertible ones. The property of noninvariance of $a(\Gamma)$ and the creation of convolutions of Γ are two aspects of the same mechanism, related to the fact that curves crossing LC_{-1} are folded along LC and are confined into the region with an higher number of preimages.

As an example, Fig. *5a* and its enlargement *5b* show the first three points of a trajectory starting from the initial condition $(Y_0, K_0) = (185.5, 103.75)$, which lies inside Γ in the region $h_0^1 = T_1^{-1}(h_1)$: the point $T(Y_0, K_0)$ belongs to the region h_1 outside the curve and the resulting trajectory remains outside the curve for a while.

******** FIG. 5 APPROXIMATELY HERE ********

The effects of noninvertibility are also evident when, by increasing σ for a fixed value of the speed of adjustment α, we analyze the qualitative changes occurring to the basins of the stable steady states before the creation of the cyclic attractor. As we have shown, in the case of invertibility (Figs. *2a, b, c*) the increasing complexity of the basins structure, leading to the appearance of the cyclic attractor, is related to the fact that the stable set of the saddle point P, which separates the basins of Q and R, becomes more and more involved as the parameter σ increases, but the basins of Q and R remain anyway connected regions, both before and after the creation of the limit cycle. In the case of noninvertibility, a different route to complexity is observed when the propensity to save σ is increased, related to global bifurcations which transform $\mathcal{B}(Q)$ and $\mathcal{B}(R)$ into non-connected regions of the phase-plane. Figs. *6a, b, c* represent these qualitative changes, obtained by increasing the propensity to save σ with a high value of the speed of adjustment $(\alpha = 12)$. The mechanism creating more and more non-connected components is the same as the one already analyzed in Figs. *4a, b* and is related to the crossing of the boundary of the region Z_3 by portions of the basins of the stable steady states. We notice that the first crossing occurs when the boundary separating the basins of the equilibria (the stable set of the saddle P) still has a very simple shape (Fig. *6a*). Fig. *6c* is obtained with $\sigma = 0.0954$, immediately after the global bifurcation which creates the attractor (chaotic in this case) on which the system shall exhibit oscillatory motion. This attractor is represented in the same figure together with the resulting non-connected basins of the stable equilibria.

******** FIG. 6 APPROXIMATELY HERE ********

5 Conclusions

In this paper we have analyzed a classical business cycle model, a discrete time Kaldor model, and we have stressed the fact that the map whose iteration simulates the time evolution of the economic system may be invertible or non invertible, according to the values of the parameters. We argued that this distinction may be important in situations of multistability, i.e. coexistence of several attractors, because different structures of the basins of attraction, as well as different kinds of basins' bifurcations, can be observed in these two regimes. Indeed, as already stressed in [2], when an economy characterized by a low propensity to save is considered - a realistic situation according to empirical data - coexistence of three distinct attractors can be obtained, two stable steady states and a cyclic attractor (which may be a closed invariant curve on which periodic or quasi-periodic motion occurs, or a chaotic ring) which surrounds both the stable equilibria. In [2] we focused our attention on the creation of such closed invariant curve, and we limited our analysis to the parameters constellations where the map is invertible. In that case, a closed invariant curve separates the phase plane into two invariant regions, namely the regions inside and outside the invariant curve respectively. This means that every trajectory which starts inside the closed invariant curve is not allowed to go out, and any trajectory starting outside remains outside forever. This property, which also holds in continuous time dynamic models, has important consequences on the structure of the basins. For example, in the case of three coexisting attractors described above, it implies that the portion of the phase space located outside the closed invariant curve cannot contain points belonging to the basins of the steady states, so that only the trajectories starting close to a stable equilibrium can converge to it. Instead, when the map is noninvertible, this invariance property is lost, due to the folding action of the noninvertible map, and this may imply that portions of the basins of the two stable equilibria also exist outside the curve. So, even starting very far from the steady states, the asymptotic dynamics may ultimately converge to one of them.

By some examples, presented throughout this paper, we have shown how the property of noninvertibility may lead to the creation of non-connected basins of attraction, and how the creation of disjoint portions of the basins can be described in terms of contact bifurcations involving critical curves, as explained in [14], [25], [1]. These situations lead to dynamic scenarios which are very different from those usually observed in continuous time dynamical systems and discrete time ones which are represented by invertible recurrences.

The importance of basins delimitation in nonlinear dynamic models was already stressed by Medio [22], where it is written that "...An economy may be stable (i.e. at a stationary equilibrium) with respect to small perturbations, but if greater perturbations are considered an irreversible departure from the stable steady state may occur after which the system enters a different dynamic regime, e.g. characterized by persistent endogenous fluctuations...".

However, even more appears from the scenarios described in this paper. Indeed, when the map is noninvertible, the appearance of non connected basins leads to situations where a perturbation causing a small displacement from a stable equilibrium, as well as a very large perturbation, do not destabilize the system, i.e. do not cause the exit of the phase point from the basin of the equilibrium, whereas a perturbation of intermediate size may move the state into a different basin, thus causing the convergence towards another attractor, a rather counterintuitive result.

From the point of view of the mathematical methods, it is worth to note that the results outlined above are obtained through an analysis which is not limited to the usual study of the local stability and local bifurcations, based on the study of the linearization of the dynamical system through the localization of the eigenvalues of the Jacobian matrix, but they require a global analysis of the properties of the dynamical system. The method used to perform this analysis is based on an interplay among analytic, geometric and numerical techniques, a "modus operandi" which is typical for the study of the global dynamic properties of nonlinear dynamical systems of dimension greater than one, as stressed in [25], [1], [5].

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A Appendix

In this appendix, following [14], [25], we give some basic definitions and a minimal vocabulary of the theory of noninvertible maps of the plane and provide the reader with some basic facts about the method of critical curves. We also describe some properties of the critical curves of our map (3).

A two-dimensional map $T : (x, y) \to (x', y')$ can be written in the form

$$(x', y') = T(x, y) = (f(x, y), g(x, y)) \tag{11}$$

where $(x, y) \in \mathbb{R}^2$ and f, g are assumed to be real valued continuous functions. The point $(x', y') \in \mathbb{R}^2$ is called rank-1 image of the point (x, y) under T. The point $(x_t, y_t) = T^t(x, y)$, $t \in \mathbb{N}$, is called image (or forward iterate) of rank-t of the point (x, y), where T^0 is identified with the identity map and $T^t(\cdot) = T(T^{t-1}(\cdot))$. The fact that the map T is single-valued does not imply the existence and the uniqueness of its inverse T^{-1}. Indeed, for a given (x', y'), several rank-1 preimages (or backward iterates) $(x, y) = T^{-1}(x', y')$ may exist, i.e. the inverse relation T^{-1} may be multivalued. In this case T is said to be a *noninvertible map*. As the point (x', y') varies in the plane $\mathbb{R}^2$ the number of its rank-1 preimages can change. According to the number of distinct rank-1 preimages associated with each point of $\mathbb{R}^2$, the plane can be subdivided into regions, denoted by Z_k, whose points have k distinct preimages. Generally pairs of real preimages appear or disappear as the point (x', y') crosses the boundary separating regions characterized by a different number of rank-1 preimages. Accordingly, such boundaries are generally characterized by the presence of two coincident (merging) preimages. This leads us to the definition of *critical curves*, one of the distinguishing features of noninvertible maps. The critical curve of rank-1, denoted by LC (from the French "Ligne Critique") is defined as the locus of points having two, or more, coincident rank-1 preimages. These preimages are located in a set called critical curve of rank-0, denoted by LC_{-1}. The curve LC is the two-dimensional generalization of the notion of critical value (local minimum or maximum value) of a one-dimensional map, and LC_{-1} is the generalization of the notion of critical point (local extremum point). As in the case of differentiable one-dimensional maps, where the derivative necessarily vanishes at the local extremum points, for a two-dimensional continuously differentiable map the set LC_{-1} is included in the set of points in which the

Jacobian determinant vanishes:

$$LC_{-1} \subseteq \left\{(x,y) \in \mathbb{R}^2 | \det DT = 0\right\} \tag{12}$$

In fact, as LC_{-1} is defined as the locus of coincident rank-1 preimages of the points of LC, in any neighborhood of a point of LC_{-1} there are at least two distinct points mapped by T in the same point near LC. This means that the map T is not locally invertible in the points of LC_{-1} and, if the map T is continuously differentiable, it follows that $\det DT$ necessarily vanishes along LC_{-1}. If the set LC_{-1} is determined by (12) then LC is simply obtained as the image of LC_{-1}, i.e., $LC = T(LC_{-1})$.

The map T defined in (3) is an invertible map for certain ranges of the parameters and a noninvertible map in other ranges.

In fact, given a point $(Y', K') \in \mathbb{R}^2$, its preimages are computed by solving the algebraic system (3), i.e.

$$\begin{cases} Y' = Y + \alpha\left[\sigma\mu + \gamma\left(\frac{\sigma\mu}{\delta} - K\right) + \arctan(Y-\mu) - \sigma Y\right] \\ K' = (1-\delta)K + \sigma\mu + \gamma\left(\frac{\sigma\mu}{\delta} - K\right) + \arctan(Y-\mu) \end{cases}$$

with respect to Y and K. From the second equation we get:

$$\arctan(Y-\mu) = K' - \sigma\mu - \gamma\left(\frac{\sigma\mu}{\delta} - K\right) - (1-\delta)K$$

and by substituting into the first equation and rearranging:

$$\begin{cases} K\alpha(1-\delta) = Y(1-\alpha\sigma) + \alpha K' - Y' \\ \arctan(Y-\mu) = (\delta+\gamma-1)K + K' - \sigma\mu\left(1+\frac{\gamma}{\delta}\right) \end{cases}$$

from which we obtain:

$$\begin{cases} K = \frac{(1-\alpha\sigma)}{\alpha(1-\delta)}Y + \frac{\alpha K' - Y'}{\alpha(1-\delta)} \\ \arctan(Y-\mu) = (\delta+\gamma-1)\frac{(1-\alpha\sigma)}{\alpha(1-\delta)}Y + (\delta+\gamma-1)\frac{\alpha K'-Y'}{\alpha(1-\delta)} + K' - \sigma\mu\left(1+\frac{\gamma}{\delta}\right) \end{cases} \tag{13}$$

By setting:

$$m = (\delta+\gamma-1)\frac{(1-\alpha\sigma)}{\alpha(1-\delta)}$$

$$q = q(Y', K') = (\delta+\gamma-1)\frac{\alpha K' - Y'}{\alpha(1-\delta)} + K' - \sigma\mu\left(1+\frac{\gamma}{\delta}\right)$$

the second equation of (13) can be rewritten as follows:

$$\arctan(Y-\mu) = mY + q(Y', K')\ , \tag{14}$$

from which we can compute the Y-coordinates of the preimages of the point (Y', K').

One can easily verify that for $m < 0$ or $m \geq 1$ equation (14) has a unique solution for any given (Y', K') and therefore in this case the map has a unique inverse. For $m = 0$ the equation (14) has no solution if $q(Y', K') \leq -\pi/2$ or $q(Y', K') \geq \pi/2$, while it admits a unique solution in the opposite case.

In the case $0 < m < 1$ one, two or three solutions may exist depending on the value of $q = q(Y', K')$. In particular, for a given m, $0 < m < 1$, equation. (14) admits two solutions if the line of equation:

$$u(Y) = mY + q(Y', K') \tag{15}$$

is tangent to S-shaped curve $v(Y) = \arctan(Y - \mu)$. Two lines satisfy the above condition. The q-values which identify the tangent lines can be found by solving, with respect to q, equation (14), where Y must satisfy the tangency condition

$$\frac{d}{dY} \arctan(Y - \mu) = m \ ,$$

i.e.:

$$(Y - \mu)^2 = \frac{1}{m} - 1 \ . \tag{16}$$

We obtain:

$$q_1 = -\arctan\left(\sqrt{\frac{1}{m} - 1}\right) - m\left(\mu - \sqrt{\frac{1}{m} - 1}\right) \ ;$$

$$q_2 = \arctan\left(\sqrt{\frac{1}{m} - 1}\right) - m\left(\mu + \sqrt{\frac{1}{m} - 1}\right) \ ,$$

with $q_1 < q_2$. The lines of equation (15) with $q(Y', K') < q_1$ or $q(Y', K') > q_2$ have unique intersection with the S-shaped curve $v(Y) = \arctan(Y - \mu)$, while the lines with $q_1 < q(Y', K') < q_2$ intersect the curve in three points. This means that the points (Y', K') of the plane for which $q(Y', K') < q_1$ or $q(Y', K') > q_2$ have a unique rank-1 preimage, while the points for which: $q_1 < q(Y', K') < q_2$ have three distinct rank-1 preimages.

Thus, following the notation used in [25], we expect this map to be, for $0 < m < 1$, of the so-called type $Z_1 - Z_3 - Z_1$, which means that the phase plane is subdivided in different regions Z_1, Z_3, whose points have, respectively, one and three distinct rank-1 preimages. Following the critical curves theory developed by Gumowski and Mira ([14], [25]) we look for the critical curves of rank-1 of the map, denoted by LC, which generally bound such regions. In our case the critical curve of rank-1 is the locus of points having exactly two *merging* rank-1 preimages, and the locus of such merging preimages is the critical curve of rank-0 of T, denoted by LC_{-1} .

Thanks to the above calculations it is easy to find, in the phase plane, the equation of the critical curve of rank-1. This set is defined as follows:

$$LC = \left\{(Y,K) \in \mathbf{R}^2 : q(Y,K) = q_1 \text{ or } q(Y,K) = q_2\right\} \tag{17}$$

and it is therefore made up of the two lines:

$$LC_a: \quad K = \frac{(\delta+\gamma-1)}{\alpha\gamma}Y + \frac{1-\delta}{\gamma}\left[q_1 + \sigma\mu\left(1+\frac{\gamma}{\delta}\right)\right] \; ;$$

$$LC_b: \quad K = \frac{(\delta+\gamma-1)}{\alpha\gamma}Y + \frac{1-\delta}{\gamma}\left[q_2 + \sigma\mu\left(1+\frac{\gamma}{\delta}\right)\right] \; .$$

Each of the *critical values* $(Y,K) \in LC$ has two merging preimages, whose Y-values satisfy the tangency condition (16). Therefore, the locus of such preimages, denoted by LC_{-1} is made up of the two lines of equation:

$$LC_{-1,a}: \quad Y = \mu - \sqrt{\frac{1}{m} - 1} \; ;$$

$$LC_{-1,b}: \quad Y = \mu + \sqrt{\frac{1}{m} - 1} \; ,$$

with $LC_a = T(LC_{-1,a})$ and $LC_b = T(LC_{-1,b})$.

In the case of the map T the critical curve LC_{-1} is also the locus of points (Y,K) of the phase plane in which the determinant of the Jacobian matrix $DT(Y,K)$ of T,

$$DT(Y,K) = \begin{bmatrix} 1 + \frac{\alpha}{1+(Y-\mu)^2} - \alpha\sigma & -\alpha\gamma \\ \frac{1}{1+(Y-\mu)^2} & -(\delta+\gamma-1) \end{bmatrix}$$

vanishes[5]. We obtain:

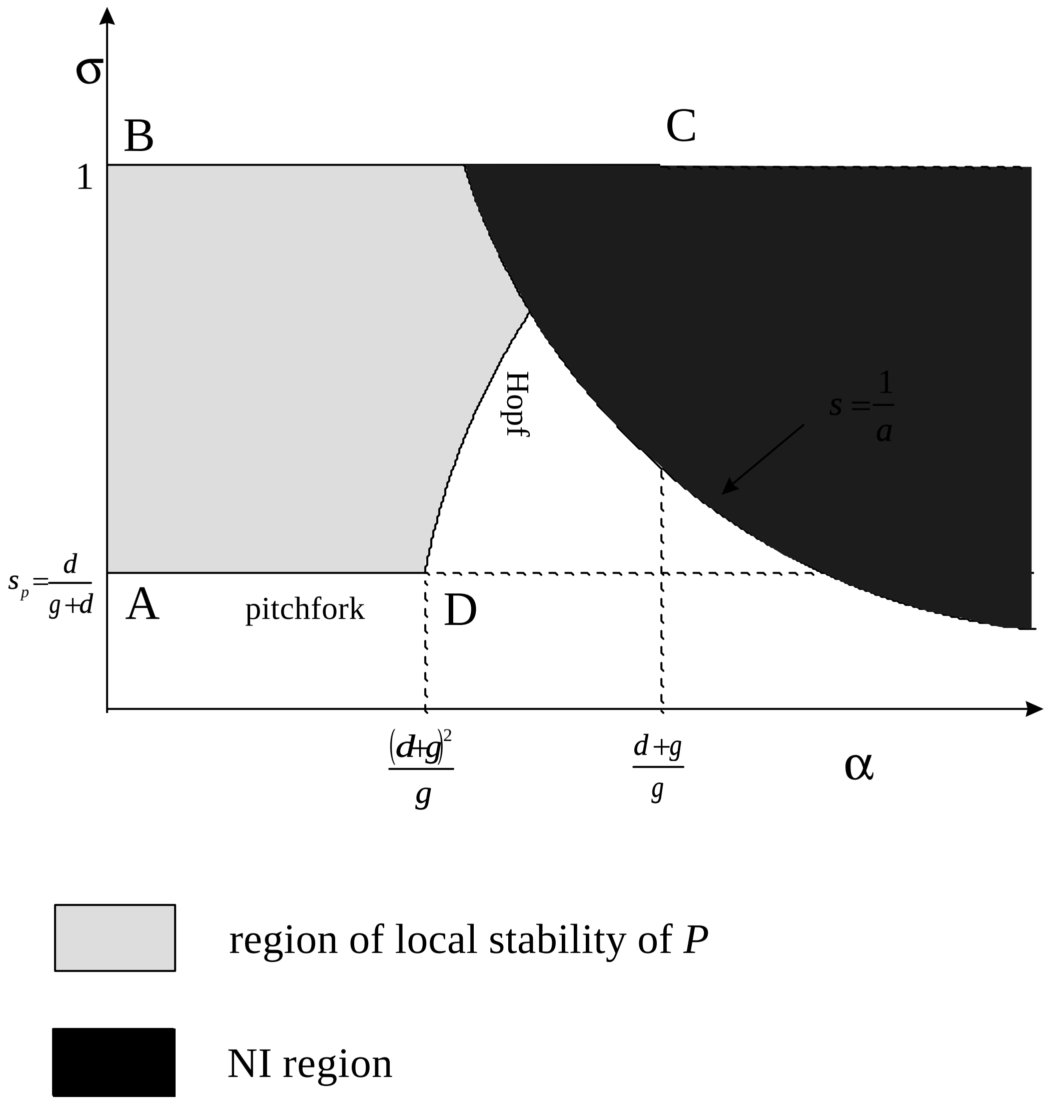
$$\det DT(Y,K) = \frac{\alpha(1-\delta)}{1+(Y-\mu)^2} - (\delta+\gamma-1)(1-\alpha\sigma)$$

and the equation $\det DT(Y,K) = 0$ can be rewritten as:

$$\frac{1}{1+(Y-\mu)^2} = \frac{(\delta+\gamma-1)(1-\alpha\sigma)}{\alpha(1-\delta)} (= m)$$

i.e. condition (16).

[5]As remarked before, for a continuously differentiable map of the plane the critical set LC_{-1} is in general a subset of the locus defined by $\det DT = 0$.



σ
B
C
1
Hopf
$s = \frac{1}{a}$
$s_p = \frac{d}{g+d}$
A
pitchfork
D
$\frac{(d+g)^2}{g}$
$\frac{d+g}{g}$
α
region of local stability of *P*
NI region

Fig. 1

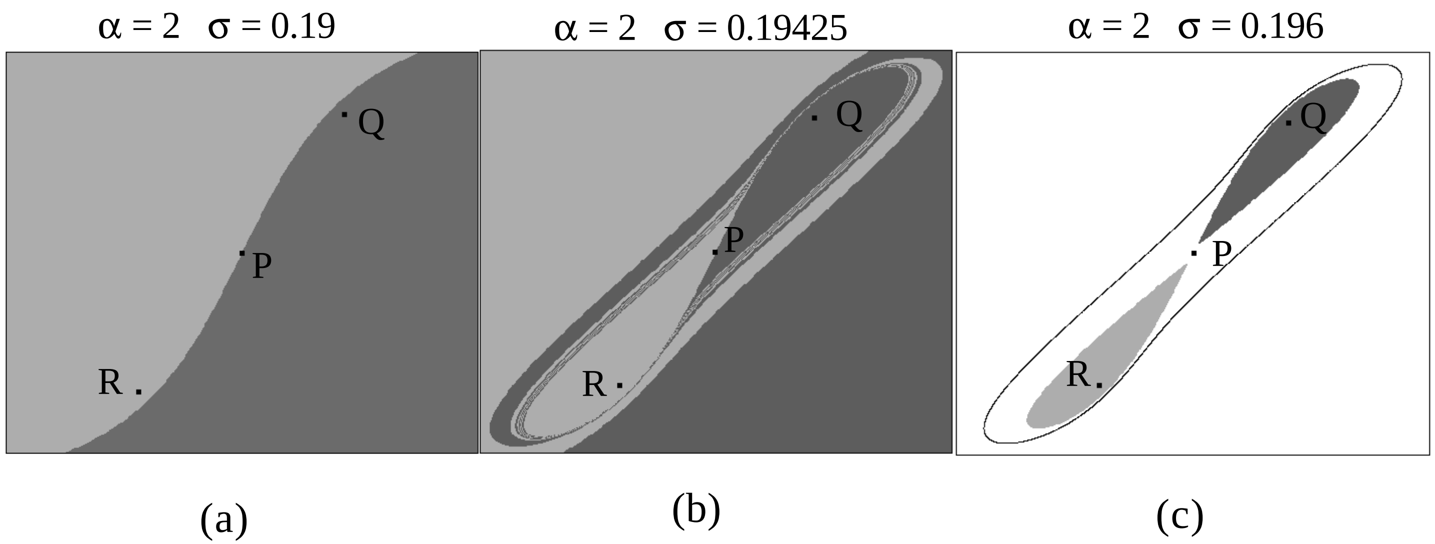


Fig. 2

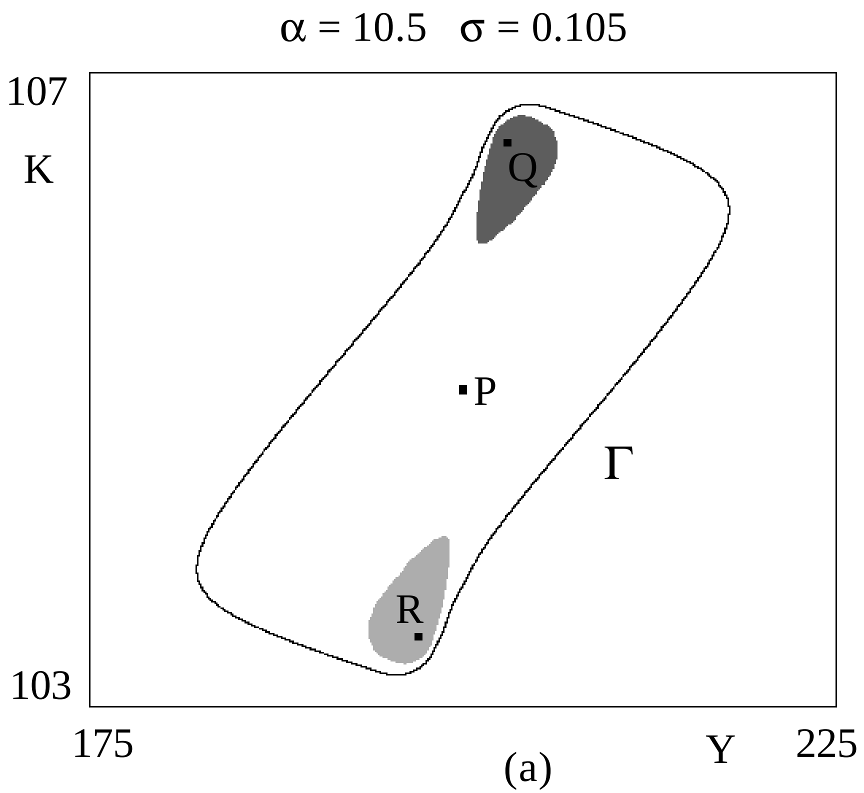
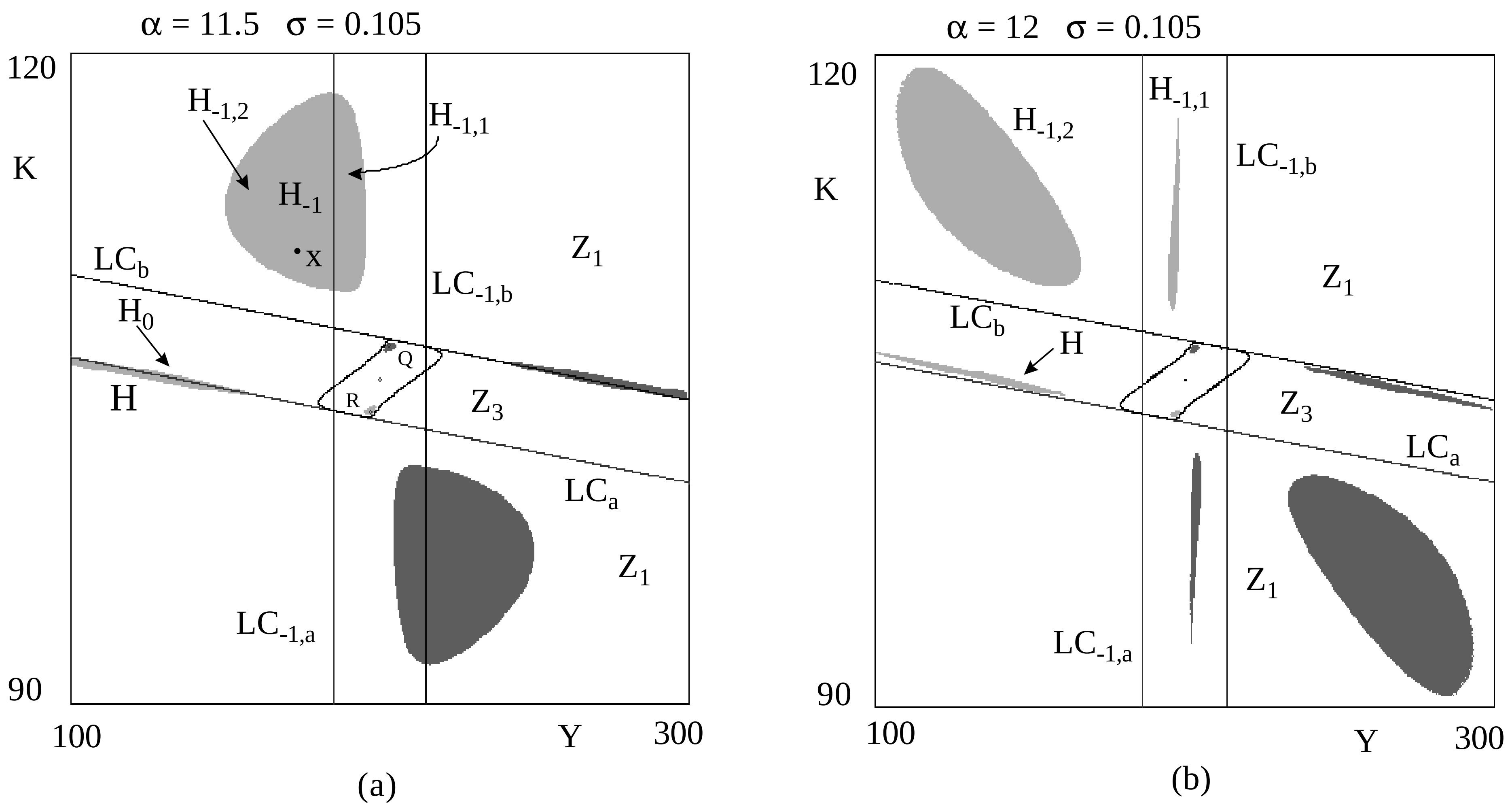

α = 10.5   σ = 0.105
107
K
Q
P
Γ
R
103
175
Y
225
(a)
zoom- out
112
K
LC_b
Z_1
LC_a
Z_3
H
LC_-1,a
LC_-1,b
Z_1
97
100
(b)
Y
300

Fig. 3



α = 11.5 σ = 0.105
120
90
100
300
K
Y
H-1,2
H-1,1
H-1
x
LCb
H0
H
LC-1,b
Z1
Q
R
Z3
LCa
Z1
LC-1,a
(a)
α = 12 σ = 0.105
120
90
100
300
K
Y
H-1,2
H-1,1
LC-1,b
Z1
LCb
H
Z3
LCa
Z1
LC-1,a
(b)

Fig. 4

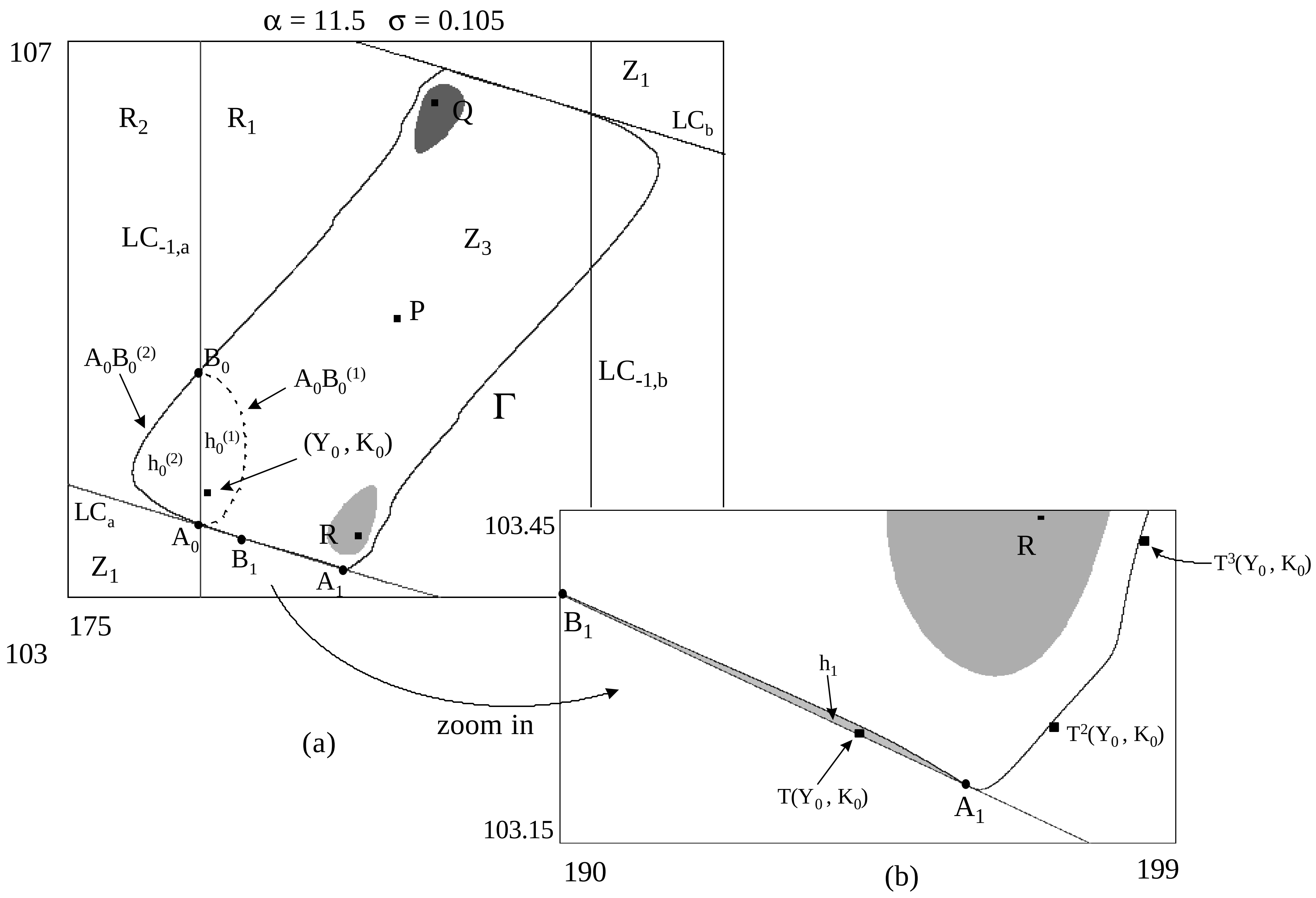

α = 11.5 σ = 0.105
107
103
175
R2
R1
Z1
Q
LCb
LC-1,a
Z3
P
A0B0(2)
B0
A0B0(1)
LC-1,b
Γ
h0(1)
h0(2)
(Y0 , K0)
LCa
A0
B1
R
A1
Z1
zoom in
(a)
103.45
103.15
190
199
R
T3(Y0 , K0)
B1
h1
T2(Y0 , K0)
T(Y0 , K0)
A1
(b)

Fig. 5

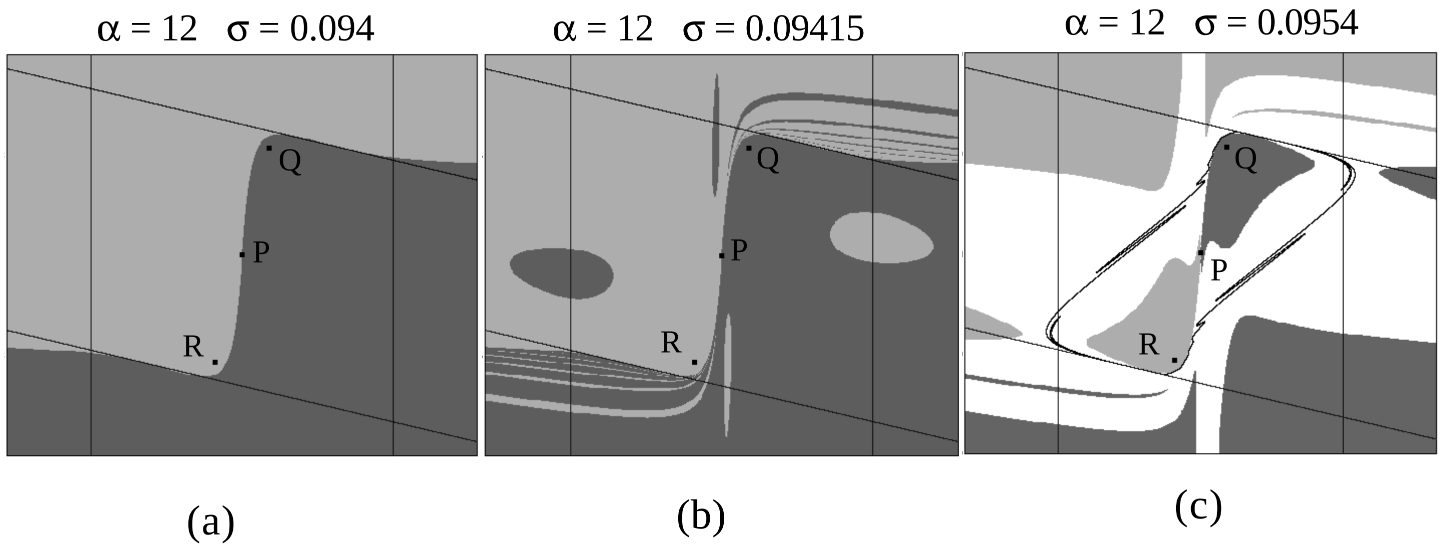


Fig. 6