PERGAMON

Chaos, Solitons and Fractals 12 (2001) 805–822

CHAOS
SOLITONS & FRACTALS

www.elsevier.nl/locate/chaos

From bi-stability to chaotic oscillations in a macroeconomic model

Roberto Dieci [a,*], Gian-Italo Bischi [b], Laura Gardini [a,b]

[a] *Istituto di Matematica "E. Levi", University of Parma, Parma, Italy*
[b] *Istituto di Scienze Economiche, University of Urbino, Urbino, Italy*

Accepted 1 March 2000

Communicated by T. Puu

Abstract

In this paper a discrete-time economic model is considered where the savings are proportional to *income* and the investment demand depends on the difference between the current income and its exogenously assumed equilibrium level, through a nonlinear S-shaped increasing function. The model can be ultimately reduced to a two-dimensional discrete dynamical system in income and *capital*, whose time evolution is "driven" by a family of two-dimensional maps of *triangular* type. These particular two-dimensional maps have the peculiarity that one of their components (the one driving the income evolution in the model at study) appears to be uncoupled from the other, i.e., an independent one-dimensional map. The structure of such maps allows one to completely understand the forward dynamics, i.e., the asymptotic dynamic behavior, starting from the properties of the associated one-dimensional map (a bimodal one in our model). The equilibrium points of the map are determined, and the influence of the main parameters (such as the *propensity to save* and the *firms' speed of adjustment* to the excess demand) on the local stability of the equilibria is studied. More important, the paper analyzes how changes in the parameters' values modify both the asymptotic dynamics of the system and the structure of the basins of the different and often coexisting attractors in the phase-plane. Finally, a particular "global" (homoclinic) bifurcation is illustrated, occurring for sufficiently high values of the firms' adjustment parameter and causing the switching from a situation of *bi-stability* (coexistence of two stable equilibria, or attracting sets of different nature) to a regime characterized by wide chaotic oscillations of income and capital around their exogenously assumed equilibrium levels.

1. Introduction

We consider a discrete-time economic model, described by a two-dimensional dynamical system in *income* and *capital*, where we assume that the savings are proportional to the income and the investment demand depends on the difference between the current income and its exogenously assumed equilibrium level, through a nonlinear S-shaped increasing function. The model we examine is a particular case of a more general *Kaldor-type* business cycle model proposed in [14] and investigated in its general dynamic behavior in [4]. Ultimately it reduces to a family of two-dimensional maps of *triangular* type. The analysis of the dynamics generated by the model is therefore a pedagogical tour through the properties of *triangular maps*. These particular two-dimensional maps have the peculiarity that one of their components (the one driving the income evolution in the model at study) is decoupled from the other, i.e., an independent one-dimensional map. As shown in [6], the particular structure of such maps allows to completely understand

* Corresponding author. Tel.: +39-0521-902387/8; fax: +39-0521-902385.
E-mail addresses: dieci@economia.econ.unipr.it, rdieci@unipr.it (R. Dieci), bischi@econ.uniurb.it (G.-I. Bischi), gardini@econ.uniurb.it (L. Gardini).

0960-0779/01/$ - see front matter
PII: S0960-0779(00)00055-2

the forward dynamics, i.e., the asymptotic dynamic behavior, thanks to the fact that the bifurcations can be deduced from the associated one-dimensional map (a bimodal one in our model).

We show that for economically meaningful values of the parameters the model always has three steady states and we study the influence of the main parameters, like the *propensity to save* and the *firms' speed of adjustment* to the excess demand, on the local stability of the equilibria. More important, we show how the dynamic behavior of the system is deeply influenced by the switching to the regime of noninvertibility of the map, leading to more complex structures of the basins of the different attracting sets. Finally, a particular "global" bifurcation is analyzed marking the switching from a situation of *bi-stability*, where the phase-plane is shared between two coexisting attractors, to a regime characterized by more complex asymptotic dynamics.

The paper is organized as follows. In Section 2, we give a description of the model and an economic interpretation of the underlying assumptions. In Section 3, we analyze in detail some important properties of the two-dimensional map driving the dynamics, such as the triangular structure of the map, the linear structure of its second component, the existence of fixed points and the conditions for their local stability, some symmetry properties and the conditions for the invertibility or noninvertibility of the map. In Section 4, by using both the analytical properties of the map and some numerical tools, we discuss the transition to more and more complex dynamic behaviors which is observed for increasing values of the firms' speed of adjustment to the excess demand. Some conclusions are contained in Section 5.

2. The model

The model we consider is a particular case of a Kaldor-type business cycle model proposed in [14] and investigated in its general dynamic behavior in [4]. The model studied in [4,14] starts from a well-known discrete-time version of the Kaldor model (see e.g., [2,10,11])

$$Y_{t+1} - Y_t = \alpha(I_t - S_t), \tag{1a}$$

$$K_{t+1} = (1 - \delta)K_t + I_t, \tag{1b}$$

where the dynamic variables Y_t and K_t represent the income (or output) value and the capital stock in period t, respectively, α ($\alpha > 0$) the firms' speed of adjustment to the excess demand, the parameter δ ($0 < \delta < 1$) the *capital stock's depreciation rate*, $I_t = I_t(Y_t, K_t)$ the investment demand in period t and $S_t = S_t(Y_t)$ are the savings in the same period.

As it is well known (see, for instance [5]), under the simple assumption that the investment demand I_t is independent of the capital stock K_t, i.e., $\partial I_t/\partial K_t = 0$, and that both I_t and S_t are linear increasing functions of Y_t, with $\mathrm{d}S/\mathrm{d}Y > \partial I/\partial Y$, the system is globally asymptotically stable, while by introducing nonlinearities into the investment demand curve, for example by assuming that I_t is a sigmoid-shaped function of Y_t, we may have a situation of bi-stability. Figs. 1(a) and (b) qualitatively represent the income adjustment process in case of disequilibria (captured by Eq. (1a)), in the linear and nonlinear cases.

The essential dynamic feature that enables the model to display cyclical behavior is the long term "shifting" of the investment function as a consequence of changes in the capital stock, as qualitatively described by Kaldor in [7]. By assuming that the investment demand curve shifts downwards (resp. upwards) when the income, and consequently the capital stock, increases (resp. decreases), cyclic movements of the level of income and capital may occur, as qualitatively shown in Fig. 2 (see again [5, pp. 122–129], for economic justifications of these assumptions and for a wider discussion).

In the form proposed in [14], savings are assumed, as usual, proportional to income

$$S_t = \sigma Y_t, \tag{2}$$

where the coefficient σ, $0 < \sigma < 1$, represents the propensity to save. On the other hand, investment demand is assumed to be an increasing and sigmoid-shaped function of income and a linear decreasing function of capital stock

$$I_t = \sigma\mu + \gamma\left(\frac{\sigma\mu}{\delta} - K_t\right) + \arctan(Y_t - \mu), \tag{3}$$

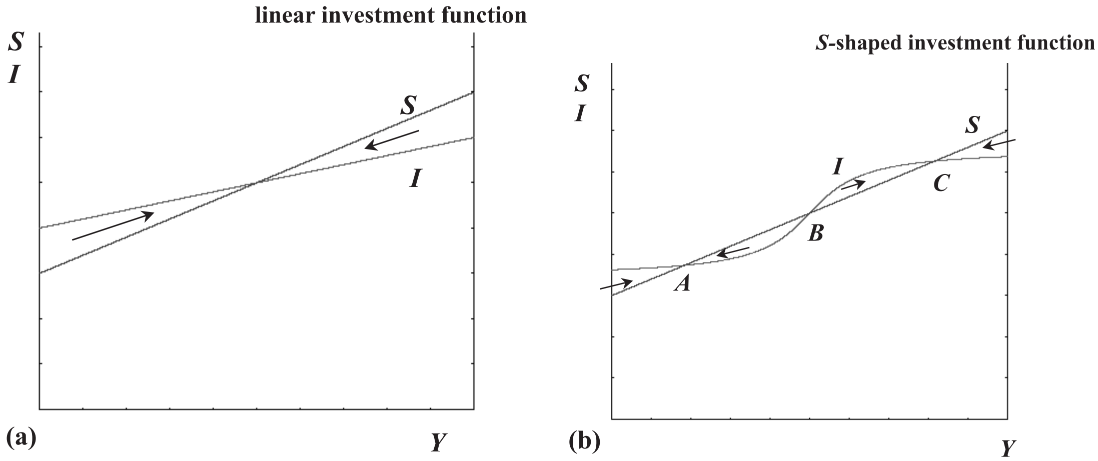

Fig. 1. Effect of the introduction of nonlinearities into the investment function: linear investment function (a) and sigmoid-shaped investment function (b).

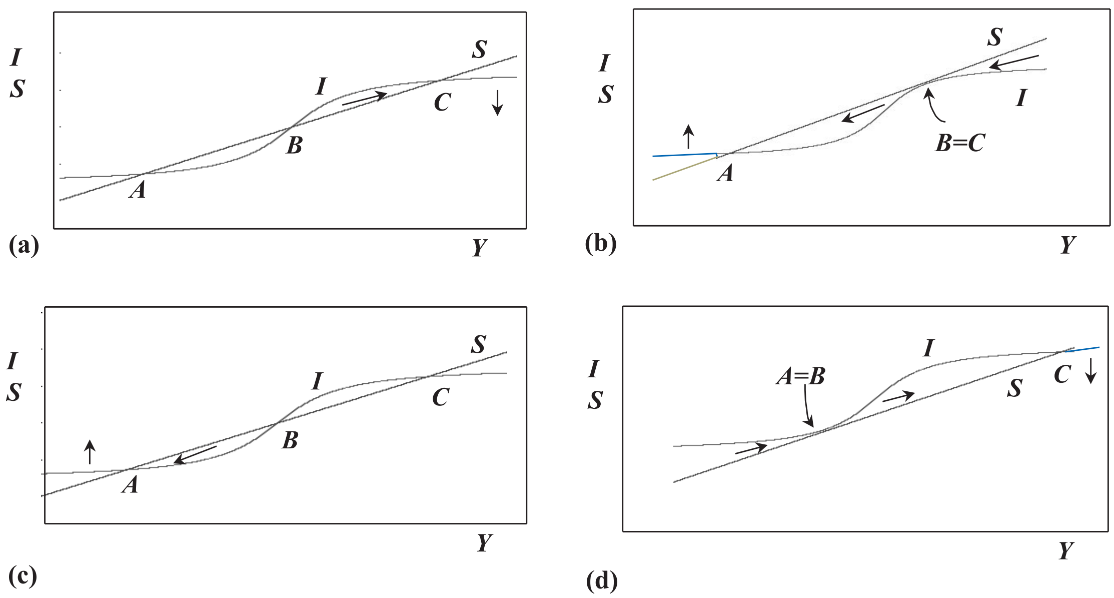

Fig. 2. Shifting of the investment function, as a consequence of changes in the income level and thus in the capital stock: the sequence (a)–(d) shows the mechanism which may generate endogenous oscillations of income.

where γ is a positive parameter, μ ($\mu > 0$) the exogenously assumed equilibrium level of income and therefore $\sigma\mu$ represents the equilibrium level of savings (and also of investment demand), while $\sigma\mu/\delta$ is the equilibrium capital stock. As usual in Kaldor business cycle models, one of three steady states may exist: in this latter case, besides the exogenously assumed equilibrium $P = (\mu, \sigma\mu/\delta)$, two more steady states exist, a "wealth" equilibrium Q, characterized by high equilibrium levels of income and capital, and a "poverty" equilibrium R, with low levels of income and capital. As shown in [4], a large variety of dynamic behaviors can occur, in particular bi-stability, with the exogenous equilibrium P acting as a watershed between the basins of attraction of the two stable steady states Q and R, and *self-sustaining oscillations*, regular or chaotic, around the equilibrium P. Moreover, regions in the space of the parameters can be determined where a large attracting limit cycle coexists with the two stable equilibria Q and R, surrounding their basins of attraction.

The aim of the present paper is to analyze in more detail the dynamic behavior of the model under the assumption that the investment demand curve is not affected by changes in the capital stock, i.e., $\partial I/\partial K = 0$. In fact, even in this case, usually described as a simple situation of bi-stability, different and more complex dynamic phenomena may occur: this complexity is related, on one hand, to the asymptotic dynamics, i.e., to the nature of the attracting sets, and on the other hand to the structure of their basins of attraction.

By setting $\gamma = 0$ in Eq. (3) and substituting into the system (1a)–(1b), the model ultimately reduces to the following two-dimensional dynamical system in income and capital:

$$Y_{t+1} = Y_t + \alpha\sigma\mu + \alpha \arctan(Y_t - \mu) - \alpha\sigma Y_t, \quad (4a)$$

$$K_{t+1} = \sigma\mu + \arctan(Y_t - \mu) + (1 - \delta)K_t. \quad (4b)$$

The study of the dynamical properties of the system (4a)–(4b) allows us to explore the long-run behavior of income and capital stock, starting from a given initial condition.

3. Some general properties

As described at the end of the previous section, the time evolution of income and capital is obtained by the iteration of a two-dimensional nonlinear map $T : (Y_t, K_t) \to (Y_{t+1}, K_{t+1})$ given by

$$T : \begin{cases} Y' = (1 - \alpha\sigma)Y + \alpha\sigma\mu + \alpha \arctan(Y - \mu), & (5a) \\ K' = (1 - \delta)K + \sigma\mu + \arctan(Y - \mu), & (5b) \end{cases}$$

where the symbol $'$ denotes the unit time advancement operator, that is, if the right-hand side variables are income and capital at time t, then the left-hand ones represent income and capital at time $t + 1$.

We shall now consider in detail some properties of the map T, such as the consequences of its triangular structure, the particular form of the second component (5b), the existence of fixed points and their local stability analysis, some symmetry property and the role of invertibility or noninvertibility of the map. In order to analyze these properties, we briefly recall the meaning of terms that will be used in the following.

Let A be a subset of the plane. Then we say that A is a trapping set of T (or T is trapping on A) if $T(A) \subseteq A$ (that is, if A is mapped into itself by T); T is *invariant* on A (or A is invariant by T) if $T(A) = A$, i.e., if A is trapping and for any $\mathbf{y} \in A$, there exists $\mathbf{x} \in A$ such that $T(\mathbf{x}) = \mathbf{y}$. A p-cycle of T is a periodic orbit of T of least period p, $p \geqslant 1$. A p-periodic point of T is a point belonging to some p-cycle of T.

In the following, we also will denote by $DT(Y, K)$ the Jacobian matrix of the map T, by T^n, $n \geqslant 1$, the nth iterated of the map T and by $DT^n(Y, K)$ the Jacobian matrix of the map T^n.

3.1. The triangular structure of the map

We can observe that the first component of the map T does not depend on K: the map is therefore, a triangular map, i.e., it has the following structure:

$$T : \begin{cases} Y' = F(Y), & (6a) \\ K' = G(Y, K). & (6b) \end{cases}$$

This means that the dynamics of the income Y are only affected by income itself, being $Y_{t+1} = F(Y_t)$, whereas the time evolution of the capital stock is also influenced by the income, being $K_{t+1} = G(Y_t, K_t)$. By using the terminology of the engineering systems (see, e.g. [15]) we may say that the one-dimensional system (6a) is the "driving system" and the capital stock is "driven" by the income dynamics. [1] As a consequence, the dynamics of the map T is deeply influenced by the dynamics of the one-dimensional

[1] In the physical and engineering literature triangular maps are often referred to as *skew products*.

map $Y' = F(Y)$. In particular, many of its bifurcations are associated to those of the one-dimensional map $Y' = F(Y)$ and all the cycles of T stem from cycles of F. Moreover, since the Jacobian matrix of the map T, given by

$$DT(Y,K) = \begin{bmatrix} 1 + \frac{\alpha}{1+(Y-\mu)^2} - \alpha\sigma & 0 \\ \frac{1}{1+(Y-\mu)^2} & 1-\delta \end{bmatrix} \tag{7}$$

is lower triangular, it cannot have complex eigenvalues and thus the occurrence of regular oscillations, similar to those usually observed in Kaldor-type models, is ruled out.

Let us briefly recall some useful properties of two-dimensional triangular maps (for a wider discussion, see [6,8,9]).

Property 1. The eigenvalues of $DT(Y,K)$ are always real, given by $z_1 = F'(Y)$ and $z_2 = G_K(Y,K)$. Any fixed point of T is therefore, either a node or a saddle.

Property 2. The eigenvalues of $DT^n(Y,K)$, for any integer $n \geqslant 1$, are real. Any cycle of T is therefore, either a node or a saddle. If $\mathscr{C}_p = \{(Y_i,K_i),\ i = 1,2,\ldots,p\}$ is a p-cycle of T, then the eigenvalues of the cycle (i.e., the eigenvalues of the Jacobian matrix of T^p in any point of the cycle) are given by $z_1 = \prod_{i=1}^{p} F'(Y_i)$ and $z_2 = \prod_{i=1}^{p} G_K(Y_i,K_i)$.

Property 3. Let $\mathscr{C}_p = \{(Y_i,K_i),\ i = 1,2,\ldots,p\}$ be a p-cycle of T. Then $\{Y_1,Y_2,\ldots,Y_p\}$ is a periodic orbit of the one-dimensional map F of least period r, where r is such that $rm = p$ for some integer $m \geqslant 1$.

Property 4. Let (Y_i,K_i), $i = 1,2,\ldots,p$, be a point of a p-cycle of T and (Y_i,K) a point on the vertical line $Y = Y_i$. Then there exists some integer $m \geqslant 1$ such that $T^r(Y_i,K)$, where $r = p/m$, is trapping on the line $Y = Y_i$ and may be considered a one-dimensional map of the state variable K.

In particular Property 4 implies that:
1. no points on the vertical line $Y = Y_i$ can belong to the stable set of some other cycle of T with periodic points all outside that line;
2. any p-periodic point of T must belong to trapping (for some T^r, with $rm = p$, $m \geqslant 1$) vertical lines $Y = Y_i$, where Y_i is a r-periodic point of the one-dimensional map F;
3. if $\mathscr{C}_r = \{Y_1,Y_2,\ldots,Y_r\}$ is an r-cycle of the map F and a p-cycle $\mathscr{C}_p$ of T exists, on the vertical lines $Y = Y_i$, $i = 1,2,\ldots,r$, of period $p = rm$ for some integer $m \geqslant 1$, then the eigenvalue z_1 of the p-cycle $\mathscr{C}_p$ is related to that of the r-cycle $\mathscr{C}_r$ of F (let us denote it by τ) as follows: $z_1 = \tau^m$;
4. if a p-cycle $\mathscr{C}_p$ of the two-dimensional triangular map T is a saddle with $|z_1| = |\prod_{i=1}^{p} F'(Y_i)| > 1$ and $|z_2| = |\prod_{i=1}^{p} G_K(Y_i,K_i)| < 1$, then the points of the local stable set of $\mathscr{C}_p$ belong to the vertical lines through the periodic points.

Property 5. If $\mathscr{C}_r = \{Y_1,Y_2,\ldots,Y_r\}$ is an r-cycle of the map F, then the restriction of the map T^r to any of the vertical lines $Y = Y_i$, $i = 1,2,\ldots,r$, is trapping on that line. If the r-cycle of F is attracting (resp. repelling), then the vertical lines $Y = Y_i$, $i = 1,2,\ldots,r$, are attracting (resp. repelling) for T^r.

As far as the bifurcations of the map T are concerned, it is easy to see from the above properties that any bifurcation of the one-dimensional map F gives a bifurcation of T. In particular, a fold bifurcation of F creates a couple of cyclical trapping lines of T (one repelling and one attracting). At a *flip* bifurcation of a cycle of F, trapping cyclical vertical lines from attracting (for T) become repelling and new cyclical attracting lines are created.

Finally, it is well known that if the two-dimensional map T is a noninvertible endomorphism, with F and G continuously differentiable, then the locus LC_{-1} (*critical curve of rank* 0) of T is generally given by $\det DT(Y,K) = 0$ (see, e.g. [13]). Therefore, for a noninvertible triangular map the following property holds:

Property 6. The locus LC_{-1} of the phase-plane is made up of curves LC_{-1,a_k} and $LC_{-1,b}$ such that
(i) LC_{-1,a_k} are vertical lines of equation $Y = c_{-1,a_k}$, where the c_{-1,a_k} satisfy $F'(c_{-1,a_k}) = 0$;
(ii) $LC_{-1,b}$ is the locus $G_K(Y,K) = 0$.
The critical curves $LC_{i,a_k} = T^{i+1}(LC_{-1,a_k})$, for $i \geqslant 0$, belong to vertical lines $x = c_{i,a_k}$, where $c_{i,a_k} = F^{i+1}(c_{-1,a_k})$ are critical points of $F(Y)$.

3.2. The structure of the second component of the map

It can be noticed that the second component of the map (5a)–(5b) is separable with respect to the variables Y and K and linear in K, i.e., it has the following structure:

$$K' = (1-\delta)K + I(Y), \tag{8}$$

where $I(Y) = \sigma\mu + \arctan(Y-\mu)$ is the investment demand function.

This property, together with the above-mentioned properties of two-dimensional triangular maps, enable us to formulate the following propositions, which are proved in Appendix A.

Proposition 1. *The fixed points and the cycles of the map* (5a)–(5b) *can only be either stable nodes or saddles.*

Proposition 2. *The stable manifold W^s of a saddle cycle of T is the union of the lines of equation $Y = Y_i$, where Y_i, $i = 1,2,\ldots,r$, are the periodic points of the corresponding cycle of the one-dimensional map F, and of the lines of equation $Y = Y_{-j}$, where Y_{-j}, $j = 1,2,\ldots$, are the preimages of any rank of the periodic points.*

3.3. Fixed points

The equilibrium points (or steady states) of the map T are the solutions of the algebraic system

$$\sigma\mu + \arctan(Y-\mu) - \sigma Y = 0,$$
$$\sigma\mu + \arctan(Y-\mu) - \delta K = 0,$$

obtained by setting $Y' = Y$ and $K' = K$ in (5a) and (5b). This system can be rewritten as

$$K = \frac{\sigma}{\delta} Y, \tag{9a}$$
$$\sigma(Y-\mu) = \arctan(Y-\mu). \tag{9b}$$

It is trivial to realize that the steady states are independent from the firms' adjustment parameter α. The first equation says that the fixed points belong to the line $K = \frac{\sigma}{\delta}Y$ in the phase-plane, and from the second equation we have that the Y-values (which are the fixed points of the one-dimensional map F) can be obtained as intersections of the two curves of equation $f(Y) = \sigma(Y-\mu)$ and $g(Y) = \arctan(Y-\mu)$. It follows that if $\sigma \geqslant 1$, then the system (9a)–(9b) admits the point $P = (\mu, \mu\frac{\sigma}{\delta})$ as unique solution, while in the case $0 < \sigma < 1$ three solutions exist, the point P and the points Q and R, which are symmetric with respect to P, as we shall see in Section 3.4. Of course, since σ represents the propensity to save and the case $0 < \sigma < 1$ includes the interval of values of interest for us, the case of three fixed points is the only one economically meaningful. The explicit coordinates of the fixed points Q and R cannot be written. We can numerically compute them as $(Y_Q, \frac{\sigma}{\delta}Y_Q)$ and $(Y_R, \frac{\sigma}{\delta}Y_R)$, where Y_Q and Y_R are obtained from Eq. (9b) and $Y_R = 2\mu - Y_Q$ due to the symmetry property (see, Section 3.4).

3.4. Symmetry property

It is worth noting that the map T is symmetric with respect to the fixed point $P = (\mu, \mu\frac{\sigma}{\delta})$. This means that symmetric points are mapped into symmetric points (with respect to P). Denote by $F(Y)$ and $G(Y,K)$ the two components of the map T

$$F(Y) = (1-\alpha\sigma)Y + \alpha\sigma\mu + \alpha \arctan(Y-\mu),$$
$$G(Y,K) = (1-\delta)K + \sigma\mu + \arctan(Y-\mu)$$

and observe that the symmetry of the point (Y,K) with respect to P is the point $(2\mu - Y, 2\frac{\sigma\mu}{\delta} - K)$. The above property, which can easily be verified, can be formalized as follows:

$$F(2\mu - Y) = 2\mu - F(Y),$$
$$G(2\mu - Y, 2\frac{\sigma\mu}{\delta} - K) = 2\frac{\sigma\mu}{\delta} - G(Y,K).$$

We can so state the following propositions:

Proposition 3. *Let* $S(Y,K) = (2\mu - Y, 2\frac{\sigma\mu}{\delta} - K)$ *be the symmetry with respect to the fixed point* $P = (\mu, \mu\frac{\sigma}{\delta})$. *Then* $\forall (Y,K)$ *we have*

$$S(T(Y,K)) = T(S(Y,K)).$$

This implies that a cycle of T is either symmetric with respect to P or admits a symmetric cycle. That is:

Proposition 4. *Let* $\mathscr{C} = \{(Y_1,K_1),\ldots,(Y_p,K_p)\}$ *be a cycle of T of period* $p \geqslant 1$. *Then*
either $S(\mathscr{C}) = \mathscr{C}$
or $S(\mathscr{C}) = \mathscr{C}' \neq \mathscr{C}$,
where $\mathscr{C}' = \{S(Y_1,K_1),\ldots,S(Y_p,K_p)\}$ *is another different cycle of T, of the same period p, with periodic points, which are symmetric with respect to P of the periodic points in* $\mathscr{C}$.

3.5. Local stability analysis of the fixed points

Let us now turn to the local stability of the fixed points $P = (\mu, \mu\frac{\sigma}{\delta})$, $Q = (Y_Q, \frac{\sigma}{\delta}Y_Q)$ and $R = (Y_R, \frac{\sigma}{\delta}Y_R)$. The results about the nature of the fixed points and their local stability analysis are summarized by the following proposition, proved in Appendix B, which defines, for each equilibrium point, the stability region in the parameters space $\Omega = \{(\alpha,\sigma) \in \mathbb{R}^2 \mid \alpha > 0,\ 0 < \sigma < 1\}$.

Proposition 5. *The equilibrium P is a saddle for each* $(\alpha,\sigma) \in \Omega$. *The equilibria Q and R are stable nodes for each* (α,σ) *in the region* $\Omega_s(Q)$ *defined as*

$$\Omega_s(Q) = \left\{(\alpha,\sigma) \in \Omega | \alpha < \alpha_f(\sigma)\right\},$$

where $\alpha_f(\sigma) = 2/\{\sigma - [1 + (Y_Q - \mu)^2]^{-1}\} > 2$. *Outside this region, Q and R are saddles.*

It is worth noting that the only way the fixed point Q (and thus R) can lose stability is through a bifurcation with $z_1(Q) = -1$, i.e., a flip (or *period doubling*) bifurcation, where the stable fixed point becomes unstable (a saddle in our case) giving rise to a stable cycle of period two. The determination of the flip-bifurcation curve of the fixed points Q and R, in the parameters' plane (α,σ), can only be done through numerical evaluation of the quantity $\alpha_f(\sigma) = 2/\{\sigma - [1 + (Y_Q - \mu)^2]^{-1}\}$.

3.6. Invertibility conditions

For some regions of the parameters' space the map T is a noninvertible map of the plane. This means that while starting from some initial values of income and capital stock, say (Y_0,K_0), the iteration of (5a)–(5b) uniquely defines the trajectory $(Y_t,K_t) = T^t(Y_0,K_0)$, $t = 1,2,\ldots$, the backward iteration of (5a)–(5b) is not uniquely defined. In fact, a point (Y',K') of the plane can have several Rank-1 preimages.

As we already pointed out, many of the properties of the map at study are related to those of the one-dimensional map

$$Y' = F(Y) = (1 - \alpha\sigma)Y + \alpha\sigma\mu + \alpha\arctan(Y - \mu).$$

It can be immediately proved that the two-dimensional map T is invertible if and only if the one-dimensional map F is. It is worth noting that this property is not simply due to the triangular structure of the map T but also due to the fact that the second component of T, i.e., the function

$$G(Y, K) = (1 - \delta)K + \sigma\mu + \arctan(Y - \mu),$$

is linear in K.

Turning to the conditions under which the one-dimensional map $F(Y)$ is invertible, it is easy to show that a point Y' has a unique preimage if and only if $\alpha\sigma \leqslant 1$, while in the opposite case, $\alpha\sigma > 1$, a point may have one, two, or three different preimages. In fact, in the case $\alpha\sigma > 1$, F is a bimodal map with a local minimum point, critical point of Rank 0 [2] denoted by $c_{-1,m}$, and a local maximum point, critical point of Rank 0 denoted by $c_{-1,M}$, where

$$c_{-1,m} = \mu - \sqrt{\frac{\alpha}{\alpha\sigma - 1} - 1}; \quad c_{-1,M} = \mu + \sqrt{\frac{\alpha}{\alpha\sigma - 1} - 1}. \tag{10}$$

The critical points of Rank 1 are given by their images

$$c_m = F(c_{-1,m}); \quad c_M = F(c_{-1,M}).$$

Thus the points Y with $Y < c_m$ or $Y > c_M$ have a unique preimage, the points satisfying $c_m < Y < c_M$ have three distinct preimages and the points $Y = c_m$ and $Y = c_M$ have two preimages *merging* in a critical point, together with another distinct preimage, called *extra-preimage*.

If we consider again the two-dimensional map T, then we can see that if Y is a point with three (resp. one, two) preimages for the one-dimensional map F, then the whole vertical line through this point has three (resp. one, two) preimages for the two-dimensional map T. Thus, following the notation used in [13], we have that the map T is, for $\alpha\sigma > 1$, of type $Z_1 - Z_3 - Z_1$, which means that the phase-plane is subdivided in different regions $Z_j (j = 1, 3)$ each point of which has j distinct Rank-1 preimages. The critical curves of Rank 1, denoted by LC generally bound such Z_j regions (see again [13]). They are defined as the locus of points having at least two merging Rank-1 preimages, and for our map T they are given by the union of two vertical lines

$$Y = c_m, \qquad Y = c_M.$$

The locus of merging Rank-1 preimages, which constitutes the critical curve of Rank 0 of T, denoted by LC_{-1}, is made up of the two lines

$$Y = c_{-1,m}, \qquad Y = c_{-1,M}.$$

We remark that we obtain the equations of LC and LC_{-1} starting from the critical points of the one-dimensional map F, i.e., by substantially applying Property 6, which holds for any two-dimensional triangular map.

4. Numerical exploration of the dynamic behaviors

In this section we explore, by numerical simulations, the dynamic behaviors of the map. We shall fix the exogenous equilibrium level of the income at the value $\mu = 100$, and the depreciation rate of capital at the value $\delta = 0.2$. This latter assumption is without loss of generality, because the stability of the equilibrium

[2] We follow the terminology of Mira et al., see [13].

points and of the cycles of the map does not depend on δ, and the same qualitative dynamics as those commented in this section can be obtained with a different value of δ, $0 < \delta < 1$.

In order to comment the local and global behaviors of the map we shall follow the particular bifurcation route, in the (α, σ) plane, obtained by assuming the propensity to save as fixed at the value $\sigma = 0.4$ and increasing the adjustment parameter α.

We will show how the bifurcations and dynamic behaviors of the two-dimensional map T can be completely described starting from those of the one-dimensional map F. It is worth noting that this result is not a consequence of the triangular structure only, but of the joint effect of the triangular structure and the linearity of the second equation of T.

In Fig. 3(a), obtained with $\alpha = 2$, we show the basins of attraction $\mathscr{B}(Q)$ and $\mathscr{B}(R)$ of the two stable nodes Q and R , separated by the stable manifold of the saddle P, which, as we know from Proposition 2, is the vertical line of equation $Y = \mu$. Fig. 3(b) shows how this simple situation of bi-stability is related to the shape of the one-dimensional map F: here it is evident that the initial conditions Y_0, with $Y_0 < \mu$, originate trajectories converging to Y_R, while the initial conditions Y_0, with $Y_0 > \mu$, originate trajectories converging to Y_Q.

The economic interpretation of the structure of the basins represented in Fig. 3 is straightforward: if the economic system starts with a high level of income ($Y_0 > \mu$), then it will maintain a high level of income over time and it will eventually settle down on the wealth equilibrium Q, while if it starts with a low level of income ($Y_0 < \mu$), then it will converge to the poverty equilibrium Q, by maintaining a low level of income during the whole time evolution.

We already know, from the local stability analysis of the fixed points, that for any given σ, $0 < \sigma < 1$, the equilibria Q and R are stable for $\alpha < \alpha_f(\sigma) = 2/[\sigma - u(Y_Q)]$, where $u(Y) = [1 + (Y - \mu)^2]^{-1}$. Numerical computations show that, when $\sigma = 0.4$, the bifurcation value is $\alpha_f \simeq 6.47$: here, as we shall see, a flip bifurcation occurs, where the stable nodes Q and R become saddles and two symmetric stable cycles of period two appear. But the simple structure of the basins shown in Fig. 3(a) changes before this bifurcation value, as soon as the map enters the regime of noninvertibility, i.e., for $\alpha > 1/\sigma = 2.5$. The main structural changes occur at this point. The basins of the two fixed points Q and R become nonconnected and structured in vertical strips and the stable manifold of the saddle P, which separates the basins of Q and R, is now made up of several vertical lines (see, Fig. 4(a)). In fact, in the regimes in which the one-dimensional map F is noninvertible, the fixed point P of the map T has three different preimages, P itself and two more

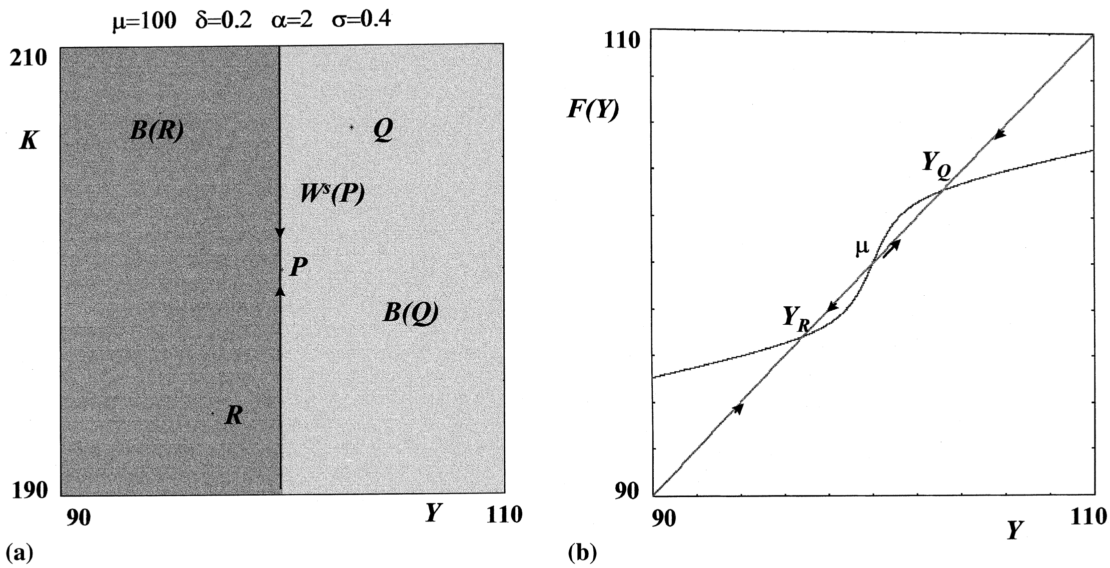

Fig. 3. Representation in the phase-plane of the basins of attraction of the stable nodes Q and R, obtained for parameters' values in the invertibility region: the stable manifold $W^s(P)$ of the saddle point P behaves like a watershed (a), and this situation of bi-stability is strictly related to the shape of the one-dimensional map driving the income evolution (b).

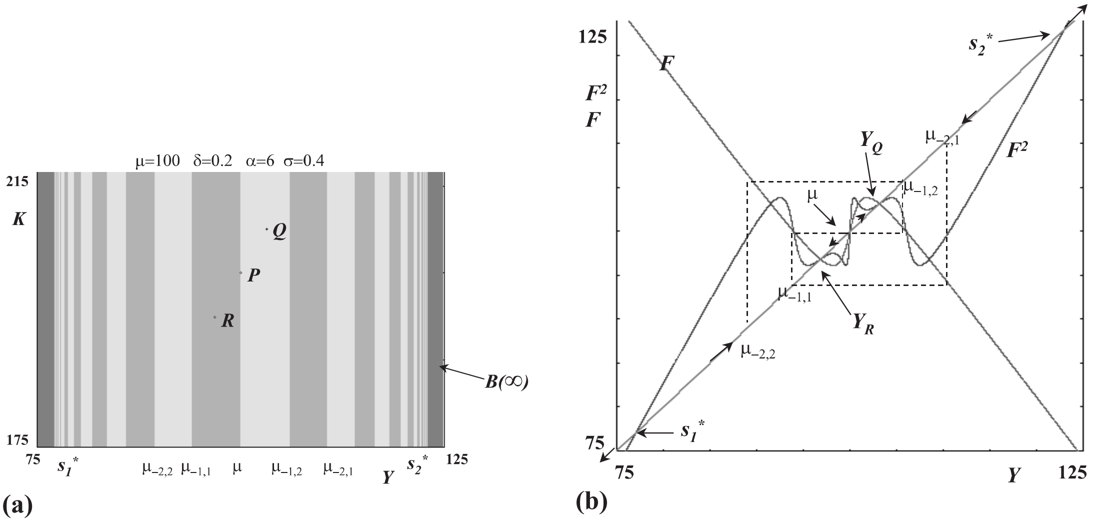

Fig. 4. Effects of the noninvertibility on the structure of the basins of the coexisting equilibria: the basins of Q (light grey) and R (intermediate grey) become nonconnected and structured in vertical strips separated by the stable manifold of the saddle P, made up of several vertical lines which accumulate on the boundary of the basin $\mathscr{B}(\infty)$ (dark grey) of divergent trajectories (a); each vertical line corresponds to a preimage (of a given rank) of the fixed point μ of the associated one-dimensional map (b).

points symmetric with respect to P, say $P_{-1,1}$ and $P_{-1,2}$ (as well as μ, $\mu_{-1,1}$, $\mu_{-1,2}$ are the preimages of the fixed point μ of the one-dimensional map F). Then by Proposition 2 the stable manifold of the saddle point P is made up of the vertical lines of equation $Y=\mu$, $Y=\mu_{-1,1}$, $Y=\mu_{-1,2}$, and of the lines $Y=\mu_{-2,1}$, $Y=\mu_{-2,2}$, where $\mu_{-2,1}$ and $\mu_{-2,2}$ are the (unique) Rank-1 preimages of $\mu_{-1,1}$ and $\mu_{-1,2}$, respectively, and of the lines corresponding to the preimages of any rank of these points (see Fig. 4(b)). In Fig. 4(b), the "immediate" basins $\mathscr{B}_0(Q)$ and $\mathscr{B}_0(R)$ of the two coexisting equilibria are given by the intervals $(\mu, \mu_{-1,2})$ and $(\mu_{-1,1}, \mu)$, respectively, but the global basin of each equilibrium is made up of infinitely many more disjoint intervals. Let us consider, for instance, the global basin $\mathscr{B}(Q)$ of the fixed point Q. Besides $\mathscr{B}_0(Q)$ and $\mathscr{B}_0(R)$, the global basins of the equilibria include all the preimages of any rank of the immediate basins: for example it can be easily seen from Fig. 4 that the set $T^{-1}(\mathscr{B}_0(Q))$ of points which are mapped by T into $\mathscr{B}_0(Q)$ by one iteration, is given by the vertical strip corresponding to the interval $(\mu_{-2,2}, \mu_{-1,1})$, while the set $T^{-1}(\mathscr{B}_0(R))$ is the strip given by $(\mu_{-1,2}, \mu_{-2,1})$. In this way, infinitely many disjoint strips $T^{-r}(\mathscr{B}_0(Q))$ and $T^{-r}(\mathscr{B}_0(R))$ are obtained, whose points are mapped into $\mathscr{B}_0(Q)$ or $\mathscr{B}_0(R)$ by r iterations.

In economic terms, this new structure of the basins of the equilibria implies that the basin of poverty includes also regions with a high level of income and vice versa, the basin of wealth includes regions with a low level of income.

The noninvertibility of F, i.e., the existence of two local extrema, also causes the appearance of diverging trajectories, due to the appearance of a repelling 2-cycle $\{s_1^*, s_2^*\}$ separating the basin of divergent trajectories from the basin of bounded trajectories (see again, Fig. 4(b), where also the map $F^2 = F \circ F$ is represented). Correspondingly, a saddle cycle for the two-dimensional map T is created, whose stable manifold is made up of the two vertical lines $Y = s_1^*$, $Y = s_2^*$ acting as a watershed between the basin $\mathscr{B}(\infty)$ of divergent trajectories and the basin of bounded trajectories in the phase-plane (Fig. 4(a)).

The following proposition, proved in Appendix C, states the existence of such a repelling 2-cycle in the region Ω^c of the parameters' space defined as

$$\Omega^c = \{(\alpha, \sigma) \in \Omega | \alpha > \alpha_c(\sigma) = 2/\sigma\},$$

included in the noninvertibility region of the map T.

Proposition 6. *By increasing the adjustment parameter* α *for a fixed value of the propensity to save* σ, *a repelling* 2-*cycle* $\{s_1^*, s_2^*\}$ *for the one-dimensional map* F *is created at* $\alpha = \alpha_c(\sigma) = 2/\sigma$.

We can see from Fig. 4(b) that the points $Y_0 \in [s_1^*, s_2^*]$ have bounded trajectories, while the points $Y_0 \notin [s_1^*, s_2^*]$ have diverging trajectories. It is also evident that when α is further increased the basin $\mathscr{B}(\infty)$ of divergent trajectories approaches the fixed points P, Q and R.

Fig. 4(a) shows that the vertical strips constituting the basins of Q and R accumulate on the vertical lines for s_1^* and s_2^*. In fact, since the vertical lines $Y = s_1^*$ and $Y = s_2^*$ constitute a repelling set for the forward iteration of T (see Property 5 of triangular maps in Section 3.1), this set behaves as an attracting set for the iteration of the inverses of T (see, for instance [1]). Such a situation leads to a loss of predictability about the long run evolution of the economy: if the initial state is for instance near the boundary of the region of bounded trajectories, then a small change in the initial state may give a completely different long run evolution of the system, if the change causes a crossing of some basin boundary. This can be observed from Figs. 5(a) and (b), which represent the income Y as a function of time in different situations. Fig. 5(a) shows two trajectories, corresponding to initial conditions slightly different from each other, one bounded and converging to the equilibrium point R, the other diverging. Fig. 5(b) illustrates a similar situation, with two initial conditions converging to different equilibria.

At $\alpha \simeq 6.47$ a flip bifurcation of the stable nodes Q and R occurs, transforming them into saddle points and creating two symmetric stable 2-cycles. We point out that such a "local" bifurcation does not modify the global structure of the basins: it simply replaces each stable steady state with an attracting 2-cycle. At $\alpha \simeq 7.6$ each attracting 2-cycle of F undergoes a flip bifurcation, transforming it into a repelling cycle and creating a new attracting 4-cycle and correspondingly the 2-cycles (nodes) of T become saddles and two new attracting 4-cycles (nodes) are created (Fig. 6).

Fig. 6(a) represents the coexisting two stable 2-cycles for $\alpha = 7.25$ and Fig. 6(b) the two coexisting stable 4-cycles for $\alpha = 7.75$. The basins of attraction of the stable cycles are contained in the vertical strips represented in Figs. 6(a) and (b), more precisely, each set of vertical strips is the closure of the basin of the corresponding cycle.

By further increasing α the map F undergoes a typical sequence of flip bifurcations leading to chaotic dynamics. Fig. 7 shows the evolution of the attracting sets sharing the phase-plane for high values of α. In Fig. 7(a) we observe two symmetric chaotic attractors, each one made up of two pieces, which merge giving rise to two disjoint one-piece chaotic attractors, increasing in size as α increases (Fig. 7(b)). These attractors in turn merge into the attractor shown in Fig. 7(c), whose shape is symmetric with respect to the saddle P. The merging of the two attractors is due to an *homoclinic bifurcation* of the saddle P. Again, due to the triangular structure of the map T, the occurrence of this global bifurcation can be completely described starting from the one-dimensional map F driving the income evolution.

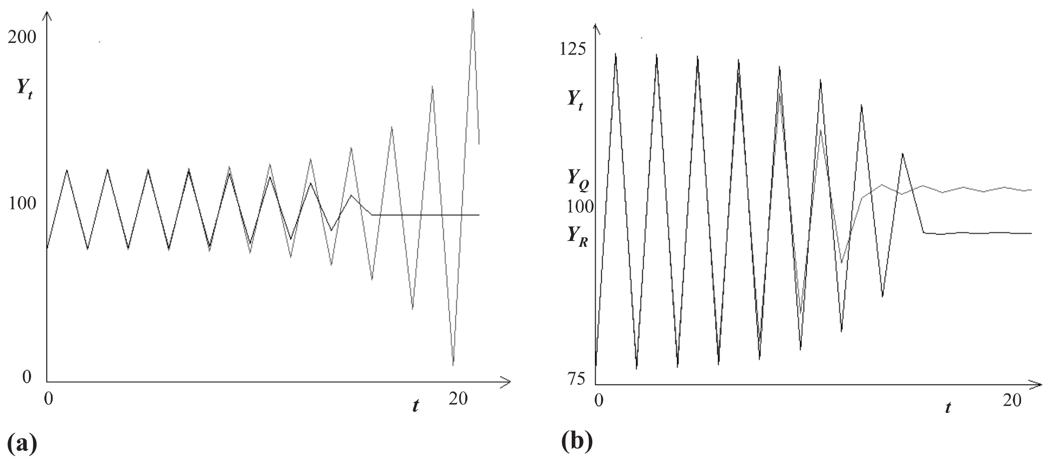

Fig. 5. Different asymptotic behaviors of income generated by slightly different initial conditions taken near the boundary of $\mathscr{B}(\infty)$: the i.c. (77.2, 175) generates a bounded trajectory (dark grey), while the i.c., (77, 175) generates a divergent trajectory (light grey) (a); the points (77.2, 175) and (77.4, 175) converge to the poverty equilibrium (dark grey) and the wealth equilibrium (light grey), respectively (b).

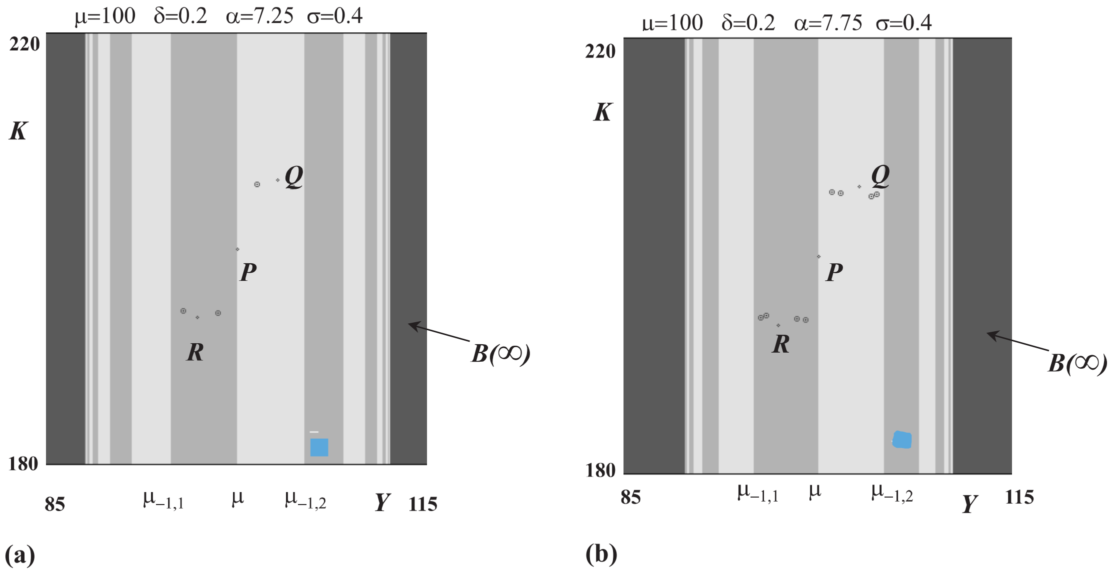

Fig. 6. Flip-bifurcations sequence of the equilibria: these local bifurcations do not affect the topological structure of the basins.

Let us consider Fig. 8 (where $c_{1,m}$ and $c_{1,M}$ denote the images of c_m and c_M, respectively). Before the homoclinic bifurcation value $\alpha_h \simeq 8.6215$ (Fig. 8(a)) the intervals $J_1 = [c_m, c_{1,m}]$ and $J_2 = [c_{1,M}, c_M]$ are invariant. The homoclinic bifurcation occurs when the critical values c_m and c_M are mapped by F into the repelling fixed point μ. After the homoclinic bifurcation we have a remarkable qualitative change in the trajectories of the system: the intervals J_1 and J_2 are no longer invariant intervals, but so is their union $J = J_1 \cup J_2$, i.e., the interval $[c_m, c_M]$ (Fig. 8(b)). This can also be observed from Figs. 9(a) and (b), which represent the *versus time* trajectories of the income Y before and after the bifurcation ($\alpha = 8$ and $\alpha = 8.65$, respectively): the bifurcation value $\alpha_h \simeq 8.6215$ marks the switching from a regime of bi-stability to a regime characterized by oscillating (although chaotic) behavior.

The local and global bifurcations described in the present section are summarized in Fig. 10. The bifurcation diagram of Fig. 10(a) describes the changes occurring to the attracting sets sharing the phase-plane when the adjustment parameter α is varied, while Fig. 10(b) represents the relevant bifurcation curves in the parameters' plane Ω.

We remark that, as it is also evident from Fig. 9(a), for each of the two disjoint invariant intervals J_1 and J_2 including the coexisting attracting sets, the map F undergoes exactly the same bifurcations which are met by the well-known one-dimensional logistic map $f(x) = ax(1-x)$ when the parameter a ranges between the values 3 and 4 (see, for instance [3,12]). Comparing again the dynamics of F with those of the logistic map, we notice that the described homoclinic bifurcation of the repelling fixed point P is equivalent to the one occurring in the logistic map when the parameter a crosses the value 4, with the difference that in the case of the logistic map after this value the generic trajectory becomes divergent, while here we still have bounded trajectories in the invariant interval $J = [c_m, c_M]$. These bounded trajectories exist as long as the invariant interval J exists. Thus, increasing α, this situation persists until the invariant interval J has a contact with the boundary of its basin of attraction (s_1^*, s_2^*): this bifurcation, called *final bifurcation* in [1,13], occurs when $c_m = s_1^*$ and $c_M = s_2^*$, being $\{s_1^*, s_2^*\}$ the repelling 2-cycle on the boundary of the basin of divergent trajectories. After this contact the generic trajectory will be divergent.

Remark. The particular shape of the attractors shown in Figs. 7(b) and 7(c) deserve some comments. The lateral borders of these attractors are very clear while their upper and lower borders are "fuzzy". This situation is typical of the so-called *mixed absorbing areas* (see [13, Ch. 4]). As well as for one-dimensional maps trapping intervals bounded by critical points can be determined, for two-dimensional maps trapping

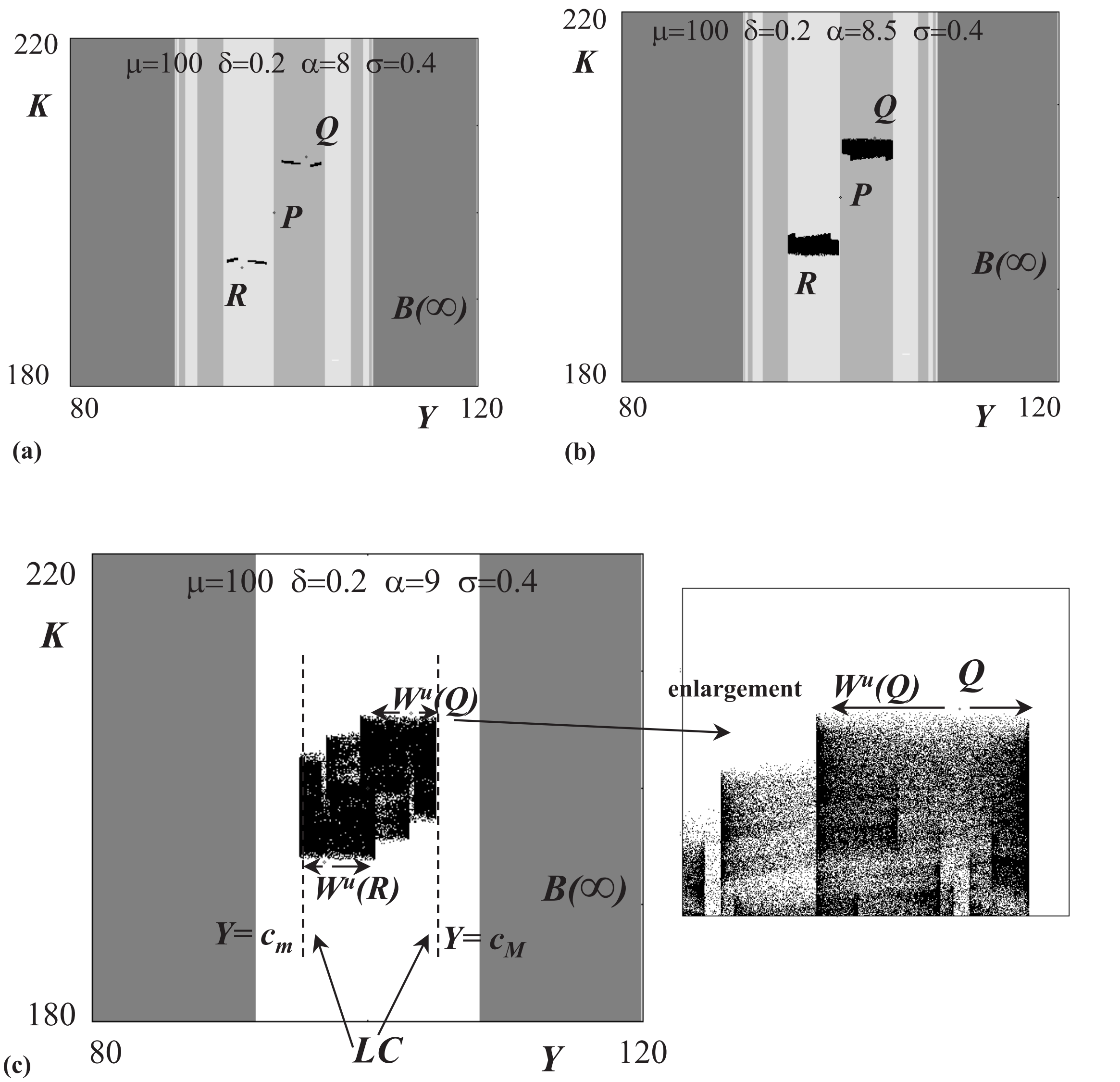

Fig. 7. Transition to complex dynamics: two symmetric chaotic attractors, each one made up of two pieces (a), merge giving rise to two disjoint one-piece chaotic attractors, increasing in size as α increases (b); these attractors in turn merge into a unique attracting set (c), whose shape is symmetric with respect to the saddle P, due to an homoclinic bifurcation of the saddle P.

regions of the phase-plane can be defined by means of critical curves: usually such a trapping region is an *absorbing area of standard type*, defined as a bounded region $\mathscr{A}$ of the plane whose boundary is given by critical curve segments (segments of the critical curve LC and its images) such that a neighborhood $U \supset \mathscr{A}$ exists whose points enter $\mathscr{A}$ after a finite number of iterations and never escape it (see again [13, Ch. 4] or [1] for a practical procedure in order to obtain the boundary of such an area). But boundaries of trapping regions can also be obtained by the union of segments of critical curves and portions of unstable sets of saddles: in this case we have a so-called *absorbing area of mixed type*. This is the case, for instance, of the area containing the one-piece chaotic attractor represented in Fig. 7(c): here the lateral borders are given by the critical lines LC_a and LC_b of equation $Y = \mathrm{c_m}$ and $Y = \mathrm{c_M}$, respectively, while the area is upperly and lowerly bounded by portions of the unstable manifolds $W^u(Q)$ and $W^u(R)$ of the saddle fixed points Q and R, and portions of the unstable sets of some other saddle cycles. This feature determines the fuzzy shape of the borders of the attractor observed in Fig. 7(c) and in its enlargement.

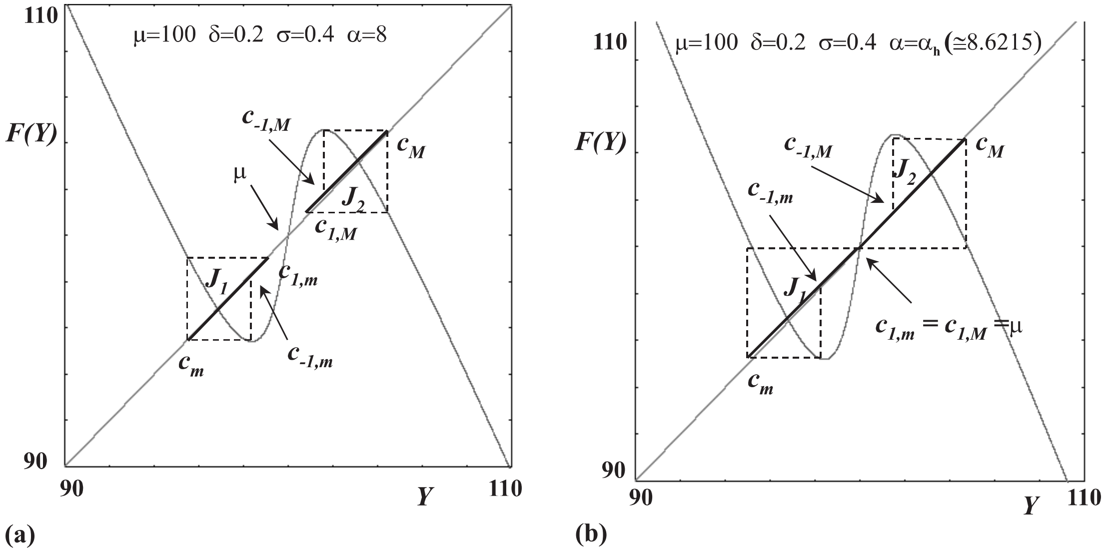

Fig. 8. Effects of the homoclinic bifurcation of the repelling fixed point μ of the one-dimensional map F on the invariancy property of the intervals J_1 and J_2.

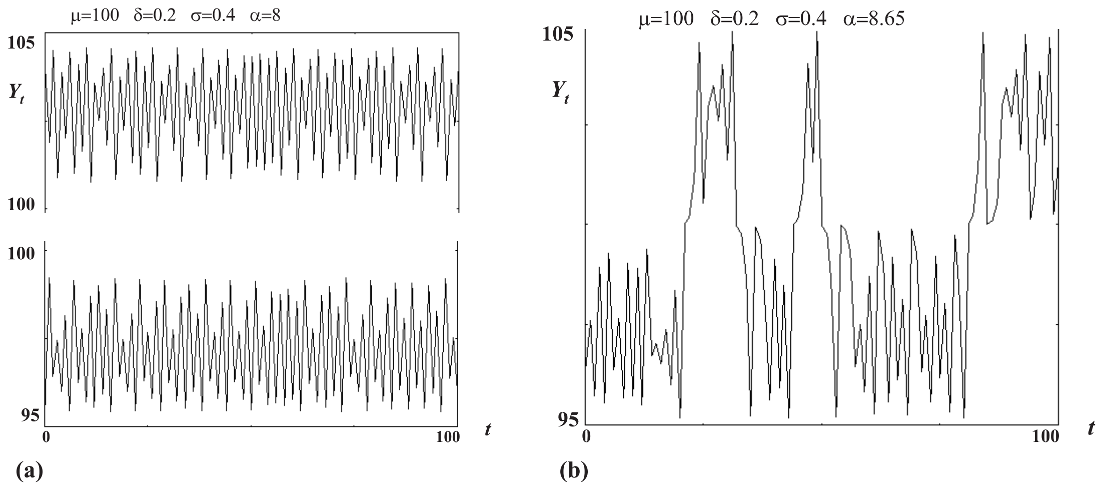

Fig. 9. Representation of the income as a function of time before (a) and after (b) the homoclinic bifurcation of the saddle P. (a) shows two different paths, corresponding to initial conditions near the wealth equilibrium and the poverty equilibrium, respectively, while the path shown in (b) is given by an initial condition near the poverty equilibrium. The homoclinic bifurcation marks the switching from a regime characterized by bi-stability to a regime characterized by wide chaotic oscillations of income and capital around their exogenously assumed equilibrium levels.

5. Conclusions

In this paper we have analyzed the dynamic behavior of a discrete-time economic model, described by a two-dimensional dynamical system in income and capital, where we have assumed that the savings are proportional to the income and that the investment demand depends only on the difference between current income and its exogenously assumed equilibrium level, through a nonlinear sigmoid-shaped increasing function, being not affected by changes in the capital stock. The model can be viewed as a particular case of a more general Kaldor-type business cycle model, proposed in [14] and analyzed in [4], where the investment demand is also a decreasing function of the level of capital: as stressed by the economic literature (see

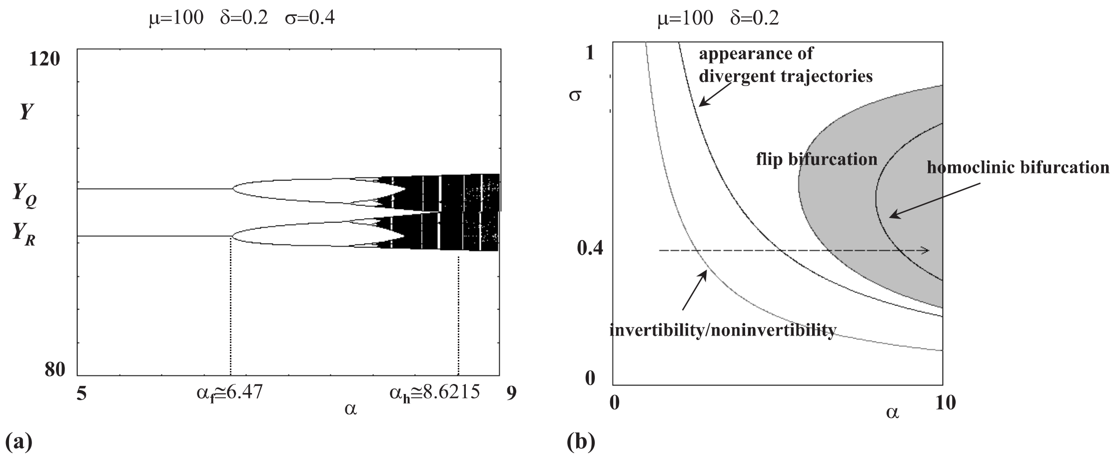

Fig. 10. Bifurcation diagram of the one-dimensional map driving the time evolution of income (a); representation of the local and global bifurcation curves in the parameters' plane Ω (the white area is the region of local stability of the equilibria) (b).

e.g. [5]), this latter property is the reason for the appearance of cyclical behavior, i.e., self-sustained oscillations of income and capital.

The time evolution of the system is driven by a family of two-dimensional maps of triangular type. These particular two-dimensional maps have the peculiarity that one of their components (the one driving the income evolution in the model at study) is an independent one-dimensional map. The structure of such maps allows to completely understand the forward dynamics, i.e., the asymptotic dynamic behavior, starting from the properties of the associated one-dimensional map (a bimodal one in our model).

We have shown that the system always has three equilibria, the central one (the saddle point P) being related to the exogenous equilibrium level of income and the others (Q and R) corresponding to a high level and a low level of the economy, respectively. Our analysis has been addressed both to the conditions for the local stability of the equilibria and to the global bifurcations, occurring for increasing values of the firms' adjustment parameter α, which cause deep changes in the nature of the existing attracting sets and in the structure of their basins. In fact, while for sufficiently low values of the adjustment parameter the dynamics are quite simple, with the saddle P acting as a watershed between the basins of the wealth and poverty equilibria Q and R, more and more complex dynamic phenomena are observed when α increases. Our analysis has shown the existence of two independent routes to complexity:

(i) the switching from invertibility to noninvertibility, which deeply modifies the structure of the basins of attraction of the stable equilibria and also creates the basin of diverging trajectories;

(ii) the homoclinic bifurcation of the saddle P, which determines a big qualitative change in the asymptotic behavior of the system, marking the switching from a regime of bi-stability (where the attractors may be fixed points, cycles or even chaotic attractors) to a regime characterized by oscillations, although not regular, but chaotic, around the saddle point P.

We stress that this kind of oscillating behavior occurs, provided that the adjustment parameter α is high enough, even though we have assumed that the investment demand is independent from the capital stock (see, Section 2), i.e., we have ruled out the possibility of cyclical shifting of the investment demand function.

Acknowledgements

The authors wish to thank Professors G. Rodano and E. Saltari for useful discussions and comments on the present work. This work has been performed under the auspices of CNR, Italy, and under the activity of the national research project "Nonlinear Dynamics and Stochastic Models in Economics and Finance", MURST, Italy.

Appendix A. Nature of the fixed points and cycles of the map T

Proof of Proposition 1. Here, we prove that the fixed points and the cycles of the map T are either stable nodes or saddles. Since $T^r, r \geqslant 1$, is a triangular map, by Properties 1 and 2 the fixed points of T^r are either nodes or saddles. The particular structure (8) of the second component of the map T implies that also the second component of the rth iteration of the map, T^r, $r > 1$, has the same structure, i.e., it is separable with respect to the variables and linear in K. More precisely, it is easy to prove by induction that the map T^r ($r \geqslant 1$) has the following form:

$$T^r : \begin{cases} Y' = F^r(Y), & \text{(11a)} \\ K' = (1-\delta)^r K + \sum_{s=1}^{r} (1-\delta)^{r-s} I(F^{s-1}(Y)), & \text{(11b)} \end{cases}$$

from which we can easily see that one of the eigenvalues of the Jacobian matrix $DT^r(Y,K)$ is constant and equal to $z_2 = (1-\delta)^r$. As in our model we have $0 < \delta < 1$, it follows that the fixed points of T^r can only be either stable nodes or saddles with $0 < z_2 < 1$. Since the points belonging to an r-cycle of T are fixed points of T^r, the statement follows. □

Proof of Proposition 2. Here, we prove that the stable manifold of a saddle cycle of T is made up of the vertical lines through the periodic points and through the preimages of any rank of the periodic points. Since T is a triangular map we know (from the implication (4) of Property 4) that the points of the local stable set of a saddle cycle belong to the vertical lines through the periodic points. We can conversely easily show that, for our particular map, any point on these vertical lines belongs to the stable set. In fact, let (Y^*, K^*) be a point belonging to a saddle cycle of period r for the map T. Since Y^* is a fixed point of the one-dimensional map $F^r(Y)$, i.e., $F^r(Y^*) = Y^*$, we can see from (11a) and (11b) that a point (Y^*, K) is mapped, after r iterations into a point (Y^*, K') lying on the same vertical line, where

$$K' = (1-\delta)^r K + \sum_{s=1}^{r} (1-\delta)^{r-s} I(F^{s-1}(Y^*)). \tag{12}$$

This means that the trajectory obtained by iterating the map T^r is driven by the one-dimensional map (12), having the only fixed point

$$K^* = \frac{\sum_{s=1}^{r} (1-\delta)^{r-s} I(F^{s-1}(Y^*))}{1-(1-\delta)^r},$$

globally stable. The speed of the dynamics of the map T^r on the vertical line $Y = Y^*$ is affected by the depreciation rate δ, $0 < \delta < 1$: the higher is δ the faster is the convergence to the fixed point.

Finally, the points with coordinates $(Y^*_{-j}, \bullet)$, where Y^*_{-j} is a preimage of Rank-j of Y^* are mapped, after j iterations, into a point $(Y^*, \bullet)$ on the vertical line $Y = Y^*$. This completes the proof. □

Appendix B. Local stability analysis of the fixed points

Here we analyze the local stability of the fixed points of the map (5a)–(5b). Since the map T is triangular, the Jacobian matrix (7) of T has real eigenvalues, located on the main diagonal, given by $z_1(Y) = 1 + \frac{\alpha}{1+(Y-\mu)^2} - \alpha\sigma$, $z_2 = 1 - \delta$, with $0 < z_2 < 1$.

Consider the fixed point $P = (\mu, \mu\frac{\sigma}{\delta})$, for which the first eigenvalue is: $z_1(P) = 1 + \alpha(1-\sigma)$. Since $z_1(P) > 1$ for each (α, σ) in the space of the parameters Ω, we can conclude that the fixed point P is always a saddle.

Let us consider the fixed points Q and R (it is enough to consider only one of them, since the symmetry property implies $DT(Q) = DT(R)$). We can rewrite the first eigenvalue as

$$z_1(Q) = z_1(R) = 1 - \alpha[\sigma - u(Y_Q)],$$

where $u(Y) = [1 + (Y-\mu)^2]^{-1}$.

We already know (Proposition 1) that the fixed points of the map T are either stable nodes or saddles. As $0 < z_2 < 1$, a sufficient condition for the local stability of Q and R is given by $|z_1(Q)| < 1$, i.e.,

$$\sigma > u(Y_Q), \tag{13a}$$
$$\sigma < 2/\alpha + u(Y_Q). \tag{13b}$$

Recalling condition (9b), we notice that for a fixed μ the equation

$$\sigma(Y_Q - \mu) - \arctan(Y_Q - \mu) = 0 \quad (Y_Q > \mu, \quad 0 < \sigma < 1) \tag{14}$$

implicitly defines the income equilibrium level Y_Q as a (differentiable) function of the parameter σ. By denoting with $h(\sigma, Y_Q)$ the left-hand side of Eq. (14), from the implicit function theorem we have

$$\frac{\mathrm{d}Y_Q}{\mathrm{d}\sigma} = -\frac{h_\sigma(\sigma, Y_Q)}{h_{Y_Q}(\sigma, Y_Q)} = -\frac{Y_Q - \mu}{\sigma - u(Y_Q)}.$$

Since it can be proved through simple geometrical considerations that the income equilibrium value Y_Q is a strictly decreasing function of σ, it follows that $\sigma - u(Y_Q) > 0$ and therefore condition (13a) is satisfied for any (α, σ) in the space of the parameters.

Condition (13b) can be rewritten as

$$\alpha < \frac{2}{\sigma - u(Y_Q)} = \alpha_f(\sigma).$$

Since $0 < u(Y_Q) = [1 + (Y_Q - \mu)^2]^{-1} < 1$ and condition (13a) holds, it follows that $0 < \sigma - u(Y_Q) < 1$ which implies: $\alpha_f(\sigma) > 2$. Of course, for $\alpha > \alpha_f(\sigma)$, the fixed points Q and R are saddles. □

Appendix C. Appearance of divergent trajectories

The existence of divergent trajectories in a particular region of the parameters space (included in the noninvertibility region) is proved by showing that, by increasing α, a repelling 2-cycle for the map F is created at the value $\alpha_c(\sigma) = 2/\sigma$. First, observe that for any σ, $0 < \sigma < 1$, the following inequality holds: $2/\sigma = \alpha_c(\sigma) < \alpha_f(\sigma) = 2/[\sigma - u(Y_Q)]$, which means that the curve of equation $\alpha = \alpha_c(\sigma)$ lies inside the stability region of Y_Q in the space Ω of the parameters. Observe also that $F^2(+\infty) = +\infty$ and $F^2(-\infty) = -\infty$. In order to prove the statement it is enough to show that, when Y_Q is stable: (a) in a right neighbourhood of the fixed point Y_Q the graph of the map F^2 is below the line $\varphi(Y) = Y$ and (b) $\lim_{Y\to\mp\infty} \frac{\mathrm{d}}{\mathrm{d}Y}(F^2(Y)) \leqslant 1$ for $\alpha \leqslant 2/\sigma$, $\lim_{Y\to\mp\infty} \frac{\mathrm{d}}{\mathrm{d}Y}(F^2(Y)) > 1$ for $\alpha > 2/\sigma$.

At the fixed point Y_Q we get

$$\frac{\mathrm{d}}{\mathrm{d}Y}(F^2(Y))_{|Y=Y_Q} = F'(F(Y_Q))F'(Y_Q) = \left[F'(Y_Q)\right]^2.$$

In the stability region of Y_Q, i.e., for $\alpha < \alpha_f(\sigma)$, $|F'(Y_Q)| < 1$ and therefore,

$$0 < \frac{\mathrm{d}}{\mathrm{d}Y}F^2(Y)_{|Y=Y_Q} = [F'(Y_Q)]^2 < 1.$$

This means that $F^2(Y)$ intersects the line $\varphi(Y) = Y$ at the point $Y = Y_Q$ with positive slope less than 1. This implies (a).

As $\lim_{Y\to\mp\infty} F'(Y) = 1 - \alpha\sigma$, we obtain

$$\lim_{Y\to\mp\infty} \frac{\mathrm{d}}{\mathrm{d}Y}(F^2(Y)) = \lim_{Y\to\mp\infty} \left[F'(F(Y))F'(Y)\right] = (1 - \alpha\sigma)^2,$$

and since $(1 - \alpha\sigma)^2 > 1$ iff $\alpha > 2/\sigma$, (b) is proved. □

As a consequence, taking into account the symmetry property of the map, two new repelling fixed points of F^2, and thus a repelling 2-cycle of F, appear as soon as α crosses the line $\alpha_c(\sigma) = 2/\sigma$.

References

[1] Abraham R, Gardini L, Mira C. Chaos in discrete dynamical system (a visual introduction in 2 dimensions). Berlin: Springer, 1997.

[2] Dana RA, Malgrange P. The dynamics of a discrete version of a growth cycle model, In: Ancot JP, editor. Analysing the structure of economic models. The Hague: Martinus Nijhoff, 1984. p. 205–22.

[3] Devaney RL. An introduction to chaotic dynamical systems, 2nd ed. Reading, MA: Addison-Wesley, 1989.

[4] Dieci R, Gardini L, Bischi GI. Global dynamics in a Kaldor-type business cycle model. Quaderni dell'Istituto di Matematica "E. Levi", Università degli Studi di Parma, vol. 4. 1998.

[5] Gabisch G, Lorenz HW. Business cycle theory, 2nd ed. Berlin: Springer, 1989.

[6] Gardini L, Mira C. On the dynamics of triangular maps, Progetto Nazionale di Ricerca M.U.R.S.T. Dinamiche non lineari e applicazioni alle scienze economiche e sociali. Quaderno 1993;9305.

[7] Kaldor N. A model of the trade cycle. Economic Journal 1940;50:78–92.

[8] Kolyada SF, Sharkovski AN. On topological dynamics of triangular maps of the plane, In: Proceedings of ECIT89. Singapore: World Scientific, 1991. p. 177–83.

[9] Kolyada SF. On dynamics of triangular maps of the square. Ergodic Theory and Dynamical Systems 1992;12:749–68.

[10] Lorenz HW. Strange attractors in a multisector business cycle model. Journal of Economic Behavior and Organization 1987;8:397–411.

[11] Lorenz HW. Multiple attractors, complex basin boundaries, and transient motion in deterministic economic models. In: Feichtinger G, editor. Dynamic economic models and optimal control. Amsterdam: North-Holland, 1992. p. 411–30.

[12] Mira C. Chaotic dynamics. Singapore: World Scientific, 1987.

[13] Mira C, Gardini L, Barugola A, Cathala JC. Chaotic dynamics in two-dimensional noninvertible maps. Singapore: World Scientific, 1996.

[14] Rodano G. Lezioni sulle teorie della crescita e sulle teorie del ciclo. Dipartimento di teoria economica e metodi quantitativi, Università di Roma "La Sapienza", 1997.

[15] Stark J. Invariant graphs for forced systems. Physica D 1997;109:163–79.