Basin boundaries and focal points in a map coming from Bairstow's method

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This paper is devoted to the study of the global dynamical properties of a two-dimensional noninvertible map, with a denominator which can vanish, obtained by applying Bairstow's method to a cubic polynomial. It is shown that the complicated structure of the basins of attraction of the fixed points is due to the existence of singularities such as sets of nondefinition, focal points, and prefocal curves, which are specific to maps with a vanishing denominator, and have been recently introduced in the literature. Some global bifurcations that change the qualitative structure of the basin boundaries, are explained in terms of contacts among these singularities. The techniques used in this paper put in evidence some new dynamic behaviors and bifurcations, which are peculiar of maps with denominator; hence they can be applied to the analysis of other classes of maps coming from iterative algorithms (based on Newton's method, or others). © *1999 American Institute of Physics.* [S1054-1500(99)02202-8]

We consider a class of two-dimensional noninvertible maps, characterized by the existence of a vanishing denominator, so that the iterative process is not defined in a zero-measure subset of the plane. This class of maps is obtained by the application of the Bairstow's method (an iterative numerical algorithm to find roots of polynomials) to a cubic equation. The study of the two-dimensional maps reveals that the choice of the initial point of the iterative process is crucial for the convergence, since very complex basins of attraction are obtained. We explain the structure of the basin boundaries and the global bifurcations which cause their qualitative changes by using concepts and techniques which have been recently proposed for the study of the global properties of maps with a vanishing denominator.

I. INTRODUCTION

The Bairstow's method is an iterative numerical algorithm to find roots of polynomials with real coefficients, proposed by Bairstow in 1914.[1] This numerical method, which involves only real arithmetic, is based on the factorization of the polynomials into products of quadratic functions, whose coefficients are the roots of algebraic equations. Application of the iterative Newton's method to find such roots gives rise to a *two-dimensional rational map*, whose fixed points corresponds to the desired coefficients of the quadratic factors. These fixed points are locally attracting, i.e., the iteration of the two-dimensional map converge to a fixed point provided that the initial condition is sufficiently close to it, but there are no general results on their global basins of attraction. Thus the main problem is to obtain the boundaries of the basins of the fixed points, and to study their qualitative changes (i.e., their bifurcations) as the coefficients of the polynomial vary.

The geometric features and dynamic behavior of a two-dimensional map coming from Bairstow's method are strongly influenced by the following two general properties: (i) it is a noninvertible map; (ii) it is a fractional rational map with vanishing denominator. From (ii) it follows that there is a subset of the plane where the map is not defined, also called *singular set* in Billings and Curry,[2] and such a subset may include points in which also a numerator vanishes, so that the map takes the form 0/0, which are candidate to be *focal points*, following the terminology introduced by Bischi, Gardini, and Mira.[3–6] Some dynamical effects of these singularities, specific to maps with a vanishing denominator, have been recently investigated (see Bischi, Gardini, and Mira[3] and references therein) and such results can be usefully applied in order to study the structure of the basin boundaries, as well as their bifurcations, for the maps coming from Bairstow's method.

As a prototype of this class of maps we shall consider the one obtained by applying Bairstow's method to the one-parameter family of cubic polynomials $P_a(x)=x^3+(a-1)x-a$. The factorization $P_a(x)=(x^2+ux+v)(x-u)$ occurs if and only if u and v are fixed points of the

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two-dimensional map $T_a:(u,v)\rightarrow(u',v')$ given by (see, e.g., Billings and Curry[2]):

$$T_a:\begin{cases} u'=\dfrac{u^3+u(v-a+1)+a}{2u^2+v} \\ v'=\dfrac{v(u^2+a-1)+2au}{2u^2+v}. \end{cases}$$

We have chosen this family for two reasons: It is the simplest map coming from the application of Bairstow's method and its dynamical properties have already been investigated by many authors (see the works of Boyd,[7] Blish and Curry,[8] Billings, and Curry,[2] Billings, Curry, and Phipps,[9] Fiedler,[10] Blish,[11] Curry, Garnet, and Sullivan[12]). However, the global structure of the basins of attraction of the fixed points of the map T_a is still an open problem, and their study is of interest both from the point of view of the application of the numerical method, since a proper choice of the initial condition is often the most challenging part of the numerical method, and from the point of view of the new and rich dynamical properties of the two-dimensional maps obtained. Particular structures and bifurcations of the basin of the fixed points of T_a have been already evidenced in Billings and Curry,[2] and Billings, Curry, and Phipps,[9] where the existence of invariant lines, which may include infinitely many repelling cycles, is shown, and the importance of the presence of the singular set is stressed. However, the basin structures are not only related to such sets. The main purpose of the present work is to show how the *global structure of the basins, as well as their main bifurcations, are related to the focal points of the map and to the corresponding prefocal sets* (a definition of these terms is given in the next section).

The plan of the work is as follows. In Sec. II we briefly deduce the two-dimensional map by the application of the Bairstow's method to a cubic polynomial and then we give some basic properties, taken from the existing literature on the Bairstow's method. Then we give a short review of some recent definitions and results concerning the maps with a vanishing denominator, in particular the definitions and the geometrical properties of the concepts of focal point and prefocal curve are given in Subsection II B, and in Subsection II C these definitions are applied to stress some properties of the inverse maps. In Sec. III the structure of the basins of the fixed points is examined in detail in different ranges of the parameter a. The main bifurcations are explained in terms of contacts of the focal points, prefocal curves, and the singular set, and many numerical explorations are used to illustrate the role of these new kinds of singularities that characterize the global dynamical properties of maps with a vanishing denominator.

II. SOME GENERAL PROPERTIES OF THE TWO-DIMENSIONAL MAP

A. Definitions and basic properties

Bairstow's method is a well-known iterative technique to determine a real quadratic factor of a polynomial $P(x)$ with real coefficients. Dividing $P(x)$ by a quadratic factor (x^2+ux+v) we get $P(x)=(x^2+ux+v)Q(x)+F(u,v)x+G(u,v)$. We then seek for a factor of $P(x)$ by using Newton's method to solve the algebraic system

$$\begin{cases} F(u,v)=0 \\ G(u,v)=0, \end{cases}$$

which gives rise to the two-dimensional map $(u',v')=B_a(u,v)$ where

$$B_a:\begin{bmatrix} u' \\ v' \end{bmatrix}=\begin{bmatrix} u \\ v \end{bmatrix}-\begin{bmatrix} F_u(u,v) & F_v(u,v) \\ G_u(u,v) & G_v(u,v) \end{bmatrix}^{-1}\begin{bmatrix} F(u,v) \\ G(u,v) \end{bmatrix}.$$

After reordering, the map B_a reads as

$$B_a:\begin{cases} u'=H(u,v)=\dfrac{N_1(u,v)}{D(u,v)} \\ v'=K(u,v)=\dfrac{N_2(u,v)}{D(u,v)}, \end{cases}$$

where both the components of the map B_a, $H(u,v)$, and $K(u,v)$ are fractional rational functions with a denominator that vanishes in the points of a one-dimensional subset of the plane, given by a set of algebraic curves that will be called *singular set*, and denoted by δ_s:

$$\delta_s=\{(u,v)\in\mathbb{R}^2:D(u,v)=0\}.$$

Let us briefly recall an important property associated with such kind of maps, which was proven by Boyd[7] (see also Blish and Curry[8]):

Property 1. *Let ξ be a real root of $P(x)$. Then the line (L_ξ) in the (u,v) plane of equation $v+\xi u+\xi^2=0$ is invariant for B_a. The restriction of B_a to the line (L_ξ) is a one-dimensional map associated with the Newton-function applied to the reduced polynomial $P_\xi(x)=P(x)/(x-\xi)$.*

We remark that, following the existing literature on the subject, the term *invariant* is used here as synonymous of trapping, i.e., $B_a(L_\xi)\subseteq(L_\xi)$, differently from the definition adopted in several texts on dynamics, where invariant means exactly mapped into itself, $B_a(L_\xi)\equiv(L_\xi)$.

In this paper we shall focus our attention on a particular map, coming from Bairstow's method applied to the cubic polynomial $P_a(x)=x^3+(a-1)x-a$. The factorization $P_a(x)=(x^2+ux+v)(x-u)$ occurs if and only if u and v are solutions of the algebraic system

$$\begin{cases} F(u,v)=u^2-v+a-1=0 \\ G(u,v)=uv-a=0, \end{cases}$$

and thus if and only if u and v are fixed points of the following two-dimensional map $T_a(u,v)=(H(u,v),K(u,v))$ given by

$$T_a:\begin{cases} u'=\dfrac{u^3+u(v-a+1)+a}{2u^2+v} \\ v'=\dfrac{v(u^2+a-1)+2au}{2u^2+v}. \end{cases} \tag{1}$$

The domain of definition of the function T_a is given by $\mathbb{R}^2\backslash\delta_s$, where δ_s is the singular set, formed by the points of the parabola

$$\delta_s=\{(u,v)\in\mathbb{R}^2:v=-2u^2\}. \tag{2}$$

So the iteration of T_a is well-defined provided that the initial condition belongs to the set E given by

$$E=\mathbb{R}^2\backslash\Lambda, \tag{3}$$

where Λ is the union of the preimages of any rank of the singular set δ_s

$$\Lambda=\bigcup_{k=0}^{\infty} T_a^{-k}(\delta_s).$$

In fact, only the points belonging to the set E generate non-interrupted trajectories

$$\tau(u_0,v_0)=\{(u_n,v_n)=T_a^n(u_0,v_0),n\geqslant 0\} \tag{4}$$

by the iteration of the map $T_a:E\rightarrow E$. We notice that, being the singular set δ_s a curve of $\mathbb{R}^2$, the set Λ of points excluded from the phase space of the recurrence has zero Lebesgue measure in $\mathbb{R}^2$.

Several properties of the map T_a have been already studied in the literature. For example, in Blish and Curry[8] it is shown that:

Property 2. *Let $R=(u^*,v^*)$ be a fixed point of T_a. Then the line $u=u^*$ is mapped by T_a into the fixed point R.*

Really, a more correct formulation of Property 2 should say that $T_a(u^*,v)=(u^*,v^*)$ for any (u^*,v) in which the map is defined, since the line $u=u^*$ always intersects the set of non definition of T_a, given by parabola (2).

In Blish and Curry[8] it is also shown that for $a<1/4$ the map T_a has three distinct fixed points (associated with three quadratic factors for $P_a(x)$), given by

$$R_1=(u_1^*,v_1^*)=(1,a),\quad R_2=(u_2^*,u_3^*),$$
$$R_3=(u_3^*,u_2^*), \tag{5}$$

where

$$u_2^*=\frac{-1+\sqrt{1-4a}}{2},\quad u_3^*=\frac{-1-\sqrt{1-4a}}{2} \tag{6}$$

are the real roots of the quadratic equation $u^2+u+a=0$.

From Property 1 we deduce that for $a<1/4$ three invariant lines exist, say L_i, $i=1,2,3$. Each invariant line L_i connects two fixed points, being R_i the excluded one. Line L_1 of equation

$$u+v+1=0 \tag{7}$$

is invariant also in the range $a>1/4$, when only the fixed point R_1 exists. The equations of the other two invariant lines L_i, $i=1,2$, existing for $a<1/4$, are

$$u=a+\lambda_i(v-1),\quad i=2,3 \tag{8}$$

where $\lambda_2=(a-u_2^*)/(1-u_3^*)$ and $\lambda_3=(a-u_3^*)/(1-u_2^*)$.

The Bairstow's method is clearly a ''local method'' since convergence to a given fixed point is only ensured for initial conditions which are sufficiently close to it (and in such a case the convergence is quadratic, because the fixed points are superstable). But a global study of the asymptotic behavior of the trajectories in Eq. (4), as the initial condition (u_0,v_0) varies in the plane, is still an open problem, studied by many authors in the recent literature (see Boyd,[7] Blish and Curry,[8] Billings and Curry,[2] Billings, Curry, and Phipps[9]). Both from a theoretical and a practical point of view, it is important to obtain an estimate of the stability extent of the fixed points, i.e., a delimitation of their basins of attraction, and to ascertain if sets of points exist that generate trajectories that do not converge to any fixed point. This requires an analysis of the global dynamical properties of the two-dimensional map, i.e., an analysis which is not limited to the study of its linear approximation. Furthermore, the global bifurcations determined by variations of the coefficients of the polynomial, which are the real parameters of the map, should also be investigated. In the following we study the boundaries of the basins of the fixed points of the map T_a and how they change as the parameter a varies.

We denote by $\mathcal{B}(A)$ the basin of attraction of a given invariant set A, defined as the set of points whose ω-limit set belongs to A

$$\mathcal{B}(A)=\{(u,v)\in E|\omega(u,v)\subseteq A\}$$

[we recall that the ω-limit set $\omega(u,v)$ of a point (u,v) is the limit set of its trajectory $\tau(u,v)$]. Considering map T_a under study, let us denote by $\mathcal{B}(R_i)$ the basin of attraction of the fixed point R_i, $i=1,2,3$. Two ranges of parameter a, with different behavior of the map, have already been noticed: $a<1/4$ and $a>1/4$.

- In the case $a<1/4$, when three fixed points R_i, $i=1,2,3$, exist, it is possible that any point $(u,v)\in E$ belongs to one of the basins $\mathcal{B}(R_i)$. However, as already argued in Blish and Curry,[8] there are regions of ''uncertainty'' with respect to the asymptotic behavior, i.e., regions in which, given an initial condition (u_0,v_0), it is difficult to decide toward which of the three fixed points the iterations will converge. In fact, as we shall prove in the present paper, there exist several regions in which the asymptotic behavior of the trajectories is extremely sensitive with respect to changes, even very small, of the initial condition. A goal of the present work is to explain the structure of such regions, and how it is modified by changing the value of the parameter a.
- In the case $a>1/4$ only the fixed point R_1 exists and, as already argued in Blish and Curry[8] and Billings and Curry,[2] in order to get the basin $\mathcal{B}(R_1)$ we have to exclude from E at least the invariant line L_1 as well as all the points whose trajectory converges toward that line, i.e., the stable set (or basin) $\mathcal{B}(L_1)$ of L_1. The set $\mathcal{B}(L_1)$ may also be a ''big'' one, in a measure theoretic sense. Indeed, as conjectured by Billings, Curry, and Phipps[9] (and we shall further motivate this conjecture) for $\frac{1}{4}<a<1$ the basin $\mathcal{B}(L_1)$ has positive Lebesgue measure, while it turns into a set of zero Lebesgue measure for $a>1$. Moreover, also in the case $a>1/4$ there is uncertainty with respect to the asymptotic behavior in a wide region of the plane, where it is very difficult to predict whether a trajectory will converge to the fixed point R_1 or if it goes to L_1 (convergence to L_1 means that the Bairstow method fails).

As stated in the Introduction, in order to understand the qualitative structure of the basins of attraction, we shall make use of the fact that the map T_a is noninvertible and that it has a vanishing denominator. We recall that a noninvertible map $T:(x,y)\rightarrow(x',y')$ is characterized by the fact that the rank-1 preimages $T^{-1}(x',y')$ of a given point (x',y'), may be more than one. This implies that the plane can be subdivided into regions Z_k, $k\geqslant 0$, whose points have k distinct rank-1 preimages. The global properties of noninvertible maps of the plane have been mainly studied by the method of critical curves (see Gumowski and Mira,[13] Mira *et al.*,[14] Abraham *et al.*[15]) defined as the locus of points having two or more merging preimages, and usually located along the boundaries that separate different regions Z_k. The role of the critical curves is analogous to that of the local maximum and minimum values in one-dimensional noninvertible maps.

Instead, the global properties of two-dimensional maps with denominator, recently considered in Bischi, Gardini, and Mira,[3] can also be characterized by the presence of other kinds of singularities, like the sets of vanishing denominator (or singular sets), the focal points, and the related prefocal sets. Indeed, as we shall see below, both the regions Z_k and the properties of the basins of the map T_a are explained by these latter kinds of singularities, specific to maps with denominator.

B. Focal points and prefocal curves

Many global dynamical properties of two-dimensional maps can be explained by the analysis of new kinds of singularities, such as the sets where a denominator of the map (or some of its inverses) vanishes, the points where the map (or some of its inverses) takes the form 0/0 in at least one component, and the prefocal curves, whose definition is here recalled from Bischi, Gardini, and Mira:[3]

Definition 1. *Consider a two-dimensional map T. A point Q belonging to the set of non definition* δ_s, *is a focal point if at least one component of the map T takes the form 0/0 in Q and there exist smooth simple arcs* $\gamma(\tau)$, *with* $\gamma(0)=Q$, *such that* $\lim_{\tau\rightarrow 0} T(\gamma(\tau))$ *is finite. The set of all such finite values, obtained by taking different arcs* $\gamma(\tau)$ *through Q, is called prefocal set* δ_Q.

In order to explain the role of a *focal point* and the related *prefocal set* in the geometric and dynamic properties of the map T_a, we consider a smooth simple arc γ transverse to δ_s and look how it is transformed by the application of the map T_a, i.e., what is the shape of its image $T(\gamma)$. In doing this we assume that the arc γ is deprived of the point in which it intersects δ_s. Let $(u_0,v_0)=(u_0,-2u_0^2)\in\delta_s$ be this point and assume that in a neighborhood of (u_0,v_0) γ is represented by the parametric equations

$$\gamma(\tau):\begin{cases} u(\tau)=u_0+\xi_1\tau+\xi_2\tau^2+\cdots \\ v(\tau)=v_0+\eta_1\tau+\eta_2\tau^2+\cdots \end{cases} \quad \tau\neq 0. \tag{9}$$

The portion of γ in such a neighborhood can be seen as the union of two disjoint pieces, say $\gamma=\gamma_-\cup\gamma_+$, where $\gamma-$ and $\gamma+$ denote the portions of γ located on opposite sides with respect to the singular curve δ_s, and are obtained from

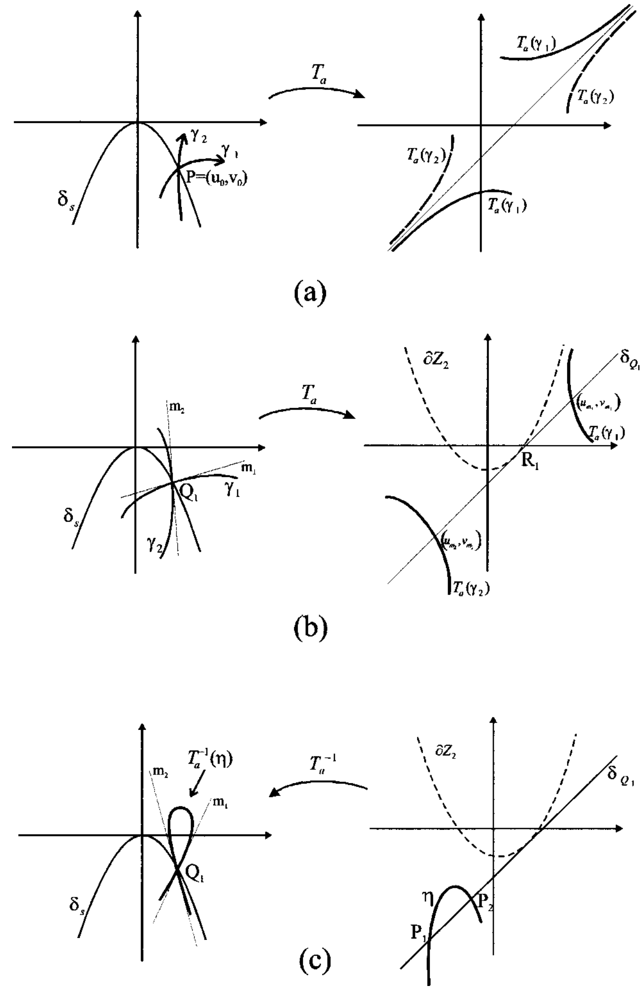

FIG. 1. Qualitative sketches that illustrate the geometric behavior of map T_a and its inverses.

Eq. (9) with $\tau<0$ and $\tau>0$, respectively. The closure $\gamma(\tau)$ is such that $\gamma_-(0)=\gamma_+(0)=(u_0,v_0)$. As $(u_0,v_0)\in\delta_s$ we have a vanishing denominator, $D(u_0,v_0)=0$, and let us first assume that the two numerators are different from zero, $N_1(u_0,v_0)\neq 0$ and $N_2(u_0,v_0)\neq 0$, then

$$\lim_{\tau\rightarrow 0} T_a(\gamma(\tau))=(\infty,\infty),$$

where ∞ means either $+\infty$ or $-\infty$. This means that the image $T_a(\gamma)$ is made up of two disjoint unbounded arcs asymptotic to a line $r(u_0,v_0)$ whose slope is given by the ratio

$$m(u_0,v_0)=\frac{N_2(u_0,v_0)}{N_1(u_0,v_0)}.$$

Different arcs through the same point are mapped into different arcs asymptotic to the same line [see Fig. 1(a)].

A different situation occurs if the point $(u_0,v_0)\in\delta_s$ is such that not only the denominator but also the numerators of the map T_a vanish in it, i.e.,

$$D(u_0,v_0)=N_1(u_0,v_0)=N_2(u_0,v_0)=0, \tag{10}$$

because in this case the limits may be finite and the image of an arc γ may be a bounded arc $T_a(\gamma)$.

It is easy to see that Eq. (10) always occurs in the point

$$Q_1=(1,-2) \tag{11}$$

and, for $a\leqslant 1/4$, also in points

$$Q_2=(u_2^*,-2u_2^{*2}),\quad Q_3=(u_3^*,-2u_3^{*2}). \tag{12}$$

Then, let us consider the point $Q_1\in\delta_s$, and take the limits which may be finite in both the components of the map, and depend on the arc γ:

$$\lim_{\tau\to 0} T_a(\gamma(\tau))=(\lim_{\tau\to 0} H(\gamma(\tau)),\lim_{\tau\to 0} K(\gamma(\tau)))=(u_\gamma,v_\gamma), \tag{13}$$

where, by denoting $\overline{N_{1,u}}=\partial N_1/\partial u(Q_1)$ and analogously for the other partial derivatives,

$$u_\gamma=\frac{\overline{N_{1,u}}\xi_1+\overline{N_{1,v}}\eta_1}{\overline{D_u}\xi_1+\overline{D_v}\eta_1},\quad v_\gamma=\frac{\overline{N_{2,u}}\xi_1+\overline{N_{2,v}}\eta_1}{\overline{D_u}\xi_1+\overline{D_v}\eta_1}. \tag{14}$$

The whole prefocal set δ_{Q_1} is given by the set of points (u_γ,v_γ) obtained on varying the slope of the arc γ through the point Q_1. In this way we obtain a one-to-one correspondence between slope $m=\eta_1/\xi_1$ of arc $\overline{\gamma}$ in Q_1 and point (u_m,v_m) in which $T(\gamma)$ crosses δ_{Q_1}. The values of the limits in Eq. (13) are finite if the denominator in Eq. (14) is different from zero, i.e., if the slope $m=\eta_1/\xi_1$ of arc $\overline{\gamma}$ in Q_1 is different from the slope of curve δ_s in the same point, given by $-\overline{D_u}/\overline{D_v}$. The parametric equation of the prefocal set δ_{Q_1} is given, as a function of the parameter m (slope of the arc) by the equations:

$$u_m=\frac{\overline{N_{1,u}}+\overline{N_{1,v}}m}{\overline{D_u}+\overline{D_v}m}=\frac{2-a+m}{4+m},$$

$$v_m=\frac{\overline{N_{2,u}}+\overline{N_{2,v}}m}{\overline{D_u}+\overline{D_v}m}=\frac{-4+2a+ma}{4+m},$$

and by eliminating the parameter m we get the equation of the prefocal line δ_{Q_1}, given by:

$$\delta_{Q_1}: v=2u-2+a. \tag{15}$$

Some arcs γ through the focal point Q_1 and their images $T_a(\gamma)$ are qualitatively shown in Fig. 1(b). However, the reasons of the terms focal and prefocal becomes clearer when the geometric behavior of the inverse(s) of T_a is considered. In fact, it is plain that given a point $(u,2u-2+a)\in\delta_{Q_1}$, with $u\neq 1$, and an arc η crossing δ_{Q_1} through it, at least one of the inverses of T_a must have a rank-1 preimage which is an arc through Q_1, with slope in Q_1 given by

$$m(u)=\frac{-4u+2-a}{u-1}. \tag{16}$$

In particular, if we consider the rank-1 preimage of an arc η crossing δ_{Q_1} in two points of abscissas u_{p_1} and u_{p_2}, as shown in Fig. 1(c), then at least one of the inverses of T_a must have a rank-1 preimage which forms a *loop* through Q_1 where the two branches have slopes $m(u_{p_1})$ and $m(u_{p_2})$ in Q_1.

For $a<1/4$, besides Q_1 the map T_a has two more focal points, given in Eq. (12). Repeating the procedure followed above with the points $Q_i=(u_i^*,-2u_i^{*2})$, $i=2,3$, we get the parametric equations of the prefocal sets δ_{Q_i} as a function of the parameter m (slope of the arc through Q_i):

$$u_m=\frac{u_i^*(m-1)+1-2a}{4u_i^*+m},$$

$$v_m=\frac{u_i^*(4a-4-m)-2a-m}{4u_i^*+m},$$

and by eliminating the parameter m we get the equations of the prefocal lines δ_{Q_i}, $i=2,3$, given by:

$$\delta_{Q_i}: v=2u_i^*u+(u_i^*+2a-1),\quad i=2,3. \tag{17}$$

It is plain that the geometric properties of the images of arcs through each focal point Q_i, as well as the preimages of arcs through each prefocal line δ_{Q_i}, are the same as those commented above for Q_1 and δ_{Q_1}.

Note that the second component of the map T_a, $K(u,v)$, also takes the form 0/0 in the origin $O=(0,0)$. However, according to definition 1, this point of δ_s is not a focal point, except for the case $a=0$, when it merges with focal point Q_2. In fact, for $a\neq 0$ the first component of the map gives $H(u,v)=a/0$, which is either $+\infty$ or $-\infty$. Thus no finite value can be obtained for $\lim_{\tau\to 0} T_a(\gamma(\tau))$, when $\gamma(0)=0$. This means that the origin $O\in\delta_s$ behaves as a generic point, not focal, of the singular set δ_s.

We can so state the following proposition:

Proposition 1. *Let T_a be the map in Eq. (1). For $a>1/4$ the map T_a has one focal point Q_1, given by Eq. (11), with related prefocal line δ_{Q_1} of Eq. (15). For $a<1/4$ the map T_a has three focal points Q_i, $i=1,2,3$, given by Eqs. (11) and (12), with related prefocal lines δ_{Q_i}, $i=1,2,3$, of Eqs. (15) and (17).*

C. Inverses of T_a and related properties

As remarked above, in order to understand the global dynamical properties of map T_a it is important to see how many inverses map T_a has and in which regions of the phase plane the inverses are defined. Let (u',v') be a given point of the plane. Then, by solving the system of equations in Eq. (1) with respect to u and v, we can get either two distinct real solutions, called rank-1 preimages of the point (u',v'), or no real solutions, depending on the point (u',v'). Let us call Z_2 and Z_0 the regions of the plane whose points have, respectively, two distinct rank-1 preimages and no preimages at all. These regions are given by

$$Z_2:\ \{(u,v):\ F(u,v)=u^2-v+a-1>0\}$$

$$Z_0:\ \{(u,v):\ F(u,v)=u^2-v+a-1<0\}. \tag{18}$$

For a point $(u',v')\in Z_2$ we denote the inverses of map T_a by $T_{a,r}^{-1}$ and $T_{a,l}^{-1}$, being the two preimages located one on the right and one on the left of the point (u',v') (symmetric with respect to that point):

$$T_{a,r}^{-1}:\begin{cases} u=u'+\sqrt{F(u',v')} \\ v=v'+\dfrac{(u'v'-a)}{\sqrt{F(u',v')}}, \end{cases} \quad T_{a,l}^{-1}:\begin{cases} u=u'-\sqrt{F(u',v')} \\ v=v'-\dfrac{(u'v'-a)}{\sqrt{F(u',v')}}. \end{cases} \tag{19}$$

In the following we shall denote by T_a^{-1} both the inverses of T_a, i.e., $T_a^{-1}(u,v)=T_{a,l}^{-1}\cup T_{a,r}^{-1}$.

We observe that in order to write the inverses as in Eq. (19), we have simplified the expressions coming from Eq. (1) by eliminating the factor $(u'+\sqrt{F(u',v')})$ to get $T_{a,r}^{-1}$ and by eliminating the factor $(u'-\sqrt{F(u',v')})$ to get $T_{a,l}^{-1}$. This is without loss of generality, as it is immediate to see that such factors only vanish on the line of equation $v'=a-1$ which belongs to Z_2 except for the point $(0,a-1)$, and the two rank one preimages of a point of such line are correctly detected by Eq. (19) [one belongs to the line $u=0$ and the other belongs to the hyperbola $v=-(2a)/u+2(a-1)$].

It is worth noticing that the regions Z_0 and Z_2 are, respectively, inside and outside the parabola of equation

$$v=u^2+a-1, \tag{20}$$

but the boundary $\partial Z_2=\partial Z_0$ between the two regions (we shall call it ∂Z_2 henceforth), given by Eq. (20), is not a locus of critical points where the two inverses are defined and merge. As explained in Bischi, Gardini, and Mira,[3] this occurrence is related to the fact that such a boundary is the singular set of at least one of the inverses of the map. In fact, in our case, from the definition Eq. (19) of the inverses, it is immediate to see that both the inverses are not defined in the set ∂Z_2, since the denominator vanishes. In other words, ∂Z_2 is the singular set of T_a^{-1}. For this reason it will also be denoted by δ_s'.

As also the inverses have a vanishing denominator we may ask for focal points and prefocal curves of the inverses. But we have now to do much less work to determine such sets. In fact, from Property 2 we know that the vertical line through a fixed point is mapped into the fixed point itself, $T_a(u_i^*,v)=R_i$, and thus T_a is not invertible in such vertical lines, where it is many-to-one (the Jacobian of T_a vanishes on the lines $u=u_i^*$). As shown in Bischi, Gardini, and Mira,[3] from these properties we can immediately state that the focal points of the inverse map T_a^{-1} are the fixed points of T_a with associated prefocal sets the vertical lines. Reassuming, for the inverse map T_a^{-1} there are three focal points for $a<1/4$: $Q_i'=R_i$, with related prefocal lines $\delta_{Q_i'}$ of equation $u=u_i^*$, for $i=1,2,3$, whereas T_a^{-1} has only the focal point Q_1', with related prefocal line $\delta_{Q_1'}$ of equation $u=u_1^*$, for $a>1/4$.

This means that if we consider a small bounded arc η which crosses ∂Z_2 in a point (u',v') which is not a fixed point, and look for the rank-1 preimages of this arc, then the points in Z_0 have no preimages, while the points in Z_2 have two distinct preimages. These preimages, $T_{a,r}^{-1}(\eta)$ and $T_{a,l}^{-1}(\eta)$, are unbounded arcs asymptotic to the straight line $u=u'$, which depends on the point in which η crosses the singular set $\delta_s'(=\partial Z_2)$ of the inverses.

We can consider such a situation similar to the one occurring for one-dimensional maps having a horizontal asymptote which separates portions of the range having different number of preimages. As a horizontal asymptote of a one-dimensional map corresponds to a vertical asymptote for at least one inverse; also in our two-dimensional map T_a we can consider the parabola $\partial Z_2\equiv\delta_s'$ as a two-dimensional analogue of a horizontal asymptote. On the basis of similar arguments we suggest the following interpretation for the sets δ_s and δ_s': *the points of the singular set δ_s of the map T_a behave as points of vertical asymptote except for the focal points Q_i, while for the map T_a the points of the singular set $\delta_s'(=\partial Z_2)$ of the inverses behave like points of horizontal asymptote, except for the points R_i.*

Moreover, straightforward algebraic computations show that the prefocal lines δ_{Qi}, $i=1,2,3$, are the tangents to ∂Z_2 in the fixed points R_i. We can so state the following proposition:

Proposition 2. *Let T_a be the map in Eq. (1), $i=1$ for $a>1/4$, $i=1,2,3$ for $a<1/4$. The fixed points belong to the boundary of the region Z_2, $R_i\in\partial Z_2$. The focal points $Q_i\in\delta_s\cap(u=u_i^*)$ have prefocal curves δ_{Qi} which are the tangents to ∂Z_2 in the fixed points R_i.*

From this proposition and the properties of the inverses given above, it follows that if we consider a small neighborhood U of the fixed point R_i, then the points belonging to $U\cap Z_0$ have no preimages, while the points in $U\cap Z_2$ have two distinct rank-1 preimages, given by an unbounded area $T_a^{-1}(U)$ (which must include the whole line $u=u_i^*$) whose qualitative shape is shown in Fig. 2. The particular shape of $T_a^{-1}(U)$ is due to the fact that the focal point Q_i belongs to the line $u=u_i^*$ and that the prefocal curve δ_{Q_i} is tangent to the singular set $\delta_s'(=\partial Z_2)$ in the fixed point R_i. This means that any neighborhood U of R_i intersects δ_{Q_i}, as well as δ_s', in two distinct points: the preimages of arcs crossing $U\cap\delta_{Q_i}$ are arcs through Q_i, hence $T_a^{-1}(U)$ must shrink into the focal point Q_i, and the preimages of points near $(u',u'^2+a-1)\in U\cap\delta_s'$ are arbitrarily large, i.e., close to infinity, asymptotic to the line $u=u'$.

D. Invariant sets of T_a

As stated above, for $a>1/4$ the line L_1 of Eq. (7) is an invariant submanifold for the map T_a, and for $a<1/4$ the three lines L_i, $i=1,2,3$, whose equations are given in Eqs. (7) and (8), are invariant submanifolds of the phase space of the map T_a. This means that if $(u,v)\in L_i$, then $(u',v')\in L_i$ and the one-dimensional dynamics embedded into the line L_i is governed by the restriction of T_a to the invariant line, which can be written as a one-dimensional map $u'=F_i(u)$. Each one-dimensional map F_i coincides with the Newton function of the reduced polynomial $P_i(u)=u^2-s_iu+p_i$, i.e., $F_i(u)=u-P_i(u)/P_i'(u)$, where the coefficients of $P_i(u)$ are given by $(s_1,p_1)=(-1,a)$, $(s_2,p_2)=(-u_2^*,u_3^*)$ and $(s_3,p_3)=(-u_3^*,u_2^*)$. Thus we get:

$$F_1(u)=\frac{u^2-a}{2u+1},\quad F_2(u)=\frac{u^2-u_3^*}{2u+u_2^*},$$

$$F_3(u)=\frac{u^2-u_2^*}{2u+u_3^*}.$$

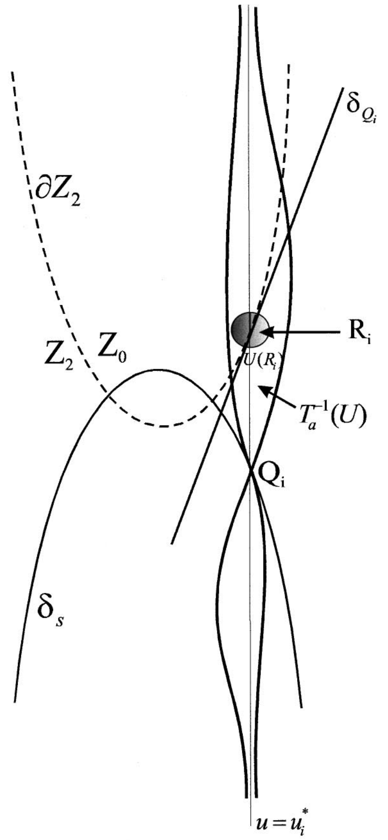

FIG. 2. Qualitative sketch of the rank-1 preimages $T_a^{-1}(U) = T_{a,l}^{-1}(U) \cup T_{a,r}^{-1}(U)$ (which is unbounded) of the neighborhood U of the fixed point R_i (the gray-shaded circular region).

It is very simple to see that for $a<1/4$ all three one-dimensional maps F_i are topologically conjugated with the map $g(x)=(x^2+1)/2x$, while for $a>1/4$ the map F_1 is topologically conjugated with the map $f(x)=(x^2-1)/2x$. This is a simple consequence of the following proposition.

Proposition 3. *Let $F(u)=(u^2-p)/(2u-s)$ be the Newton-function associated with the polynomial $P(u)=u^2-su+p$. When $P(u)$ has real roots then the map $F(u)$ is topologically conjugated with the map $g(x)=(x^2+1)/2x$ via the homeomorphism $u=h_g(x)=(\sqrt{s^2-4p}/2)x+s/2$. When $P(u)$ has complex roots then the map $F(u)$ is topologically conjugated with the map $f(x)=(x^2-1)/2x$ via the homeomorphism $u=h_f(x)=(\sqrt{4p-s^2}/2)x+s/2$.*

In our case, the three polynomials associated with the restrictions F_i have real roots for $a<1/4$ and complex roots for $a>1/4$. Thus for $a<1/4$ all the three maps F_i on the invariant lines L_i have the simple dynamic behavior of the map $g(x)=(x^2+1)/2x$, and their graphs have the same characteristics of the one shown in Fig. 3(a). R_i is the fixed point not belonging to L_i, and thus we have denoted by R_j and R_k the two fixed points belonging to L_i which are separated by a point of vertical asymptote A_i, clearly due to the intersection of L_i with the singular set δ_s. However, each invariant line L_i intersects the singular set δ_s in two points, the vertical asymptote A_i and the focal point Q_i. Thus the dynamics of each one-dimensional map F_i *is not completely equivalent to the restriction* of the two-dimensional map T_a

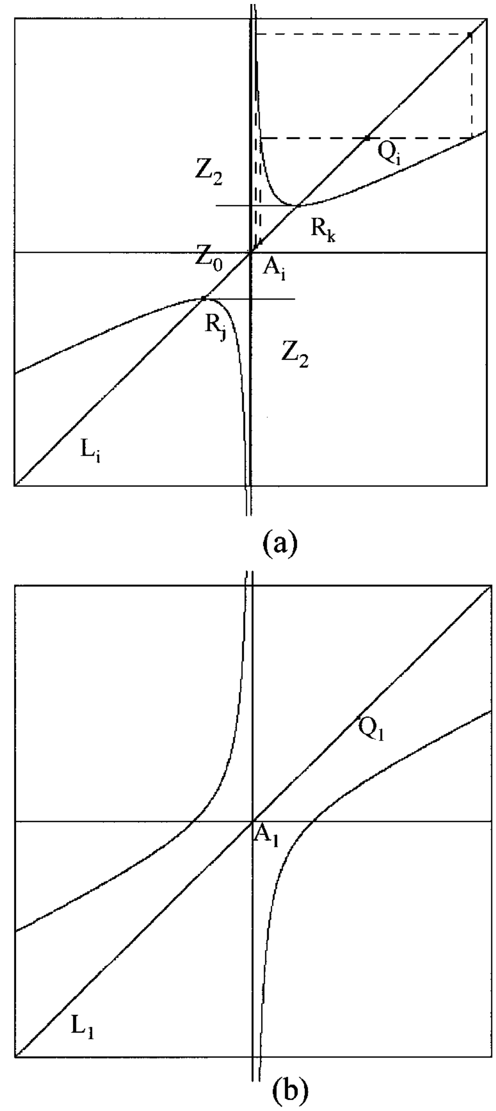

FIG. 3. One-dimensional restriction of map T_a to invariant lines L_i. For $a<1/4$ they have the same shape as the graph shown in (a), where R_k and R_j are the fixed points belonging to L_i, A_i is the vertical asymptote, and Q_i is the focal point belonging to L_i. Some preimages of Q_i are also shown in (a). For $a>1/4$ the restriction to the only invariant line L_1 has the same shape as the graph shown in (b).

to the invariant line, since the equivalence does not hold for the focal point and its preimages of any rank. In fact, the reduced polynomial is obtained by dividing by the root u_i^* (which has the same u-coordinate of the focal points Q_i and of the fixed points R_i), thus we are assuming $u \neq u_i^*$, as well as u different from all the preimages of u_i^* in the case of iterated application of the map.

For example, each restriction F_i has two superstable fixed points, whose basins of attraction are separated by the point of non definition A_i (i.e., the vertical asymptote): we denote by $B(R_j)=]-\infty, A_i[$ the one-dimensional basin of one fixed point, and $B(R_k)=]A_i,+\infty[$ the one-dimensional basin of the other one. The set $B(R_k)$ is not exactly the intersection of the two-dimensional basin $\mathcal{B}(R_k)$, of the map T_a, with the invariant line L_i, because we have to exclude the focal point Q_i and its preimages $T_a^{-1}(Q_i)$: Those obtained by the iterated application of $T_{a,r}^{-1}$ constitute a diverging sequence on the right, and those obtained by $T_{a,l}^{-1}$, constitute a sequence which tends toward the vertical asymptote A_i (i.e., the other singular point on the invariant line L_i). These preimages play an important role, which will be evidenced in the next section.

For $a>1/4$ only one invariant line L_1 exists, and the polynomial $P_1(u)$ associated with the Newton-function $F_1(u)$ has complex roots, hence $F_1(u)=(u^2-a)/(2u+1)$ has no fixed points, and its dynamics are chaotic. In fact $F_1(u)$ is conjugate to the sawtooth map

$$g(z)=\begin{cases}2z & \text{for } 0\leqslant z\leqslant 1/2\\ 2z-1 & \text{for } 1/2<z\leqslant 1\end{cases} \tag{21}$$

by the homeomorphism $z=h(u)=\frac{1}{2}+(1/\pi)\arctan[(2u+1)/(\sqrt{4a-1})]$, and the dynamics of Eq. (21) are well known, from both a topological and a measure theoretical point of view: it has chaotic dynamics in the interval $[0, 1]$ with an absolutely continuous invariant measure associated to it. This implies that *the preimages of* Q_1 *of any rank are dense in the whole line* L_1.

We can so state the following proposition:

Proposition 4. *Let* T_a *be the map in (1),* $i=1$ *for* $a>1/4$, $i=1,2,3$ *for* $a<1/4$. *The invariant lines* L_i *of the map, whose equations are given in Eq. (7) or Eqs. (7) and (8), respectively, do not include the fixed points* R_i *and include the focal points* Q_i:

$$R_i\notin L_i, \quad Q_i\in L_i.$$

The restriction of the two-dimensional map T_a *to a line* L_i *has the same dynamic behavior of the following one-dimensional maps*

$$u'=F_1(u)=\frac{u^2-a}{2u+1} \text{ on } L_1$$

$$u'=F_2(u)=\frac{u^2-u_3^*}{2u+u_2^*} \text{ on } L_2$$

$$u'=F_3(u)=\frac{u^2-u_2^*}{2u+u_3^*} \text{ on } L_3,$$

except for the point Q_i *and its preimages of any rank.*

III. THE BASINS OF MAP T_a

In this section we describe the qualitative shape of the basins of attraction $\mathcal{B}(R_i)$, $i=1,2,3$ and their bifurcations as the parameter a varies. It is clear that strong qualitative changes in the basins must occur at $a=1/4$, when the fixed points R_2 and R_3 (existing for $a<1/4$) merge and then disappear for $a>1/4$. We also note that at $a=1/4$ we have $R_2=R_3=Q_2=Q_3$, that is, the merging and disappearance also holds for two focal points of the map.

A second bifurcation value can be predicted. At $a=-2$ we have again the merging of two fixed points with two focal points, that is, $R_1=R_2=Q_1=Q_2$, and this causes a drastic change in their basins, as we shall see below.

A third bifurcation value is given by $a=1$, because as the parameter a crosses the value $a=1$ we have a qualitative change in the shape of the singular set δ_s and its preimages. In fact, set $\Lambda=\cup_{n\geqslant 0}T_a^{-n}(\delta_s)$, excluded from the set of points of the plane in which the recurrence is well defined, is obtained by taking the preimages of any rank of δ_s. Thus only the portion $\delta_s\cap Z_2$ of δ_s is important to obtain the set Λ. From the equation of δ_s and that of ∂Z_2, it is easy to see that for $a<1$ δ_s intersects ∂Z_2 in two points, and in this case $\delta_s\cap Z_2$ is made up of two disjoint pieces, say $\delta_s\cap Z_2=\delta_{s,l}\cup\delta_{s,r}$, whereas for $a>1$ we have $\delta_s\cap Z_2=\emptyset$, i.e., the singular set δ_s is entirely included in Z_2. This implies a change in the structure of Λ, and thus of $E=\mathbb{R}^2\backslash\Lambda$, and consequently a qualitative change also in the basin $\mathcal{B}(R_1)$.

The basins' boundaries are sets which are invariant under inverse iteration of the map, that is $T_a^{-1}(\partial\mathcal{B}(R_i))=\partial\mathcal{B}(R_i)$. Generally a basin's boundary is a set which includes some repelling cycles and their stable sets. However, for $a<1/4$ we have not found any other cycle except for the stable fixed points R_i. But in two-dimensional maps with vanishing denominator it may occur that the set Λ of the preimages of any rank of the singular set also behaves as a frontier between two or more basins of attraction. A one-dimensional analogue is the map $g(x)$ shown in Fig. 3(a), where the vertical asymptote separates the basins of the two fixed points. A two-dimensional example where a similar property holds is given in the Bischi, Gardini, and Mira.[3] The map T_a given in Eq. (1) is another example of this remarkable property, specific to maps with denominator.

A. Basins structures for $a<1/4$

As remarked above, in order to describe in detail the basins' boundaries in this range of the parameter, we have to consider the singular set δ_s and its preimages. The two parabolas δ_s and ∂Z_2 intersect in two points: The portion of δ_s located inside Z_0 has no preimages, whereas the portion located inside Z_2, say $\delta_s\cap Z_2=\delta_{s,l}\cup\delta_{s,r}$, is made up of two disjoint branches, each one having two rank-1 preimages. Thus $T_a^{-1}(\delta_s)$ is made up of four branches, as shown in Fig. 4(a). We remark that even if Fig. 4 has been obtained for the particular value $a=0.15$ of the parameter, the qualitative structure of the preimages shown in this figure may be considered as emblematic of the whole range $a<1/4$.

As $\delta_{s,r}$ is an unbounded curve that intersects the prefocal line δ_{Q_1} in a point with abscissa u_1, then its rank-1 preimage $T_{a,r}^{-1}(\delta_{s,r})$ is an unbounded arc crossing through Q_1 with slope $m(u_1)$ given by Eq. (16). Furthermore, since $\delta_{s,r}$ includes Q_1, its preimages must include the corresponding preimages of that focal point. Analogously, as $\delta_{s,r}$ intersects also the prefocal lines δ_{Q_2} and δ_{Q_3} the rank-1 preimage on the left, $T_{a,l}^{-1}(\delta_{s,r})$, must be an unbounded arc crossing through the two focal points Q_2 and Q_3, with known slopes. Following similar arguments, as the arc $\delta_{s,l}$ intersects the prefocal lines δ_{Q_1} and δ_{Q_2} its rank-1 preimage $T_{a,r}^{-1}(\delta_{s,l})$ must be an unbounded arc crossing through Q_1 and Q_2 with given slopes, and the rank-1 preimage on the left, $T_{a,l}^{-1}(\delta_{s,l})$, must be an unbounded arc crossing through the focal point Q_3 and its preimage $Q_{3,-1}$. These four branches are represented in Fig. 4(a). Many portions of the rank-1 preimages of δ_s belong to the region Z_2, so preimages of higher rank exist and so on. Indeed, the process never ends, and infinitely

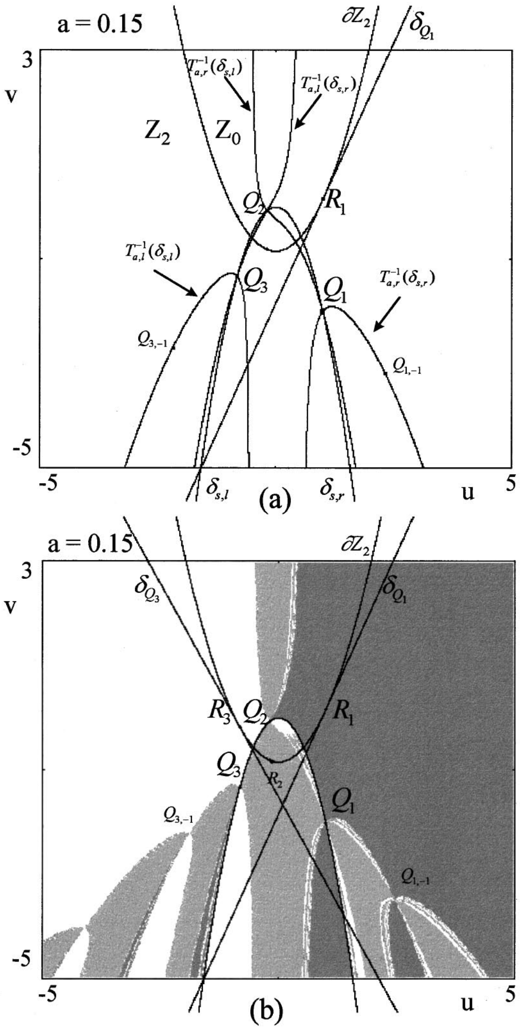

FIG. 4. $a=0.15<1/4$. In (a) the rank-1 preimages of $\{\delta_{s,l},\delta_{s,r}\}=\delta_s\cap Z_2$ are represented. In (b) three different gray tones are used to represent the three basins of attraction of the fixed points. The dark gray represents the points belonging to $\mathcal{B}(R_1)$, the intermediate gray represents the points belonging to $\mathcal{B}(R_2)$, and the light gray represents the points belong to $\mathcal{B}(R_3)$. The points Q_1, Q_2, and Q_3 are the focal points.

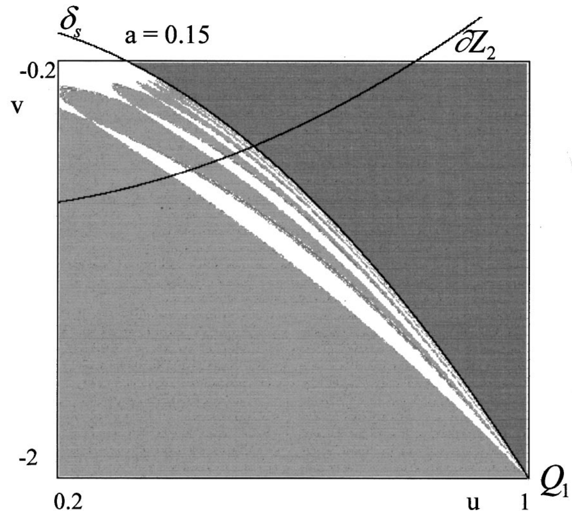

FIG. 5. $a=0.15<1/4$. Enlargement of a portion of Fig. 4(b).

many preimages exist. Note that whenever an arc is unbounded and intersects the set ∂Z_2 then it also intersects all the three prefocal lines (due to the fact that the prefocal lines are tangent to ∂Z_2 in the fixed points R_i), thus its rank-1 preimage is made up of two branches, $T_a^{-1}=T_{a,l}^{-1}\cup T_{a,r}^{-1}$, which cross through the three focal points Q_i.

It is a numerical evidence that all these arcs, that constitute the set Λ, also separate the three basins of attraction [compare Fig. 4(a) with Fig. 4(b)]. But more, each branch of the preimages determined as described above is, from one side, a limit set of other preimages, i.e., a limit set of portions of the three different basins. This numerical result can be explained by the properties of the focal points and prefocal lines. For example, consider in Fig. 4(a) the arc $T_{a,l}^{-1}(\delta_{s,l})$. It crosses the prefocal line δ_{Q_1} in two points (only the upper one is visible in the figure). Thus its rank-1 preimage on the right must be a ''loop'' issuing from the focal point Q_1, as qualitatively explained in Sec. II. The same property holds for the infinite sequence of preimages having a ''parabolic shape'' that are located on the left (i.e., in the half-plane $u<0$). This gives infinitely many ''lobes'' of different basins issuing from the focal point Q_1, as shown in the enlargement of Fig. 5. Note also that all the lobes issuing from Q_1 intersect both ∂Z_2 and δ_{Q_1}, so that the infinitely many preimages ''on the right'' must include a ''fan'' of unbounded arcs issuing from Q_1 and crossing through $Q_{1,-1}$, issuing from $Q_{1,-1}$ and crossing through $Q_{1,-2}$, and similar fans of strips of different colors (i.e., whose points belong to different basins) issue from all the infinitely many preimages of Q_1.

Analogously, the preimage $T_{a,r}^{-1}(\delta_{s,r})$ (or, better, the fan issuing from $Q_1,Q_{1,-1}$, etc.) intersects the prefocal line δ_{Q_3} in two points, forming an arc, and its rank-1 preimage has a ''loop'' issuing from the focal point Q_3, and so on for the other curves, constituting a fan, intersecting the prefocal line δ_{Q_3} in ''arcs,'' giving rise to infinitely many lobes of the three basins issuing from the focal point Q_3, shown in Fig. 6(a). A further enlargement is shown in Fig. 6(b), which puts in evidence the invariant line L_1 and some preimages of the focal point Q_1 on it, which accumulate toward the point A_1, the other intersection of L_1 with the singular set δ_s, and constituting the vertical asymptote for the one-dimensional restriction $F_1(u)$ [see Fig. 3(b)]. It is clear that the two-dimensional map is not defined in such points, which are the loci from which infinitely many arcs (and a kind of crescents for the basins) are issuing.

From Fig. 4 it can be seen the different role played by the focal point Q_2 with respect to the other two focal points. This is due to the fact that $Q_2\in Z_0$, so that it has no preimages. It follows that whenever some arc crosses through the prefocal line δ_{Q_2} its preimage gives rise to an arc through the focal point Q_2 and here the sequence of preimages stops.

As we shall see, the property that $Q_2\in Z_0$ holds for $-2<a<1/4$, while for $a<-2$ we have $Q_2\in Z_2$ and $Q_1\in Z_0$, i.e., the role of these two focal points is swapped at $a=-2$. In fact, as the parameter a is decreased, at the value $a=0$ we have $Q_2=(0,0)\in Z_0$, which is also a value with a particular symmetric structure in the basins [as the basins

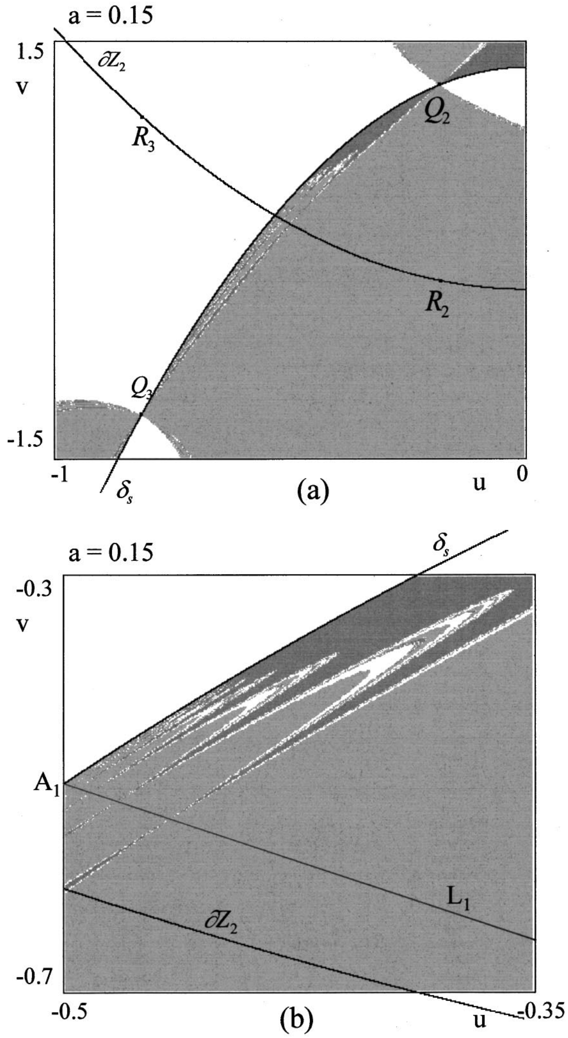

FIG. 6. $a=0.15<1/4$. Enlargements of two portions of Fig. 4(b).

$\mathcal{B}(R_1)$ and $\mathcal{B}(R_3)$ become symmetric with respect to the vertical axis $u=0$, as shown in Fig. 7], and then, for $a<0$, the focal point Q_2 moves on the right side toward Q_1 (see Fig. 8).

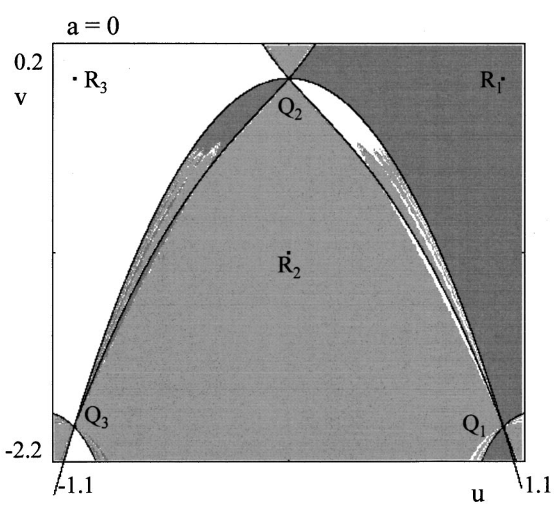

FIG. 7. $a=0$. Basins of attraction of the fixed points.

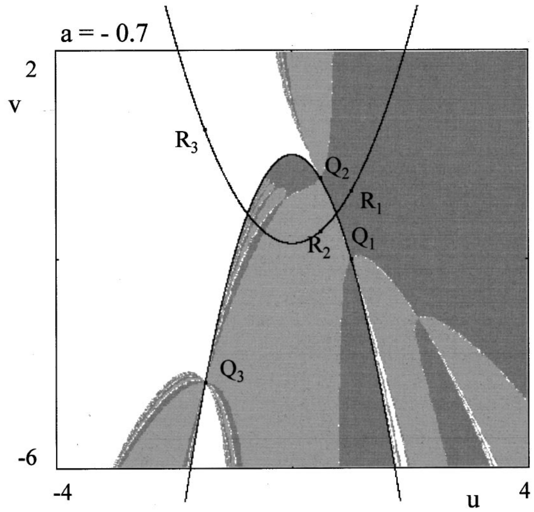

FIG. 8. $a=-0.7$. Basins of attraction of the fixed points.

At $a=-2$ we have $R_1=R_2=Q_1=Q_2$. Then, for $a<-2$ a ''change of role'' between the couple R_1,Q_1 and R_2,Q_2 occurs, as now it is Q_1 the focal point in Z_0 (with no preimages) while $Q_2\in Z_2$ has infinitely many preimages. In Fig. 9(a) the structure of the basins just before the bifurcation is shown, while in Fig. 9(b), obtained just after the bifurcation, it is evident that a generic point (u_0,v_0) which belongs to the basin $\mathcal{B}(R_1)$ for $a>-2$ will belong to the basin $\mathcal{B}(R_2)$ for $a<-2$.

The situation appearing in Fig. 10, obtained with $a=-5$, can be considered a generic representation of the qualitative structure of the basins for any value of a below $a=-2$. The structure of the basins is the same as that commented above, with the only change of the index 1 with 2.

B. Basin structure for *a*>1/4

As remarked in Sec. II, as the parameter a approaches the value 1/4 the fixed points R_2 and R_3 approach each other [see Fig. 11(a)], and at $a=1/4$ they merge with the two focal points Q_2 and Q_3. In this case there is not a ''change of role,'' because the fixed points disappear for $a>1/4$, and only the fixed point R_1 survives. Also the focal points Q_2 and Q_3 disappear at $a=1/4$, and for $a>1/4$ only the focal point Q_1 exists with the related prefocal line δ_{Q_1} (which is tangent to ∂Z_2 in R_1).

In this situation the only invariant line is L_1 which includes the focal point Q_1 and does not contain any fixed point in it [the restriction of T_a to L_1 is the map F_1, whose graph has the qualitative shape represented in Fig. 3(b)]. The boundaries of the basin $\mathcal{B}(R_1)$ are now changed with respect to those described in the previous subsection, as the preimages of the singular set δ_s are different. In fact, as long as $a<1$, δ_s intersects ∂Z_2 in two points, and $\delta_s\cap Z_2=\delta_{s,l}\cup\delta_{s,r}$. Thus $T_a^{-1}(\delta_s)$ is still made up of four branches, but having a different structure with respect to the case considered in the previous subsection. The branch $\delta_{s,r}$ intersects only the prefocal line δ_{Q_1} in a point (u_1,v_1), and includes Q_1 [see Fig. 11(b)]. Then its rank-1 preimage $T_{a,r}^{-1}(\delta_{s,r})$ is an

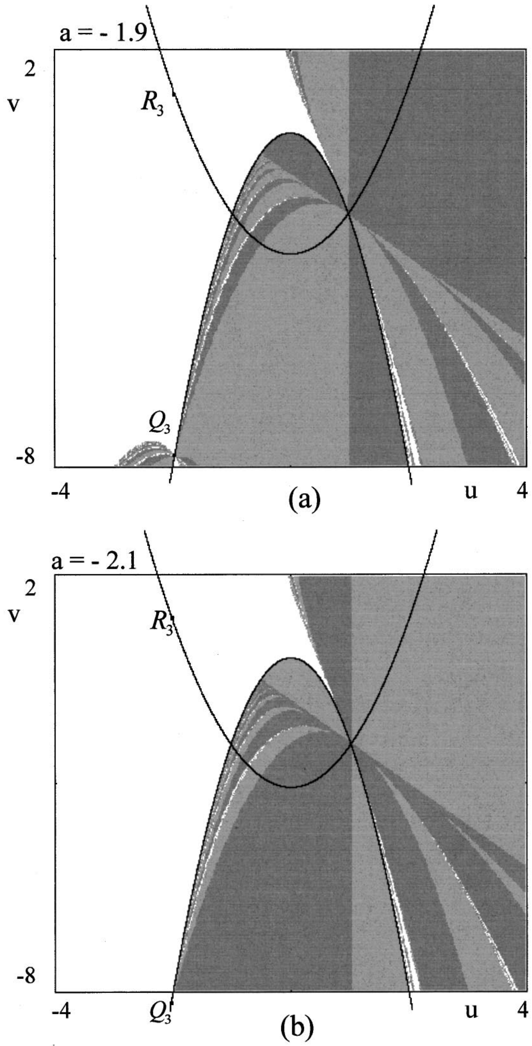

FIG. 9. Basins of attraction of the fixed points. (a) $a=-1.9$, just before the bifurcation occurring at $a=-2$. (b) $a=-2.1$, just after the bifurcation occurring at a $a=-2$.

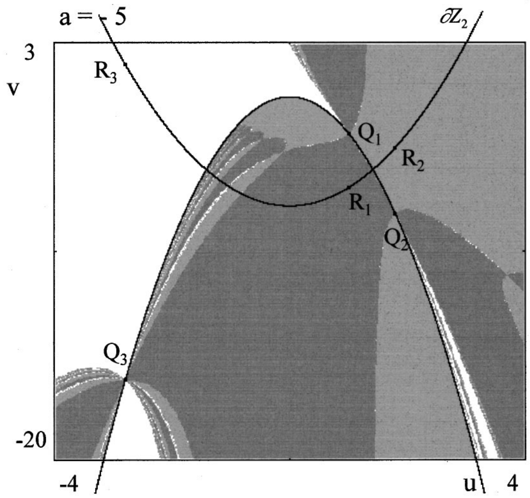

FIG. 10. $a=-5$. Basins of attraction of the fixed points.

arc crossing through Q_1 with slope $m(u_1)$ and includes the preimage $T_{a,r}^{-1}(Q_1)$ of the focal point. Its left preimage $T_{a,l}^{-1}(\delta_{s,r})$ is an unbounded arc crossing the invariant line L_1 in the other rank-1 preimage of $Q_1, T_{a,l}^{-1}(Q_1)$. Similar properties hold for the rank-1 preimages of the other arc, $\delta_{s,l}$, of the singular set δ_s. In fact, $\delta_{s,l}$ intersects the prefocal line δ_{Q_1} in a point (u_2,v_2) and the invariant line L_1 in the point corresponding to the vertical asymptote for the restriction $F_1(u)$. Then its rank-1 preimage $T_{a,r}^{-1}(\delta_{s,l})$ is an arc crossing through Q_1 with slope $m(u_2)$ and intersecting the invariant line L_1 in the right preimage $T_{a,r}^{-1}(A_1)$, while the left preimage $T_{a,l}^{-1}(\delta_{s,l})$ is an unbounded arc crossing the invariant line L_1 in the left preimage $T_{a,l}^{-1}(A_1)$ [see Fig. 11(b)].

Also in this case infinitely many preimages exist, because all the preimages of increasing rank have some points belonging to the region Z_2. It is evident that, with respect to the situation described in the previous subsection, now the preimages of the singular set, located on the invariant line L_1, are very different. In fact, the restriction of the map T_a to the invariant line L_1, given by the one-dimensional map $F_1(u)$, is chaotic on the whole line with absolutely continuous invariant measure on it, and this implies that the preimages of any point, and in particular the preimages of the vertical asymptote and of the focal point Q_1, are dense on that line. However, while on the restriction F_1 only the vertical asymptote plays a role (as that point and all its dense preimages of any rank are to be excluded from the domain of definition), for the two-dimensional map T_a also the focal point Q_1 play an important role. In fact, the focal point Q_1 is crossed by a ''fan'' of curves belonging to Λ, and the same occurs to all its preimages of any rank, which are dense on the line. Such curves are on the boundary of the domain of definition of the map T_a and also on the boundary of the basin $\mathcal{B}(R_1)$. That the basin $\mathcal{B}(R_1)$ must cross L_1 on the focal point Q_1 in a particular way is already known. Considering a neighborhood U of R_1 the geometric behavior of its rank-1 preimage is schematically shown in Fig. 2, and the same qualitative picture must hold also in all the preimages of Q_1 which are dense on L_1. Thus the basin shown in Fig. 11(b), where it is evident that also in this situation the set Λ of all the preimages of δ_s form the basin boundary, is only a rough approximation of the true basin $\mathcal{B}(R_1)$, which must include infinitely many ''fan'' issuing from the preimages of Q_1. This structure becomes more evident as a is increased. An example is shown in Fig. 12(a), and a few preimages of δ_s (up to those of rank 4) are shown in Fig. 12(b).

Thus it is easy to conclude that the invariant line L_1 cannot be an attracting set for $a>1/4$, although infinitely many cycles exists on it and all are transversally attracting. That is, the invariant line L_1 may only be an attractor in Milnor sense (see Milnor,[16] Alexander *et al.*,[17] Buescu[18]). This bifurcation was already considered in Billings and Curry,[2] where it was called ''eruption,'' due to the fact that on the invariant line L_1 we have the sudden appearance of infinitely many cycles, of any period (except for the period 1) which are repelling for the map T_a. In fact such periodic points are all expanding for the restriction on L_1 and transversally attracting at least as a is slightly above 1/4, and loose transverse stability as a is increased as shown in Billings and Curry.[2]

In Billings, Curry, and Phipps,[9] it is shown that for

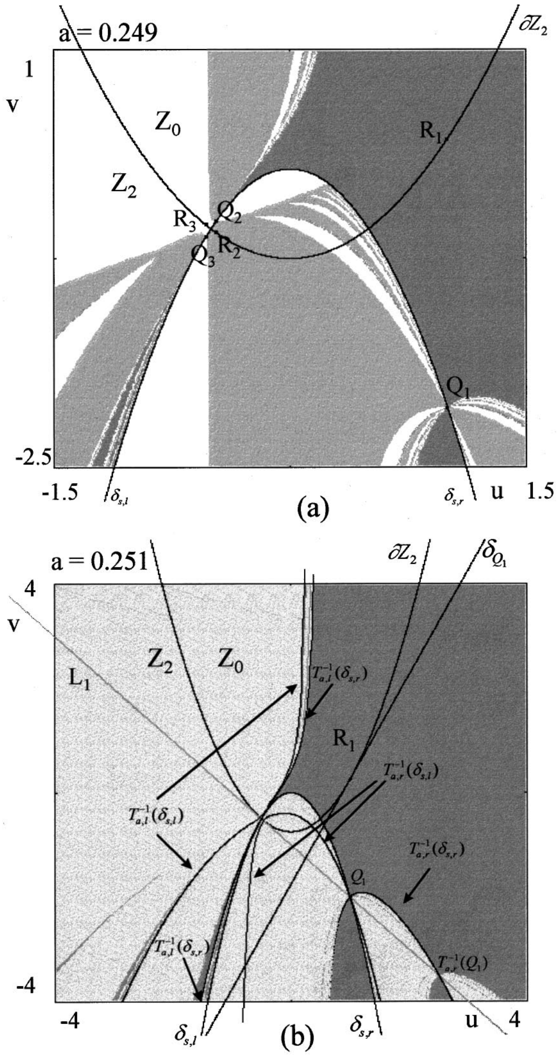

FIG. 11. (a) $a=0.249<1/4$. Fixed points R_1 and R_2, as well focal points Q_1 and Q_2, are very close, and they are going to merge at $a=1/4$. (b) $a=0.251>1/4$. Only fixed point R_1 exists, and only focal point Q_1 exists. The dark gray region represents the basin of the fixed point R_1, as in the previous figures, whereas now the light-gray region represents the set of points which are attracted to the chaotic set embedded into the invariant line L_1.

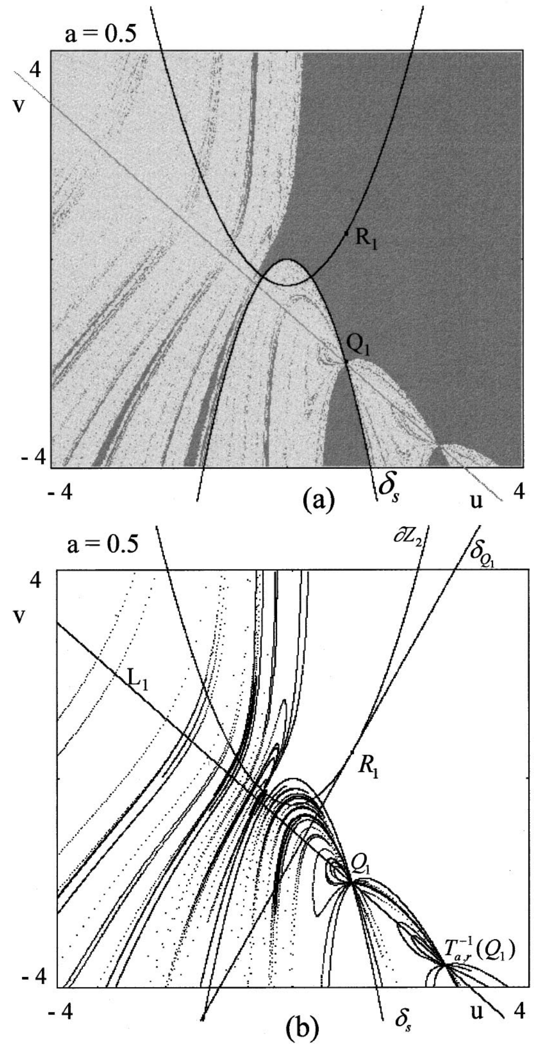

FIG. 12. $a=0.5$. (a) Basins of attraction of fixed point R_1 and of the chaotic set embedded inside the invariant line L_1, represented by dark gray and light gray points, respectively. (b) Preimages, up to rank-4, of δ_s.

$1/4<a<1$ *the invariant line* L_1 *is probably a Milnor attractor, with the peculiarity that the ''tongues'' issuing from it, made up of points whose trajectory escape from a neighborhood of the line, are not issuing from a repelling cycle but from the focal point and its preimages, which are densely distributed along* L_1.

Moreover, as already proved by Boyd,[7] the one-dimensional map F_1 is purely chaotic and an ergodic invariant measure exists. According to Alexander *et al.*,[17] the presence of such set of tongues issuing from points dense on L_1 and transverse to it, whose points go away from L_1 to reach the fixed point, is an essential feature for the existence of riddled basins. Thus as argued by Billings, Curry, and Phipps,[9] the bifurcation occurring at $a=1/4$ gives rise to a Milnor attractor with a basin $\mathcal{B}(L_1)$ which is riddled by the basin $\mathcal{B}(R_1)$.

We note that in Billings and Curry[2] it was conjectured that such a kind of ''eruption'' is likely to occur whenever two fixed points (of a map coming from Bairstow's method) merge into a unique one. However, as is shown by this example (map T_a), it is not the case when two fixed points merge and then exchange their position on the invariant line, as occurs when the parameter a crosses the value $a=-2$. While *when two fixed points merge on an invariant line and then disappear*, leaving a one-dimensional restriction which is the Newton-function associated with a reduced polynomial having only complex roots, then it is certainly true that an explosion of infinitely many repelling cycles occur, because the one-dimensional restriction suddenly becomes a chaotic map.

C. Basin structure for *a*>1

The qualitative shape of the basin $\mathcal{B}(R_1)$ after the riddling bifurcation, as shown in Fig. 12(a), is that of a ''wide'' basin having a fractal boundary whose closure, however, is not the whole plane, because the stable set $\mathcal{B}(L_1)$ is a set with positive Lebesgue measure, as long as $1/4<a<1$. How-

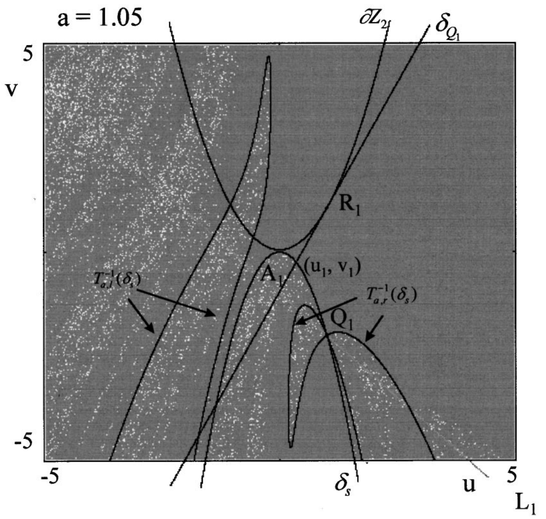

FIG. 13. $a=1.05>1$. The basin $\mathcal{B}(L_1)$ has now zero Lebesgue measure. The two rank-1 preimages of δ_s, which is now entirely included inside Z_2, are also represented.

ever, as noticed in Billings and Curry,[2] the cycles embedded inside L_1 become less and less attracting in the direction transverse to the invariant line L_1 as a approaches the value $a=1$. In a measure theoretic sense, the stable set $\mathcal{B}(L_1)$ becomes smaller and smaller, as also revealed by the natural transverse Lyapunov exponent (here explicitly known, and given in Billings, Curry, and Phipps[9]). In fact this is negative for $a<1$, but it increases, and crosses the value 0 at $a=1$, becoming positive for $a>1$. This bifurcation was already noticed by Boyd.[7] Thus $a=1$ marks the transition from a Milnor attractor L_1 with a basin $\mathcal{B}(L_1)$ of positive Lebesgue measure, into a chaotic saddle with a stable set $\mathcal{B}(L_1)$ of zero Lebesgue measure. This corresponds to a *blow-out bifurcation* of the Milnor attractor (see Ashwin *et al.*,[20] Ott *et al.*[21]).

As noticed at the beginning of Sec. II, we could predict the occurrence of some qualitative change in the basins $\mathcal{B}(R_1)$ and $\mathcal{B}(L_1)$ from the fact that the set Λ has a drastic qualitative change as a crosses 1. In fact $\delta_s \cap \partial Z_2 = \emptyset$ for $a>1$, which means that δ_s is entirely included in the region Z_2 (see Fig. 13). Thus its rank-1 preimage is made up of two pieces: $T_a^{-1}(\delta_s)=T_{a,r}^{-1}(\delta_s)\cup T_{a,l}^{-1}(\delta_s)$. Noticing that now δ_s intersects the prefocal line δ_{Q_1} in two points (u_1,v_1) and (u_2,v_2), and the arc of δ_s connecting these points is entirely included in the region Z_2, we have that the rank-1 preimage $T_{a,r}^{-1}(\delta_s)$ is an arc with a loop issuing from the focal point Q_1 [with slopes $m(u_1)$ and $m(u_2)$], crossing L_1 in $T_{a,r}^{-1}(A_1)$ and $T_{a,r}^{-1}(Q_1)$ [compare the qualitative Fig. 1(c) with the preimages of δ_s shown in Fig. 13]. While the left preimage $T_{a,l}^{-1}(\delta_s)$ is a single ''folded'' arc, crossing L_1 in the left preimages of the vertical asymptote of F_1 and in the left preimages of the focal point.

The preimage $T_{a,r}^{-1}(\delta_s)$ is again completely included in Z_2, while $T_{a,l}^{-1}(\delta_s)$ has a portion in Z_0, without preimages. However, two unbounded branches are still in Z_2, and we continue the process of computing the preimages of any rank of δ_s. These preimages of any rank constitute the set Λ, and constitute the boundary of the basin $\mathcal{B}(R_1)$.

The important feature in this regime is that although R_1 is not globally attracting, the basin $\mathcal{B}(R_1)$ is such that its closure covers the whole (u,v) plane, as the points excluded from $\mathcal{B}(R_1)$ constitute a set of zero Lebesgue measure.

In fact the invariant line L_1 and all its preimages are to be excluded, and also, as proved in Billings and Curry,[2] repelling cycles outside the invariant line L_1, which make their appearance via transverse bifurcation of the cycles on L_1. These, on their turn, may undergo further bifurcations leading to more and more cycles outside L_1. However, all of these belong to a set of zero measure as well as the stable set $\mathcal{B}(L_1)$, the natural transverse Lyapunov exponent being now positive.

IV. CONCLUSIONS

Many numerical iterative methods to find the roots of equations, being based on the Newton's method, require the iteration of maps with denominator. Among these, one of the most known and used in practice is the Bairstow's method.

In this paper we have considered a two-dimensional noninvertible map arising from the application of Bairstow's method to a cubic polynomial, and we have shown how the complex structures of the basin boundaries, as well as the global bifurcations that change the qualitative properties of the basins, can be explained in terms of new kinds of singularities specific to maps with denominator, recently introduced in the literature, such as singular sets, focal points and prefocal curves.

Although our analysis is limited to map T_a, it is likely that similar properties hold also in other maps deduced in the same way (see Grau,[22] Fiala and Krebsz,[23] besides the references given in the Introduction), because they are based on two properties: (i) the map is noninvertible; and (ii) the map has a vanishing denominator. These properties are a common feature also in maps coming from more general division algorithms (see also Blish and Curry,[8] Henrici,[24] Traub[25]), and the same techniques and mathematical tools used in the present paper can be used to study the global dynamic behavior also in other families of maps.

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