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On new phenomena in dynamic promotional competition models with homogeneous and quasi-homogeneous firms

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Abstract

In this paper we study a class of dynamic promotional competition models, in which firms compete for market share by expending marketing effort. We investigate two main issues. First, we answer the question if it is possible to give a global characterization of the stability of the steady state effort allocation. We show that by using the concept of critical curves and an invariance property of the coordinate axes a characterization of the set of feasible points (points that generate positive trajectories converging to the steady state allocation) and its changes can be given. Second, we deal with the assumption of homogeneous firms, which is often made in the literature. We demonstrate that the symmetric model which derives from this assumption exhibits, in many situations, non-generic dynamical behavior. New phenomena, like Milnor attractors and synchronization of trajectories, arising in the homogeneous case are illustrated. The introduction of small heterogeneities into the model invalidates many of the conclusions derived under the hypothesis of homogeneous firms.

Keywords: promotional competition, homogeneous and quasi-homogeneous firms, global dynamics, Milnor attractors, synchronization, symmetry breaking.
JEL Classification: E32, M30

1 Introduction

Market share attraction models specify that the market share of a competitor is equal to its attraction divided by the total attraction of all the competitors in the market, where the firm's attraction is given in terms of competitive effort allocations. Consider the case of two firms, which compete against each other in a market on the basis of both the quality and the magnitude of the marketing effort expended by each competitor. Let B denote the sales potential of the market (in terms of customer market expenditures). If firm 1 expends x dollars of effort and firm 2 expends

y dollars, then the share of the market (sales revenue) accruing to firm 1 and to firm 2 is Bs_1 and $Bs_2 = B - Bs_1$, respectively, where

$$
\begin{aligned}
s_1 &= \frac{ax^{\beta_1}}{ax^{\beta_1} + by^{\beta_2}} \qquad (1)\\
s_2 &= \frac{by^{\beta_2}}{ax^{\beta_1} + by^{\beta_2}}
\end{aligned}
$$

The terms $A_1 = ax^{\beta_1}$ and $A_2 = by^{\beta_2}$ are the attractions of customers to firm 1 and 2, respectively, given the expenditures of x and y units of effort[1]. The parameters a and b denote the relative effectiveness of effort expended by the firms. Since $\frac{dA_1}{dx}\frac{x}{A_1} = \beta_1$ and $\frac{dA_2}{dx}\frac{x}{A_2} = \beta_2$ the parameters β_1 and β_2 denote the elasticity of the attraction of firm (or brand) i with regard to the effort of firm i. Note that the payoff of each firm depends on the actions of both firms. This type of model is theoretically appealing because it is logically consistent: it yields market shares that are between zero and one, and sum to one across all the competitors in the market. Market share attraction models have been used frequently in empirical work; see, e.g., Bultez and Naert (1975), Naert and Weverbergh (1981). Moreover, they are prevalent in the economics, game theory, operations research and marketing literature; see, for example, Monahan and Sobel (1994), Monahan (1987), Friedman (1958), Schmalensee (1976), Case (1979, Ch. 4), Cooper and Nakanishi (1988).

In the existing literature predominantly *static* market share attraction models are used. Questions like the existence and uniqueness of (Nash) equilibria (Friedman 1958, Mills 1961, Schmalensee 1976), and their structure (Schmalensee 1976, Monahan 1987, Karnani 1985) are investigated. Few authors also study the local stability properties of these equilibria (Schmalensee 1976, Balch M. 1971), but global stability properties are completely neglected. This is quite in contrast to the recent interest on global phenomena in the economics literature. See, for example, Brock and Hommes (1997), Kopel (1996), de Vilder (1996). Furthermore, it is often assumed that the elasticities of the attractions of the firms with respect to effort, given by the parameters β_1 and β_2 in (1) are the same for all firms in the industry (often assumed to be equal to one, see Friedman 1958 and Mills 1961). The same can be said for the relative effectiveness of efforts, measured by the parameters a and b. This restrictive assumption of *homogenous* firms is only made to keep the models analytically tractable, but oftentimes lacks empirical evidence.

In this paper several open questions are addressed related to the issues raised in the previous paragraph. In order to do this we use (1) and introduce a dynamic version of a market share attraction model with adaptive adjustment of competitive effort allocations. The first topic we then briefly cover is in line with recent research agendas in economics, namely the characterization of the global properties of the (symmetric) model. We will be concerned with the question of how to describe the set of initial effort allocations which will converge to a competitive steady-state effort allocation, and the changes of this set when parameters (slightly) change. This topic has not been covered in the literature and will be one of the main points in the paper. The second issue we will address is the importance of the assumption of homogenous firms (or brands). That is, we ask the question if the introduction of *small heterogeneities* matter or not. If they do (and it will be shown that they do under certain circumstances), then the conclusions derived in the literature under the assumption of symmetry should be applied with caution.

[1] In marketing theory market share attraction models are used to describe the competition between several brands of a product in the market. The expressions then describe the attractions of the individual brands.

2 A brand competition model for market share

In this section we introduce a dynamic version of a market share attraction model with adaptive adjustment of competitive effort allocations following Bischi et al. (1998a). In this model it is assumed that the two competitors change their marketing efforts adaptively in response to the profits achieved in the previous period. In particular, the marketing efforts in period $t+1$ are determined by

$$\begin{aligned} x_{t+1} &= x_t + \lambda_1(Bs_{1t} - x_t)x_t \\ y_{t+1} &= y_t + \lambda_2(Bs_{2t} - y_t)y_t \end{aligned} \tag{2}$$

where the market shares s_{1t} and s_{2t} are determined by (1). The decision rule the firms use is a type of anchoring and adjustment heuristic (Tversky and Kahneman 1975), and is widely used in decision theory (see Wansink et al. 1998, Sterman 1989). The marketing efforts x_{t+1} and y_{t+1} of period $t+1$ are determined by, first, recalling an anchor - the previous allocations x_t and y_t - and then adjusting for the achieved results of the previous period, $Bs_{1t} - x_t$ and $Bs_{2t} - y_t$. Note that this adjustment also depends on how much effort has been expended before. The parameters $\lambda_i > 0, i = 1, 2$, measure the extent of the adjustment or the adjustment speed. If we replace the expressions for the market shares s_{1t} and s_{2t} in (2) by the expressions in (1), the dynamic market share attraction model

$$T: \begin{cases} x_{t+1} = x_t + \lambda_1 x_t \left(B \frac{x_t^{\beta_1}}{x_t^{\beta_1} + k y_t^{\beta_2}} - x_t \right) \\ \\ y_{t+1} = y_t + \lambda_2 y_t \left(B \frac{k y_t^{\beta_2}}{x_t^{\beta_1} + k y_t^{\beta_2}} - y_t \right) \end{cases} \tag{3}$$

where $k := b/a$, describes the evolution of the marketing efforts and the corresponding market shares of the two firms over time. The local and global properties of the map (3) for the general case of non-homogeneous firms have been studied in Bischi et al. (1998a). Here we are more concerned with the case of homogeneous and almost homogeneous firms, where the firms' parameters differ only slightly. We will describe which new phenomena arise in such situations and how they can be studied.

3 Homogeneous firms: general properties

In what follows we will be mainly interested in the homogeneous case of identical firms

$$\lambda_1 = \lambda_2 = \lambda > 0; \quad \beta_1 = \beta_2 = \beta > 0; \quad k = 1 \tag{4}$$

in which the map T in (3) assumes the *symmetric* form

$$T_s: \begin{cases} x_{t+1} = x_t + \lambda x_t \left(B \frac{x_t^{\beta}}{x_t^{\beta} + y_t^{\beta}} - x_t \right) \\ \\ y_{t+1} = y_t + \lambda y_t \left(B \frac{y_t^{\beta}}{x_t^{\beta} + y_t^{\beta}} - y_t \right) \end{cases} \tag{5}$$

This map is symmetric in the sense that it remains the same if the variables x and y are swapped. We will later deal with the case of quasi-homogeneous firms, where the parameters of the firms are only slightly different. As an example of quasi-homogeneous firms, one might imagine that firms differ only slightly in their relative effectiveness of efforts, captured by a value of the parameter k close to, but different from, one. Hence, k is in effect a measure of the degree of heterogeneity of the two firms. Note that the response parameter β measures the degree of competition in model (5), see also Hibbert and Wilkinson (1994). If $\beta = 0$, there is no competition between the firms and the attractiveness of each firm is constant. The two firms act independently and the market shares are equal. The larger the parameter β, the larger is the effect of an increase in marketing effort exerted by the competitor on the other firms market share and, hence, the larger is the degree of competition. We will use this simple model as a vehicle to cover the topics described at the end of the Introduction.

3.1 The Feasible Set

First note that the map (5) is defined only for nonnegative values of the marketing efforts x and y, because of the presence of the real exponent β. Starting from a given initial effort allocation (x_0, y_0), a *feasible* time evolution of the system is obtained only if the corresponding trajectory $\left\{(x_t, y_t) = T_s^t(x_0, y_0),\ t = 0, 1, 2...\right\}$ is entirely contained in the positive orthant. Such a trajectory has been called *feasible trajectory* in Bischi et al. (1998a), and the *feasible set* has been defined as the subset of $\mathbb{R}_+^2$ whose points generate feasible trajectories. The delimitation of the feasible set is a prerequisite for any study of models like (3) and (5). This point has not been addressed in the literature so far, but it has been studied in Bischi et al. (1998a). In that paper it is shown there that for the model (3) the invariant coordinate axes and their preimages of any rank form the boundary of the feasible set. We briefly and informally repeat the argument for the case of homogeneous firms. For analytical details (in the non-symmetric case) we refer to Bischi et al. (1998a). An important feature of the model (5) is that the two coordinate lines are invariant, i.e. $x_t = 0$ implies $x_{t+1} = 0$ and $y_t = 0$ implies $y_{t+1} = 0$. This means that if one of the firms expends no resources, it cannot achieve a positive market share, hence, earn any profit, and will not have anything to expend in the next period. If, at the same time, the competitor expends positive marketing effort, it captures the whole market. The decision rule then determines if the marketing effort from one period to the next is raised or lowered, depending on the fact if the competitor made a profit or a loss. Accordingly, the dynamics of the model (5) restricted to one of the axis is governed by a one-dimensional system, $s_{t+1} = f(s_t)$, where

$$f(s) = (1 + \lambda B)s - \lambda s^2 \tag{6}$$

The map f generates the same dynamics as the so-called logistic map $z_{t+1} = h(z_t) = \mu z_t(1 - z_t)$, where $\mu = 1 + \lambda B$ and the relation between the two systems is $s = \frac{1+\lambda B}{\lambda} z$. This feature enables us to deduce the possible dynamics of the time evolution of the marketing expenditures along the invariant axes from the well-known properties of the logistic map[2]. Bounded and feasible trajectories along the invariant axes are obtained when $\lambda B \leq 3$ (corresponding to $\mu \leq 4$), provided that the initial effort allocations lie in the segments $\omega_i = 00_{-1}^{(i)}$, $i = x, y$, where $0_{-1}^{(x)} = (\frac{1+\lambda B}{\lambda}, 0)$ and $0_{-1}^{(y)} = (0, \frac{1+\lambda B}{\lambda})$

[2] The logistic map has been the object of interest for researchers from various fields for many years, and it is frequently used in applications in economics, see e.g., Day (1994), Baumol and Benahbib (1987). The dynamics generated by the logistic map are well understood, see e.g. Mira (1987) or Devaney (1989).

are the rank-1 preimages[3] of the origin on the corresponding axis computed using the map f (corresponding to the unit interval for the quadratic map). If the initial effort expenditures along the axes are taken outside the segment ω_i, unfeasible trajectories are obtained. Now consider the region bounded by the segments ω_x and ω_y and their rank-1 preimages $\omega_x^{-1} = T_s^{-1}(\omega_x)$ and $\omega_y^{-1} = T_s^{-1}(\omega_y)$.

Following Bischi et al. (1998a), these preimages can be analytically computed as follows. Let $X = (p, 0)$ be a point of ω_x. Its preimages are the real solutions of the algebraic system obtained from (5) with $(x', y') = (p, 0)$, and it is easy to see that the preimages of the point X are either located on the same invariant axis $y = 0$ (in the points whose coordinates are the solutions of the equation $f(x) = p$, with f given in (6)) or on the curve of equation

$$x = \left[ky^{\beta}\left(\frac{\lambda B - \lambda y + 1}{\lambda y - 1}\right)\right]^{\frac{1}{\beta}}. \tag{7}$$

Analogously, the preimages of a point $Y = (0, q)$ of ω_y belong to the same invariant axis $x = 0$, in the points whose coordinates are the solutions of the equation $f(y) = q$, or lie on the curve of equation

$$y = \left[\frac{x^{\beta}}{k}\left(\frac{\lambda B - \lambda x + 1}{\lambda x - 1}\right)\right]^{\frac{1}{\beta}}. \tag{8}$$

These two curves intersect the axes in the points $0_{-1}^{(i)}$ and intersect each other in the point $0_{-1}^{(d)}$, located on the diagonal (see fig. 1). All points outside the region bounded by ω_x, ω_y, ω_x^{-1} and ω_y^{-1} cannot generate feasible trajectories.

This process can now be iterated: in general, the boundary of the feasible set is given by the union of all the preimages of ω_x and ω_y of any rank. However, as shown in the next subsection, as long as $\lambda B \leq 3$, the boundary of the feasible set has the simple shape shown in fig. 1. This is due to the fact, that the preimages ω_x^{-1} and ω_y^{-1} are entirely contained in a region where points have no preimages. To gain more information about these regions with different numbers of preimages, we have to introduce the concept of critical curves.

FIG. 1 APPROXIMATELY HERE

3.2 Critical Curves

If we consider a two-dimensional system (3), then the fact that the map T is single-valued does not imply the existence and the uniqueness of its inverse T^{-1}. For a given (x', y') the rank-1 preimage (or backward iterate) $(x, y) = T^{-1}(x', y')$, obtained by solving the system with respect to the unknowns x and y, may not exist or it may be multivalued. In other words, there might be several effort allocations of the two competitors leading to the same marketing expenditures in the following period, or there may be none. In such cases T is said to be a noninvertible map, and the plane can be subdivided into regions Z_k, $k \geq 0$, whose points have k distinct rank-1 preimages. As the point (x', y') varies in the plane $\mathbb{R}^2$, pairs of preimages appear or disappear as this point crosses

[3] A preimage of a point $P = (x_p, y_p)$ is a point $P_{-1} = (x, y)$ such that $T_s(x, y) = P$. A point P may have more than one preimages (or no preimages) which are obtained by solving the system, with respect to the unknowns x and y, for given values of x_p and y_p.

the boundaries which separate regions of different numbers of preimages. Hence, such boundaries are characterized by the presence of at least two coincident (merging) preimages. This leads to the definition of the critical curves, one of the distinguishing features of noninvertible maps. Following the notations of Gumowski and Mira (1980), Mira et al. (1996), Abraham et al. (1997), the *critical set* LC (from the French "Ligne Critique") is defined as the locus of points having two, or more, coincident rank-1 preimages, located on a set (*set of merging preimages*) called LC_{-1}. LC is the two-dimensional generalization of the notion of critical value (a local minimum or maximum value) of a one-dimensional map, LC_{-1} is the generalization of the notion of critical point (a local extremum point)[4]. Arcs of LC separate the regions of the plane characterized by a different number of real rank-1 preimages. The critical sets of rank k are the images of rank k of LC_{-1} denoted by $LC_{k-1} = T^k(LC_{-1}) = T^{k-1}(LC)$, LC_0 being LC. Points of LC_{-1} in which the map is differentiable are necessarily points where the Jacobian determinant vanishes: in any neighborhood of a point of LC_{-1} there are at least two distinct points which are mapped by T in the same point (near LC), hence the map is not locally invertible in these points. This implies, for a differentiable map T, that

$$LC_{-1} \subseteq J_0 = \left\{(x,y) \in \mathbb{R}^2 | \det DT(x,y) = 0\right\}. \tag{9}$$

For the symmetric model (5) the locus of points for which $\det DT(x,y) = 0$ is given by the union of two branches, denoted by $LC_{-1}^{(a)}$ and $LC_{-1}^{(b)}$ in fig. 2a. Also LC is the union of two branches, denoted by $LC^{(a)} = T(LC_{-1}^{(a)})$ and $LC^{(b)} = T(LC_{-1}^{(b)})$, see fig. 2b. The branch $LC^{(b)}$ separates the region Z_0, whose points have no preimages, from the region Z_2, whose points have two distinct rank-1 preimages. $LC^{(a)}$ separates the region Z_2 from Z_4, where the points in Z_4 have four distinct preimages. It is then said that (5) is a noninvertible map of $Z_4 - Z_2 - Z_0$ type. Using the critical curves it is now possible to understand why the feasible set has the simple shape as shown in fig. 1 as long as $\lambda B \leq 3$. The branch $LC^{(b)}$ intersects the axes in the points $((1+\lambda B)^2/4\lambda, 0)$ and $(0, (1+\lambda B)^2/4\lambda)$ respectively, where the value $(1+\lambda B)^2/4\lambda$ is obtained as the image of the critical point $1+\lambda B/2\lambda$ of the map f in (6). Recall, on the other hand, that $\omega_i, i = x, y$ intersect the axes in the points $0_{-1}^{(x)} = (\frac{1+\lambda B}{\lambda}, 0)$ and $0_{-1}^{(y)} = (0, \frac{1+\lambda B}{\lambda})$. These points - and in fact the whole segments $\omega_i, i = x, y$ - lie above $LC^{(b)}$ (and hence in the region Z_0) as long as $\lambda B \leq 3$.

The question naturally arises, what happens when $\lambda B > 3$? An answer is given in Bischi et al. (1998a) where the critical curves of the map (3) are used in order to study the global bifurcations that change the qualitative structure of the boundaries of the feasible set. In that paper it is shown that when a portion of the boundary of the feasible set crosses the critical curve LC passing from Z_0 to Z_2 or from Z_2 to Z_4, new portions of the boundaries are created, resulting in a fractal structure of the boundary.

FIG. 2 APPROXIMATELY HERE

3.3 Steady state effort allocations

We are now ready to get to the first main point in the paper. Our economic interest in studying systems like (3) and (5) is two-folded. First, we want to find out if there is something like a steady

[4] This terminology, and notation, originates from the notion of critical points as it is used in the classical works of Julia and Fatou. For the logistic map the critical point is $c_{-1} = 1/2$, and the critical value $c = h(c_{-1}) = \mu/4$.

state effort allocation, so that we can safely forget studying the transient phase, and investigate the steady states and their structure instead. Second, we are interested in the delimitation of the set of points which converge to it, to gain some insights on the robustness of the model and the dependence of the model's behavior on the initial effort allocations.

Inside the feasible set we described above, one or more attractors of the dynamical system, e.g. several fixed points, cycles of different periods, or more complex attractors, may exist. It can be shown that for the model (3) for $\beta_i \in (0,1)$[5], $i = 1, 2$, a fixed point

$$E^* = (x^*, B - x^*) . \tag{10}$$

exists inside the feasible set, where $x^* \in (0, B)$ is the unique positive solution of the equation $k^{\frac{1}{1-\beta_2}} x^{\frac{1-\beta_1}{1-\beta_2}} + x - B = 0$ (see Bischi et al., 1998a), and it is unique. A particularly simple solution is obtained in the case of homogeneous firms

$$E^* = \left(\frac{B}{2}, \frac{B}{2} \right) \tag{11}$$

Note that this steady state allocation belongs to the diagonal $\Delta = \{(x, y) \,|x = y\}$. This yields the sensible result for the symmetric case that two homogeneous firms competing in the same market split the market equally. In order to determine the local stability properties of the steady state allocation, we consider the Jacobian matrix, computed in a point on the diagonal, which becomes

$$DT(x, x; \lambda, B, \beta,) = \begin{bmatrix} 1 - 2\lambda x + \frac{\lambda B(\beta+2)}{4} & -\frac{\lambda B \beta}{4} \\ -\frac{\lambda B \beta}{4} & 1 - 2\lambda x + \frac{\lambda B(\beta+2)}{4} \end{bmatrix} \tag{12}$$

The eigenvalues are

$$\lambda_{||} = 1 + \frac{1}{2}\lambda B - 2\lambda x \text{ , with eigendirection along } \Delta; \tag{13}$$

$$\lambda_{\perp} = 1 + \frac{1}{2}\lambda B(1+\beta) - 2\lambda x \text{ , with eigendirection orthogonal to } \Delta. \tag{14}$$

It is easy to see that the steady state allocation E^* is locally asymptotically stable for $\lambda B < 4$. Furthermore, the results of the previous subsections enable us to say also something about the global behavior of the model. Numerical results indicate that all the trajectories with initial effort allocations inside the feasible set converge to E^* (i.e., there is no evidence of other attractors). Accordingly, we have a global stability result, which says that all the points inside the feasible set converge to E^* as long as $\lambda B < 4$. One further point deserves mentioning: recall that the feasible set has a simple shape only as long as $\lambda B < 3$. For $\lambda B > 3$ portions of the boundary of the feasible set cross the critical curve $LC^{(b)}$, passing from Z_0 to Z_2 (see the portions near the axes indicated the arrow in fig.3a). Hence, portions of the set of unfeasible points enter the region Z_2. That means that all the points belonging to these portions suddenly have two preimages instead of none. These preimages lie in regions with two and four preimages respectively, and lead to further preimages in these regions. This cascade of preimages lead to a fractal structure of the (boundary of the) feasible set, which can be clearly observed in the enlargement of fig.3b (only the region around the

[5] Only values of the response parameter belonging to this range are meaningful in applications, see Cooper and Nakanishi (1988).

y axis is enlarged). It is a rather surprising fact that the set of points which converge to the steady state effort allocation may have fractal boundaries. On the other hand, the segments ω_x and ω_y and their preimages ω_x^{-1} and ω_y^{-1} still give a rather good approximation of the feasible set.

FIG. 3 APPROXIMATELY HERE

4 Homogeneous firms and synchronization

If homogeneous firms characterized by identical parameters (4) are considered, the evolution of the effort allocations over time is given by (5). In this case it is easy to check that the diagonal Δ is invariant, i.e., $T_s(\Delta) \subseteq \Delta$: in a deterministic framework, identical firms starting with identical initial effort allocations behave identically over time. Formally, $x_0 = y_0$ implies $x_t = y_t$ for all $t \geq 0$. We call such trajectories, which are embedded into Δ, *synchronized trajectories*. The dynamics on the diagonal are governed by a one-dimensional dynamical system $s_{t+1} = f^d(s_t)$, where $f^d = T_s|_\Delta : \Delta \to \Delta$ is the restriction of the two-dimensional map T_s to the invariant submanifold Δ. For our model this restriction yields

$$f^d(s) = (1 + \frac{1}{2}\lambda B)s - \lambda s^2. \tag{15}$$

Again, this one-dimensional system exhibits the same dynamics as the standard logistic map $z = \mu z(1-z)$, with

$$\mu = 1 + \frac{1}{2}\lambda B \tag{16}$$

by the linear transformation $s = \frac{1+0.5\lambda B}{\lambda} z$. Note that the coordinates of the steady state effort allocation E^* are given by the fixed point of the map (15). This simpler model can be interpreted as the model of the so-called *representative agent*: it captures the dynamical behavior of both of the two homogeneous firms when the dynamics of these two firms are synchronized.

If the two homogeneous firms start out with different initial effort allocations, the question arises if the trajectories synchronize over time, i.e., if $|x_t - y_t| \to 0$ as $t \to +\infty$. In this case the initial difference between the marketing efforts of the two firms, $|x_0 - y_0,| > 0$, would cancel out in the long-run by the endogenous dynamics of the system, and the asymptotic behavior of the two homogeneous competitors is well-represented by the simpler one-dimensional model (15). If synchronization occurs within a reasonably short time span, we can safely ignore the transient dynamics of the two-dimensional system, and consider the model of the representative firm instead. If synchronization takes very long or does not occur at all, then the concept of the representative firm becomes meaningless. This leads to the second main point in our analysis: under which conditions do the trajectories of marketing efforts of identical competitors which start from different initial effort choices synchronize, and how does this depend on the difference $|x_0 - y_0|$ of the initial effort allocations? Starting from this question, we may then ask, if small heterogeneities between the two firms - a small mismatch of some of the parameters - matter for synchronization or not (see the next section). Answering these questions is not easy, since new dynamic phenomena may appear, especially when the one-dimensional model (15) exhibits chaotic behavior. In this case *chaotic synchronization* may occur, a phenomenon that has been extensively studied in the recent physical and mathematical literature(see e.g. Fujisaka and Yamada (1983), Pecora and Carrol(1990), Pikovsky and Grassberger (1991), Ashwin et al. (1994, 1996), Hasler et al. (1997), Maistrenko et al. (1998))[6]. Before we go on to study our model, we briefly introduce some (mathematical) definitions and notions and present some of the existing results.

4.1 Synchronization in symmetric dynamic models

The question of asymptotic synchronization of the marketing efforts x_t and y_t of the two-dimensional dynamical system (5), possessing a one-dimensional invariant submanifold, can be rephrased as follows. Let $A_s \subseteq \Delta$ be an attractor of the one-dimensional map (15): is it also an attractor of the two-dimensional map T_s? As pointed out above, an answer to this question is not immediate, because measure theoretic attractors, which are not stable according to the usual Lyapunov (or topological) definition, arise quite naturally in this context, and create the conditions for the occurrence of new kinds of dynamic phenomena and bifurcations. Obviously, an attractor A_s of the restriction f^d is stable with respect to perturbations along the invariant diagonal Δ. Accordingly, an answer to the question addressed above can only be given through a study of the stability of A_s with respect to perturbations *transverse* to Δ (*transverse stability*). If $A_s = \{x_1, ..., x_k\}$ is a stable k-periodic cycle of the map f^d then an answer to the question addressed above is very simple: in this case $\mathbf{A}_s = \{(x_1, x_1), ..., (x_k, x_k)\}$ is a k-cycle, embedded into the diagonal Δ, of the two-dimensional map T_s and, due to the symmetry of T_s, the Jacobian matrix computed in a point $(x, x) \in \Delta$ is symmetric, with eigenvalues

$$\begin{array}{l} \lambda_{||}(x), \text{ with eigenvector } \mathbf{v}_{||} = (1,1) \text{ along } \Delta \\ \lambda_{\perp}(x), \text{ with eigenvector } \mathbf{v}_{\perp} = (1,-1) \text{ orthogonal to } \Delta \end{array} \tag{17}$$

Therefore, the multipliers of the k-cycle $\mathbf{A}_s$ are

$$\lambda_{||}(\mathbf{A}_s) = \prod_{i=1}^{k} \lambda_{||}(x_i), \text{ with } \mathbf{v}_{||} = (1,1) \text{ and } \lambda_{\perp}(\mathbf{A}_s) = \prod_{i=1}^{k} \lambda_{\perp}(x_i), \text{ with } \mathbf{v}_{\perp} = (1,-1) \tag{18}$$

Since we assumed that $\mathbf{A}_s$ is attracting along the diagonal Δ, i.e. $\left|\lambda_{||}(\mathbf{A}_s)\right| \leq 1$, a sufficient condition for its asymptotic stability[7] in the two-dimensional phase space of the map T_s is $|\lambda_{\perp}(\mathbf{A}_s)| < 1$.

The situation becomes more complex when $\mathbf{A}_s$ is a chaotic attractor of f^d, i.e., when chaotic synchronization is considered. In this case results on the transverse stability are given in terms of the so-called *transverse Lyapunov exponents*

$$\Lambda_{\perp}(\mathbf{A}_s) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N} \ln |\lambda_{\perp}(x_i)| \tag{19}$$

where $\{x_i,\, i \geq 0\}$ denotes a trajectory embedded into $\mathbf{A}_s$. If x_0 belongs to a k-cycle then $\Lambda_{\perp} = \ln\left|\lambda_{\perp}^k\right|$, so that for each k-cycle embedded into $\mathbf{A}_s$ a particular of $\Lambda_{\perp}$ is obtained. In this case $\Lambda_{\perp} < 0$ if and only if $\left|\lambda_{\perp}^k\right| < 1$, that is, if the corresponding cycle is transversely stable. Instead,

[6] For a first application of these concepts to a dynamic Cournot duopoly game with boundedly rational agents, see Bischi et al. (1998b).

[7] By asymptotic stability we refer to the usual topological definition: (i) $\mathbf{A}_s$ must be Lyapunov stable, i.e., for every neighborhood U of $\mathbf{A}_s$ there exists a neighborhood V of $\mathbf{A}_s$ such that $T_s^t(V) \subset U\ \forall t \geq 0$ and (ii) for each $x \in V$ $T_s^t(x) \to \mathbf{A}_s$ as $t \to +\infty$ must hold.

if x_0 belongs to a generic aperiodic trajectory of $\mathbf{A}_s$ then $\Lambda_\perp$ is independent of x_0, provided that $\mathbf{A}_s$ is an ergodic chaotic attractor, with absolutely continuous invariant measure. In this case $\Lambda_\perp$ is called *natural transverse Lyapunov exponent*[8], denoted by $\Lambda_\perp^{nat}$. Since infinitely many cycles, all unstable along Δ, are embedded inside a chaotic attractor $\mathbf{A}_s$, a spectrum of transverse Lyapunov exponents can be defined, see Buescu (1997)

$$\Lambda_\perp^{\min} \leq .... \leq \Lambda_\perp^{nat} \leq ... \leq \Lambda_\perp^{\max} \tag{20}$$

The meaning of the inequalities in (20) can be intuitively understood on the basis of the property that a chaotic attractor $\mathbf{A}_s$ includes within itself infinitely many periodic orbits which are unstable in the direction along Δ, and $\Lambda_\perp^{nat}$ expresses a sort of "weighted balance" (see Nagai and Lai 1997) between transversely stable cycles (characterized by $\Lambda_\perp < 0$) and transversely unstable ones (characterized by $\Lambda_\perp > 0$).

The one-dimensional chaotic invariant set $\mathbf{A}_s \subseteq \Delta$ is asymptotically stable (in the usual topological sense) for the two-dimensional dynamical system if all the cycles embedded in it are transversely stable (or, equivalently, if $\Lambda_\perp^{\max}$ <0). However, it may occur that some cycles embedded into the chaotic set $\mathbf{A}_s$ become transversely repelling ($\Lambda_\perp^{\max}$ >0) even if the *natural transverse Lyapunov exponent* $\Lambda_\perp^{nat}$ is still negative; this is due to the presence of many other transversely attracting orbits embedded inside A_s. In this case $\mathbf{A}_s$ is no longer a Lyapunov attractor: a two-dimensional neighborhood U of A_s exists such that in any neighborhood $V \subset U$ there are points (really a set of points of positive measure) that exit U after a finite number of iterations. However, $\mathbf{A}_s$ continues to be attracting "on average". More precisely, it is an attractor in Milnor sense (see Milnor 1985, Ashwin et al. 1996), which means that it attracts a set of points of the two-dimensional phase space of positive (Lebesgue) measure. The transitions between the two different situations, as some parameter is changed, define new kinds of local bifurcations. The change from asymptotic stability to attractiveness only in Milnor sense, occurring when $\Lambda_\perp^{\max}$ becomes positive, is denoted as *riddling bifurcation* in Lai, Grebogi and Yorke (1996) or *bubbling bifurcation* in Ashwin et al. (1994) and in Venkataramani (1996). Furthermore, when also $\Lambda_\perp^{nat}$ becomes positive, due to the fact that the transversely unstable periodic orbits embedded into A_s have a greater weight with respect to the transversely attracting ones (see Nagai and Lai 1997) a so-called *blowout bifurcation* occurs, at which a Milnor attractor becomes a chaotic saddle. In what follows we will mainly focus on the local and global phenomena occurring after riddling and before blowout bifurcations, that is, at a range of parameters in which a non topological Milnor attractor exists. Note that even if the occurrence of riddling and blowout bifurcations is detected through the transverse Lyapunov exponents, that is, by a local analysis of the linear approximation of the map near Δ , their effects are determined by the global properties of the map. The fate of the locally repelled trajectories is determined by the nonlinearities acting far from the diagonal. In fact, in such a situation, two possible scenarios can be observed depending on the evolution of the trajectories that are locally repelled along (or near) the local unstable manifolds of the transversely repelling cycles:

(**L**) the trajectories may be folded back towards Δ by the action of the nonlinearities acting far from Δ, so that the dynamics are characterized by some bursts far from Δ before the trajectories synchronize on the diagonal (a very long sequence of such bursts, which can be observed when $\Lambda_\perp^{nat}$ is close to zero, has been called *on-off intermittency* in Ott and Sommerer 1994);

[8]By the term "natural Lyapunov exponent" we mean the Lyapunov exponent associated with the natural (or SBR) measure, computed for a typical trajectory along the chaotic attractor A_s.

(**G**) the trajectories may belong to the basin of another attractor, in which case the phenomenon of *riddled basins* is obtained (see Alexander et al. 1992).

The distinction between the two different scenarios (**L**) and (**G**) described above depends on the global properties of the dynamical system[9]. The global dynamical properties can be usefully studied by the method of *critical curves*, which we introduced above. The reinjection of the locally repelled trajectories occurring in local riddling may be described in terms of their folding action[10]. This idea has been recently proposed in Bischi et al. (1998b) for the study of symmetric maps arising in game theory, and in Bischi et al. (1998c) for the study of the effects of small asymmetries due to mismatches of the parameters. In these two papers the geometric properties of the critical curves have been used to obtain the boundary of a compact trapping region, called *absorbing area* (see Mira et al. 1996a), inside which intermittency and blowout phenomena are confined. In other words, the critical curves are used to bound a compact region of the phase plane that acts as a trapping bounded vessel inside which the trajectories starting near the diagonal are confined. For further details on the concept of *minimal* and *invariant* absorbing area and its use to give a global characterization of the different dynamical scenarios, see Bischi and Gardini (1998).

4.2 Synchronization and synchronization failure of homogeneous firms

After these preparations we can now turn back to our model (5). Recall that the steady state allocation E^* is locally asymptotically stable for $\lambda B < 4$. In what follows we are more interested in the situation when E^* is unstable and we investigate the question of synchronization of the marketing efforts of the two competing firms. The point E^* loses stability along Δ (via a so-called period doubling bifurcation) at $\lambda B = 4$. For $\lambda B > 4$ and $1 - \frac{4}{\lambda B} < \beta < 1$ it is a saddle point, with unstable set along Δ and stable set transverse to it. At $\beta = 1 - \frac{4}{\lambda B}$ it also loses transverse stability (again via a period doubling bifurcation) that creates a stable cycle of period 2 out of the diagonal, with periodic points located symmetrically with respect to Δ. In order to determine the transverse stability of a trajectory $\{x_n, x_n\} \in \Delta$ we consider the transverse Lyapunov exponent for the map (5), readily obtained from (19) with (14):

$$\Lambda_{\perp}^{(nat)} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} \ln \left| 1 + \frac{1}{2}\lambda B(1+\beta) - 2\lambda x_n \right| .$$

It is important to note that in our case only the orthogonal eigenvalue (14) depends on the response parameter β, i.e. β is a *normal parameter*: it has no influence on the dynamical properties of the restriction along the invariant submanifold Δ, and only influences the transverse stability[11]. This feature enables us to consider a certain attractor along the diagonal and observe for which values of β the evolution of marketing efforts of the two firms synchronize or not, and which kind of transient phenomena occur. Recall that β is also a measure of the degree of competition in the market we try to capture with our model and, accordingly, we can determine how the degree of competition affects the dynamical properties, in particular, the synchronization properties, of the model (5). Will the trajectories synchronize for lower or higher degrees of competition between the two firms? Is the

[9] The term "global" refers in this context to "not in a neighborhood of the diagonal Δ".

[10] See Mira et al. (1996a) or Mira et al. (1996b) for a description of the geometric properties of a noninvertible map related to the folding (or foliation) of its phase space.

[11] This is a typical property of coupling parameters in symmetrically coupled maps; see Buescu (1997), Maistrenko et al. (1998), Hasler and Maistrenko (1997).

relation between the degree of competition and synchronization properties of the system monotone in the sense that higher/lower values of β lead to synchronization/synchronization failure?

We now consider fixed values of the parameters λ and B, such that a chaotic attractor $\mathbf{A}_s \subseteq \Delta$ of the map (15) exists, with absolutely continuous invariant measure on $\mathbf{A}_s$, and we study the transverse stability of $\mathbf{A}_s$ as the degree of competition between the two firms, measured by the parameter β, varies. Suitable values of the aggregate parameter λB, at which chaotic intervals for the restriction (15) exist, are obtained from the relation (16)[12]. In the examples given below we let $\overline{\mu}_2 = 3.5748049387592....$ Using (16) this yields $\lambda B = 2(\overline{\mu}_2 - 1)$, and the attractor $\mathbf{A}_s$ along the diagonal Δ is in this case a four-band chaotic set[13]. Figure 4 shows the result of the computation of the natural transverse Lyapunov exponent $\Lambda_\perp^{nat}$ as β varies in the interval $(0, 0.2)$, where we chose the interval for the values of the response parameter to be line with empirical evidence; see, e.g., Bultez and Naert, (1975). Observe that in fig. 4 a "window" of negative values of $\Lambda_\perp^{nat}$ is visible for $0.0575... < \beta < 0.1895....$

FIG. 4 APPROXIMATELY HERE

Before discussing the effects of the changes of the sign $\Lambda_\perp^{nat}$, we first show, by numerical simulation, that for very small degrees of the competition β the evolution of the marketing effort x_t and y_t of two firms appear to be totally uncorrelated over time. This is no surprise since for $\beta = 0$ the payoffs of the two firms only depend on the firm's own marketing effort and, hence, the firms act independently of each other. Fig. 5 has been obtained with the same parameters λ, B, and k as those used in fig.4, and we set $\beta = 0.0001$. Due to the particular value of the parameter $\lambda = \lambda_* = 2(\overline{\mu}_2 - 1)/B = 0.5149609877518401...$ both of the effort time series exhibit a chaotic pattern, as shown in fig. 5a, where the early 300 values of x_t and y_t are represented versus time. The initial effort allocation is $(x_0, y_0) = (5, 5.001)$: the firms start out with almost identical initial marketing efforts and very close to the steady state allocation $E^* = (5, 5)$. Nevertheless, no synchronization takes place. Moreover, the two time patterns are totally uncorrelated, as shown by the graph of fig. 5b, where the difference between the marketing efforts of the competitors, $(x_t - y_t)$, is represented versus time. It is evident that after a very short transient (approximately 20 iterations) the difference between the two variables is of the same order of magnitude as the single variables, even if they are identical and start from quasi-identical initial choices.

FIG. 5 APPROXIMATELY HERE

A quite different situation is obtained for slightly higher degrees of competition β, where the natural transverse Lyapunov exponent $\Lambda_\perp^{nat}$ is negative. For example, for $\beta = 0.09$ we have $\Lambda_\perp^{nat} = -8.36 \times 10^{-2} < 0$ (see fig. 4), and we expect that synchronization of the marketing efforts of the two firms occurs for a set of initial conditions of positive Lebesgue measure (this implies that trajectories that synchronize, even starting out of the diagonal, can be numerically observed). The

[12] Recall that the mathematical properties of the logistic map are well understood. Hence, we can use these results if we take into account that the relation between the parameter values and the state variables of the two systems is as described by the equations above.

[13] At the parameter value $\overline{\mu}_2$ the period-4 cycle of the quadratic map undergoes the first homoclinic bifurcation, and four cyclic chaotic intervals are obtained by the merging of 8 cyclic chaotic intervals.

issue of synchronization gets more complex in this case, however, because for this values of the parameters two coexisting attractors inside the feasible set can be numerically observed: the 4-cyclic chaotic set $\mathbf{A}_s \subset \Delta$ and an attracting cycle of period 2 with periodic points located out of Δ. The cycle $C_2 = ((5.975, 3.371), (3.371, 5.975))$ has been created at $\beta = 1 - \frac{4}{\lambda B} = 0.22$, via a (period doubling) bifurcation of E^* in the transverse direction as explained above. In fig. 6 the coexisting attractors are represented by black points, each with its own basin of attraction: the white points represent the basin $\mathcal{B}(\mathbf{A}_s)$ of the points generating trajectories that synchronize along $\mathbf{A}_s$, whereas the light grey points represent the basin $\mathcal{B}(C_2)$ whose points generate trajectories converging to the stable cycle C_2. The dark-grey region represents the set of points which are not feasible, i.e., which generate unfeasible trajectories. Observe that the issue of synchronization becomes quite complicated now without having any knowledge of the global behavior of the model (5). If we do not have fig. 6 available, it is hard to predict from which initial effort allocations synchronized marketing efforts over time are obtained and for which initial outlays marketing efforts would (two-) cycle. Hence, it is hard to decide when the lower-dimensional model of a representative firm would be a reasonable substitute for the higher-dimensional model (5), and when such a substitution would be misleading. It is interesting to note that (long-run) synchronization of the marketing efforts can also occur starting from initial allocations located very far from the diagonal. In other words, even starting from fairly heterogeneous choices of the two identical competitors, the firms may end up with perfectly synchronized marketing efforts over time, if the initial allocations happen to lie in the white region in fig. 6. On the other hand, and quite counter to one's intuition, even if the initial effort allocations are very close to the diagonal, i.e., $x_0 \cong y_0$, they may not synchronize because they generate trajectories converging to the cycle C_2. Actually, in this case the evolution of marketing efforts exhibits conditions of asynchronous behavior (*phase opposition* between the choices of the two competitors). The reason for this *synchronization failure* is that near the steady state effort allocation E^*, and its preimages along Δ, there are "tongues" formed by initial outlays such that the corresponding trajectory converges to the cycle of period 2. Another important feature to notice is the complex structure of the boundaries that separate $\mathcal{B}(\mathbf{A}_s)$ from $\mathcal{B}(C_2)$. In particular, $\mathcal{B}(\mathbf{A}_s)$ is a non connected set with a fractal structure (self-similarity), a situation which is peculiar of dynamical systems represented by noninvertible maps. Although the feasible set still has a shape similar to the one obtained for $\lambda B < 3$, inside the feasible set we now have two coexisting attractors and, accordingly, two situations might arise - synchronization or synchronization failure of the marketing efforts of the two firms - depending on the fact if initial allocations are chosen from the white or the grey region. This feature is only revealed if we look at the global properties of the system.

For the set of parameters used in preparing fig. 6 the four-band chaotic set $\mathbf{A}_s$, embedded into the invariant diagonal Δ, is not a topological attractor however. In fact, an 8-cycle C_8 embedded inside the diagonal exists, which is transversely repelling[14]. This means that trajectories starting along the local unstable set $W_{\perp}^{u}(C_8)$, issuing from the periodic points of C_8, as well as those starting from narrow tongues along $W_{\perp}^{u}(C_8)$ and from all the infinitely many preimages of the periodic points of C_8 (such preimages are densely distributed along $\mathbf{A}_s$ due to the fact that $\mathbf{A}_s$ is a chaotic set with absolutely continuous invariant measure) are repelled away from the diagonal. These locally repelled trajectories are then folded back by the action of the global dynamical properties of the map (5), and after a transient with some bursts away from Δ occurring, they synchronize in the long-

[14] The 8-cycle is $C_8 = (5.588, 3.894, 6.112, 2.612, 5.824, 3.352, 6.197, 2.378)$ of the map (15) and it has the transverse multiplier $\lambda_{\perp}(C_8) = -3.0$, as can be easily computed from (18) with (14).

run. The time evolution of the difference of the marketing efforts, $(x_t - y_t)$, during the transient portion of a typical trajectory, starting from the initial allocations $(x_0, y_0) = (6, 6.01)$, is shown in fig. 7, where the early 300 iterates are represented. After about 40 periods the evolution of the system seems to have reached almost complete synchronization. During the next 40 periods the two competitors behave practically in the same way. At this point the trajectory seems to have definitively settled down on the attractor $\mathbf{A}_s$ (this would be the case for a topological attractor), and we would tend to conclude that the two-player-model can be replaced by a one-player-model. However, the trajectory then moves again far away from the diagonal, and the two competitors now act again in a very different fashion. Several bursts of the trajectory, out of Δ, are observed until perfect synchronization of the marketing efforts is eventually obtained. Such an intermittent behavior is a typical characteristic of the convergence to a non-topological Milnor attractor. The pattern of the time series resembles that of a system which is subject to exogenous random shocks, even if the dynamical system that generates such a pattern is completely deterministic. This peculiar dynamical behavior is related to the fact that even if the Milnor attractor attracts "on average" according to the fact that $\Lambda_{\perp}^{nat} < 0$, the presence of some transversely repelling cycles (even if less influent than the transversely attracting ones) causes sudden bursts when the trajectories happen to get close to them.

FIGURES 6 and 7 APPROXIMATELY HERE

The locally repelled trajectories cannot reach the other attractor C_2 however, i.e., the scenario (**L**) of locally riddling (or intermittency) occurs. This is due to the presence of a so-called absorbing area $\mathcal{A}$ around $\mathbf{A}_s$, from which the trajectory starting close to $\mathbf{A}_s$ cannot escape[15]. We briefly describe now how the boundary $\partial\mathcal{A}$ of such an absorbing area can be easily obtained (see Bischi and Gardini 1998 for more details). The boundaries of the region in which the asymptotic dynamics are confined (absorbing and chaotic areas) can be obtained by segments of critical curves and their iterates. It can be used to obtain minimal and invariant absorbing areas which include the Milnor attractor where chaotic synchronization takes place. A practical procedure to obtain the boundary of an absorbing area makes use of the concept of critical curves and can be outlined as follows: starting from a portion of LC_{-1}, approximately taken in the region occupied by the area of interest, its images by T_s of increasing rank are computed until a closed region is obtained. When such a region is mapped into itself, then it is an absorbing area $\mathcal{A}$. The length of the initial segment must be taken, in general, by a trial and error method, although several suggestions are given in the books referenced above. Once an absorbing area $\mathcal{A}$ is found, in order to see if it is invariant or not, the same procedure must be repeated by taking only the portion

$$\gamma = \mathcal{A} \cap LC_{-1} \tag{21}$$

as the starting segment. In order to obtain the boundary of the absorbing area $\mathcal{A}$ shown in fig. 8, six images of the generating arc $\gamma = \mathcal{A} \cap LC_{-1}$ are sufficient. However, only the portion of γ belonging to the branch $LC_{-1}^{(b)}$ has been used because the images of the other portion, the one

[15] An absorbing area $\mathcal{A}$ is a bounded region of the plane, whose boundary is given by critical curves segments of finite rank (segments of the critical curve LC and its images), such that the successive images of the points of a neighborhood of $\mathcal{A}$, say $\mathcal{U}(\mathcal{A})$, enter inside $\mathcal{A}$ after a finite number of iterations, and never exit, being $T(\mathcal{A}) \subseteq \mathcal{A}$. See, e.g., Gumowski and Mira (1980), Mira et al. (1996), Abraham et al. (1997).

belonging to the upper branch $LC_{-1}^{(a)}$, are always inside the absorbing area, so that they do not form part of the boundary. Hence in fig. 8 we have $\gamma = \mathcal{A} \cap LC_{-1}^{(b)}$ and $\partial\mathcal{A} \subset \bigcup_{k=1}^{6} T^k(\gamma)$.

We remark that $\mathcal{A}$ includes the Milnor chaotic attractor $\mathbf{A}_s \subset \Delta$ (see fig. 6), and all the trajectories starting from a neighborhood of $\mathbf{A}_s$ cannot go out of $\mathcal{A}$. Loosely speaking $\partial\mathcal{A}$ behaves as a bounded vessel for the intermittency phenomena related to the presence of the transversely repelling cycles embedded inside $\mathbf{A}_s$. The local unstable sets of these cycles are folded back (reinjected) by the folding action of the critical curves that form the $\partial\mathcal{A}$. A similar transient behavior is observed with lower values of the degree of competition β such that $\Lambda_{\perp}^{nat} < 0$. The only difference is that the absorbing area is smaller (so that the bursts are of smaller amplitude) and longer transients, characterized by intermittency, are observed before the marketing efforts of the two firms synchronize along the diagonal. This is due to the fact that for values of $\Lambda_{\perp}^{nat}$ closer to zero (but negative) the influence of the transversely repelling cycles is stronger and, consequently, the bursts are more frequent and persist longer before the trajectories are eventually captured by $\mathbf{A}_s$ in the long run.

The bottom-line of the investigation so far is this, given the initial allocations are in $\mathcal{B}(\mathbf{A}_s)$: first, the size of the absorbing area containing the Milnor attractor $\mathbf{A}_s$ gives us an idea of the maximal difference between the marketing efforts of the two firms. Second, there is an inverse relationship between the longevity of transients and the values of the natural Lyapunov exponent $\Lambda_{\perp}^{nat}$. For values of the degree of competition β for which $\Lambda_{\perp}^{nat}$ is strongly negative, the absorbing area is large (and, hence, the possible difference between the marketing efforts is large), but the transient phase where bursts occur before the trajectories of marketing efforts settle down along the diagonal is relatively short. Neglecting this relatively short transient period we can conclude that the model of the representative player is a good approximation. On the other hand, if $\Lambda_{\perp}^{nat}$ is close to zero but negative, then the transient phase is rather long. Frequent and persistent bursts occur before the marketing efforts of the competitors synchronize. However, in this case the absorbing area is (very) small, which means that the difference between the marketing effort is (very) small. Neglecting this small difference, again we can conclude that the model of a representative player is a good approximation even in the transient phase. It might seem that this justifies the assumption often made in economic and game theory models, where for analytical convenience it is often assumed that firms are homogeneous. Our analysis so far has shown that even if we consider a dynamic promotional competition model there is either only a relatively short transient before the firms behave in a similar way, or the difference between the choices of the two competitors in the transient phase (which might be long) is negligibly small. Of course, we still have to assume that the initial effort allocations are located inside the basin of attraction of the Milnor attractor.

FIG. 8 APPROXIMATELY HERE

5 Quasi-Homogeneous Firms and Symmetry Breaking

In the previous subsection we made the very restrictive assumption that the firms' structural parameters are the same and the difference between the competitors lies only in their initial choices of the effort allocations. Although synchronization does not necessarily occur for all initial effort allocations (namely those in the grey region) we can determine for which initial marketing outlays the two-dimensional model (5) can be substituted by the model of a representative player. One question, however, raised in the Introduction, has not been answered yet: Is the assumption of homogeneous firms which is so predominant in the literature an innocuous one? Or do small heterogeneities matter. This is the topic we will now turn to. If a *small heterogeneity* due to a small parameter mismatch is introduced, additional interesting phenomenons occur. Let us assume, for example, that there is a small difference between the two response parameters β_1 and β_2 of the two competitors in the model (3), that is

$$\lambda_1 = \lambda_2 = \lambda; \; k = 1; \text{ and } \beta_2 = \beta_1 + \varepsilon \tag{22}$$

where ε is small with respect to β_1, i.e. $\varepsilon/\beta_1 \ll 1$. Such an assumption should not invalidate the conclusions made in the previous subsection unless these conclusions where only valid under the restrictive assumptions that these results only hold for parameter values which are *exactly* equal. If this would be the case, then these results are not robust to small parameter perturbations and can be questioned on empirical grounds. For any practical purpose we have to make sure that the insights derived from the symmetric model carry over to the model with slightly perturbed parameter values in order to show the robustness of our findings. Unfortunately, as it will turn out, in general the symmetric model does not give rise to a generic behavior. That is, if a small heterogeneity is introduced into the model (3) the evolution of the marketing efforts of the two firms over time may be quite different.

Note, first of all, that such a mismatch of structural parameters causes the destruction of the invariance of Δ, due to the fact that the map is no longer symmetric (this kind of perturbation has been called *symmetry breaking* in Bischi et al. (1998c)). The fact that the diagonal is no longer an invariant set also causes the disappearance of the one-dimensional Milnor attractor $\mathbf{A}_s$ along the diagonal. In effect, such a small perturbation may lead to quite different dynamics, since after the symmetry breaking synchronization can no longer occur, and the bursts never stop. The generic trajectory fills up the absorbing area, which now appears to be a two-dimensional chaotic area. Figure 9a is obtained after the introduction of a very small difference between the response parameters of the firms with respect to the set of parameters used in figures 6, 7 and 8: $\beta_2 = 0.09001$ ($\varepsilon = 0.00001$). The evolution of the system (3) starting from the initial effort allocation $(x_0, y_0) = (3.5, 3.5) \in \Delta$, i.e., from homogeneous initial choices, is represented in the phase space (x, y). As in the homogeneous case we have two coexisting attractors, but the two attractors are now a two-dimensional chaotic area and the cycle C_2. In other words, after an apparently negligible heterogeneity has been introduced, the dynamical behavior of the resulting model is quite different: the Milnor chaotic attractor on which asymptotic synchronization occurs is replaced by a two-dimensional chaotic attractor on which on-off intermittency occurs, i.e., bursts never stop. This is clearly visible in fig. 9b, where the difference of the marketing efforts over time, $(x_t - y_t)$, is represented over 10000 periods. It is evident that long time intervals exist in which the two firms show quasi-synchronized behavior, but in-between such intervals asynchronous behavior emerges with an apparently random pattern. As suggested in Bischi et al. (1998c), if the attractor $\mathbf{A}_s$ embedded in the diagonal in the symmetric case is a topological attractor, i.e., no transverse repelling cycles exist, then the introduction of small heterogeneities does not have such a disruptive effect. In this case the symmetric model still serves as a good approximation of the behavior of the two firms.

From an economic point of view, the results of this section make us aware how restrictive the assumptions made in (or almost throughout) the literature are. If the assumption of homogeneity is made for analytical tractability, we should be aware that we solve the model for a very special case. The reason is that for dynamic models the symmetric case is often non-generic, i.e., it exhibits a behavior which is quite different from the model with heterogeneous agents. On the other hand, parameter regions may exist, where the assumptions of homogeneity does not matter at all (see the last remark in the previous paragraph). If the attractor of the symmetric model is a topological attractor, i.e., if all the cycles embedded into the diagonal are attractive, then even after the introduction of a small heterogeneity the evolution of the (now asymmetric) model would still lead to almost perfectly synchronized trajectories. In other words, model builders have to be aware when the assumption of homogeneous players is justified and when it is not. For certain ranges of the structural parameters this assumption might be sensible and valid, whereas for other regions it might be simply wrong and misleading. Assuming homogeneity among all players would in this case give a wrong idea of the variety of dynamical phenomena which can be observed for given model.

FIG. 9 APPROXIMATELY HERE

6 Concluding remarks and further developments

So far we have presented two main ideas. First, we have demonstrated that by using the concept of critical curves and segments on the invariant coordinate axes we can determine the feasible set and the changes of it as some parameters are varied. This gives us the opportunity to derive global stability results, which tells us something about the conditions under which convergence to a steady state allocation is achieved and for which set of initial allocations. Second, we have argued that the assumption of homogeneity, which is so often made in the literature, may lead to wrong conclusions about the resulting dynamical behavior of a model for certain values of the models' parameters.

However, the study of dynamical phenomena of the symmetric model can be continued. As the degree of competition β spans the whole interval $(0,1)$ other global bifurcations can be evidenced that cause strong qualitative changes of the structure of the set of initial condition which generate trajectories that synchronize. This will be object of further researches, and we just give here a brief description of some phenomena that occur as β is further increased. For slightly increased values of the degree of competition β (with respect to the value $\beta = 0.09$), a transition from the scenario (**L**) of locally riddling to the scenario (**G**) of globally riddling is observed. This occurs because the absorbing area $\mathcal{A}$ that included the Milnor chaotic attractor $\mathbf{A}_s$ where synchronization takes place (fig. 8) becomes larger as β increases, so that finally it has a contact with the boundary that separates $\mathcal{B}(\mathbf{A}_s)$ from $\mathcal{B}(C_2)$. After this contact the absorbing area $\mathcal{A}$ is destroyed, and some trajectories that are locally repelled from $\mathbf{A}_s$ can reach the basin $\mathcal{B}(C_2)$. This leads to a situation of additional uncertainty about the fate of a trajectory starting from a given initial effort allocation, due to the creation of a riddled basin. Given an initial allocation in the feasible set, the evolution of the system may lead to the 2-cycle or it may synchronize, converging to $\mathbf{A}_s$ after a short[16] transient with intermittent behavior. Since the model is deterministic, the fate of the trajectory of marketing efforts is uniquely determined by the initial allocation, but due to the riddled structure of $\mathcal{B}(\mathbf{A}_s)$ the presence of arbitrarily small perturbations or of arbitrarily small errors in measuring the initial outlays makes it practically impossible to forecast the long-run behavior. This would also

[16] Given an initial allocation which belongs to $\mathcal{B}(A_s)$, the transient before synchronization is shorter with respect to that obtain for $\beta = 0.09$ because the natural Lyapunov exponent is smaller.

mean that it is practically impossible to decide, when the model of a representative agent might be used to replace the two-dimensional model. We can say that the contact bifurcation described in this situation marks a transition from a situation in which the model allows to make reasonable forecasts about the long-run behavior of the system (fig. 6) to a situation of complete uncertainty, in which a forecast of the long-run synchronization behavior is impossible. We remark that such a strong qualitative change in the predictability of the time evolution of the system is not related to local properties of the system near the invariant diagonal (i.e., the Lyapunov exponent) but it is due to a global bifurcation, a contact occurring far from Δ.

If the natural Lyapunov exponent $\Lambda_{\perp}^{nat}$ changes sign from negative to positive, the chaotic set $\mathbf{A}_s$ becomes a chaotic saddle, i.e., $\mathcal{B}(\mathbf{A}_s)$ has zero measure. This means that the probability for emergence of synchronized behavior is zero, even when the homogeneous firms are starting out with initial allocations arbitrarily close to the diagonal, i.e., $x_0 \cong y_0$. Trajectories after the so-called blowout bifurcation fill up a large chaotic area. Again, this chaotic area is bounded by critical curves: it is the minimal invariant absorbing area that already existed around the Milnor attractor before the occurrence of the blowout bifurcation. However, even if the marketing efforts of the two firms never synchronize, their behavior is not totally uncorrelated. The trajectories remain close to the line of equal choices Δ quite often, that is, the probability that the decisions made by the two firms are similar is higher with respect to a totally uncorrelated competitive system[17].

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[17]From a dynamic point of view this is due to the fact that even if $\mathbf{A}_s$ is now a chaotic saddle, i.e., transversely repelling "on average", infinitely many transversely stable cycles still exist embedded into it. $\mathbf{A}_s$ is a chaotic saddle, but not a normally repelling chaotic saddle.

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Figure captions

Fig. 1: The boundary of the feasible set is given by the segments $\omega_x = 00_{-1}^{(x)}$ and $\omega_y = 00_{-1}^{(y)}$ on the invariant axes, and their rank-1 preimages ω_x^{-1} and ω_y^{-1}. The curves on which the preimages ω_x^{-1} and ω_y^{-1} are located intersect the diagonal in the point $0_{-1}^{(d)}$.

Fig. 2: (a) Critical curves of rank-0, obtained as the locus of points where the Jacobian $det(DT_s(x, y)) = 0$. (b) Critical curves of rank-1, obtained as $LC = T_s(LC_{-1})$. These curves separate the plane into three distinct regions: Z_0, Z_2, Z_4, whose points have no, two, and four rank-1 preimages respectively. For this parameter setting the preimages ω_x^{-1} and ω_y^{-1} are competely contained in the region Z_0, hence they have no further preimages.

Fig. 3: (a) Feasible set for $3 < \lambda B < 4$. The steady state allocation E^* is asymptotically stable. Numerical evidence suggests that all the feasible trajectories converge to it. The feasible set has fractal boundaries due to the fact that a portion of the unfeasible set, indicated by the arrow, entered the region Z_2. (b) Enlargement of (a), obtained by zooming along the x axis of a factor 20, in order to see the fractal structure of the boundary of the feasible set near ω_y. Similar structures also exist along the other portions of the boundaries, i.e. along ω_x, ω_x^{-1} and ω_y^{-1}.

Fig. 4: Natural Lyapunov Exponent $\Lambda_{\perp}^{(nat)}$ as a function of the degree of competition β, with β ranging from 0 to 0.2, $B = 10, \lambda = 0.5149609877518401$... Each point is obtained by iterating the map (starting from an initial condition along the diagonal) 10.000 times to eliminate transient behavior, and then averaging, according to (19) over another 100.000 iterations.

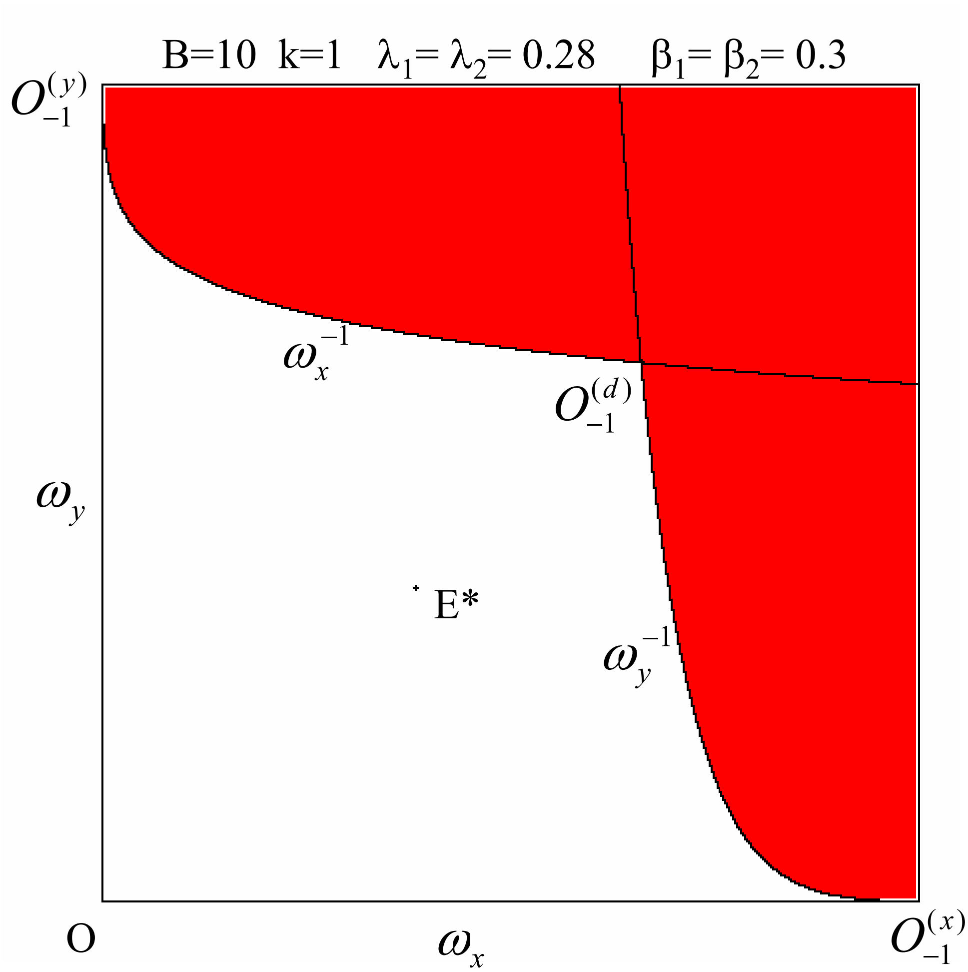
Fig. 5: (a) Effort allocations of the firm 1 and 2 respectively over time, shown for the first 300 periods. (b) Difference between the marketing efforts of the two firms for small values of the degree of competition. The decisions of the two firms seem to be independent of each other.

Fig. 6: Two coexisting attractors in the feasible set, a Milnor attractor along the diagonal $\triangle$ and a stable period two cycle symmetric with respect to $\triangle$. The white regions indicate points which converge to the Milnor attractor, whereas points in the light-grey region converge to the two-cycle.

Fig. 7: Typical trajectory converging to a Milnor attrator along the diagonal. Before the marketing efforts of the two firms finally synchronize (i.e. $x_t - y_t = 0$), several bursts can be observed in the transient phase.

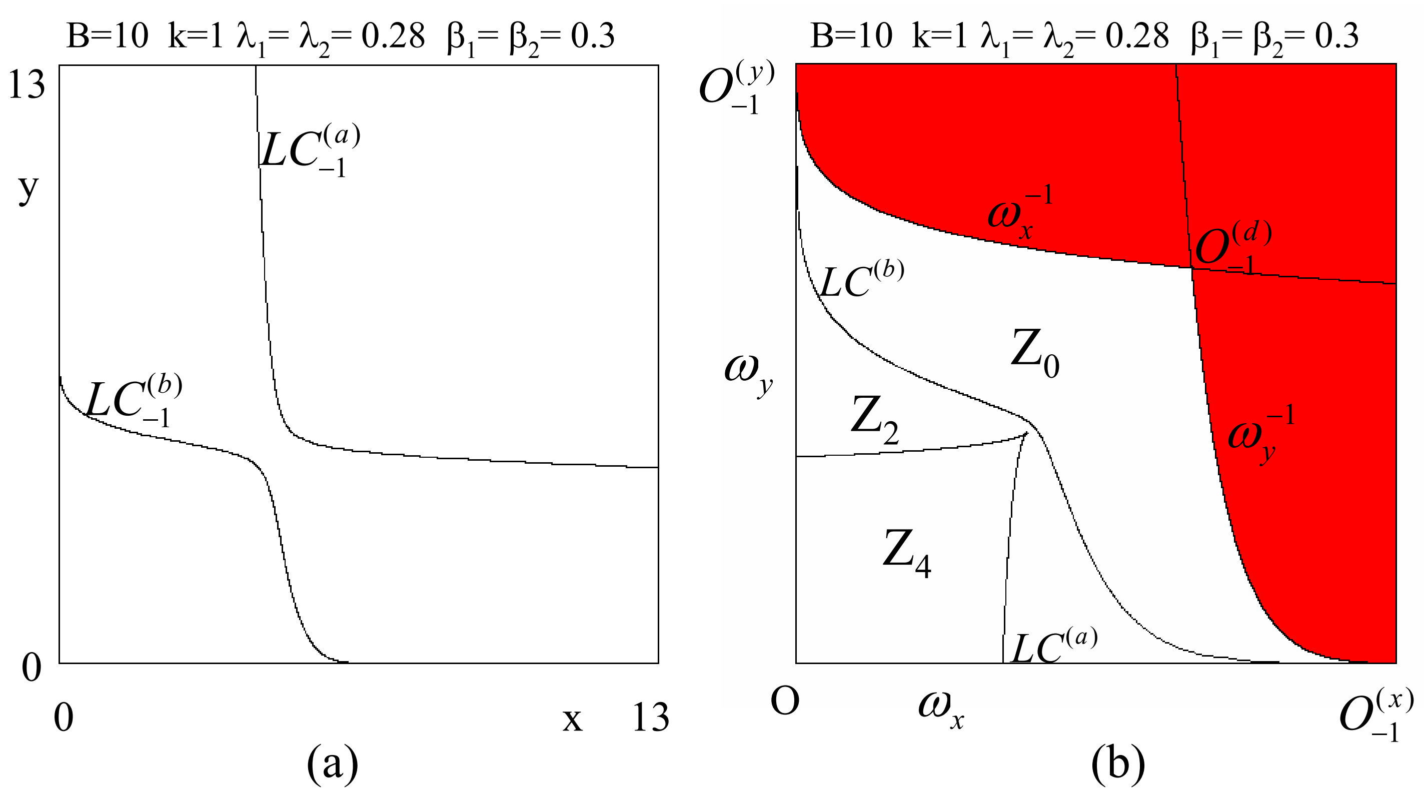
Fig. 8: Boundary of the absorbing are around the Milnor attractor along the diagonal, obtained by arcs of critical curves: $L = T(\gamma)$, with $\gamma \in LC_{-1}$, $L_1 = T(L)$,...$L_k = T(L_{k-1})$, with $k = 2, ..., 5$.

Fig. 9: The introduction of a small heterogeneity in the form of a mismatch of the parameters causes the disappearance of the Milnor attractor along the diagonal; symmetry breaking occurs. Marketing efforts of the two firms no longer synchronize and bursts never stop.



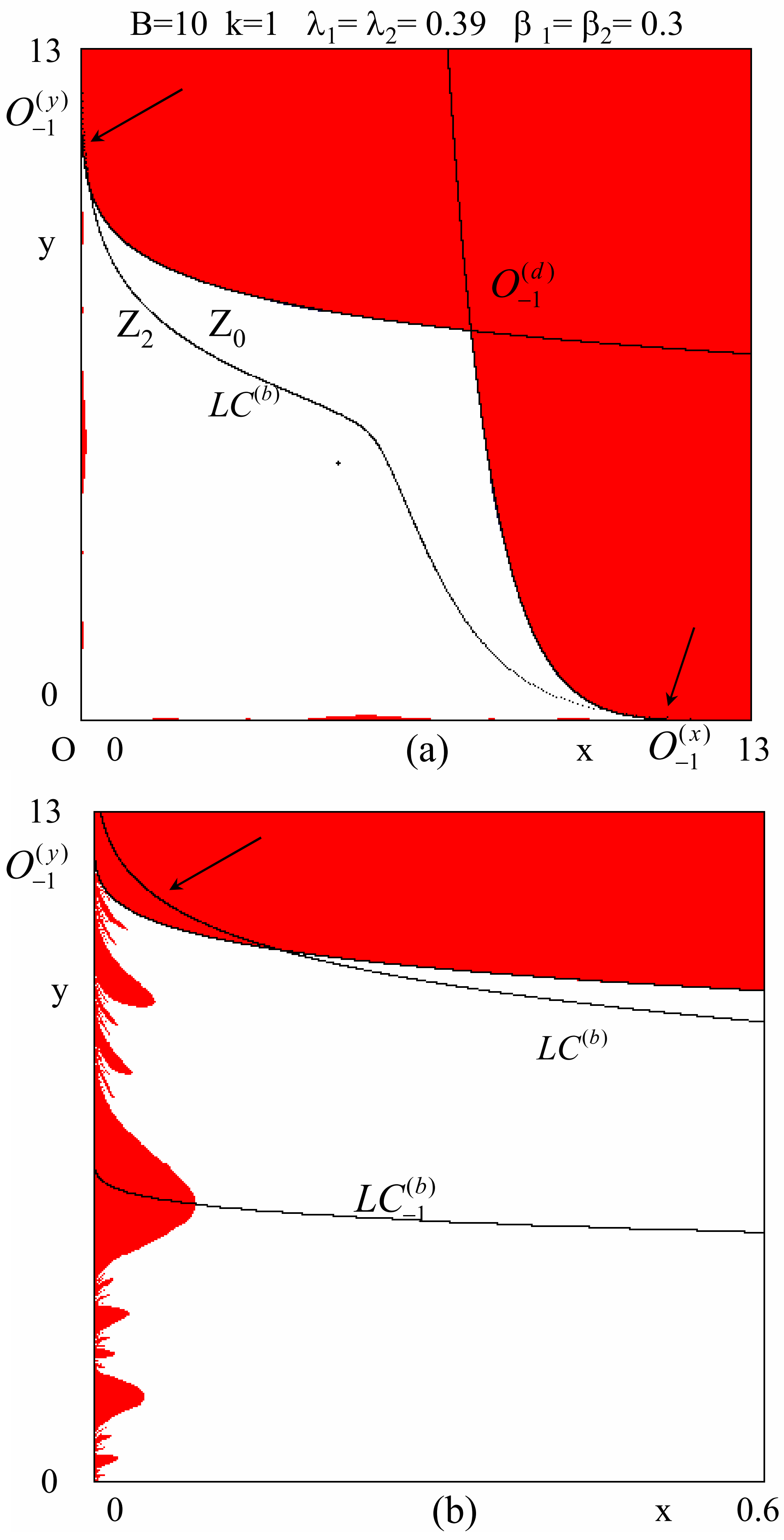
B=10 k=1 λ1= λ2= 0.28 β1= β2= 0.3
$O_{-1}^{(y)}$
ω_x^{-1}
$O_{-1}^{(d)}$
ω_y
E*
ω_y^{-1}
O
ω_x
$O_{-1}^{(x)}$

Fig.1



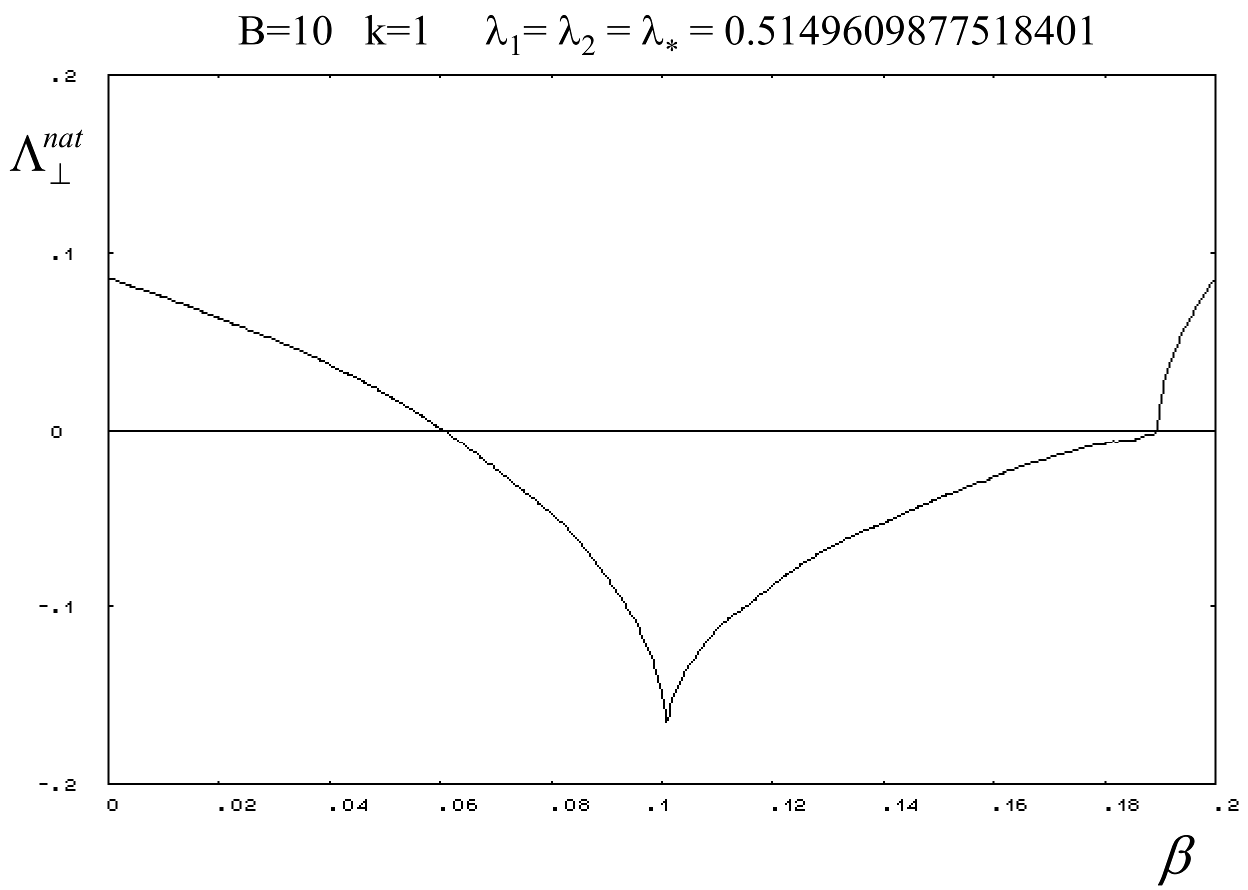
B=10 k=1 λ1= λ2= 0.28 β1= β2= 0.3
13
y
0
0
x 13
$LC_{-1}^{(a)}$
$LC_{-1}^{(b)}$
(a)
B=10 k=1 λ1= λ2= 0.28 β1= β2= 0.3
$O_{-1}^{(y)}$
ω_x^{-1}
$O_{-1}^{(d)}$
$LC^{(b)}$
Z_0
ω_y
Z_2
ω_y^{-1}
Z_4
$LC^{(a)}$
O
ω_x
$O_{-1}^{(x)}$
(b)

Fig.2



B=10 k=1 λ1= λ2= 0.39 β 1= β2= 0.3
13
$O_{-1}^{(y)}$
y
$O_{-1}^{(d)}$
Z_2
Z_0
$LC^{(b)}$
0
O 0
(a)
x
$O_{-1}^{(x)}$
13
13
$O_{-1}^{(y)}$
y
$LC^{(b)}$
$LC_{-1}^{(b)}$
0
0
(b)
x
0.6

Fig.3



B=10   k=1 $\lambda_1 = \lambda_2 = \lambda_* = 0.5149609877518401$
$\Lambda_\perp^{nat}$
.2
.1
0
-.1
-.2
0
.02
.04
.06
.08
.1
.12
.14
.16
.18
.2
β

Fig.4

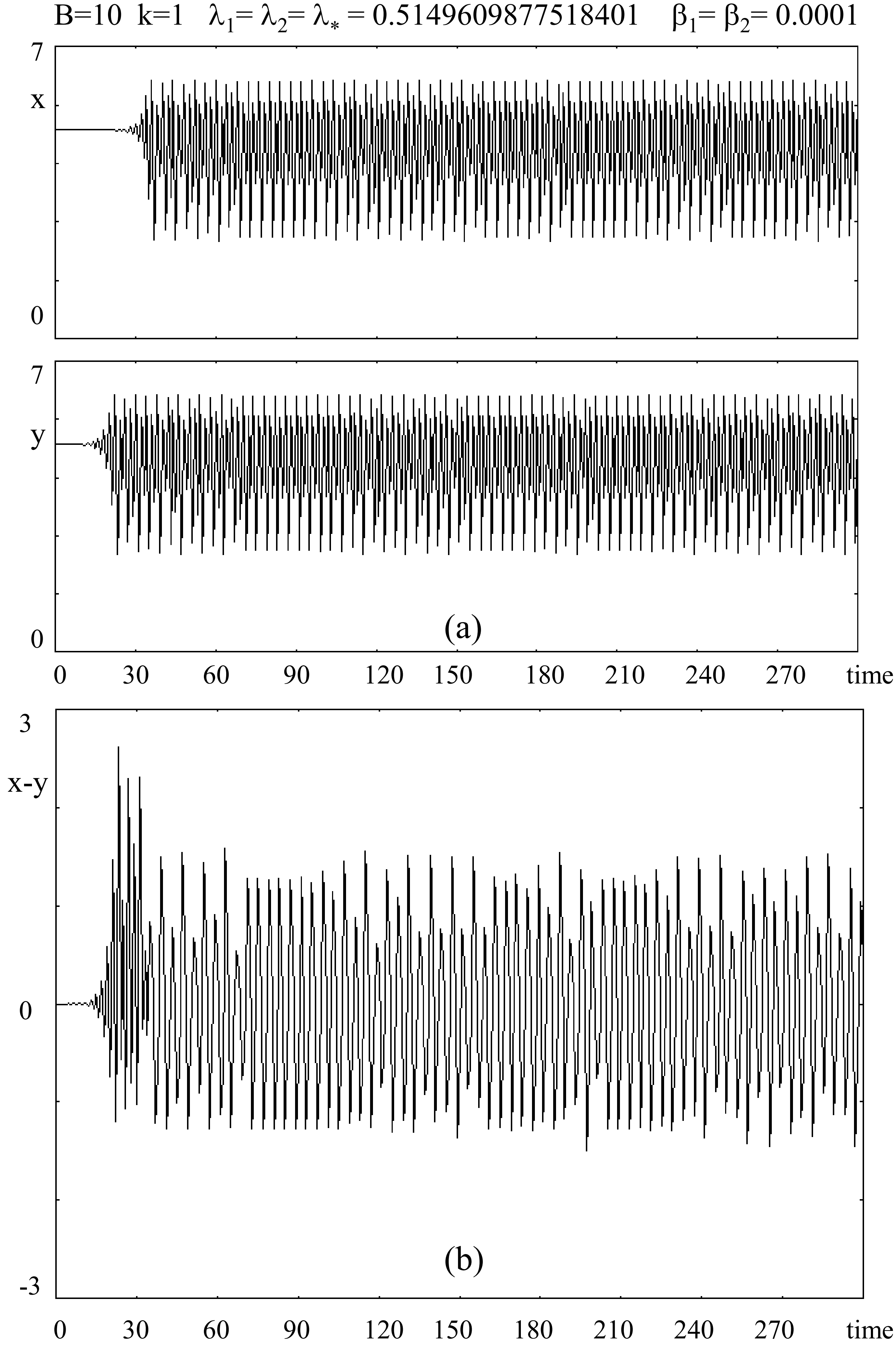

B=10 k=1 λ1= λ2= λ* = 0.5149609877518401 β1= β2= 0.0001
7
x
0
7
y
0
(a)
0 30 60 90 120 150 180 210 240 270 time
3
x-y
0
-3
(b)
0 30 60 90 120 150 180 210 240 270 time

Fig.5

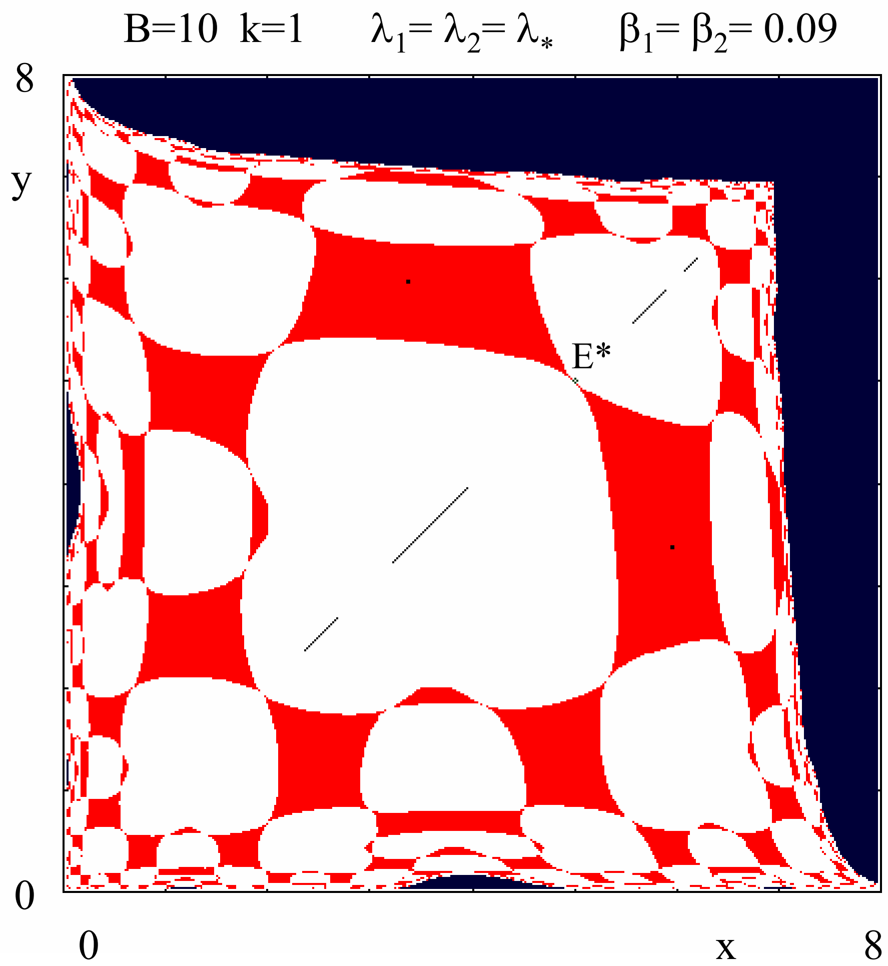
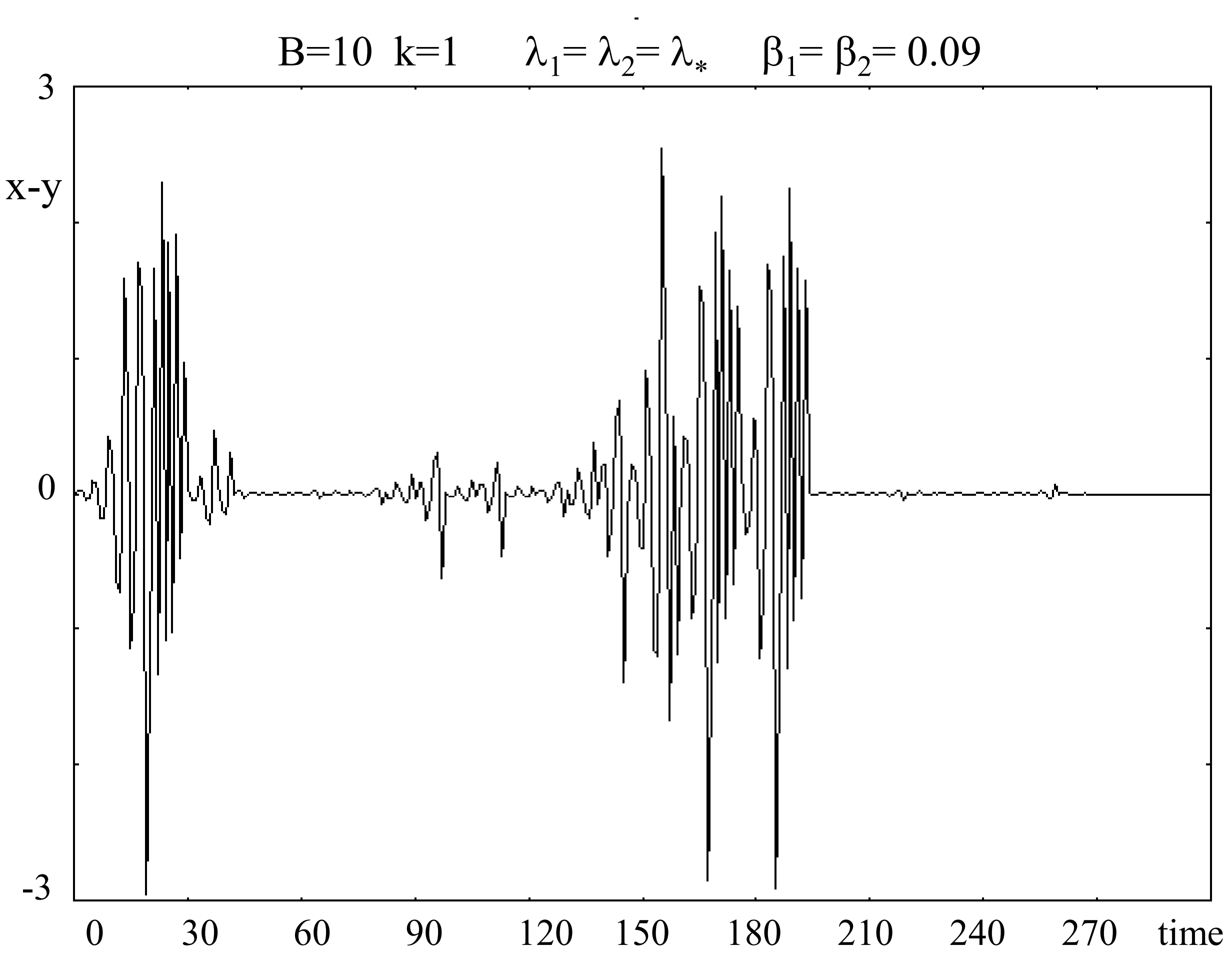


Fig.6



B=10 k=1 λ1= λ2= λ* β1= β2= 0.09
3
x-y
0
-3
0
30
60
90
120
150
180
210
240
270
time

Fig.7

B=10 k=1 $\lambda_1 = \lambda_2 = \lambda_*$ $\beta_1 = \beta_2 = 0.09$

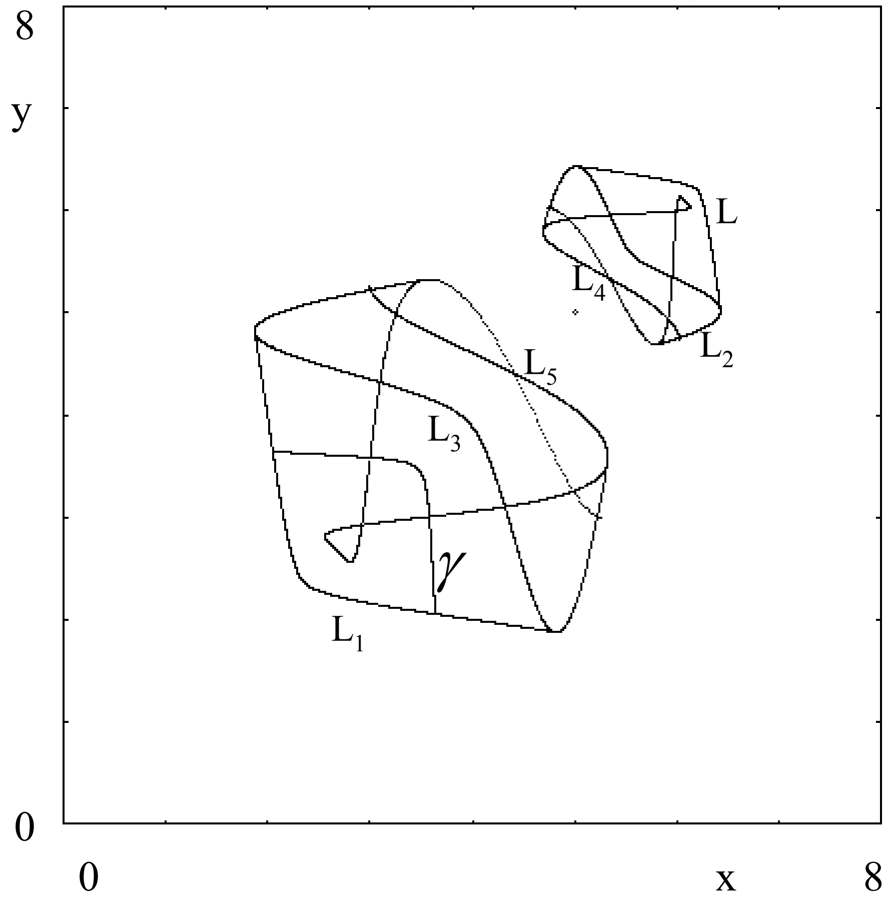
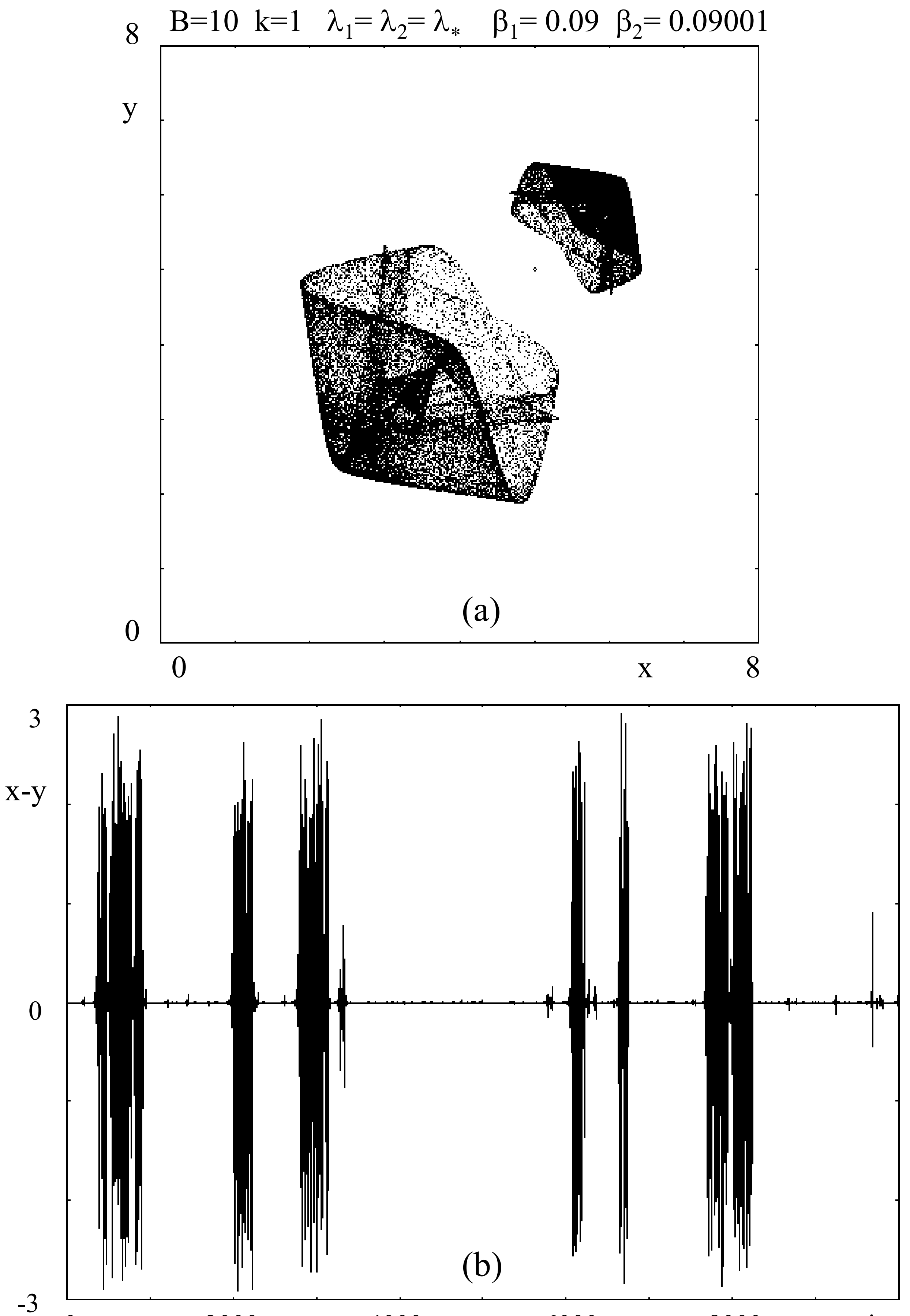


Fig.8



B=10 k=1 λ1= λ2= λ* β1= 0.09 β2= 0.09001
8
y
0
(a)
0
x
8
3
x-y
0
-3
(b)
0
2000
4000
6000
8000
time

Fig.9