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Cyclicity of chaotic attractors in one-dimensional discontinuous maps

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Abstract

A chaotic attractor may consist of some number of bands (disjoint connected subsets). In continuous maps multi-band chaotic attractors are cyclic, that means every generic trajectory visits the bands in the same order. We demonstrate that in discontinuous maps multi-band chaotic attractors may be acyclic. Additionally, a simple criterion is proposed which allows to distinguish easily between cyclic and acyclic chaotic attractors.

Keywords: Chaotic attractors; Piecewise smooth maps; Discontinuous maps

1. Introduction

It is well-known that *piecewise smooth* dynamical systems show many interesting phenomena which can not occur in smooth systems. In particular, occurrence of border-collision bifurcations, sliding and grazing bifurcations, robust chaos, etc., which are characteristic namely for piecewise smooth systems, is studied nowadays by many researchers from different theoretical and applied fields of science (see, e.g., the books [21,7,19] and references therein). Indeed, these and several other phenomena are definitely related to the presence of so-called *switching manifolds*, or borders, in the phase space of the system, along which the system function changes its definition. For piecewise smooth *one-dimensional* maps these borders are just break or discontinuity points, so that, for example, a *border-collision bifurcation* of a fixed point is caused by its collision with one of such points. The term border-collision bifurcation is introduced in [17], and nowadays it is well-known that such a bifurcation may result in transition from, for example, an attracting fixed point to an attracting cycle of any period, or even to a chaotic attractor directly (see also [13,8,12,14,18]).

Although piecewise smooth systems have been a focus of investigation since at least the mid 1990s, the differences between dynamics of continuous and discontinuous piecewise smooth systems are still far away from being understood completely. The goal of this paper is to draw attention to a significant difference between these two classes of dynamical

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systems with respect to the cyclicity of chaotic attractors. For simplicity we restrict ourselves to 1D maps, keeping however in mind that similar considerations can be repeated for multi-dimensional maps.

Regarding chaotic attractors in piecewise smooth systems it is already known that they may be *robust*. This phenomenon, defined as persistence of chaotic attractors under parameter perturbations [5], is quite important for several applications (see for example [4]). In the present work we demonstrate one more peculiarity of chaotic attractors in discontinuous maps: We show that they may be *acyclic* and that this phenomenon is related to the behavior of the maps close to the switching manifold.

In multi-dimensional systems (both continuous or discrete in time) chaotic attractors[1] are known to posses complicated geometrical shapes, both interesting from the mathematical point of view and sometimes also from the aesthetic one (so that they are frequently used as some kind of advertisement for the nonlinear dynamics). By contrast, the geometrical shape of a chaotic attractor in 1D maps is quite trivial: it may represent either an interval or a finite collection of intervals[2]. In the first case the attractor is called a *one-band attractor*, and in the second case an *m-band attractor*, where the number m refers to the number of disjoint intervals (strongly connected components) separated from each other by $(m-1)$ finite-sized gaps[3]. A simple but important question arises whether there are some regularities in the order in which these bands are visited by the orbits, namely, if the band are mapped cyclically onto each other or not. The aim of this paper is to demonstrate that continuous and discontinuous maps behave differently in that respect.

The paper is organized as follows. First, in §2 we give the basic definitions and recall some quite obvious results related to cyclicity of chaotic attractors in continuous maps. Then, in §3 we show that several of the results well-known for continuous one-dimensional maps may be no longer valid for discontinuous maps. Finally, in §4 we demonstrate that the introduced concepts are applicable not only for one-dimensional maps. We show examples of cyclic and acyclic chaotic attractors in two-dimensional discontinuous maps, give a criterion how to distinguish between them, and outline the direction of the future work in this field.

2. Continuous maps

Let a chaotic attractor $\mathcal{A}$ of a map $x_{n+1}=f(x_n)$ consist of $m>1$ disjoint bands $\mathcal{B}_0, \ldots, B_{m-1}$, then $\mathcal{A}$ is called *cyclic* if each of its bands is mapped by f onto the next one:

$$\forall i = 0, \ldots, m-1: \quad f(\mathcal{B}_i) = \mathcal{B}_{(i+1) \bmod m} \tag{1}$$

This definition implies several obvious properties. If $\mathcal{A}$ is a cyclic m-band chaotic attractor of a map $x_{n+1}=f(x_n)$, then

Property 1. *Each band of $\mathcal{A}$ has exactly one successor and exactly one predecessor band.*

Property 2. *Each orbit of f started at a typical initial value converging to $\mathcal{A}$ visits the bands in the same order.*

Property 3. *Each band of $\mathcal{A}$ represents a one-band attractor for the mth iterate f^m.*

As an example let us consider the four-band attractor of the logistic map

$$x_{n+1} = \alpha x_n(1-x_n) \tag{2}$$

shown in Fig. 1a. Its bands are given by

$$\begin{aligned} &\mathcal{B}_0 = \left[f^7(c), f^3(c)\right], \qquad &&\mathcal{B}_1 = \left[f^4(c), c\right], \\ &\mathcal{B}_2 = \left[f(c), f^5(c)\right], \qquad &&\mathcal{B}_3 = \left[f^2(c), f^6(c)\right] \end{aligned} \tag{3}$$

[1] Note that we mean here an attractor defined as an undecomposable attracting set, which attracts *all* points from its neighborhood (for a discussion on different definitions of an attractor see [15].

[2] As shown in [6], 1D maps may possess unbounded attractors. However, this does not represent any difficulty for the purposes of our work. Indeed, an unbounded interval $[\cdot, \infty)$ is still an interval and all arguments presented below are valid for such intervals too.

[3] When speaking about the gaps we mean the intervals which are located inside the minimal invariant absorbing interval but do not belong to the attractor. So, for example, the chaotic attractor of the logistic map (2) shown in Fig. 1a is located within the minimal absorbing interval $[f(c), c]$ with $c = f(\frac{1}{2})$ and has three gaps. The intervals $(-\infty, f(c))$ and (c, ∞) located outside the minimal absorbing interval do not matter for the counting of the gaps

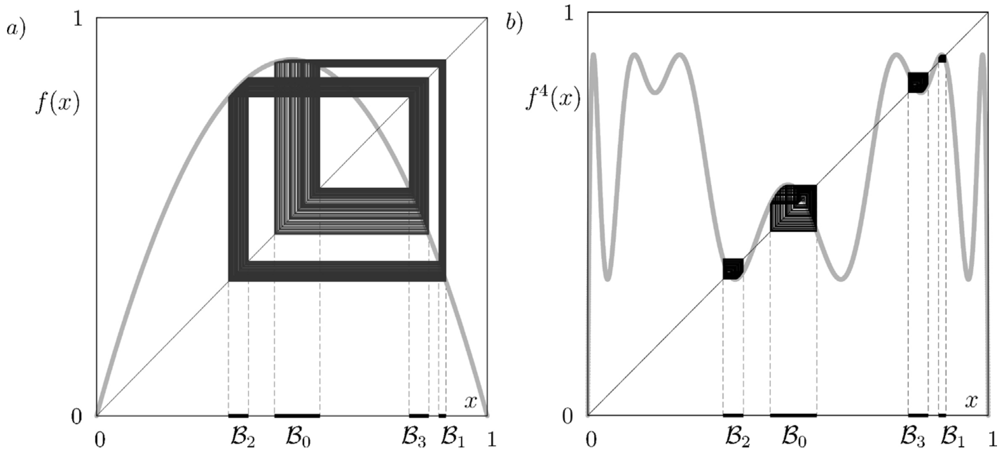


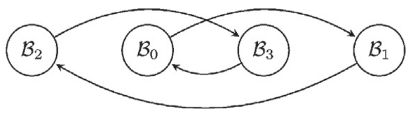
Fig. 1. Cyclic four-band chaotic attractor of the logistic map (2) at $\alpha \approx 3.578$ (a) and four coexisting attractors of the fourth iterate (b).

where c is the critical point given by the maximal value of the function, $c = f(\frac{1}{2}) = \frac{1}{4}\alpha$. As one can easily see both from Fig. 1a and from Eq. (3), the bands are mapped onto each other in the following way:

$$f(\mathcal{B}_0) = \mathcal{B}_1, \quad f(\mathcal{B}_1) = \mathcal{B}_2, \quad f(\mathcal{B}_2) = \mathcal{B}_3, \quad f(\mathcal{B}_3) = \mathcal{B}_0 \tag{4}$$

which can also be represented by the following directed graph:

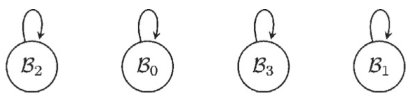
G:



(5)

where each node corresponds to one band of the attractor and the edges show how the bands are mapped onto each other. As a consequence of this cyclicity, the fourth iterate of the logistic map has four coexisting one-band chaotic attractors, as shown in Fig. 1b. Indeed, the fourth iterate[4] of the graph G in (5) is given by

G^4:



(6)

As one can see, the graph G^4 contains four *strongly connected components*[5] and reflects the situation shown in Fig. 1b: For the fourth iterate of the logistic map the bands $B_0, \ldots, \mathcal{B}_3$ are disconnected and represent four coexisting one-band chaotic attractors.

Note that the cyclicity of multi-band chaotic attractors is a useful property, which allows, for example, to develop a numerical technique for automatic counting the number of bands, as suggested in [1]. However, the question arises in which cases multi-band chaotic attractors are cyclic and in which cases they are not. Indeed, in the literature such attractors are often denoted as *cyclic chaotic intervals*, so the question is whether this notation is applicable for any multi-band chaotic attractor. The answer is provided by the following

Theorem 1. *Let $\mathcal{A}$ be a multi-band chaotic attractor of the map $x_{n+1} = f(x_n)$ where the function f is continuous. Then $\mathcal{A}$ is cyclic.*

For the proof of this theorem we refer to [1]. Note that the proof uses explicitly the definition of the continuity of a function, but does not require the function to be smooth. As a consequence, the multi-band chaotic attractors are cyclic in both cases of smooth as well as piecewise smooth continuous maps, and hence the notation "cyclic chaotic

[4] Recall that for a directed graph $G = (V, E)$ with the set of nodes $V = \{v_1, v_2, \ldots\}$ and the set of edges $E = \{e_1, e_2, \ldots\} \subset V \times V$ the nth iterate is defined as a directed graph $G^n = (V, E')$ where an edge $e = (v_i, v_j)$ belongs to the set E' iff there exists a path of the length n from the node v_i to the node v_j in the graph G.

[5] Recall that for a graph $G = (E, V)$ a subset of its nodes $V' \subset V$ is called a strongly connected component if there exists a path in G from each node $v_i \in V'$ to each other node $v_j \in V'$.

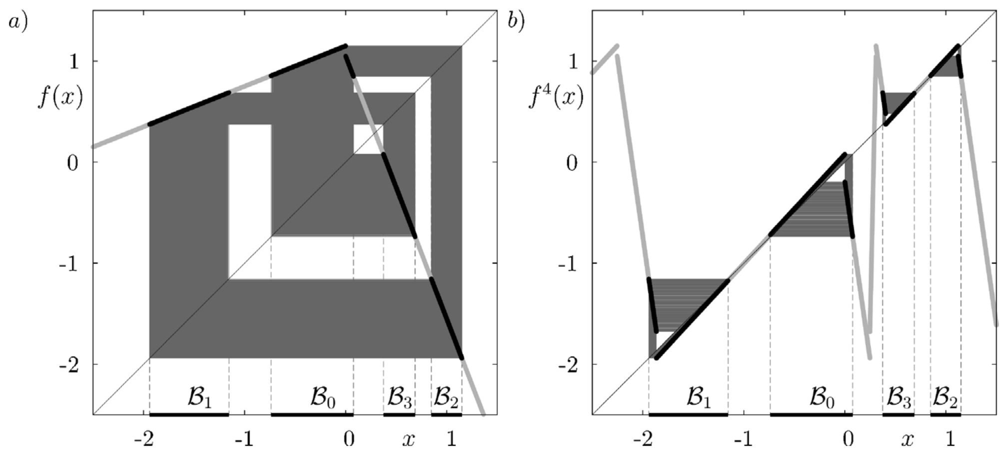


Fig. 2. Cyclic four-band chaotic attractor of map (7) at $a_\ell = 0.4$, $a_r = -2.6$ $\mu_\ell = 1.15$ $\mu_r = 1.05$ (a) and four coexisting attractors of the fourth iterate (b).

intervals" is applicable for them, but not necessarily for multi-band chaotic attractors in discontinuous maps, as it will be shown in the next section.

3. Discontinuous maps

When dealing with discontinuous maps, the situation is more complicated than in the continuous case because cyclic as well as acyclic chaotic attractors are possible. To demonstrate this fact, let us consider the piecewise linear map with one point of discontinuity in the general form given by

$$x_{n+1} = \begin{cases} f_\ell(x_n) = a_\ell x_n + \mu_\ell & \text{if} \quad x_n < 0 \\ f_r(x_n) = a_r x_n + \mu_r & \text{if} \quad x_n > 0 \end{cases} \tag{7}$$

In the next paragraphs we discuss some characteristic examples of chaotic attractors of the map (7).

3.1. Examples

Example 1. The four-band attractor of map (7) shown in Fig. 2a resembles very much the situation of the four-band attractor of the logistic map discussed above. Indeed, the mapping of the bands onto each other is the same as given by the graph (5) and hence the attractor is cyclic. To explain this fact note that in the considered case the jump of the system function at the discontinuity point $\Delta = |\mu_\ell - \mu_r|$ is quite small compared with the size of the bands of the attractor. For $\Delta = 0$ the system function is continuous and the attractor must be cyclic by Theorem 1. However, it is also not surprising, that for Δ small enough this property persists (below we will specify a criterion for that more precisely).

Example 2. As a next example let us consider the three-band chaotic attractor of map (7) shown in Fig. 3a. In this case we have

$$f(\mathcal{B}_1) = \mathcal{B}_2 \quad \text{and} \quad f(\mathcal{B}_2) = \mathcal{B}_0 \tag{8}$$

similar to the case of the logistic map considered above, but for the band $\mathcal{B}_0$ it can clearly be seen that

$$f(\mathcal{B}_0) \subsetneq \mathcal{B}_0 \cup \mathcal{B}_1. \tag{9}$$

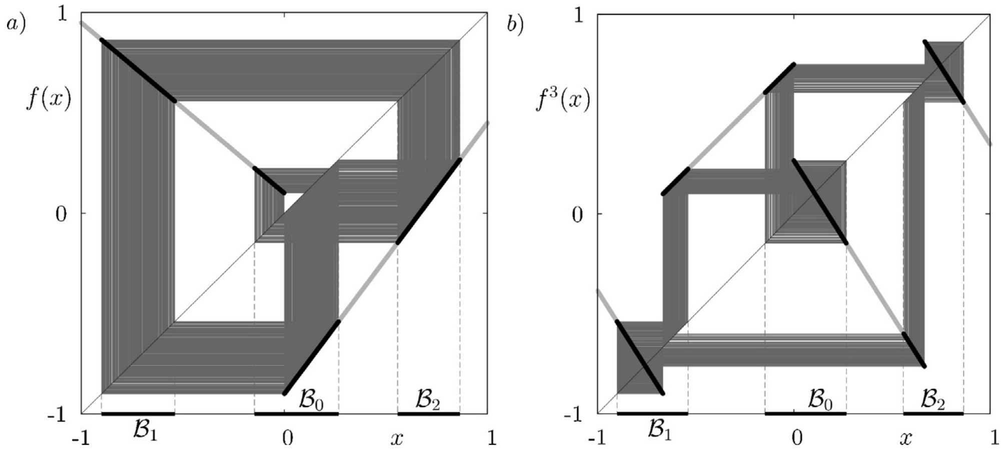
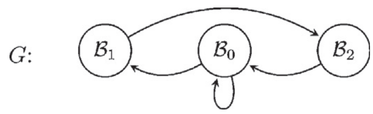


Fig. 3. An acyclic three-band chaotic attractor of map (7) at $a_\ell = -0.85$, $a_r = 1.35$ $\mu_\ell = 0.1$ $\mu_r = -0.9$ (a) and the corresponding acyclic three-band chaotic attractor of the third iterate (b).

As a consequence of that, the attractor is not cyclic, since the band $\mathcal{B}_0$ is mapped not only onto $\mathcal{B}_1$ but partially into itself. Hence, the mapping of the bands of this attractor onto each other can in principle be illustrated by the following graph:



(10)

Unfortunately, this representation is not completely correct. The problem is that in this graph the loop at the node $\mathcal{B}_0$ can be repeated several times, which corresponds to an orbit visiting the bands in the following way:

$$\ldots \to \mathcal{B}_2 \to \underbrace{\mathcal{B}_0 \to \ldots \to \mathcal{B}_0}_{k\ \text{times}} \to B_1 \to \ldots \tag{11}$$

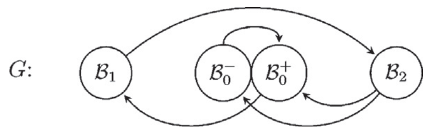
However, it can immediately be seen in Fig. 3a that such orbits are possible only for $k = 1$ and $k = 2$. This can easily be explained taking into account that the band B_0 consists of two parts

$$\mathcal{B}_0^- := \mathcal{B}_0 \cap \{x < 0\} \quad \text{and} \quad \mathcal{B}_0^+ := \mathcal{B}_0 \cap \{x > 0\} \tag{12}$$

which behave differently. The right part $\mathcal{B}_0^+$ is mapped directly onto $\mathcal{B}_1$. By contrast, the left part $\mathcal{B}_0^-$ is mapped into (but not onto) the right part $\mathcal{B}_0^+$, corresponding to the case $k = 2$ in Eq. (11).

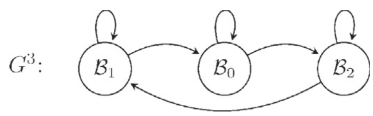
$$f(\mathcal{B}_0^-) \subsetneq \mathcal{B}_0^+, \quad f(\mathcal{B}_0^+) = \mathcal{B}_1 \tag{13}$$

Therefore, the mapping of the bands onto each other must be represented as follows:



(14)

This graph can be used to predict the behavior of the third iterate. Recall that for a continuous map a three-band chaotic attractor must correspond to three coexisting attractors of the third iterate. In the discontinuous case this does not need to be true. Indeed, considering the third iterate of the graph given in (14) and merging for simplicity the nodes corresponding to $\mathcal{B}_0^-$ and $\mathcal{B}_0^+$ to a single node corresponding to $\mathcal{B}_0$, we obtain the graph



(15)

As this graph has a single connected component, the acyclic three-band chaotic attractor of map (7) shown in Fig. 3a corresponds to a single acyclic three-band chaotic attractor of the third iterate of map (7), as shown in Fig. 3b.

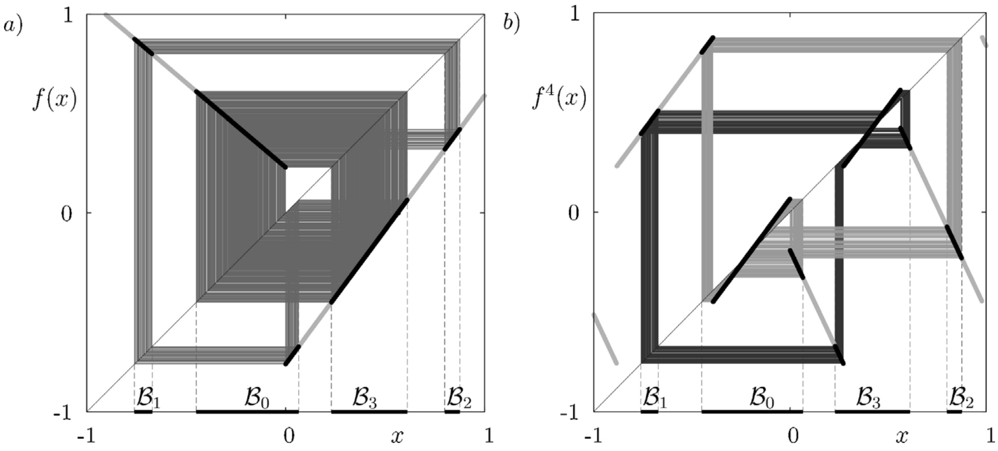
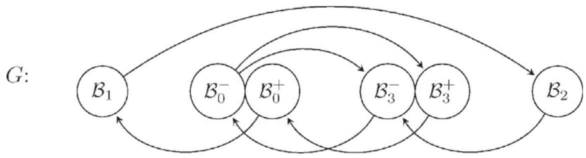


Fig. 4. An acyclic four-band chaotic attractor of map (7) at $a_\ell = -0.85$, $a_r = 1.35$ $\mu_\ell = 0.23$ $\mu_r = -0.76$ (a) and two coexisting acyclic attractors of the fourth iterate (b).

Example 3. Note also that not every acyclic m-band chaotic attractor corresponds to a single m-band chaotic attractor of the mth iterate. As an example Fig. 4a shows a four-band chaotic attractor of map (7). The following graph illustrates the mapping of its bands into respectively onto each other:

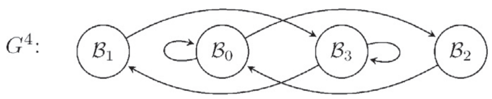


(16)

Note that for the construction of this graph it is necessary to split not only the node $\mathcal{B}_0$ as before but also the node B_3, namely, as follows:

$$\mathcal{B}_3^- := \mathcal{B}_3 \cap \{x < f_r^{-1}(0)\} \quad \text{and} \quad \mathcal{B}_3^+ := \mathcal{B}_3 \cap \{x > f_r^{-1}(0)\} \tag{17}$$

This is necessary because a point $x \in \mathcal{B}_3^+$ will be mapped to $\mathcal{B}_0^+$ and reaches $\mathcal{B}_1$ within two iteration steps, whereas a point $x \in \mathcal{B}_3^-$ will be mapped to $\mathcal{B}_0^-$ and needs at least four steps to reach $\mathcal{B}_1$. Performing the steps described above we obtain for the fourth iterate of the graph given in (16) the graph



(18)

which has two strongly connected components. As a consequence, the fourth iterate of map (7) has two coexisting two-band attractors, as shown Fig. 4b.

3.2. Point of discontinuity

The examples shown above lead us to the following observation. One can easily verify that in all examples the bands of acyclic attractors of map (7) behave differently depending on the fact whether they contain the point of discontinuity $x = 0$ or not. In fact, the bands not containing the point of discontinuity ($\mathcal{B}_1$ and $\mathcal{B}_2$ in Example 2, as well as $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ in Example 3) have one successor band. By contrast, the band containing the point of discontinuity ($\mathcal{B}_0$ in both examples) has more than one successor bands and this breaks the cyclicity. Therefore, it is natural to ask whether the point of discontinuity needs to be located inside the chaotic attractor. It turns out that this is indeed the case:

Property 4. *For any chaotic attractor of the piecewise linear map (7) the point of discontinuity $x = 0$ is located inside the attractor.*

This property can easily be proved by contradiction: Suppose, the attractor does not contain the point of discontinuity. Note that in this case the chaotic intervals cannot contain any of its preimages, of any rank (as otherwise in a finite number of step a point of a chaotic interval is mapped in the discontinuity point, against the assumption). Then, when we apply the system function to some band of the attractor, we have to apply a linear function, and this is the case for each band. However, the composition of linear functions is a linear function as well, and a linear function cannot have chaotic dynamics. Therefore, our assumption can not be correct and the attractor must contain the point of discontinuity.

Remarkably, this property can also be generalized for maps with a system function which has one point of discontinuity and is monotone on both sides of this point (*piecewise monotone* maps for short):

Theorem 2. *For any chaotic attractor of a piecewise monotone discontinuous map with one point of discontinuity, this point is located inside the attractor.*

The proof is analogous, using the fact that a continuous invertible map can not have a chaotic attractor.

3.3. Check for cyclicity

Now, as we know that the point of discontinuity is located inside one of the bands of any multi-band chaotic attractor, we can also introduce a practical criterion which allows us to distinguish between cyclic and acyclic attractors:

Theorem 3. *A multi-band chaotic attractor of a piecewise monotone discontinuous map with one point of discontinuity is cyclic if the band containing the point of discontinuity has one successor band, and acyclic otherwise.*

If the band containing the point of discontinuity, say B_0, has more than one successor band, the attractor is acyclic per definition. If B_0 has one successor band, say $B_1 = [a, b]$, it is enough to note that on the complete interval $[a, b]$ the system function is monotone. Due to this monotonicity, the image of this interval is again an interval, and hence the band B_1 has also one successor band. The same argument applies for all further bands until the band B_0 is reached again, and therefore the attractor is cyclic.

Note that theorems 2 and 3 can easily be generalized for maps with an arbitrary finite number of discontinuity points. We can prove that for any chaotic attractor of a piecewise monotone discontinuous map with any number of discontinuity points, at least one of these points belongs to the attractor. Hereby, a multi-band chaotic attractor of a piecewise monotone discontinuous map with any number of discontinuity points is cyclic if every band containing a point of discontinuity has only one successor band, and acyclic otherwise.

3.4. Coexistence of attractors

Theorem 2 has a consequence for the number of coexisting chaotic attractors in piecewise monotone maps with one point of discontinuity. In general, the coexistence of two periodic attractors, as well as a periodic and a chaotic attractor, is a usual phenomenon in piecewise monotone 1D maps (see, e.g., [3,20,9,2]). However we can prove the following property:

Property 5. *Two chaotic attractors cannot coexist in a piecewise monotone discontinuous map with one point of discontinuity.*

The proof is obvious: Suppose, two chaotic attractors coexist, then the point of discontinuity must be contained in both of them, and therefore the attractors overlap and represent one attractor.

3.5. Number of attractors of the mth iterate

If an m-band chaotic attractor $\mathcal{A}$ of a discontinuous map f is proved to be acyclic, the question may arise: how many coexisting chaotic attractors has the mth iterate f^m? Clearly, as $\mathcal{A}$ can not coexist with any further chaotic attractor, all chaotic attractors of the mth iterate correspond to some bands of $\mathcal{A}$. For a cyclic attractor this number is m, while for an acyclic one it can be a smaller number. To answer this question, we can proceed as it was shown in the examples and determine the number of strongly connected components of the directed graph G^m associated with the mth iterate by the following algorithm:

step 1. Start with the set of m nodes corresponding to m bands of the attractor $\mathcal{A}$.
step 2. Split the node corresponding to the band $\mathcal{B}_0$ containing the point of discontinuity in two nodes corresponding to $\mathcal{B}_0^-$ and $\mathcal{B}_0^+$ as it was done in Example 1, see Eq. (12).
step 3. Construct the edges of the graph G according to the mapping of the bands of $\mathcal{A}$ to each other. Start with the two nodes corresponding to sub-bands $\mathcal{B}_0^-$ and $\mathcal{B}_0^+$ and proceed *backward*, determining for each node its predecessors. Hereby the nodes corresponding to the bands which are mapped (directly or indirectly) to only one of the sub-bands $\mathcal{B}_0^-$ and $\mathcal{B}_0^+$ must be split too. The splitting points are given by preimages of zero (see Example 3, Eq. (17)).
step 4. For the resulted graph G determine the mth iterate G^m.
step 5. For the graph G^m merge the nodes corresponding to the parts of the same band (for example, the nodes corresponding to $\mathcal{B}_0^-$ and $\mathcal{B}_0^+$ will be merged to one node corresponding to $\mathcal{B}_0$).
step 6. After the nodes are merged, count the strongly connected components in the resulting graph. Their number represents the number of coexisting attractors of f^m, and the number of nodes in each of them represents the number of bands in the corresponding attractor.

The advantage of this algorithm is that it allows to predict the behavior of the mth iterate f^m by investigation of the function f only, using simple tools from the graph theory.

4. Two-dimensional maps

So far we investigated the cyclicity of chaotic attractors in 1D maps. But, as we claimed at the beginning, the results are valid for multi-dimensional systems as well. Indeed, many of them can immediately be generalized[6]. The fact that in continuous (smooth and piecewise smooth) maps chaotic attractors are cyclic follows directly from the proof of Theorem 1. This proof, given in [1], does not require the map to be one-dimensional but is valid for any continuous maps in $\mathbb{R}^n$. On the other hand, the fact that chaotic attractors in multi-dimensional maps may be acyclic is obvious since any 1D map possessing an acyclic chaotic attractor can easily be embedded in a two- or higher-dimensional state space. Therefore the remaining task is to demonstrate how to distinguish between cyclic and acyclic chaotic attractors, or in other words, how Theorem 3 can be extended for multi-dimensional maps. For sake of simplicity we restrict ourselves in the following by 2D maps and present only a few final results for this case, leaving the intermediate steps and all the proofs for a forthcoming publication.

Let $T:\mathbb{R}^2\mapsto\mathbb{R}^2$ be a piecewise smooth map defined on two partitions which are separated by a critical set (switching manifold) LC_{-1} on which T is discontinuous[7]. It can be shown that if T is invertible, then any chaotic attractor of T is cyclic. If T is noninvertible, then it can be shown that any chaotic attractor of T which does not intersect LC_{-1} is cyclic. Therefore the only case which requires a more detailed consideration is that the map T is noninvertible and the attractor $\mathcal{A}$ intersects the discontinuity boundary LC_{-1}. For this case the following result can be proved:

Theorem 4. *Let $\mathcal{A}$ be an n-band chaotic attractor ($n>1$) of a 2D map T which is discontinuous at the critical set LC_{-1} and let $g=LC_{-1}\cap\mathcal{A}\neq\emptyset$ be the intersection of the attractor with the critical set LC_{-1} consisting of one or more connected components. Then $\mathcal{A}$ is cyclic if for each continuous component of g the image $T(g_i)$ by both components of T represents a connected set, and acyclic otherwise.*

[6] It is worth to emphasize that when dealing with multi-dimensional systems it is preferable to associate the concept of cyclicity with attracting absorbing sets which contain chaotic attractors and not with attractors themselves. The difference plays a role for chaotic attractors possessing a Cantor-set structure, especially in discontinuous maps.

[7] The notation LC goes back to the French term *ligne critique* (critical line), used by Mira and co-workers (see [10,11]), who developed a theory of critical lines very useful for investigation of dynamics of noninvertible, as well as piecewise smooth maps. In particular, they recognized already in the mid 1970s that the boundaries of chaotic attractors in 2D maps are given by images of some critical lines.

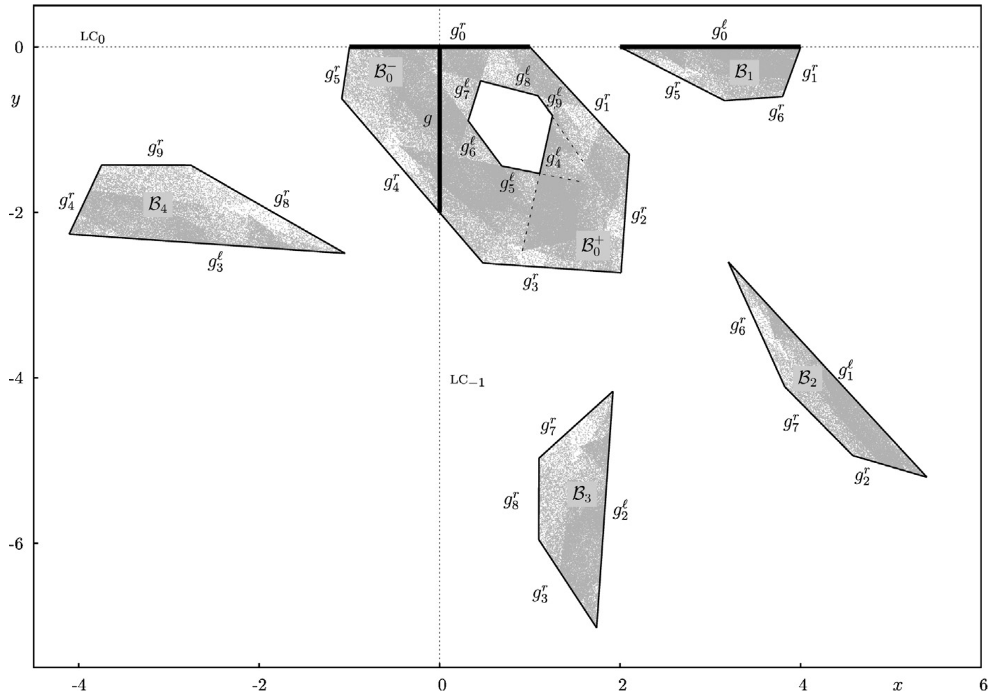


Fig. 5. Acyclic five-band chaotic attractor of map (19). Analytically calculated boundaries of the attractor are labeled. Thick lines mark the interval $g = LC_{-1} \cap \mathcal{B}_0$ and its two images $T^{\ell}(g)$, $T^{r}(g)$. Parameters: $\tau_\ell = 0.2$, $\tau_r = 1.1$, $\delta_\ell = -0.6$, $\delta_r = 1.3$, $\mu_\ell = 4$, $\mu_r = 1$.

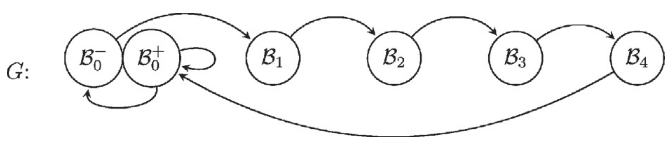
For a proof of this theorem we refer to the forthcoming work. Below we restrict ourselves by presenting a few examples only. To this end let us consider the following discontinuous modification of the well-known piecewise linear 2D border collision normal form:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{cases} T^{\ell}\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} \tau_\ell & 1 \\ -\delta_\ell & 0 \end{pmatrix}\begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} \mu_\ell \\ 0 \end{pmatrix}, & \text{if} x_k < 0 \\ T^{r}\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} \tau_r & 1 \\ -\delta_r & 0 \end{pmatrix}\begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} \mu_r \\ 0 \end{pmatrix}, & \text{if} x_k > 0 \end{cases} \qquad (19)$$

For the continuous case $\mu_\ell = \mu_r$ map (19) represents one of the most frequently investigated models in the theory of piecewise smooth maps (see [17] and references in [7,19]).

As a first example let us consider the 5-band attractor of map (19) shown in Fig. 5. Note that by a simple look on the phase space it is not immediately clear whether this attractor is cyclic or acyclic. However, Theorem 4 provides an answer to this question. The critical set LC_{-1} is given for map (19) by the vertical axis $x = 0$. In the shown example, its intersection with the band $\mathcal{B}_0$ is given by the segment g (marked as a thick line in Fig. 5). It can immediately be calculated that this segment is mapped by the functions T^{ℓ} and T^{r} onto two segments g_1^{ℓ} and g_1^{r}, respectively, located at the horizontal axis $y = 0$. As one can see, these segments are disjoint, and thus the chaotic attractor is acyclic.

Note also that the segment g plays an important role for the structure of the state space, as its images form all boundaries of the chaotic attractor (see [16], pp. 208–246 and pp. 276–299 for more details). Tracking these images, we can see how the bands $\mathcal{B}_0, \ldots, \mathcal{B}_4$ are mapped to each other:



(20)

Similarly to the examples discussed above, the band B_0 intersecting the discontinuity line LC_{-1} must be split by the segment g in two parts B_0^- and B_0^+ which are mapped differently. Hence, we can conclude that the situation is similar

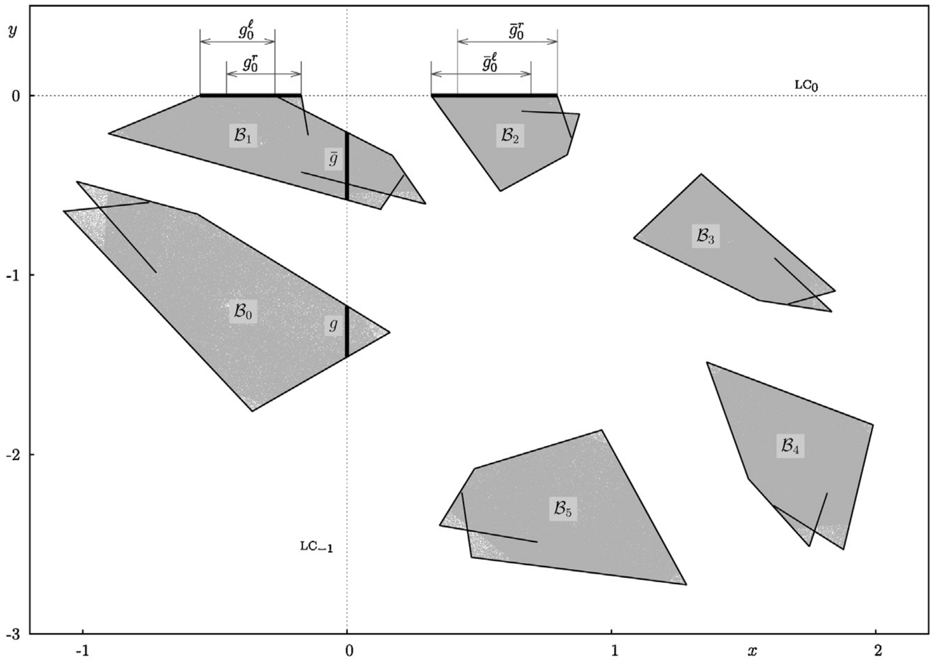


Fig. 6. Cyclic six-band chaotic attractor of map (19). Thick lines mark the intervals $g = LC_{-1} \cap \mathcal{B}_0$, $\overline{g} = LC_{-1} \cap \mathcal{B}_1$, and their first images $g_0^\ell = T^\ell(g)$, $g_0^r = T^r(g)$, $\overline{g}_0^\ell = T^\ell(\overline{g})$, $\overline{g}_0^r = T^r(\overline{g})$. Further images of these intervals defining the boundaries of the attractors are shown without labels. Parameters: $\tau_\ell = 0.12$, $\tau_r = 1.065$, $\delta_\ell = -0.591$, $\delta_r = 1.37$, $\mu_\ell = 0.9$, $\mu_r = 1$.

to the one discussed above for 1D maps: each band which does not intersect the discontinuity boundary LC_{-1} (that means, $\mathcal{B}_1, \ldots, \mathcal{B}_4$) has one successor band, and the band $\mathcal{B}_0$ is partially mapped into itself, and partially to $\mathcal{B}_1$.

As a second example let us consider the 6-band attractor of map (19) shown in Fig. 6. In this case the intersection of the attractor with the critical set LC_{-1} is given by two segments g and $\overline{g}$. As one can see, the images of both segments by the functions T^ℓ and T^r overlap pairwise. Therefore the attractor is cyclic.

5. Summary

In this paper we considered multi-band chaotic attractors in piecewise smooth 1D maps and demonstrated that some results well-known for piecewise smooth continuous maps are not valid for piecewise smooth discontinuous maps. Especially, in piecewise smooth continuous maps multi-band chaotic attractors are cyclic. Due to this cyclicity, it is a correct and commonly used way to investigate instead of an m-band attractor of a map f one of the coexisting one-band attractors of the mth iterate f^m and it is also correct to denote such attractors as cyclic chaotic intervals.

We demonstrated that in discontinuous maps the situation is different. It is shown that a multi-band chaotic attractor in discontinuous maps may be cyclic or acyclic. We introduced also a criterion which allows to distinguish between these two types of multi-band attractors in piecewise monotone maps with one point of discontinuity. This criterion is based on the fact that in such maps the point of discontinuity is always located inside some band of the attractors and this band represents the only part of the attractor which may destroy the cyclicity. We provided also an algorithm which allows to determine the number of coexisting attractors for the mth iterate.

Additionally, we demonstrated that the introduced concepts are applicable not only for 1D maps. We presented examples of cyclic and acyclic chaotic attractors of a piecewise linear discontinuous 2D map, and provided (without a proof) a theorem, which makes it possible to distinguish between them. This outlines the direction of the future research which will continue and extend the results of the present work.

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