

Contents lists available at [ScienceDirect](#)

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

The bull and bear market model of Huang and Day: Some extensions and new results

Fabio Tramontana^{a,*}, Frank Westerhoff^b, Laura Gardini^c^a Department of Economics and Management, University of Pavia, Via S.Felice 5, 27100 Pavia, Italy^b Department of Economics, University of Bamberg, Feldkirchenstrasse 21, 96045 Bamberg, Germany^c Department of Economics, Society, Politics, University of Urbino, Via Saffi 42, 61029 Urbino, Italy

ARTICLE INFO

Article history:

Received 14 November 2012

Received in revised form

8 May 2013

Accepted 10 June 2013

Available online 19 June 2013

JEL classification:

C02

D84

G12

G14

Keywords:

Heterogeneous interacting agents

Bull and bear market dynamics

Piecewise-linear maps

Border collision bifurcations

ABSTRACT

We develop a financial market model with interacting chartists and fundamentalists that embeds the famous bull and bear market model of Huang and Day as a special case. Their model is given by a one-dimensional continuous piecewise-linear map. Our model, on the other hand, is more flexible and is represented by a one-dimensional discontinuous piecewise-linear map. Nevertheless, we are able to provide a more or less complete analytical treatment of the model dynamics by characterizing its possible outcomes in parameter space. In addition, we show that quite different scenarios can trigger real-world phenomena such as bull and bear market dynamics and excess volatility.

© 2013 Elsevier B.V. All rights reserved.

1. Introduction

The development and analysis of financial market models with heterogeneous interacting agents began with the seminal paper by Day and Huang (1990).¹ Based on a few stylized institutional and behavioral facts, Day and Huang proposed a simple financial market model with three types of agent: a market maker who adjusts prices with respect to excess demand; chartists who believe in the persistence of bull and bear markets; and fundamentalists who bet on mean reversion. In their model, market participants' transactions may cause apparently unpredictable price dynamics with randomly alternating periods of generally rising or generally falling prices, so-called bull and bear market dynamics. Since then, literally hundreds of follow-up papers have been presented, deepening on our understanding of how financial markets function. For surveys of this burgeoning research field see, for instance, Chiarella et al. (2009), Hommes and Wagener (2009), Lux (2009), and Westerhoff (2009).

The one-dimensional map studied in Day and Huang (1990) is nonlinear since fundamentalists become increasingly aggressive as the price runs away from its fundamental value. Huang and Day (1993) reformulated their original model such that it corresponds to a one-dimensional continuous piecewise-linear map by assuming that fundamentalists only start

* Corresponding author. Tel.: +39 0382 986224.

E-mail addresses: fabio.tramontana@unipv.it (F. Tramontana), frank.westerhoff@uni-bamberg.de (F. Westerhoff), laura.gardini@uniurb.it (L. Gardini).

¹ As usual, there are a number of predecessors, e.g. Zeeman (1974), Beja and Goldman (1980), and Frankel and Froot (1986).

trading if the distance between the price and its fundamental value exceeds a critical threshold value. Otherwise, speculators' trading strategies are linear and thus the model is represented by three connected linear branches. Of course, simplifying assumptions must be made to derive such a piecewise-linear map. Due to the model's piecewise-linearity, however, [Huang and Day \(1993\)](#) were able to study how certain model parameters, e.g. chartists' and fundamentalists' reaction parameters, affect the distribution of prices. They found, for instance, that an increase in fundamentalists' aggressiveness tends to fatten the tails of the distribution of prices. Obviously, such insights are quite important for our understanding of financial markets.

In addition, the simplified model of [Huang and Day \(1993\)](#) initiated a substantial number of studies, some of which are described below. For instance, [Gu \(1993\)](#) managed to estimate the model of [Huang and Day \(1993\)](#) on the basis of monthly S&P 500 data, and discovered that the distribution of model-generated price changes does not differ statistically from its empirical counterpart. In addition, the autocorrelation functions of simulated and actual returns are strikingly similar. Gu concludes that although random elements play a role in the price formation process, the majority of the dynamics is due to intrinsic market forces, as represented by the simple model of [Huang and Day \(1993\)](#), for instance.

[Gu \(1995\)](#) further extended the analysis of [Huang and Day \(1993\)](#) by deriving closed form density functions of prices. In particular, he used this result to study how aggressively the market maker has to adjust prices with respect to excess demand to make profits (density functions depend on market maker's price adjustment behavior and control her/his profits). Gu showed that the market maker has to churn the market to make a living. Although it is undesirable for the market maker to destabilize the market too considerably, she/he evidently has an incentive to keep prices moving. Gu, too, admitted that the underlying piecewise-linear model is based on simplifying assumptions. Again, however, these assumptions enabled rigorous results to be derived that have real practical relevance.

In reality, of course, there are more than two different types of speculator. Speculators may differ with respect to degree of aggressiveness, their trading horizons, market entry levels, perceptions of the fundamental value, and so on. [Day \(1997\)](#) thus considered the case where there are different types of fundamental and technical traders, and presented an example of a one-dimensional continuous piecewise-linear map with 11 branches. As it turns out, this model is capable of generating quite exciting dynamics. For instance, bull and bear markets may emerge at quite different levels and the price dynamics may become even more intricate.

[Tramontana et al. \(2010a\)](#) developed a model in which chartists and fundamentalists react asymmetrically to bull and bear market situations. For instance, market participants may trade more/less aggressively in overvalued markets than in undervalued markets. As a result, they obtain a one-dimensional discontinuous piecewise-linear map (where the discontinuity point is given by the market's fundamental value). Although the mathematical tools for studying such models are still underdeveloped, they offer a detailed analytical treatment of the underlying map, which can produce all kinds of dynamic behavior including fixed point dynamics, cycles, quasi-periodic motion, chaos, and exploding trajectories. This model was further developed and investigated in [Tramontana et al. \(2010b, 2011a\)](#). Besides identifying economic mechanisms that can produce endogenous bull and bear market dynamics, these papers enrich our knowledge about how to deal with one-dimensional discontinuous piecewise-linear maps.²

[Huang and Zheng \(2012\)](#) also presented a model in the spirit of [Huang and Day \(1993\)](#) to explain different types of financial crisis, in particular, the sudden crisis, the disturbing crisis, and the smooth crisis. In their model, chartists hold regime-dependent beliefs about the fundamental value, motivated by psychologically grounded support and resistance levels; the different types of crisis are explained by a constant switching between regimes. In addition, their model is also able to explain a number of stylized facts concerning financial markets. The underlying map is essentially represented by a one-dimensional discontinuous piecewise-linear map where all branches have a positive slope. [Huang et al. \(2010\)](#) modified the latter model by allowing traders to switch between technical and fundamental trading strategies according to an evolutionary fitness measure. On the basis of this model, [Huang et al. \(2012\)](#) demonstrated, amongst other things, that major stock market movements and returns are asymmetric, i.e. that bubbles emerge gradually and crash suddenly and that drawdowns tend to be more dramatic than drawups.

Another model inspired by [Huang and Day \(1993\)](#) is that by [Venier \(2008\)](#). Despite its deterministic nature, his model, represented by a saw-tooth map, was able to reproduce some stylized facts concerning financial markets, such as fat tails and volatility clustering. By buffeting their one-dimensional discontinuous piecewise-linear bull and bear market model with dynamic noise, [Tramontana and Westerhoff \(2013\)](#) obtained a good match of the statistical properties of actual financial market prices, also from a quantitative perspective. [Westerhoff and Franke \(2012\)](#) provided an even simpler but nonetheless powerful stochastic bull and bear market approach. Further empirical evidence for regime-dependent trading behavior of chartists and fundamentalists is provided by [Chiarella et al. \(2012\)](#), [Manzan and Westerhoff \(2007\)](#), and [Vigfusson \(1997\)](#), amongst others.

To sum up, simple piecewise-linear models enable us to derive new analytical insights into how financial markets function. Moreover, these models – despite their simplicity – are not at odds with reality. Firstly, their main building blocks are based on stylized institutional and behavioral facts. For instance, the survey study conducted by [Menkhoff and Taylor \(2007\)](#) revealed that market participants do indeed rely on technical and fundamental analysis to predict financial market

² The formal analysis of one-dimensional discontinuous piecewise-linear models with one discontinuity can be traced back to [Leonov \(1959, 1962\)](#). Recently, this research field has gained new momentum; see, for instance, [Gardini et al. \(2010\)](#), [Avrutin et al. \(2010\)](#), [Gardini and Tramontana \(2010\)](#).

prices (this has also been found in laboratory experiments; see, e.g. Hommes, 2011). Secondly, these models are able to explain some important stylized facts concerning financial markets such as bubbles and crashes, excess volatility, and long-memory effects. Given the repeated emergence of severe financial crises and the herewith associated risk to the real economy, we believe that more work should be undertaken in this exciting research direction.

In this paper, we thus continue this line of research by proposing a financial market model that may be regarded as a generalization of the model of Huang and Day (1993). Let us briefly recall why their one-dimensional piecewise-linear map is continuous. Close to the fundamental value, only chartists are active in the market. Hence, the slope of the map's inner branch is larger than one. If the price deviates too far from the fundamental value, additional fundamentalists enter the market. Since their demand is zero when they enter the market, all three branches of the map are connected, and the slope of the two outer branches is lower than the slope of the inner branch. Instead, we assume that a number of chartists and fundamentalists are always active in the market, that additional chartists and additional fundamentalists may enter the market when the distance between the price and its fundamental value exceeds a critical level, and that new traders' demand may be non-zero at the market entry level. As a result, the dynamics of our model is due to a quite flexible one-dimensional discontinuous piecewise linear map. In particular, the three branches are typically disconnected, and there are no restrictions to the values their slopes may assume.

Nevertheless, we are able to provide a comprehensive analysis of the model dynamics. For instance, we are able to determine the frontiers, in parameter space, that separate bounded dynamics from divergent dynamics. This analysis demonstrates that both chartists and fundamentalists can contribute to or prevent market stability, making the regulation of financial markets a delicate issue. Moreover, we find that quite different scenarios can lead to intricate bull and bear market dynamics. As in Huang and Day (1993), we observe repeated price rallies and subsequent market crashes if the slope of the two outer branches is negative, due to fundamentalists' aggressive trading behavior. However, we also observe such – and even more intricate – boom-bust dynamics if the slope of the two outer branches is positive, due to chartists' aggressive trading behavior. Overall, the emergence of endogenous bull and bear market dynamics may thus be regarded as a robust and characteristic feature of speculative markets.

Our model results in a discontinuous map with two discontinuity points. Exploration of the so-called *border collision bifurcations* that occur in this case is in its infancy. A few results can be found in Tramontana et al. (2012), in which, however, only stable regimes are investigated (i.e. the slopes of all three branches are between 0 and 1). Clearly, such a scenario does not occur in our model. Thus, from a mathematical point of view, our model reveals exciting dynamics and interesting bifurcation structures, some of which have never yet been documented in the literature. As we will see, our analysis will also highlight a number of open problems, presenting non-trivial (mathematical) challenges for future research. We also hope that the detailed formal analysis of our model will help other researchers to explore their own discontinuous piecewise-linear maps. Unfortunately, as already noted, our knowledge about such maps is as yet rather limited. Variations of our model (still with two discontinuity points) are studied in Tramontana et al. (2011b, in press). However, these models do not embed the model of Huang and Day (1993) as a special case.

The remainder of our paper is organized as follows. In Section 2, we present our financial market model and show that it corresponds to a one-dimensional discontinuous piecewise-linear map. In Section 3, we give an overview of the potential model dynamics and illustrate its economic implications. Section 4 has a stronger mathematical orientation and explores some interesting parameter regions in further detail. In Section 5, we conclude the paper and highlight avenues for further research.

2. A simple financial market model

Our model is closely related to those of Day and Huang (1990) and Huang and Day (1993), and may even be interpreted as a generalization of the latter. Following the original papers, we assume that a market maker adjusts prices with respect to excess demand in the usual way. In our case, excess demand comprises the transactions of four different types of market participant. So-called type 1 chartists and type 1 fundamentalists are always active in the market. Type 1 chartists bet on the persistence of bull and bear markets while type 1 fundamentalists expect the price to return to its fundamental value. So-called type 2 chartists and type 2 fundamentalists share the same general trading philosophy as their type 1 counterparts, but only enter the market if mispricing exceeds a critical threshold value. Huang and Day (1993) argue that fundamentalists may perceive the chance of gains as small or zero near to the fundamental value and thus may opt against the participation in the trading process. For chartists, it can be argued that they only learn about (or trust) bull and bear markets if mispricing is sufficiently extreme. Of course, these are simplifying assumptions. Nonetheless, it appears natural to us to assume that not all traders react immediately to their trading signals. We will now present the building blocks of our model and show that they constitute a one-dimensional discontinuous piecewise-linear map.

2.1. The setup

Within our model, the price formation is due to a (standard) log-linear price adjustment rule according to which excess buying drives prices up and excess selling drives them down. Such a rule may also be interpreted as the stylized price

setting behavior of a risk-neutral market maker. Let P be the log of the price. Then we have

$$P_{t+1} = P_t + a (D_t^{C,1} + D_t^{F,1} + D_t^{C,2} + D_t^{F,2}). \tag{1}$$

The four terms in brackets on the right-hand side of (1) capture the transactions conducted by type 1 chartists, type 1 fundamentalists, type 2 chartists, and type 2 fundamentalists, respectively. Parameter a is a positive price adjustment parameter which we set, without loss of generality, equal to $a=1$.

Type 1 chartists optimistically buy (pessimistically sell) if prices are in the bull (bear) market, that is, if log price P is above (below) its log fundamental value F . Their orders are formalized as

$$D_t^{C,1} = c^1(P_t - F). \tag{2}$$

Reaction parameter c^1 is positive and indicates how aggressively type 1 chartists react to their perceived price signals. The larger the c^1 is, the more aggressive type 1 chartists are.

Type 1 fundamentalists always trade in the opposite direction of type 1 chartists. Orders placed by type 1 fundamentalists are written as

$$D_t^{F,1} = f^1(F - P_t), \tag{3}$$

where reaction parameter f^1 is positive. In an overvalued market, therefore, type 1 fundamentalists sell; in an undervalued market they buy.

Type 2 speculators are only active if prices are at least a certain distance from their fundamental value. The critical distance for type 2 speculators is given by z , a positive constant.³ The orders of type 2 chartists are thus expressed as

$$D_t^{C,2} = \begin{cases} c^2(P_t - F) + c^3 & \text{for } P_t - F \geq z \\ 0 & \text{for } -z < P_t - F < z \\ c^2(P_t - F) - c^3 & \text{for } P_t - F \leq -z. \end{cases} \tag{4}$$

Again, reaction parameter c^2 is positive, i.e. the trading intensity of type 2 chartists increases in line with the distance between prices and fundamentals. However, their transactions can be adjusted using parameter c^3 for which we assume $c^3 \geq -c^2(z - F)$. Therefore, type 2 chartists' transactions are non-negative in the bull market and non-positive in the bear market.

Orders placed by type 2 fundamentalists are based on the same principles, i.e. we have

$$D_t^{F,2} = \begin{cases} f^2(F - P_t) - f^3 & \text{for } P_t - F \geq z \\ 0 & \text{for } -z < P_t - F < z \\ f^2(P_t - F) + f^3 & \text{for } P_t - F \leq -z, \end{cases} \tag{5}$$

where restrictions $f^2 > 0$ and $f^3 \geq f^2(F - z)$ hold.

2.2. The model's law of motion

To simplify the notation, let us define

$$S^1 = c^1 - f^1, \quad S^2 = c^2 - f^2 \quad \text{and} \quad M = c^3 - f^3.$$

Note first that there are no restrictions on these aggregate parameters (they can take any values). Combining (1)–(5) then yields

$$P_{t+1} = \begin{cases} P_t + (S^1 + S^2)(P_t - F) + M & \text{if } P_t - F \geq z \\ P_t + S^1(P_t - F) & \text{if } -z < P_t - F < z \\ P_t + (S^1 + S^2)(P_t - F) - M & \text{if } P_t - F \leq -z. \end{cases} \tag{6}$$

Furthermore, it is convenient to express the model in terms of deviations from the fundamental value by defining $\tilde{P}_t = P_t - F$. We then have

$$\tilde{P}_{t+1} = \begin{cases} (1 + S^1 + S^2)\tilde{P}_t + M & \text{if } \tilde{P}_t \geq z \\ (1 + S^1)\tilde{P}_t & \text{if } -z < \tilde{P}_t < z \\ (1 + S^1 + S^2)\tilde{P}_t - M & \text{if } \tilde{P}_t \leq -z, \end{cases} \tag{7}$$

i.e. prices are driven by a one-dimensional piecewise linear map that has three separate linear branches.

³ Two aspects deserve some more attention. First, many cases discussed in our paper are consistent with a model in which there are either only type 2 chartists or only type 2 fundamentalists. Second, different market entry levels for type 2 chartists and type 2 fundamentalists, say z^c and z^f (instead of z), would result in a one-dimensional discontinuous piecewise-linear map with four discontinuity points and five linear branches. We leave this mathematically intricate scenario for the future.

Note that this map embeds the famous models of Day and Huang (1990) and, in particular, Huang and Day (1993) as special cases. We obtain their model if we assume that $f^1 = 0$, i.e. there are no type 1 fundamentalists; $c^2 = c^3 = 0$, i.e. there are no type 2 chartists; and $f^3 = f^2(F-z)$, i.e. type 2 fundamentalists' demand is zero at the market entry level. In this case, the two outer branches are connected with the inner branch. The inner branch has a slope that is always larger than one. However, interesting bull and bear market dynamics only occurs in their model if the slope of the two outer branches is less than minus one. Note that the latter condition requires that fundamentalists trade quite aggressively. Our model is more flexible by imposing less restrictive assumptions on traders' behavior. In general, the slopes of its outer and inner branches can be positive or negative, and the map is discontinuous. As we will show in the next section, this leads to quite interesting bifurcation properties and new bull and bear market scenarios.

3. Dynamics of the model: a first overview

In this section, we present an overview of the model dynamics. In Section 4, we will proceed to study certain aspects of the model dynamics in further detail. Following Huang and Day (1993), we assume from here on that $(1 + S^1) > 1$. Economically, this implies that type 1 chartists react more aggressively to a given price signal than type 1 fundamentalists (i.e. $c^1 > f^1$ and thus $S^1 > 0$).⁴ Apart from this assumption, both M and $(1 + S^1 + S^2)$ can take any positive or negative value, and $z > 0$.

Note first that market entry level z can be considered as a scale variable. By using the change of variable $x = \tilde{P}/z$ and by setting $m = M/z$, we obtain

$$F: x' = \begin{cases} F_R(x) = (1 + S^1 + S^2)x + m & \text{if } x > 1 \\ F_C(x) = (1 + S^1)x & \text{if } -1 < x < 1 \\ F_L(x) = (1 + S^1 + S^2)x - m & \text{if } x < -1, \end{cases} \quad (8)$$

where “'” denotes the unit time advancement operator.

One useful property of the model is that its dynamics is symmetric with respect to the origin $x=0$. Using the definition $y = -x$ results in

$$y' = \begin{cases} (1 + S^1 + S^2)y + m & \text{if } y > 1 \\ (1 + S^1)y & \text{if } -1 < y < 1 \\ (1 + S^1 + S^2)y - m & \text{if } y < -1, \end{cases} \quad (9)$$

which is identical to F in (8), meaning that the following property holds:⁵

Property 1 (Symmetry). Any invariant set of the map F in (8) is either symmetric with respect to $x=0$ or the symmetric set also exists.

A direct consequence of Property 1 is that cycles of odd periods are never unique since the symmetric cycle also exists. That is, if a cycle with periodic points (x_1, \dots, x_k) exists where k is odd (and whatever is the sign of x_i , positive or negative), then the cycle $(-x_1, \dots, -x_k)$ also necessarily exists. As we shall see, this property will lead to a peculiar dynamic behavior described in Section 4.

In general, map F is discontinuous, with two discontinuity points in $x = -1$ and $x = 1$. In fact, map F is continuous only if:

$$F_C(1) = F_R(1). \quad (10)$$

With respect to the model parameters, this condition can be rewritten as

$$(C): S^2 = -m, \quad (11)$$

and always holds in the framework of Huang and Day (1993).

Let us next consider the two-dimensional bifurcation diagram in Fig. 1. Here we fix $S^1 = 0.5$ and vary S^2 and m as indicated on the axes.⁶ The parameter conditions for which our model is continuous are represented by the straight line (C). Again, all possible parameter combinations of the (continuous) model of Huang and Day (1993) are located on this line. Since Huang and Day (1993) only consider a market entry of additional fundamentalists, S^2 is in their model, by definition, negative. Hence, only part of the parameter combinations given by line (C) corresponds to their model. As will become clearer below, the model of Huang and Day (1993) may produce two symmetric stable fixed points (line (C), yellow region in Fig. 1), chaotic dynamics (line (C), white region in Fig. 1), which can be with two disjoint chaotic intervals (see Fig. 2a for an example) or in a unique chaotic interval (see Fig. 2b for an example), and divergent dynamics (line (C), gray region in Fig. 1).

⁴ As a result, type 1 chartists' transactions drive the price monotonically away from the fundamental value. Of course, for $(1 + S^1) < -1$ the inner regime also becomes unstable. Then, however, type 1 fundamentalists' transactions cause explosive improper oscillations. A detailed mathematical analysis of this case is quite involved and will necessitate a separate paper.

⁵ Recall that an invariant set is a set A that satisfies $F(A) = A$.

⁶ In all of our simulations, we set $S^1 = 0.5$. However, we obtain similar results for any other value of S^1 .

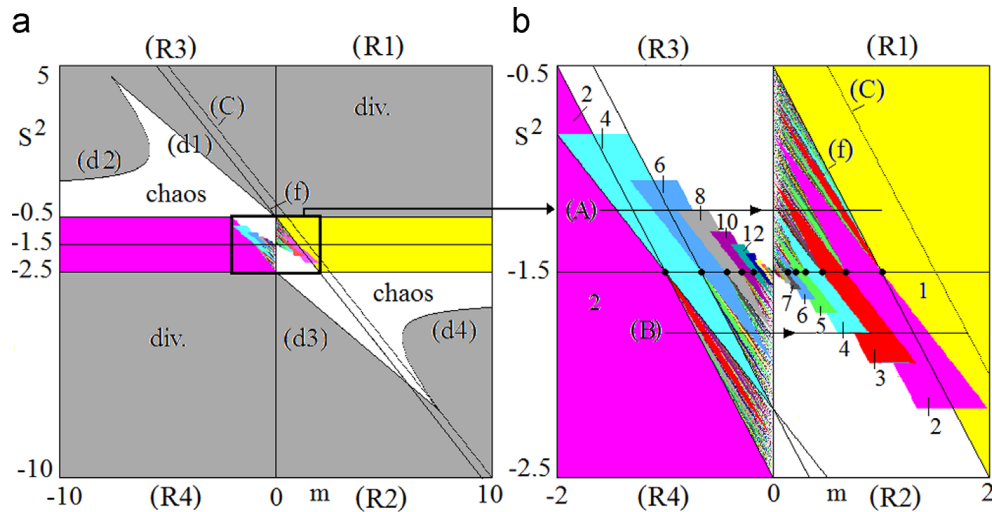


Fig. 1. Two-dimensional bifurcation diagram in the parameter plane (m, S^2) at $S^1 = 0.5$ fixed. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

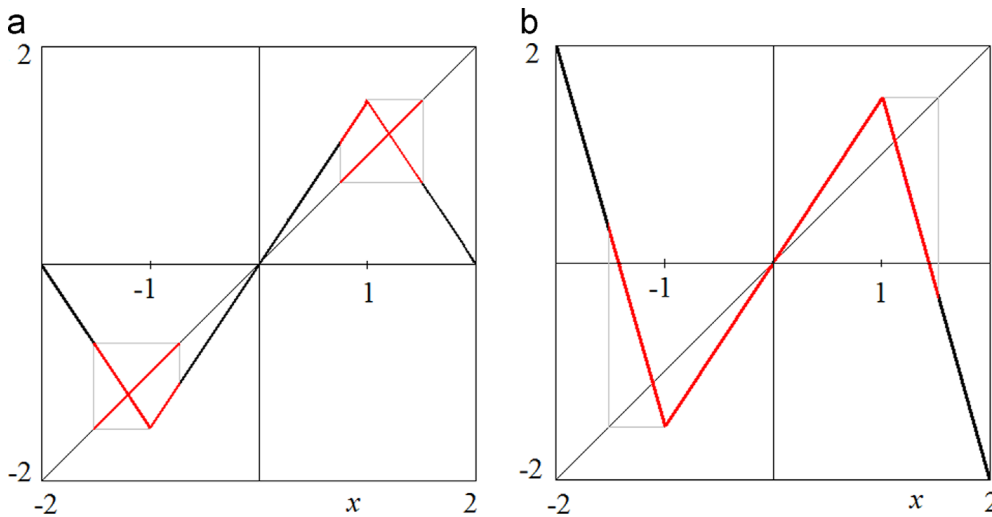


Fig. 2. In (a) $m=3, S^2=-3$; in (b) $m=5, S^2=-5$.

Naturally, our model includes a much larger parameter space and, as already visible from Fig. 1, many different dynamic peculiarities.

Since the central branch $F_c(x)$ of map (8) is a straight line running through the origin with a slope always greater than 1, the dynamic properties of our model strongly depend on external branches $F_L(x)$ and $F_R(x)$, that is on parameter m and $(1 + S^1 + S^2)$. As we will see in further detail in Section 4, the parameter regions

- (R1) : $(1 + S^1 + S^2) > 0$ and $m > 0$;
 - (R2) : $(1 + S^1 + S^2) < 0$ and $m > 0$;
 - (R3) : $(1 + S^1 + S^2) > 0$ and $m < 0$;
 - (R4) : $(1 + S^1 + S^2) < 0$ and $m < 0$;
- (12)

produce quite different dynamics. In Fig. 1, we thus draw the lines $m=0$ and $S^2 = -1.5$ (which results from $(1 + S^1 + S^2) = 0$ and $S^1 = 0.5$) to mark the four regions.

Economically, $(1 + S^1 + S^2) < 0$ implies that the joint price-dependent trading impact of type 1 and type 2 fundamentalists strongly overcompensates the joint price-dependent trading impact of type 1 and type 2 chartists. Similarly, it may be argued that for $0 < (1 + S^1 + S^2) < 1$ the joint price-dependent trading impact of type 1 and type 2 fundamentalists weakly overcompensates the joint price-dependent trading impact of type 1 and type 2 chartists while for $(1 + S^1 + S^2) > 1$ the joint price-dependent trading impact of type 1 and type 2 fundamentalists is overcompensated by the joint price-dependent trading impact of type 1 and type 2 chartists. Moreover, $m > 0$ ($m < 0$) can be interpreted in the sense that the

price-independent trading impact of type 2 chartists is stronger (weaker) than the price-independent trading impact of type 2 fundamentalists.⁷

Since our map is piecewise-linear and since the slope of the central branch is higher than 1, it follows that stable cycles can only exist in the strip

$$-1 < (1 + S^1 + S^2) < 1, \tag{13}$$

that is

$$-2 - S^1 < S^2 < -S^1. \tag{14}$$

This can also be seen in Fig. 1 (where $S^1 = 0.5$ turns (14) into $-1.5 < S^2 < -0.5$).

The fixed points of map (8), apart from $x=0$ which is always unstable, are associated with the symmetric external branches. Solving $F_L(x) = x$ and $F_R(x) = x$ leads to

$$x_L^* = \frac{m}{S^1 + S^2}, \quad x_R^* = -\frac{m}{S^1 + S^2}. \tag{15}$$

Of course, the fixed points only exist if $x_L^* < -1$ and $x_R^* > 1$. For instance, for x_R^* we thus have $-m/(S^1 + S^2) > 1$, which occurs for (a) $S^2 < -m - S^1$ if $m < 0$ and $S^1 + S^2 > 0$, or (b) $S^2 > -m - S^1$ if $m > 0$ and $S^1 + S^2 < 0$. Thus in Fig. 1 we also show the straight line

$$(f): \quad S^2 = -m - S^1. \tag{16}$$

Hence, the symmetric fixed points are stable only for $m > 0$ in the strip $-2 - S^1 < S^2 < -S^1$ (the yellow area in Fig. 1).⁸ As we will study in further detail below, in a portion of this area the fixed points coexist with a stable period 2-cycle. For $m < 0$, the fixed points, if they exist, are always unstable.

From an economic perspective, a fixed point implies that the market maker adjusts the price no further. The reason for this is, obviously, that excess demand is zero. For $x=0$, excess demand is zero since the price is equal to its fundamental value, and thus speculators do not perceive any trading signals. For x_L^* and x_R^* , however, excess demand is zero since speculators' buying and selling orders cancel each other out.

Outside the strip $-2 - S^1 < S^2 < -S^1$, the dynamics is either chaotic (as all the slopes are in modulus larger than 1) or the generic trajectory explodes (marked by white and gray in Fig. 1, respectively). As it turns out, it is possible to determine the frontiers separating bounded dynamics from the regions of divergent trajectories analytically. Consider the case of unstable fixed points for $m < 0$ (the qualitative shape of map F is as shown in Fig. 3a).

As long as the dynamics is bounded, the unstable fixed points are on the boundary of the basin of attraction:

$$\mathcal{B}(A_c) =]x_L^*, x_R^*[\tag{17}$$

and a condition leading to a so-called final bifurcation is the homoclinic bifurcation of the unstable fixed points, which occurs when the offsets of the central branch reach the fixed points, i.e. when

$$F_C(1) = x_R^* \tag{18}$$

and clearly also at the same time $F_C(-1) = x_L^*$, which leads to the condition

$$(d1): \quad S^2 = -\frac{m}{1 + S^1} - S^1 \tag{19}$$

(straight line (d1) in region $m < 0$ in Fig. 1a).

However, a homoclinic bifurcation of the fixed points may also occur due to a change in the slopes and offsets of the external branches, when the condition

$$F_L(-1) = x_R^* \tag{20}$$

occurs, which leads to the following condition to be satisfied:

$$(S^2)^2 + S^2(1 + 2S^1 + m) + (S^1(1 + S^1) + mS^1 - m) = 0, \tag{21}$$

the solution of which is expressed by the two branches of equation

$$(d2): \quad S^2 = \frac{1}{2}(1 + 2S^1 + m) \pm \sqrt{(1 + 2S^1 + m)^2 - 4(S^1(1 + S^1) + mS^1 - m)}, \tag{22}$$

shown as (d2) in Fig. 1a.

⁷ Note that for $S^2 > -1.5$ ($S^2 < -1.5$) we have that right (left) of the line $m=0$ even small parameter changes may lead to many different cycles. As pointed out by one of the referees, this aspect may have interesting economic implications. Suppose, for instance, that the model parameters are changing in this area from time to time. Then, the model dynamics alternates between different cycles, and the overall dynamics may become quite intricate. This parameter space may be a good candidate if one seeks to bring the model closer to the data.

⁸ Note that fixed point dynamics, yellow area in Fig. 1, may instantaneously turn into chaotic motion, white area in Fig. 1. Hence, even a tiny change in a model parameter, respectively a tiny change in the trading behavior of the market participants, may result in a completely different dynamic outcome. In contrast, nonlinear smooth maps usually produce more gradual "routes to chaos".

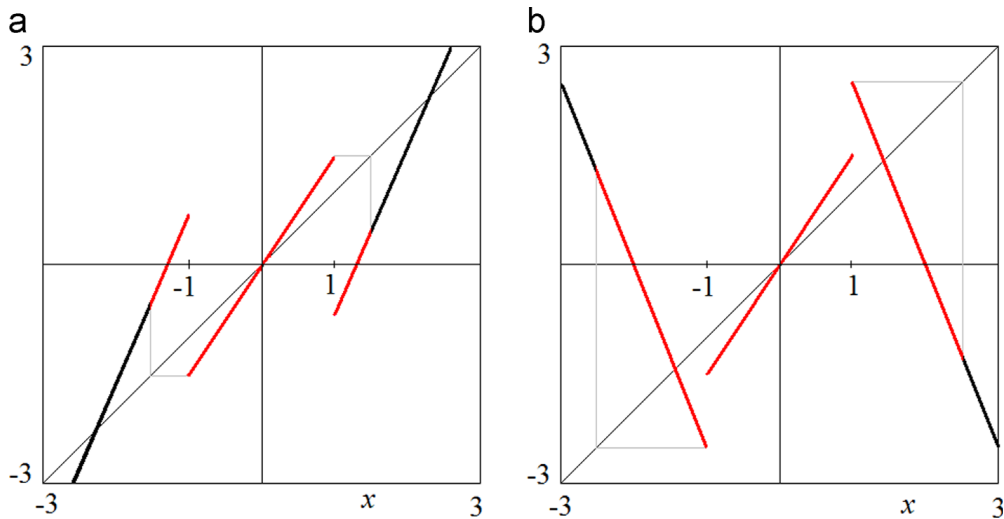


Fig. 3. In (a) $m = -3, S^2 = 0.8$; in (b) $m = 5, S^2 = -4$.

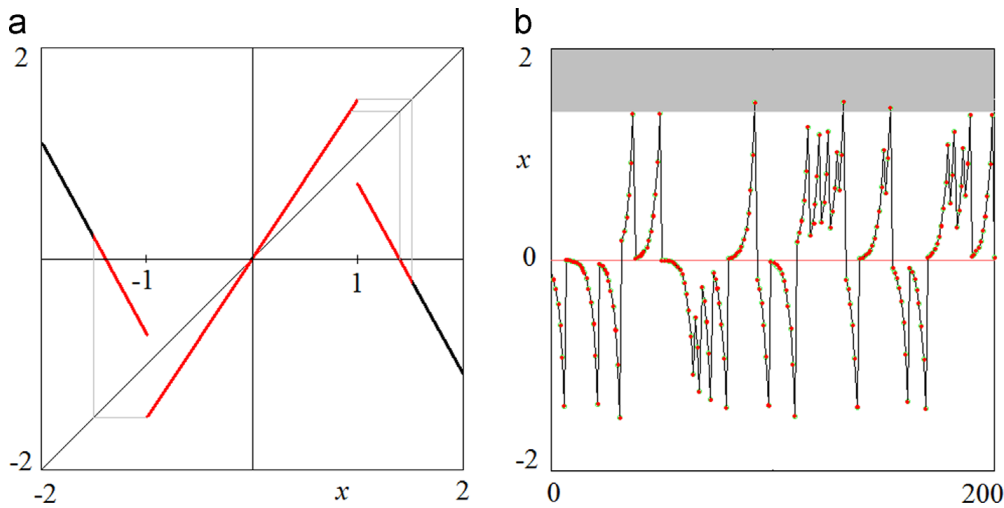


Fig. 4. In (a) $m = 2.5, S^2 = -3.3$; in (b) versus time trajectory. Points belonging to the grey strip on the bull region are mapped in one iteration in the bear region. A symmetric strip exists in the bear region, mapping points in the bull region in one iteration.

We have to reason differently in the case $m > 0$. If the fixed points do not exist (an example is shown in Fig. 4), then an unstable 2-cycle exists, with periodic points x_R and $x_L = -x_R$ in the external branches of F (i.e. with symbolic sequence RL), where x_R satisfies $F_L \circ F_R(x_R) = x_R$. This leads to

$$x_R = \frac{-m}{2 + S^1 + S^2}. \quad (23)$$

This unstable 2-cycle bounds the basin of attraction of the existing chaotic set:

$$\mathcal{B}(A_c) =]x_L, x_R[\quad (24)$$

and the bifurcation leading to divergent trajectories occurs at the homoclinic bifurcation of this 2-cycle, which occurs when

$$F_C(1) = x_R. \quad (25)$$

This leads to the condition

$$(d3): \quad S^2 = \frac{-m}{1 + S^1} - 2 - S^1, \quad (26)$$

which is the straight line (d3) in Fig. 1a.

A second way to obtain the homoclinic bifurcation of the unstable 2-cycle on the basin boundary is due to the external branches (see the qualitative shape of the map in Fig. 3b, and the unstable fixed points are included in the chaotic intervals). This occurs when the condition

$$F_R(1) = x_R \quad (27)$$

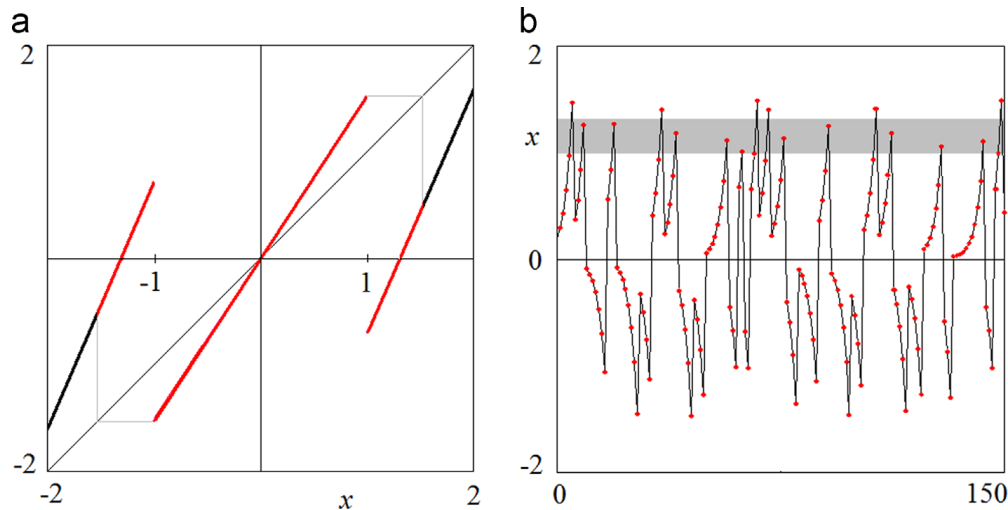


Fig. 5. In (a) $m = -3$, $S^2 = 0.8$; in (b) versus time trajectory. Points belonging to the gray strip in the bull region are mapped in one iteration in the bear region. A symmetric stripe exists in the bear region, and points in this region are mapped in one iteration in the bull region.

is satisfied, which leads to

$$(S^2)^2 + S^2(2(1 + S^1) + 1 + m) + (1 + S^1)^2 + (1 + S^1)(1 + m) + 2m = 0. \tag{28}$$

The solution of this is expressed by the two branches of equation

$$(d4): \quad S^2 = \frac{1}{2}(-2(1 + S^1) + 1 + m) \pm \sqrt{(2(1 + S^1) + 1 + m)^2 - 4((1 + S^1)^2 + (1 + S^1)(1 + m) + 2m)}, \tag{29}$$

shown as (d4) in Fig. 1a. Thus we have proved the following

Property 2 (divergence). For any value of m , outside the region satisfying $-2 < S^1 + S^2 < 0$ either chaotic dynamics exist or the generic trajectory is divergent. The region with divergent trajectories is bounded by the surfaces given in (19) and (22) for $m < 0$, and in (26) and (29) for $m > 0$. Chaos occurs in robust chaotic intervals of map F .

Let us take a closer look at Fig. 1 and the region bounded by (d1)–(d4). Naturally, one important goal of a central authority would be to prevent a market from collapsing, i.e. to prevent the price trajectory from settling on an explosive course. However, as is clear from Fig. 1, the regulation of financial markets is a non-trivial, challenging task. Note that either an increase or decrease in parameter S^2 and/or parameter m , and thus an increase or decrease in the aggressiveness of type 2 chartists and type 2 fundamentalists, may benefit or harm market stability. On the other hand, we learn from Fig. 1 that markets at least do not explode as long as the condition $-2 - S^1 < S^2 < -S^1$ is met. Given our model, a central authority may thus seek to manipulate the slopes of the outer branches of the underlying map such that they fall into this corridor, for instance by following appropriate intervention strategies (e.g. Wieland and Westerhoff, 2005 studied the impact of chaos control-inspired intervention strategies within the original model of Day and Huang, 1990).

A number of examples of chaotic dynamics outside the strip $-2 < S^1 + S^2 < 0$ are shown in Fig. 4 for $m > 0$ and decreasing external branches, in Fig. 5 for $m < 0$ and increasing branches. As can be seen, both parameter combinations, despite being quite different, can produce complex bull and bear market dynamics, which underlines the robustness of this finding. In the following, we sketch how these price fluctuations come about. However, let us first return to Fig. 2, which illustrates the dynamics of the original model of Huang and Day (1993). Suppose, for a start, that the price is slightly above the fundamental value. Since only chartists are active in that market area, prices are driven upwards, period for period, until fundamentalists enter the market. Since fundamentalists trade quite aggressively, prices are pushed downwards. Fig. 2 reveals that as long as the price drop is not too extreme, i.e. as long as the market remains in its bull state, prices will rise again, due to chartists' transactions. This pattern repeats itself in a complex manner, enabling us to observe irregular bull market dynamics. However, when prices reach a very high level, fundamentalists' transactions may become so forceful that the resulting price drop turns the market into a bear market. From then on, the chartists' previously optimistic mood turns pessimistic, and they expect a further price decrease. And indeed, prices decrease – until fundamentalists re-enter the market again and push prices upwards. We may now observe a longer bear market episode or, alternatively, a faster comeback of a bull market. It is virtually impossible to predict the duration of bull and bear markets, which is one of the aspects that make the model of Huang and Day so interesting.

Let us now take a look at Fig. 4. Essentially, the dynamics is similar to that just discussed. In our generalized model, we also see more or less randomly alternating periods of complex bull and bear market dynamics, despite the map's two discontinuities. Depending on the size of parameter $m > 0$, however, we may observe larger price drops more/less frequently. Recall that in our model, chartists and fundamentalists are always jointly active. In the inner regime, the trading activity of type 1 chartists is stronger than that of type 1 fundamentalists. For this reason, the slope of the map's inner branch is higher than one, and prices are driven away from the fundamental value. At some point, however, additional

speculators enter the market. Depending on the relative strength of type 2 chartists and type 2 fundamentalists, the slope and location of the two outer branches change. The specification in Fig. 4 is such that the aggregate transactions of all speculators produce more rapid price corrections than observed in Huang and Day's original model.

The situation differs in Fig. 5, where a new price pattern also emerges. What do the parameters now imply? Note that $m < 0$ can be interpreted in the sense that the price-independent trading behavior of type 2 fundamentalists is quite pronounced, while $S^2 > 0$ can be interpreted in the sense that type 2 chartists trade rather aggressively on their price signals. The new feature of Fig. 5 is as follows: observe, for instance, that a transition from a bull market to a bear market now emerges for more intermediate price levels rather than for very high price levels. Here we have a situation in which fundamentalists' selling orders exceed chartists' buying orders to such an extent that the consequent price drop triggers a bear market. If the price had been somewhat higher, however, we would still have observed a price drop. Due to the increased number of buying orders placed by chartists and the higher price level, however, the bull market would have survived. Hence, our model is able to generate a different regime-shifting pattern, and it may be argued that the price dynamics becomes even more erratic. Other examples of complex bull and/or bear market dynamics are depicted in Figs. 7–9, 11 and 12, further supporting the robustness of bull and bear market dynamics.

Apart from the regions with chaotic dynamics outside the strip $-2 < S^1 + S^2 < 0$, another quite interesting region is given exactly by this strip. As we will see, attracting cycles of any period can exist in this particular strip, and the boundaries can also be analytically determined. In the enlargement in Fig. 1b we illustrate the periodicity regions associated with stable cycles using different colors. We can also see from the structure of the regions that we have different dynamic properties in regions $m < 0$ and $m > 0$ and also, depending on the sign of the slope $(1 + S^1 + S^2)$, of the external branches, i.e. in the four regions defined in (12), as described in the next section. First of all, however, the separating case occurring at value $m = 0$ deserves special mention. Recall that a very important property of piecewise-linear maps is whether or not they are invertible. In fact, when a piecewise-linear map is invertible inside an invariant interval (i.e. with a unique inverse in the invariant interval), then only stable dynamics can exist, i.e. no unstable cycle or chaotic dynamics can occur, as proved in Keener (1980). However, when the map is non-invertible, stable and unstable cycles can exist, as well as chaos. In our case, considering the discontinuity point on the positive side, where the map is increasing/decreasing (i.e. all slopes are positive), the invertibility condition, given by

$$F_C \circ F_R(1) > F_R \circ F_C(1) \tag{30}$$

reads as

$$(1 + S^1)m > m, \tag{31}$$

which is always satisfied in region (R1). Moreover, the condition

$$F_C \circ F_R(1) = F_R \circ F_C(1) \tag{32}$$

is satisfied for $m = 0$, which implies that for $m = 0$ map F is conjugated with a circle map and, in particular, with a linear rotation, whose dynamics are well known. That is, either all points of the absorbing interval are periodic (of the same period) or any trajectory is quasiperiodic and dense in the absorbing interval (see also Gardini et al., 2010; Gardini and Tramontana, 2010). This depends on the rotation number, rational or irrational, respectively, and can also be observed numerically: the points associated with rational rotation numbers are the points on $m = 0$; the periodicity regions in Fig. 1b emanate from there. Thus we can state the following:

Property 3 (Linear rotation). For $m = 0$ and $-2 < S^1 + S^2 < 0$ map F is conjugated with a linear rotation.

4. Dynamics of the model in regions (R1)–(R4)

In this section, we characterize the model dynamics in parameter regions (R1)–(R4). The dynamics in regions (R1) and (R2) can easily be related to well-known results of discontinuous maps with only one discontinuity point (see Gardini et al., 2010; Avrutin et al., 2010; Gardini and Tramontana, 2010). However, new dynamic behaviors and new bifurcation structures may occur when all three branches of the map start to interact (although a few results exist for maps with two discontinuity points by Tramontana et al., 2011b, 2012, in press, they cannot be applied in our case).

To make our mathematical analysis as stark as possible, we decided to refrain from giving a deeper economic discussion of our results in Section 4. In doing so, we hope that our analysis will be easier to follow and may encourage the study of other economic models and further discontinuous piecewise-linear maps. So far, our knowledge about such systems is, unfortunately, quite limited. In any case, the economic meaning of parameters S^1 , S^2 and m is well defined and thus the economic implications of our study should be straightforward to deduct.

Due to the symmetry of our map, and the useful notation in terms of symbolic sequences to represent existing cycles, we shall use R and L to describe the right-most and left-most branches of our map F , while the central branch is separated by the origin on the right and the left, denoted by C_+ and C_- , respectively.

4.1. Region (R1): adding structure and disjoint symmetric attractors

All the branches of map F are increasing in region (R1). It is clear that for $F_R(1) > 1$, i.e. for $S^2 > -m - S^1$ (as we have seen in (16) and (15)), a stable fixed point exists on the right, and similarly on the left. It follows that x_R^* attracts all initial conditions $x_0 > 0$, while x_L^* attracts all negative ones.

Regarding the remaining portion, in (R1) and $m < -(S^1 + S^2)$, the map is uniquely invertible in two disjoint (symmetric) absorbing intervals, one on the positive side and one on the negative side, and without fixed points. Any initial condition $x_0 > 0$ converges to the unique attractor existing in

$$I_R = [F_R(1), F_C(1)] = [1 + S^1 + S^2 + m, 1 + S^1], \tag{33}$$

and similarly in the negative one in the absorbing interval

$$I_L = [F_C(-1), F_L(-1)] = [-(1 + S^1), -(1 + S^1 + S^2 + m)]. \tag{34}$$

As already noted, in this range we can take advantage of the properties of a map with one discontinuity point. Hence we can state that all dynamics is regular. Stable cycles of any period exist. Applying the formulas reported in several papers (see, for example, Gardini et al., 2010), we can also write the explicit analytic equations of BCB curves (really surfaces in our case, since the parameter space is three-dimensional), leading to a stable cycle in any period. Such cycles are organized in complexity levels. As we can see from Fig. 1b, the so-called regions of cycles of the first complexity level (or maximal cycles) of period $(n+1)$ for any $n \geq 1$ have symbolic sequences C_+R^n above the region of the attracting 2-cycle, and symbolic sequences RC_+^n below the region of the 2-cycle. Then, between any two consecutive regions of cycles of the first complexity level, for example between C_+R^k and C_+R^{k+1} , we can detect two infinite families of periodicity regions of cycles of the second complexity level with symbolic sequence $(C_+R^k)^j C_+R^{k+1}$ and $C_+R^k (C_+R^{k+1})^j$ for any $j \geq 1$, and so on: between any two consecutive periodicity regions of cycles of complexity level r we can find two infinite families of cycles of complexity level $r+1$ by using the same adding rule in the symbolic sequences. That is, the adding mechanism leads to the structure of the attracting cycles and their periodicity regions, as evident in Fig. 1b.

A one-dimensional bifurcation diagram showing the dynamic behavior of our variable x is illustrated in Fig. 6a for $m > 0$ at fixed parameters for S^1 and S^2 (along the horizontal path (A) shown in Fig. 1b). The attracting set on the positive side is shown in black. The symmetric dynamics clearly also exists on the negative side (shown in red in Fig. 6a). The symbolic sequence of the orbits is obtained from those on the right, substituting C_+ and R with C_- and L , respectively. The adding mechanism in the structure of the attracting cycles is clearly evident in Fig. 6a. We also recall that for $0 < m < -(S^1 + S^2)$ the intervals associated with stable cycles are dense in the interval $(0, -(S^1 + S^2))$. The parameter points that do not belong to the closure of a periodicity region of a certain cycle are associated with stable non-periodic dynamics: the trajectory ultimately converges to a non-chaotic Cantor set attractor, on which the trajectories are quasiperiodic and dense.

4.2. Region (R2): increment structure and chaotic intervals

The external branches of F are decreasing in the region (R2). For $F_R(1) > 1$, i.e. for $m > -(S^1 + S^2)$ (as we have seen in (16) and (15)), a stable fixed point may exist on the right, and similarly on the left. However, the stable fixed point may now no longer be the unique attractor. In fact, when the fixed points exist, the shape of the map is increasing/decreasing in the two disjoint absorbing intervals on the two opposite sides. For such kinds of maps, with one discontinuity only in each invariant

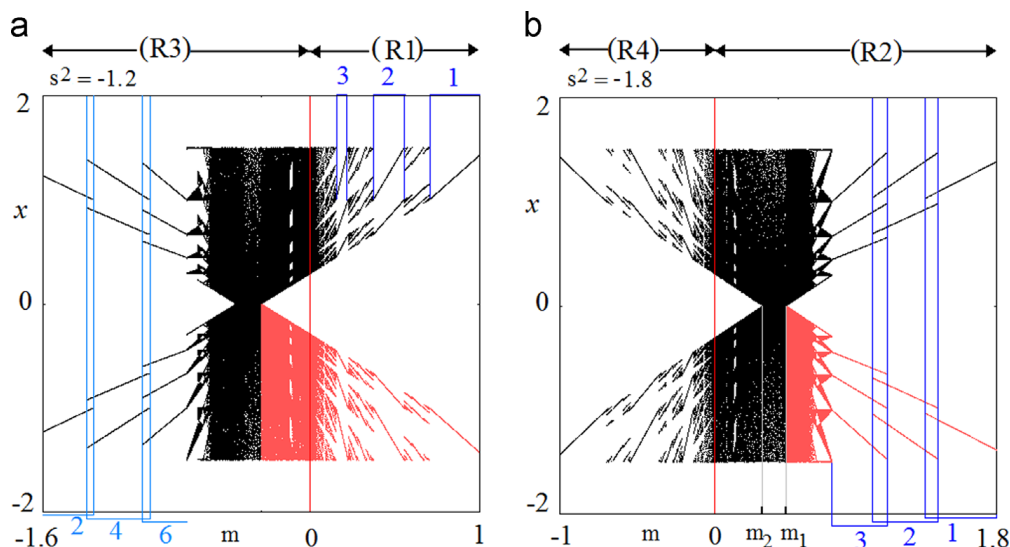


Fig. 6. One-dimensional bifurcation diagram as a function of m . In (a) along path (A) of Fig. 1b, at $S^2 = -1.2$; in (b) along path (B) of Fig. 1b, at $S^2 = -1.8$. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

interval, we know that there exists a regime of bistability, and a so-called increment structure for existing cycles (see Gardini and Tramontana, 2010). Thus we have to separate the cases in which two disjoint absorbing intervals exist, or only a unique one. For the particular value $(1 + S^1 + S^2) = 0$ the external branches are flat, which corresponds to the transition between regions (R1) and (R2). This case is also well known: the interval $(0, -(S^1 + S^2)) = (0, -1)$ consists of consecutive segments associated with one unique cycle in each absorbing interval; on the right they have the symbolic sequence RC_+^n (on the left LC_-^n). The boundary points between one period and the other correspond to so-called big bang bifurcation points in the parameter space (see Fig. 1b). They correspond to the intersection of two different border collision bifurcation curves. Infinitely many BCB curves are issued on one side from those points (in our case for $(1 + S^1 + S^2) > 0$), associated with the adding mechanism. On the other side (in our case for $(1 + S^1 + S^2) < 0$), we have the overlapping of two periodicity regions of cycles with symbolic sequence RC_+^k and RC_+^{k+1} on the right, and cycles with symbolic sequence LC_-^k and LC_-^{k+1} on the left. A few regions can be seen in Fig. 6b for $m > 0$ (those on the positive side are black those on the opposite side are red).

Thus as long as we have two absorbing intervals, on the right the existing attracting cycle is the unique attractor or two attracting cycles coexist (in which case, the two basins of attraction within the absorbing interval are separated by discontinuity point $x=1$ and its preimages). The eigenvalue of a cycle with symbolic sequence RC_+^n is given by

$$\lambda(RC_+^n) = (1 + S^1 + S^2)(1 + S^1)^n. \tag{35}$$

Note that this holds for any $n \geq 0$, as for $n=0$ we get fixed point x_R^* . It follows that a flip bifurcation⁹ of these cycles occurs when $\lambda(RC_+^n) = -1$ at

$$S^2 = -\frac{1}{(1 + S^1)^n} - (1 + S^1) \tag{36}$$

(see the horizontal segments that form the lower bounds of the colored periodicity regions in Fig. 1b for $m > 0$).

In the piecewise linear case we have chaotic dynamics in invariant intervals for discontinuous maps after the flip bifurcation of the maximal cycles. Note that the chaotic intervals may be or not cyclically invariant (as remarked in Avrutin et al., in press) or may not be made up by disjoint absorbing intervals. Now (contrary to case (R1)) we have

$$I_R = [F_R \circ F_C(1), F_C(1)] = [(1 + S^1)(1 + S^1 + S^2) + m, 1 + S^1] \tag{37}$$

(due to the negative slope of the external branches), and

$$I_L = [F_C(-1), F_L \circ F_C(-1)] = [-(1 + S^1), -(1 + S^1)(1 + S^1 + S^2) - m], \tag{38}$$

so that when

$$m = -(1 + S^1)(1 + S^1 + S^2) \tag{39}$$

occurs, the two intervals merge into one unique interval:

$$I = [F_C(-1), F_C(1)] = [-(1 + S^1), 1 + S^1]. \tag{40}$$

One example of the dynamics in two disjoint intervals (i.e. before the occurrence of the bifurcation described above, and given in (39)) is shown in Fig. 7, where the dynamics is either in the bull market or in the bear market, depending on the initial condition. The dynamic behavior after the bifurcation is shown in Fig. 8: a unique absorbing interval I exists, and the dynamics are oscillating (in a chaotic regime) between the two markets.¹⁰

A one-dimensional bifurcation diagram showing the dynamic behavior of our variable x is illustrated in Fig. 6b for $m > 0$ along horizontal path (B) shown in Fig. 1b. In Fig. 6b, the point corresponding to the bifurcation described above, given in (39), is marked m_1 . In the same figure, we also see that the dynamics is only chaotic throughout interval $I = [F_C(-1), F_C(1)] = [-(1 + S^1), 1 + S^1]$ in the range $m_2 < m < m_1$, while for $0 < m < m_2$, even if the dynamics involve all three branches of map F , the asymptotic behavior is confined in two symmetric intervals

$$[F_C(-1), F_R(1)] \cup [F_L(-1), F_C(1)] \subset I = [F_C(-1), F_C(1)] \tag{41}$$

and the chaotic intervals are bounded by the iterates of the offsets at the discontinuity points. The second bifurcation of the chaotic intervals occurs when the signs of the offsets of the external branches change, that is when

$$F_R(1) = 0, \tag{42}$$

which occurs at

$$m = -(1 + S^1 + S^2). \tag{43}$$

In Fig. 6b, the point corresponding to the bifurcation given in (43) is marked m_2 .

⁹ To be precise, a degenerate flip bifurcation, see Sushko and Gardini (2010).

¹⁰ Note that the dynamics depicted in Fig. 8 resembles the dynamics depicted in Fig. 2 of the asset-pricing model of Brock and Hommes (1998). Indeed, if the intensity of choice parameter goes to plus infinity and if the prediction rules of the traders are linear, the dynamics of their model is driven by a discontinuous piecewise-linear map. A similar remark also holds for other discrete-choice models, e.g. the cobweb approach of Brock and Hommes (1997). We hope that the mathematical tools developed in this and related papers make it possible to study the dynamics of such models in more detail.

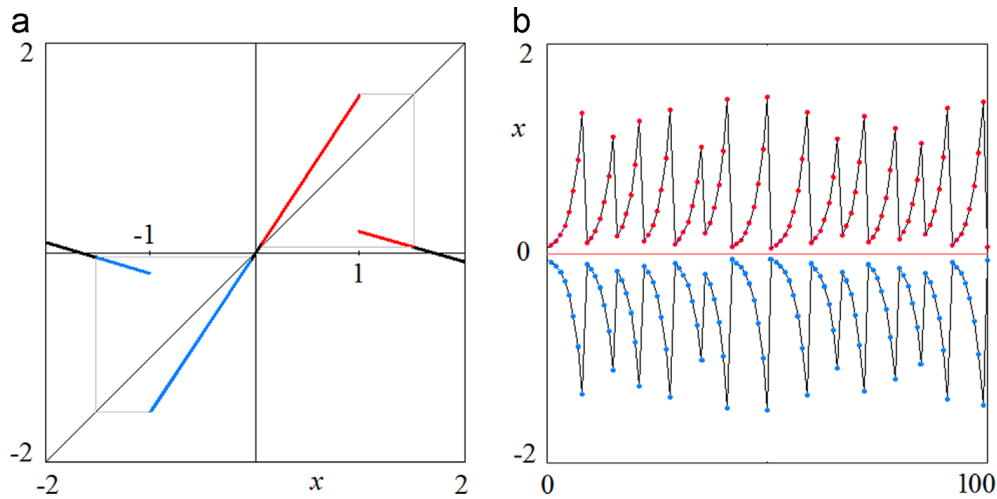


Fig. 7. In (a) $m=0.5$, $S^2 = -1.8$; in (b) versus time trajectory.

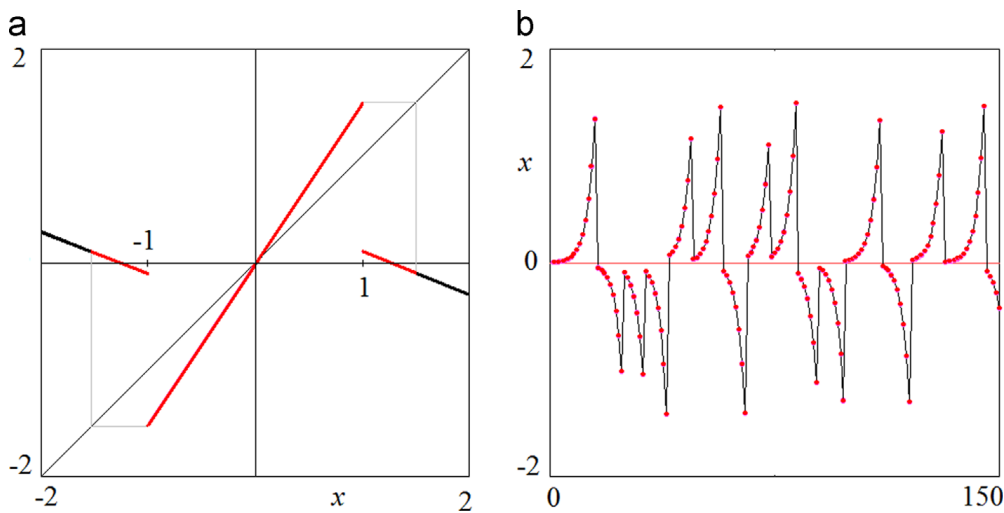


Fig. 8. In (a) $m=0.5$, $S^2 = -1.9$; in (b) versus time trajectory.

Thus we have proved the following:

Property 4. Let the parameters belong to the chaotic region in (R2), then

- for $m > -(1 + S^1)(1 + S^1 + S^2)$ two disjoint invariant intervals exist, on opposite sides with respect to the origin (disjoint bull and bear regimes);
- for $-(1 + S^1 + S^2) < m < -(1 + S^1)(1 + S^1 + S^2)$ the dynamics is chaotic in a unique invariant interval $I = [-(1 + S^1), 1 + S^1]$ including the origin (bull and bear intermingled regimes);
- for $0 < m < -(1 + S^1 + S^2)$ the dynamics is chaotic in an invariant set that does not include the origin (but still involves bull and bear intermingled regimes).

One example of the chaotic dynamics occurring in the third case, after the bifurcation given in (43) for $0 < m < m_2$ is illustrated in Fig. 9.

4.3. Region (R3): even increment structure and chaotic intervals

In region (R3) the external branches of F are increasing $(1 + S^1 + S^2) > 0$. For values of this slope close to 0, taking $m < 0$ such that $F_R(1) = (1 + S^1 + S^2) + m < 0$, it is possible to have stable cycles of any even period, involving all three branches of F except for the 2-cycle. In fact, the stable 2-cycle existing for $m < 0$ is that with symbolic sequence RL (one point of which has been already determined in (23), $x_R = -m/(2 + S^1 + S^2)$ and $x_L = -x_R$), so that its region of existence is bounded by the BCB curve of equation

$$S^2 = -m - S^1 - 2 \tag{44}$$

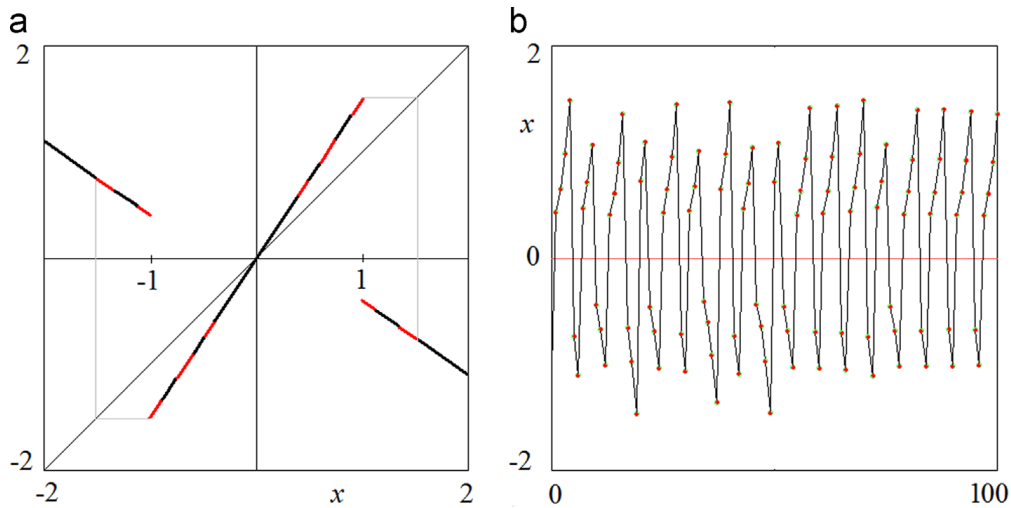


Fig. 9. In (a) $m = -0.3$, $S^2 = -2.2$; in (b) versus time trajectory.

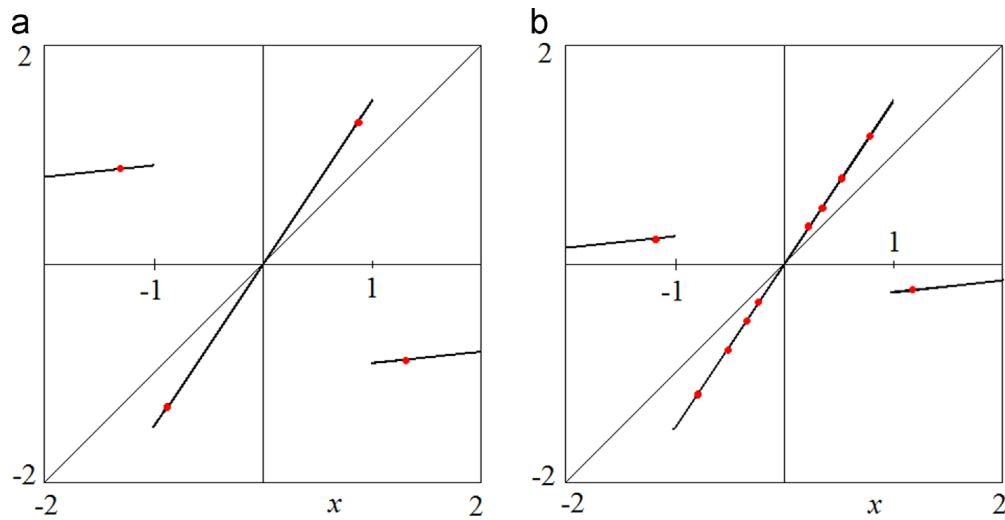


Fig. 10. In (a) $m = -1$, $S^2 = -1.4$; in (b) $m = -0.35$, $S^2 = -1.4$.

(see the straight line in Fig. 1b for $m < 0$ at the boundary of the region of the 2-cycle). All other stable cycles that may be observed in Fig. 1b in an increment structure with even periods have the symbolic sequence $RC_-^n LC_+^n$ for any $n \geq 0$ ($n=0$ leading to the 2-cycle RL).

One example of the 4-cycle is shown in Fig. 10a; a 10-cycle is given in Fig. 10b. Moreover, as in an usual period increment structure, there is a region of bistability between any two consecutive pairs $RC_-^n LC_+^n$ and $RC_-^{n+1} LC_+^{n+1}$ for any $n \geq 0$. This is a totally new increment scenario, associated with the existence of two discontinuity points, that leads to cycles with periodic points in all partitions. The BCB curves leading to their appearance and disappearance is due to the collision with the discontinuity points as follows. For any $n \geq 1$ one boundary (that on the left side in Fig. 1b for $m < 0$) is determined via the condition

$$F_C^{n-1} \circ F_L \circ F_C^n \circ F_R \circ F_C(1) = 1 \tag{45}$$

leading to the equation, for any $n \geq 1$:

$$m = \left[\frac{1}{(1+S^1)^{n-1}} - (1+S^1+S^2)^2(1+S^1)^{n+1} \right] / [(1+S^1+S^2)(1+S^1)^n - 1] \tag{46}$$

while the other boundary (that on the right side in Fig. 1b for $m < 0$) is determined via the condition

$$F_C^n \circ F_L \circ F_C^n \circ F_R(1) = 1 \tag{47}$$

leading to the equation, for any $n \geq 1$:

$$m = [1 - (1+S^1+S^2)^2(1+S^1)^{2n}] / [(1+S^1+S^2)(1+S^1)^{2n} - (1+S^1)^n]. \tag{48}$$

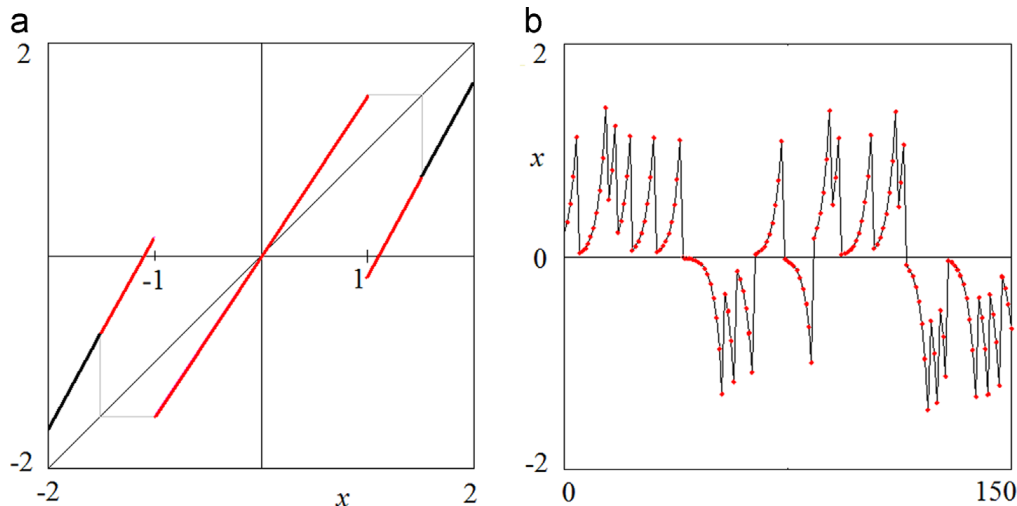


Fig. 11. In (a) $m = -2$, $S^2 = 0.319$; in (b) versus time trajectory.

The boundaries of the 4-cycle obtained for $n = 1$ are shown in Fig. 1b. This region overlaps with the region of the 2-cycle RL , leading to the related region of bistability.

The eigenvalue associated with these cycles of symbolic sequence $RC_-^n LC_+^n$ is

$$\lambda(RC_-^n LC_+^n) = (1 + S^1 + S^2)^2 (1 + S^1)^{2n} \tag{49}$$

so that they are stable as long as $\lambda(RC_-^n LC_+^n) < 1$ after which, for

$$S^2 > \frac{1}{(1 + S^1)^n} - (1 + S^1), \tag{50}$$

the cycle of period $(2n + 2)$ becomes unstable and we detect $2(2n + 2)$ chaotic intervals. Thus we have proved the following:

Property 5. Let the parameters belong to region (R3), then

- the existence region of a 2-cycle with symbolic sequence RL , $x_R = -m/(2 + S^1 + S^2)$ and $x_L = -x_R$, is bounded by the border collision bifurcation curve given in (44);
- the existence region of a cycle with symbolic sequence $RC_-^n LC_+^n$ for any $n \geq 1$ is bounded by two border collision bifurcation curves given in (46) and in (48);
- for $S^2 < 1/(1 + S^1)^n - (1 + S^1)$ cycle $RC_-^n LC_+^n$ for any $n \geq 0$ is stable ($n = 0$, corresponding to the 2-cycle RL);
- overlapping regions leading to bistability occur for each pair $RC_-^n LC_+^n$ and $RC_-^{n+1} LC_+^{n+1}$ for any $n \geq 0$.

One example of the dynamics as a function of m along path (A) of Fig. 1b is shown in Fig. 6a: a few periodicity regions (of cycles of period 2, 4, 6 and overlapping by pair) are crossed before chaotic dynamics emerges. In the white region in Fig. 1b we have attracting chaotic intervals, and we can observe that, as m increases, the dynamics first involves all three branches of the map, then two disjoint invariant absorbing intervals appear. That is, following arguments similar to those used in the previous region, it is easy to see that, as long as we have $F_R \circ F_C(1) < 0$, then the dynamics are invariant in two intervals that do not include the origin, that is, inside the set

$$[F_C(-1), F_R \circ F_C(1)] \cup [F_L \circ F_C(1), F_C(1)]$$

and a bifurcation occurs when $F_R \circ F_C(1) = 0$ at

$$m = -(1 + S^1)(1 + S^1 + S^2) \tag{51}$$

since then, as long as $F_R \circ F_C(1) > 0$ and $F_R(1) < 0$, the dynamics is chaotic in the interval

$$I = [F_C(-1), F_C(1)] = [-(1 + S^1), (1 + S^1)].$$

In both cases, the chaotic dynamics involves all branches of F , so that bull and bear dynamics are intermingled, an example of which is shown in Fig. 11.

This occurs as long as bifurcation $F_R(1) = 0$ takes place, when

$$m = -(1 + S^1 + S^2), \tag{52}$$

leading to two disjoint absorbing intervals, still with chaotic dynamics, but one in the bull region and the other in the bear region. One example of the disjoint chaotic dynamics is shown in Fig. 12.

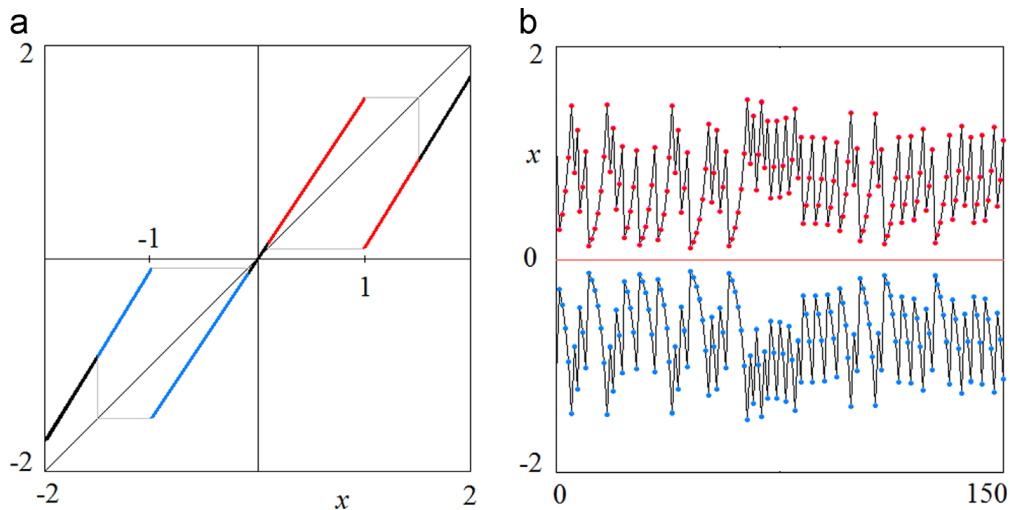


Fig. 12. In (a) $m = -1.5$, $S^2 = 0.1$; in (b) versus time trajectory.

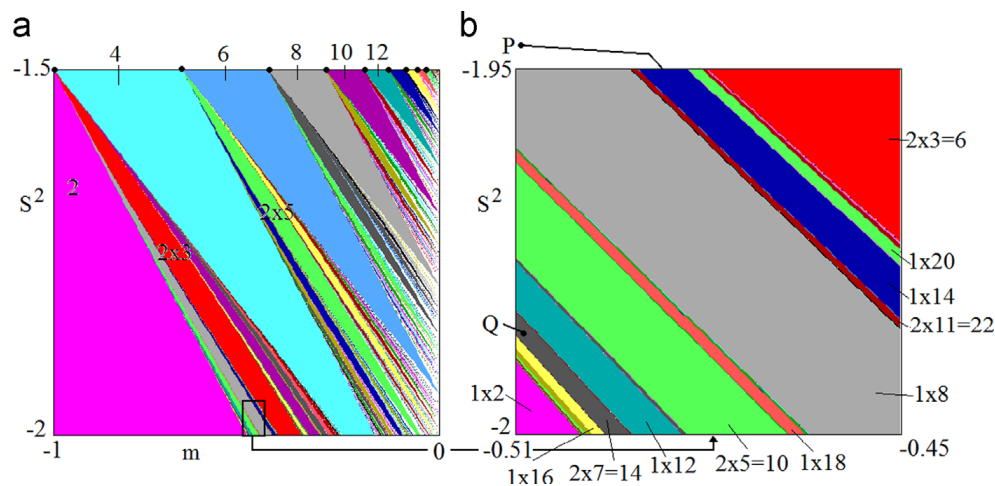


Fig. 13. Two-dimensional bifurcation diagram. In (a) enlarged portion of Fig. 1b in region (R4). In (b) enlarged portion of the rectangle in (a).

Thus we have proved the following:

Property 6. Let the parameters belong to the chaotic region in (R3), then

- for $m < -(1 + S^1)(1 + S^1 + S^2)$ the dynamics is chaotic in an invariant set not including the origin (but bull and bear intermingled regimes);
- for $-(1 + S^1)(1 + S^1 + S^2) < m < -(1 + S^1 + S^2)$ the dynamics is chaotic in a unique invariant interval $I = [-(1 + S^1), 1 + S^1]$ including the origin (bull and bear intermingled regimes);
- for $-(1 + S^1 + S^2) < m < 0$ the dynamics is chaotic in two disjoint invariant intervals, on opposite sides with respect to the origin (disjoint bull and bear regimes).

4.4. Region (R4): even adding structure and bistability

In this region (R4) with $m < 0$ we are interested in the dynamics inside the strip $-2 < S^1 + S^2 < -1$, i.e. $-1 < (1 + S^1 + S^2) < 0$. After all, we know that below, for $(1 + S^1 + S^2) < -1$, only divergent dynamics occur, as described in Section 3, and as is visible in Fig. 1. Here, as for region (R3), in the stability strip we have a new dynamic behavior of the map, that has never been investigated before. The branches of the map are increasing/decreasing in the positive discontinuity point and decreasing/increasing in the negative discontinuity; all three branches are involved in the periodic points of the existing cycles.

In Fig. 13a we have enlarged the portion of the parameters' plane associated with a new adding mechanism. The overlapping regions associated with stable cycles determined in (R3) and described in the previous subsection intersect at points of line $1 + S^1 + S^2 = 0$, creating infinitely many big bang bifurcation points. In fact, we can see that between the

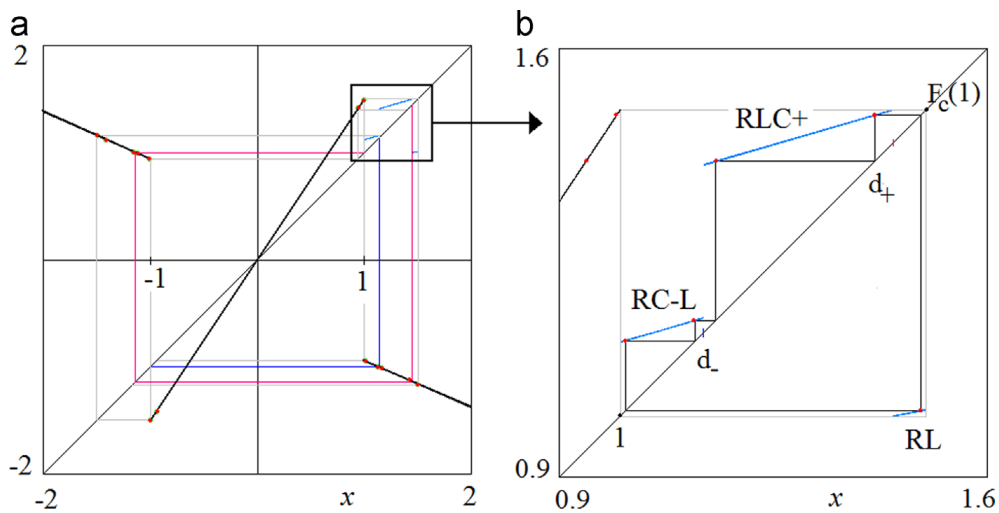


Fig. 14. In (a) shape of the map in P : ($m = -0.5, S^2 = -1.94$) with a stable 14-cycle; in (b) enlarged part with the first return map T_r , as defined in (54).

principal tongues (or main cycles) of symbolic sequence $RC_-^k LC_+^k$ and $RC_-^{k+1} LC_+^{k+1}$ for any $k \geq 0$ (whose existence has been determined in region (R3)) are now, in the region with $(1 + S^1 + S^2) < 0$, disjoint from one another, and we have an adding bifurcation structure. However, the adding mechanism is totally new (since it involves three branches).

The main property that characterizes the map in the stability region considered here (the strip defined above) is that for the invariant absorbing interval which includes the asymptotic behaviors of the map, intervals which are bounded by the critical points (iteration of the offsets in the discontinuity points), *the map has a unique inverse*. Thus, the map cannot have complex behaviors, and the invariant sets may only be stable cycles (when the dynamics are associated with a rational rotation number) or Cantor set attractors (when the rotation number is irrational). It is worth noticing that having two discontinuity points, the concept of “rotation number” is not yet very well defined although it has already been used for this kind of map, for example in Tramontana et al. (2012). However, the property of “invertibility”, which characterizes this regime, can be used to show that, although we have branches with negative slopes, all cycles can only have a positive eigenvalue. This is due to the fact that we can properly investigate the dynamics of the map making use of its first return map at interval $J = [1, F_c(1)]$, say T_r :

$$T_r(x) : J \rightarrow J, \quad J = [1, F_c(1)] \tag{53}$$

as any trajectory of F must necessarily visit this interval J , and the first return map T_r completely determines the dynamics of F . It is clear that the problem is not very simplified, as also the first return map in this interval J may be a discontinuous map with two discontinuity points: point d_- , which is the preimage in J of the discontinuity $x = -1$ of F , and point d_+ , which is the preimage in J of the discontinuity point $x = +1$. However, the advantage is that in the first return map we must have all positive slopes. This is because in order for a trajectory to return in interval J , the branch with a negative slope must necessarily be applied an even number of times.

As an example, consider parameter point P in Fig. 13b P : ($m = -0.5, S^2 = -1.94$), which is associated with a unique cycle of period 14. Fig. 14a shows the map and the points of the cycle of period 14; the enlarged part shows the first return map at interval J , and the three branches are defined by the symbolic sequence evidenced there, that is $T_r(x)$ is defined as follows:

$$T_r(x) : x' = \begin{cases} F_L \circ F_{C_-} \circ F_R(x) & \text{if } 1 < x < d_- \\ F_{C_+} \circ F_L \circ F_R(x) & \text{if } d_- < x < d_+ \\ F_L \circ F_R(x) & \text{if } d_+ < x < F_c(1), \end{cases} \tag{54}$$

where

$$d_- = F_R^{-1}(-1), \quad d_+ = F_R^{-1} \circ F_L^{-1}(1).$$

We can see that the symbolic sequence of the cycle is $RC_-LRC_-LRLC_+RLC_+RLRL$. However, in the adding mechanism we can also see 14 periodic points belonging to two coexisting symmetric cycles of period 7, which occurs when the parameters are at point Q of Fig. 13b, Q : ($m = -0.5, S^2 = -1.993$), as illustrated in Fig. 15. The definition of the first return map (53) is the same as before in (54). We can see that two symmetric cycles exist, with symbolic sequence RC_-LRLRL and the symmetric cycle, whose symbolic sequence is obtained by changing R, L and C_- into L, R and C_+ , respectively.

We recall that for a map with only one discontinuity point, in the summation rule of two cycles, the new one has the symbolic sequence obtained by concatenation of the symbols of the two starting cycles, so that the number of periodic points on the two sides of the discontinuity (say L and R) are given by the sum of letters L and R of the two starting cycles, and the period is the sum of the periods of the two starting cycles. In our case, we can reason similarly. However, we add cycles with four symbols (L, C_- and R, C_+), and in the summation rule of two cycles we have points in the four partitions. Regarding the result of the summation rule between two cycles $RC_-^k LC_+^k$ and $RC_-^{k+1} LC_+^{k+1}$, we have the symbolic sequences of

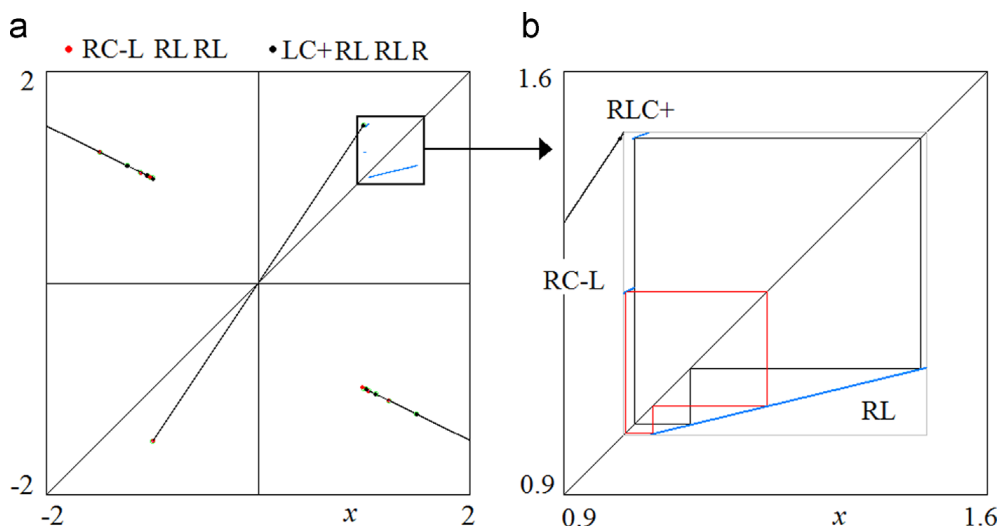


Fig. 15. In (a) the shape of the map in $Q : (m = -0.5, S^2 = -1.993)$ with two symmetric stable 7-cycles; in (b) an enlarged part with the first return map T_r , as defined in (54).

type $(RC_-^k LC_+^k)^n (RC_-^{k+1} LC_+^{k+1})$ and $(RC_-^k LC_+^k) (RC_-^{k+1} LC_+^{k+1})^n$ and we can either have a unique cycle or a pair of symmetric cycles (with odd period), which is a feature of the symmetry property and the two discontinuity points. What is relevant (in order to have one case or the other) is the number of periodic points existing in the external branches (L and R) which, due to the symmetry of the map, are the same in number and symmetric with respect to $x=0$. Hence, if the summation of two cycles leads to an *odd number of periodic points* on the L side (and, equivalently, on the R side), then a unique cycle exists (as in the example of the 14-cycle in Fig. 14a, where we have 5 periodic points in branches L and R). If the summation of the two cycles leads to an *even number of periodic points* on the L side (and, equivalently, on the R side), then two symmetric cycles appear, with an odd period (as in the example in Fig. 15a, the summation rule leads to 6 periodic points in branches L and R , so that a pair of symmetric cycles exist).

Although we are unable to give rigorous proof of this empirical result (which has been left for further research work), it has been numerically verified in our simulations and bifurcation diagrams (see Fig. 13b). As the principal cycles are those of symbolic sequence $RC_-^k LC_+^k$ having only one point in the external branches, we have that the adding of two consecutive sequences, say $RC_-^k LC_+^k$ and $RC_-^{k+1} LC_+^{k+1}$, leads to 2 points in the external branches, which is even. Hence two symmetric cycles exist (with odd period). However, this is only one step of the adding mechanism. For example, for $k=0$, between the periodicity regions of 2-cycle RL and of 4-cycle $RC_- LC_+$, we have families $(RL)^n (RC_- LC_+)$ and $(RL) (RC_- LC_+)^n$ for any $n \geq 1$ and at $n=1$ we have the pair of 3-cycles, as commented above. Then in family $(RL)^n (RC_- LC_+)$ as n increases, we add only one point in the external branches, so that we have alternating single cycles or pairs of cycles of odd periods (see the enlargement in Fig. 13b).

5. Conclusions

In our paper we aimed to generalize the famous bull and bear market models of Day and Huang (1990) and, in particular, Huang and Day (1993) and to improve our understanding of the dynamics of discontinuous piecewise-linear maps. Indeed, by relaxing some restrictive behavioral assumptions about speculators' trading activities, we succeeded in extending the model of Huang and Day such that it changes from a continuous piecewise-linear map to a discontinuous one. Since the map has two discontinuity points, little is known about its dynamic properties. Nevertheless, we have managed to provide a more or less complete analytical and numerical characterization of our model.

From a mathematical point of view, we describe a number of dynamics and bifurcation structures that have never been investigated before. While standard results can be used to explain the dynamic behaviors in regions ($R1$) and ($R2$), the dynamics occurring in region ($R3$) is due to a new kind of phenomenon. In Section 4.3 we introduced a totally new period increment scenario, with cycles that have periodic points in all three partitions of the map. The symbolic sequences and bifurcation structure depend on the existence of two discontinuity points. We analytically determined the equations of the border collision bifurcation curves associated with their existence, the region of bistability between any two consecutive pair of cycles $RC_-^n LC_+^n$ and $RC_-^{n+1} LC_+^{n+1}$ for any $n \geq 0$, as well as their degenerate flip bifurcation. The conditions under which chaotic regimes are intermingled in the bull and bear regions were also determined. Moreover, in the stability strip observed in region ($R4$) we described a new period adding structure. The peculiarity, due to the symmetry property and to the two discontinuity points, is that in the adding mechanism we can have either a unique cycle or a pair of symmetric cycles (with odd period). Bistability cannot occur in maps with one discontinuity only and increasing branches, and we showed that our map can be studied making use of the first return map in a suitable interval. The increasing branches and invertibility of the

first return map explain the existence of the adding structure. We also empirically understood the peculiarity of the bistability, as described in Section 4.4, although rigorous proof has been left for further studies.

From an economic point of view, we conclude that the alternating switching between irregular bull and bear markets, as first reported by Day and Huang, is a rather robust finding for such types of model. Moreover, we detect a number of boom-bust cycles that are even more intricate than those observed in Day and Huang's original models. For instance, while in their model a bull market only turns into a bear market if prices have reached certain extremely high price levels, our model may also produce such market crashes for intermediate price levels. In addition, we determined, in parameter space, the border that separates bounded dynamics from divergent dynamics. Our analysis reveals that both chartists and fundamentalists may contribute to market stability/instability—which makes the regulation of financial markets such a delicate issue.

Our model can be extended in many ways. For instance, one could assume that the market entry level of type 2 chartists differs from that of type 2 fundamentalists, where the result would be a map with five branches. It could also be assumed that speculators react differently to overvalued and undervalued market situations, leading to an asymmetric map. Note, however, that the current map has not even been fully explored yet. For instance, the case where the inner regime is unstable due to aggressive transactions by fundamentalists has not yet been addressed (as far as we know, the same is true for a pure mathematical analysis).

Instead of considering a deterministic model, as in our paper, a stochastic model specification could also be explored. Preliminary work we have performed reveals that our setup, if polluted with random noise, is quite capable of mimicking some important stylized facts of financial markets. For instance, extreme price changes may be found close to the market entry levels of type 2 speculators, even when prices reach quite high or low levels. Note that deterministic skeletons of such stochastic models are an excellent way of developing a better understanding of their complex price dynamics.

Both deterministic and stochastic model versions of our setup may also be used to enhance our knowledge about how regulatory measures function. For instance, one may incorporate a central authority in our model and study how certain intervention (feedback) strategies, such as targeting long-run fundamental rules or leaning against the wind rules, influence the model dynamics. Given the instability of financial markets, this may be a worthwhile endeavor.

Of course, many other extensions of our work are possible. We hope that our paper stimulates further work in the exciting analysis of bull and bear market models and the rather novel use of piecewise-linear maps to investigate them. In addition, the study of discontinuous piecewise-linear maps seems to us to be important per se, as many economic problems entail natural breaks or hard boundaries.

Acknowledgments

We thank two anonymous referees and the editor, Cars Hommes, for helpful comments and suggestions. This work was performed within the national research Project PRIN-2009NZNM7C “Local interactions and global dynamics in economics and finance: models and tools”, MIUR, Italy, and under the auspices of COST Action IS1104 “The EU in the new complex geography of economic systems: models, tools and policy evaluation”.

References

- Avrutin, V., Schanz, M., Gardini, L., 2010. Calculation of bifurcation curves by map replacement. *International Journal of Bifurcation and Chaos* 20 (10), 3105–3135.
- Avrutin, V., Sushko, I., Gardini, L. Cyclicity of chaotic attractors in one-dimensional discontinuous maps. *Mathematics and Computers in Simulation*, <http://dx.doi.org/10.1016/j.matcom.2012.07.019>, in press.
- Beja, A., Goldman, B., 1980. On the dynamic behavior of prices in disequilibrium. *Journal of Finance* 35, 235–248.
- Brock, W., Hommes, C., 1997. A rational route to randomness. *Econometrica* 65, 1059–1095.
- Brock, W., Hommes, C., 1998. Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *Journal of Economic Dynamics and Control* 22, 1235–1274.
- Chiarella, C., He, X.-Z., Huang, W., Zheng, H., 2012. Estimating behavioural heterogeneity under regime switching. *Journal of Economic Behavior and Organization* 83, 446–460.
- Chiarella, C., Dieci, R., He, X.-Z., 2009. Heterogeneity, market mechanisms, and asset price dynamics. In: Hens, T., Schenk-Hoppé, K.R. (Eds.), *Handbook of Financial Markets: Dynamics and Evolution*, North-Holland, Amsterdam, pp. 277–344.
- Day, R., Huang, W., 1990. Bulls, bears and market sheep. *Journal of Economic Behavior and Organization* 14, 299–329.
- Day, R., 1997. Complex dynamics, market mediation and stock price behavior. *North American Actuarial Journal* 1, 6–21.
- Frankel, J., Froot, K., 1986. Understanding the U.S. dollar in the eighties: the expectations of chartists and fundamentalists. *Economic Record* 62, 24–38.
- Gardini, L., Tramontana, F., Avrutin, V., Schanz, M., 2010. Border Collision Bifurcations in 1D PWL map and Leonov's approach. *International Journal of Bifurcation and Chaos* 20 (10), 3085–3104.
- Gardini, L., Tramontana, F., 2010. Border Collision Bifurcations in 1D PWL map with one discontinuity and negative jump. Use of the first return map. *International Journal of Bifurcation and Chaos* 20 (11), 3135–3529.
- Gu, M., 1993. An empirical examination of the deterministic component in stock price volatility. *Journal of Economic Behavior and Organization* 22, 239–252.
- Gu, M., 1995. Market mediating behavior: an economic analysis of the security exchange specialists. *Journal of Economic Behavior and Organization* 27, 237–256.
- Hommes, C., Wagener, F.O.O., 2009. Complex evolutionary systems in behavioral finance. In: Hens, T., Schenk-Hoppé, K.R. (Eds.), *Handbook of Financial Markets: Dynamics and Evolution*, North-Holland, Amsterdam, pp. 217–276.
- Hommes, C., 2011. The heterogeneous expectations hypothesis: some evidence from the lab. *Journal of Economic Dynamics and Control* 35, 1–24.
- Huang, W., Day, R., 1993. Chaotically switching bear and bull markets: the derivation of stock price distributions from behavioral rules. In: Day, R., Chen, P. (Eds.), *Nonlinear Dynamics and Evolutionary Economics*, Oxford University Press, Oxford, pp. 169–182.

- Huang, W., Zheng, H., Chia, W.M., 2010. Financial crisis and interacting heterogeneous agents. *Journal of Economic Dynamics and Control* 34, 1105–1122.
- Huang, W., Zheng, H., Chia, W.M., 2012. Asymmetric returns, gradual bubbles and sudden crashes. *European Journal of Finance* 19, 420–437.
- Huang, W., Zheng, H., 2012. Financial crisis and regime-dependent dynamics. *Journal of Economic Behavior and Organization* 82, 445–461.
- Keener, J.P., 1980. Chaotic behavior in piecewise continuous difference equations. *Transactions of the American Mathematical Society* 261 (2), 589–604.
- Leonov, N.N., 1959. On a pointwise mapping of a line into itself. *Radiofizika* 2 (6), 942–956.
- Leonov, N.N., 1962. On a discontinuous pointwise mapping of a line into itself. *Doklady Akademii Nauk SSSR* 143 (5), 1038–1041.
- Lux, T., 2009. Stochastic behavioural asset-pricing models and the stylized facts. In: Hens, T., Schenk-Hoppé, K.R. (Eds.), *Handbook of Financial Markets: Dynamics and Evolution*, North-Holland, Amsterdam, pp. 161–216.
- Manzan, S., Westerhoff, F., 2007. Heterogeneous expectations, exchange rate dynamics and predictability. *Journal of Economic Behavior and Organization* 64, 111–128.
- Menkhoff, L., Taylor, M., 2007. The obstinate passion of foreign exchange professionals: technical analysis. *Journal of Economic Literature* 45, 936–972.
- Sushko, I., Gardini, L., 2010. Degenerate bifurcations and border collisions in piecewise smooth 1D and 2D maps. *International Journal of Bifurcation and Chaos* 20 (7), 2045–2070.
- Tramontana, F., Westerhoff, F., Gardini, L., 2010a. On the complicated price dynamics of a simple one-dimensional discontinuous financial market model with heterogeneous interacting traders. *Journal of Economic Behavior and Organization* 74, 187–205.
- Tramontana, F., Gardini, L., Westerhoff, F., 2010b. Intricate asset price dynamics and one-dimensional discontinuous maps. In: Puu, T., Panchuck, A. (Eds.), *Nonlinear Economic Dynamics*, Nova Science Publishers, pp. 43–57.
- Tramontana, F., Gardini, L., Westerhoff, F., 2011a. Heterogeneous speculators and asset price dynamics: further results from a one-dimensional discontinuous piecewise-linear map. *Computational Economics* 38, 329–347.
- Tramontana, F., Westerhoff, F., Gardini, L., 2011b. A simple financial market model with chartists and fundamentalists: market entry levels and discontinuities. *Mathematics and Computers in Simulation*, in press.
- Tramontana, F., Gardini, L., Avrutin, V., Schanz, M., 2012. Period adding in piecewise linear maps with two discontinuities. *International Journal of Bifurcation and Chaos* 22 (3), 1250068, 1–30.
- Tramontana, F., Westerhoff, F., Gardini, L. One-dimensional maps with two discontinuity points and three linear branches: mathematical lessons for understanding the dynamics of financial markets. *Decisions in Economics and Finance*, <http://dx.doi.org/10.1007/s10203-013-0145-y>, in press.
- Tramontana, F., Westerhoff, F., 2013. One-dimensional discontinuous piecewise-linear maps and the dynamics of financial markets. In: Bischi, G.I., Chiarella, C., Sushko, I. (Eds.), *Global Analysis of Dynamic Models in Economics and Finance*, Springer, New York, pp. 205–227.
- Venier, G., 2008. A new model for stock price movements. *Journal of Applied Economic Sciences* 3, 329–350.
- Vigfusson, R., 1997. Switching between chartists and fundamentalists: a Markov regime-switching approach. *International Journal of Finance and Economics* 2, 291–305.
- Westerhoff, F., 2009. Exchange rate dynamics: a nonlinear survey. In: Rosser Jr, J.B. (Ed.), *Handbook of Research on Complexity*, Edward Elgar, Cheltenham, pp. 287–325.
- Westerhoff, F., Franke, R., 2012. Converse trading strategies, intrinsic noise and the stylized facts of financial markets. *Quantitative Finance* 12, 425–436.
- Wieland, C., Westerhoff, F., 2005. Exchange rate dynamics, central bank intervention and chaos control methods. *Journal of Economic Behavior and Organization* 58, 117–132.
- Zeeman, E., 1974. On the unstable behavior of stock exchanges. *Journal of Mathematical Economics* 1, 39–49.