# Asset Price Dynamics and Diversification with Heterogeneous Agents

Carl Chiarella[1], Roberto Dieci[2], and Laura Gardini[3]

[1] School of Finance and Economics, University of Technology Sydney, PO Box 123 Broadway NSW 2007, Australia carl.chiarella@uts.edu.au
[2] Dipartimento di Matematica per le Scienze Economiche e Sociali, University of Bologna, I-40126 Bologna, Italy rdieci@rimini.unibo.it
[3] Istituto di Scienze Economiche, University of Urbino, I-61029 Urbino, Italy gardini@econ.uniurb.it

**Summary.** A discrete-time dynamic model of a financial market is developed, where two types of agents, *fundamentalists* and *chartists*, allocate their wealth between two risky assets and a safe asset, according to one-period mean-variance maximization. The two groups of agents form different expectations about asset returns and their variance/covariance structure, and this results in different demand functions. At the end of each trading period, agents' demands are aggregated by a *market maker*, who sets the next period prices as functions of the excess demand. The model results in a high-dimensional nonlinear discrete-time dynamical system, which describes the time evolution of prices and agents' beliefs about expected returns, variances and correlation. It is shown that the unique steady state may become unstable through a Hopf-bifurcation and that an attracting limit cycle, or more complex attractors, exist for particular ranges of the key parameters. In particular, the two risky assets may exhibit "coupled" long-run price fluctuations and time-varying correlation of returns.

## 1 Introduction

A key assumption in modern portfolio theory is that of rational, homogeneous agents who have complete knowledge of the distribution of future asset returns. However, both the homogeneity and the rationality assumptions have started to look tenuous, as shown by several theoretical and empirical studies (see [14], [12], [10]), and interest has grown in recent years in models of financial market dynamics based on the interaction of heterogeneous agents, who seek to learn about the future distribution of asset returns using different information sets (see e.g. [9], [3], [13], [7], [8], [5]). These models in general consider a financial market with one risky asset and one riskless asset and analyze the dynamic effect of agents' heterogeneous beliefs about expected return and volatility [4]. One of the main findings of such models is that the interaction of heterogeneous beliefs with the market trading mechanism can generate sustained deviations of the price away from the "fundamental" equilibrium, as well as more complex dynamic scenarios, even without the intervention of external random events. In addition, the interaction of the underlying deterministic dynamics with simple noise processes is able to generate the fat tails and volatility clustering that are a key feature of asset returns in financial markets. One of simplest ways to model heterogeneity in financial markets is to consider two groups of agents', *fundamentalists* and *chartists* (see e.g. [9], [4], [5]). The models with fundamentalists and chartists show that the former act in general as a stabilizing force, because their demand brings back the price to its fundamental value, while the latter may have a destabilizing role because their demand pushes the price in the direction of the current trend: chartists can thus cause wide price fluctuations, especially when their beliefs are sensitive to the most recent price history.

The basic case with one risky and one risk-free asset is only a first step to understanding the effect of heterogeneous agents interaction on asset price dynamics. In a multiple risky asset framework, the way agents form and update their beliefs about correlation also becomes an important factor in the investors' decision process. A natural question that arises in this context is whether these beliefs can cause "coupled" fluctuations of the prices of the risky assets, and to what extent the assets become "interdependent" due to agents' portfolio diversification.

In this paper we develop a discrete time model of financial market dynamics, which combines the essential elements of the heterogeneous agents paradigm with the classical model of *diversification* between two risky assets and a risk-free asset. In common with the earlier cited literature, we assume that the market consists of two types of traders: *fundamentalists*, who hold an estimate of the fundamental values of the risky assets and whose demand for each asset is a function of the deviation of the current price from the fundamental, and *chartists*, whose trading strategies are based on an extrapolation of the observed price trends, as well as of the past volatility and correlation of returns. At the end of each trading period a *market maker* aggregates agents' demand for each asset, and announces the next period prices on the basis of the excess demand. The model is reduced to a 7-dimensional nonlinear dynamical system that describes the time evolution of prices and agents' beliefs about expected returns, variances and correlation. The local stability analysis of the unique equilibrium, together with numerical simulations, allows to understand how interdependent fluctuations of prices may arise, due to agents' time varying beliefs and demands.

[4]See however [2] for a heterogeneous agent framework that allows for multiple risky assets. [15] also considers a fundamentalist-chartist model with multiple assets, but his framework is quite different from that studied in this paper

The structure of the paper is as follows. Sect. 2 derives the agents' demand functions for each asset. Sect. 3 describes the schemes used by each group to revise expectations. Sect. 4 describes the price adjustment rules and the resulting dynamical system for prices, expected returns, variances and correlation. Sect. 5 outlines the main analytical results about the local asymptotic stability of the unique steady state of the model and its dependence on the key parameters (chartist extrapolation parameter, risk aversion coefficients, price reaction parameters). Sect. 6 explores the out-of-equilibrium dynamics and shows how "coupled" long-run fluctuations of prices may emerge. Sect. 7 highlights the role of agents' beliefs in determining time varying correlation of returns. Sect. 8 contains some conclusions and final remarks.

## 2 Asset Demand

We derive the asset demands in a standard one period mean variance framework, but we assume that agents have heterogeneous beliefs about the distribution of future returns and update dynamically their beliefs as a function of observed returns. Our starting point is the fundamentalist/chartist model studied in [5], whose antecedents are [4], [9], and [1].

We denote by $P_{i,t}$ the log price of the $i$-th risky asset at time $t$ $(i = 1, 2)$, and use the subscript $j \in \{f, c\}$ to denote fundamentalists or chartists. In each time period each group of agents is assumed to invest some of its wealth in the risky assets and some in the risk-free asset. Denote, respectively, by $\Omega_t^{(j)}$ and $Z_{i,t}^{(j)}$ the wealth of agent $j$ at time $t$ and the fraction that agent $j$ decides to invest in the $i$-th risky asset. The evolution of the wealth of agent $j$ can then be written

$$\Omega_{t+1}^{(j)} = \Omega_t^{(j)} + \Omega_t^{(j)}(1 - Z_t^{(j)})r + \Omega_t^{(j)} \sum_{i=1}^{2} Z_{i,t}^{(j)}(P_{i,t+1} - P_{i,t} + D_{i,t+1})$$

where $Z_t^{(j)} = Z_{1,t}^{(j)} + Z_{2,t}^{(j)}$ is the fraction invested in the risky assets, $r$ is the (constant) risk-free rate of return, $D_{i,t+1}$, $(P_{i,t+1} - P_{i,t})$ and $(P_{i,t+1} - P_{i,t} + D_{i,t+1})$, are the *dividend yield*, the *capital gain* and the *return* of the $i$-th asset in period $(t, t+1)$, respectively. We denote by $E_t^{(j)}$, $Var_t^{(j)}$, $Cov_t^{(j)}$ the "beliefs" of investor type $j$, at time $t$, about conditional expectation, variance, and covariance, respectively. We assume that investor type $j$ has CARA utility of wealth function $u(\Omega) = -\exp(-\alpha^{(j)}\Omega)$, where $\alpha^{(j)}$ is agent $j$'s risk aversion coefficient. Agent $j$ seeks the fractions $Z_{i,t}^{(j)}$, so as to maximize expected utility of wealth at time $t+1$, $E_t^{(j)}[-\exp(-\alpha^{(j)}\Omega_{t+1}^{(j)})]$. Assuming that returns are conditionally normally distributed in agents' beliefs, the problem becomes

$$\max_{Z_{1,t}^{(j)}, Z_{2,t}^{(j)}} \left\{ E_t^{(j)}[\Omega_{t+1}^{(j)}] - \frac{\alpha^{(j)}}{2} Var_t^{(j)}[\Omega_{t+1}^{(j)}] \right\}$$

The first order conditions of the foregoing optimization problem lead to the demand functions for the risky assets, $\zeta_{i,t}^{(j)} \equiv Z_{i,t}^{(j)} \Omega_t^{(j)}$, $i = 1, 2$, given by

$$\zeta_{1,t}^{(j)} = \frac{1}{(1-(\rho_t^{(j)})^2)} \frac{(m_{1,t}^{(j)} + d_{1,t}^{(j)} - r)}{\alpha^{(j)} V_{1,t}^{(j)}} - \frac{\rho_t^{(j)} \sqrt{V_{2,t}^{(j)}}}{(1-(\rho_t^{(j)})^2)\sqrt{V_{1,t}^{(j)}}} \frac{(m_{2,t}^{(j)} + d_{2,t}^{(j)} - r)}{\alpha^{(j)} V_{2,t}^{(j)}} \quad (1)$$

$$\zeta_{2,t}^{(j)} = \frac{1}{(1-(\rho_t^{(j)})^2)} \frac{(m_{2,t}^{(j)} + d_{2,t}^{(j)} - r)}{\alpha^{(j)} V_{2,t}^{(j)}} - \frac{\rho_t^{(j)} \sqrt{V_{1,t}^{(j)}}}{(1-(\rho_t^{(j)})^2)\sqrt{V_{2,t}^{(j)}}} \frac{(m_{1,t}^{(j)} + d_{1,t}^{(j)} - r)}{\alpha^{(j)} V_{1,t}^{(j)}} \quad (2)$$

where $m_{i,t}^{(j)} \equiv E_t^{(j)}[P_{i,t+1} - P_{i,t}]$, $d_{i,t}^{(j)} \equiv E_t^{(j)}[D_{i,t+1}]$, $V_{i,t}^{(j)} \equiv Var_t^{(j)}[P_{i,t+1} - P_{i,t} + D_{i,t+1}]$, and $\rho_t^{(j)}$ is agent $j$'s "belief", at time $t$, about the correlation between the risky returns over the next trading period i.e.

$$\rho_t^{(j)} = Cov_t^{(j)}[(P_{1,t+1} - P_{1,t} + D_{1,t+1}), (P_{2,t+1} - P_{2,t} + D_{2,t+1})] / \sqrt{V_{1,t}^{(j)} V_{2,t}^{(j)}}$$

The demand for each risky asset is a linear combination of the expected risk adjusted excess returns, with coefficients being determined by the expected correlation, and has a fairly standard interpretation as a *direct* demand (the first term in (1) and (2)) and a *hedging* demand (the second term). In the next section we describe how agents update these "beliefs" about future returns and so generate different demand functions.

## 3 Agents' Heterogeneous Beliefs

The two groups of agents differ in the way they update their beliefs about expected returns, and variances and correlation of returns over successive time intervals. For simplicity it is assumed that the dividend yields are i.i.d. and uncorrelated with price changes in agents' beliefs, and that agents share the same beliefs about the dividend yields, with $E_t(D_{i,t+1}) \equiv d_i$, $Var_t(D_{i,t+1}) \equiv \sigma_i^2$, $i = 1, 2$, $Cov_t(D_{1,t+1}, D_{2,t+1}) \equiv \delta\sigma_1\sigma_2$. The common beliefs about variances ($\sigma_i^2$, $i = 1, 2$) and correlation ($\delta$) of the dividend yields determine the "long-run" or "equilibrium" variance/covariance structure of returns in this model. Agents' heterogeneity is introduced by assuming that fundamentalists and chartists have different beliefs about the "price" component of the return ($P_{i,t+1} - P_{i,t}$), $i = 1, 2$, i.e. about expected values, variances and correlation of "capital gains".

### 3.1 Fundamentalist Beliefs

We assume the *fundamental values* of the risky assets as exogenously given, constant over time or growing at constant rates, so that log fundamental values evolve according to

$$W_{i,t+1} = W_{i,t} + g_i \tag{3}$$

where $g_i$ ($g_i \geq 0$) represents the underlying growth rate of the "equilibrium" price of asset $i$, $i = 1, 2$. Fundamentalists are assumed to know both $W_{i,t}$ and $g_i$. They believe that the expected capital gain of asset $i$ contains a "long-run" or "equilibrium" component and a "short-run" component, the latter being proportional to the deviation of the log fundamental from the log price. Hence they calculate the expected change of log-price according to

$$m_{i,t}^{(f)} \equiv E_t^{(f)}[P_{i,t+1} - P_{i,t}] = \eta_i(W_{i,t} - P_{i,t}) + g_i$$

where $\eta_i > 0$ represents the fundamentalist estimate of the speed of reversion to the fundamental price. We also assume that fundamentalist beliefs about variances and correlation do not vary over time, being given by the long-run variance/covariance structure that is determined by that of the dividend yield process. Thus we set $V_{i,t}^{(f)} = \sigma_i^2$, $\rho_t^{(f)} = \delta$, so that $Cov_t^{(f)} = \delta\sigma_1\sigma_2$. The fundamentalist demand functions thus become

$$\zeta_{1,t}^{(f)} = a_1(W_{1,t} - P_{1,t}) - b_2(W_{2,t} - P_{2,t}) + h_1 \tag{4}$$

$$\zeta_{2,t}^{(f)} = a_2(W_{2,t} - P_{2,t}) - b_1(W_{1,t} - P_{1,t}) + h_2 \tag{5}$$

where $a_i = \eta_i/[\alpha^{(f)}(1-\delta^2)\sigma_i^2]$, $b_i = \delta\eta_i/[\alpha^{(f)}(1-\delta^2)\sigma_1\sigma_2]$, $i = 1, 2$, and

$$h_1 \equiv \frac{1}{(1-\delta^2)}\frac{\pi_1}{\alpha^{(f)}\sigma_1^2} - \frac{\delta\sigma_2}{(1-\delta^2)\sigma_1}\frac{\pi_2}{\alpha^{(f)}\sigma_2^2} \tag{6}$$

$$h_2 \equiv \frac{1}{(1-\delta^2)}\frac{\pi_2}{\alpha^{(f)}\sigma_2^2} - \frac{\delta\sigma_1}{(1-\delta^2)\sigma_2}\frac{\pi_1}{\alpha^{(f)}\sigma_1^2} \tag{7}$$

The symbol $\pi_i \equiv (g_i + d_i - r)$, denotes the long-run expected excess return (risk premium) of asset $i$, determined by the growth of fundamental and the dividend yield. Notice also that $h_1 \equiv \overline{\zeta}_1^{(f)}$ and $h_2 \equiv \overline{\zeta}_2^{(f)}$ can be interpreted as constant "long-run" or "equilibrium" components of fundamentalist demands.

### 3.2 Chartist Beliefs

We assume that chartists compute expected price changes for each asset by looking at the past price trends; in a similar way chartists form their beliefs about variances and correlation of future price changes by looking at past deviations from expected trends. In chartists' computations, the past data are extrapolated according to time averages with exponentially decreasing weights. Chartists' beliefs about expected values, variances and covariance of log-price changes of the next period are thus given by:

$$m_{i,t}^{(c)} \equiv E_t^{(c)}[P_{i,t+1} - P_{i,t}] = \sum_{s=0}^{\infty} c(1-c)^s(P_{i,t-s} - P_{i,t-s-1}) \tag{8}$$

$$v_{i,t} \equiv Var_t^{(c)}[P_{i,t+1} - P_{i,t}] = \sum_{s=0}^{\infty} c(1-c)^s (P_{i,t-s} - P_{i,t-s-1} - m_{i,t}^{(c)})^2 \quad (9)$$

$$K_t \equiv Cov_t^{(c)}[(P_{1,t+1} - P_{1,t}), (P_{2,t+1} - P_{2,t})] = \\ = \sum_{s=0}^{\infty} c(1-c)^s (P_{1,t-s} - P_{1,t-s-1} - m_{1,t}^{(c)})(P_{2,t-s} - P_{2,t-s-1} - m_{2,t}^{(c)}) \quad (10)$$

which result in the following adaptive updating rules (see [6] for details)

$$m_{i,t}^{(c)} = (1-c)m_{i,t-1}^{(c)} + c(P_{i,t} - P_{i,t-1}) \quad (11)$$

$$v_{i,t} = (1-c)v_{i,t-1} + c(1-c)(P_{i,t} - P_{i,t-1} - m_{i,t-1}^{(c)})^2 \quad (12)$$

$$K_t = (1-c)K_{t-1} + c(1-c)(P_{1,t} - P_{1,t-1} - m_{1,t-1}^{(c)})(P_{2,t} - P_{2,t-1} - m_{2,t-1}^{(c)}) \quad (13)$$

The chartist *extrapolation parameter* $c$ $(0 < c < 1)$ represents the weight given to the most recent price change in the computation of the time average: the higher is $c$, the more sensitive are chartists to recent data in updating their beliefs. Finally, the chartists' conditional variances and correlation of asset returns will be given by

$$V_{i,t}^{(c)} = v_{i,t} + \sigma_i^2 \qquad \rho_t^{(c)} = \frac{K_t + \delta\sigma_1\sigma_2}{\sqrt{(v_{1,t} + \sigma_1^2)(v_{2,t} + \sigma_2^2)}} \quad (14)$$

where $\sigma_i^2$ and $\delta\sigma_1\sigma_2$ are the constant, long-run components of the conditional variances and covariance, respectively, determined by the assumed common beliefs about the dividend yield processes, while $v_{i,t}$ and $K_t$ are time varying components that are updated in each period according to the observed volatility and correlation of price changes. The chartist demand functions thus become

$$\zeta_{1,t}^{(c)} = \frac{1}{(1-(\rho_t^{(c)})^2)} \frac{(m_{1,t}^{(c)} + d_1 - r)}{\alpha^{(c)}(v_{1,t} + \sigma_1^2)} - \frac{\rho_t^{(c)}\sqrt{(v_{2,t} + \sigma_2^2)}}{(1-(\rho_t^{(c)})^2)\sqrt{(v_{1,t} + \sigma_1^2)}} \frac{(m_{2,t}^{(c)} + d_2 - r)}{\alpha^{(c)}(v_{2,t} + \sigma_2^2)} \quad (15)$$

$$\zeta_{2,t}^{(c)} = \frac{1}{(1-(\rho_t^{(c)})^2)} \frac{(m_{2,t}^{(c)} + d_2 - r)}{\alpha^{(c)}(v_{2,t} + \sigma_2^2)} - \frac{\rho_t^{(c)}\sqrt{(v_{1,t} + \sigma_1^2)}}{(1-(\rho_t^{(c)})^2)\sqrt{(v_{2,t} + \sigma_2^2)}} \frac{(m_{1,t}^{(c)} + d_1 - r)}{\alpha^{(c)}(v_{1,t} + \sigma_1^2)} \quad (16)$$

Similarly to the case of fundamentalist demand, the quantities

$$\overline{\zeta}_1^{(c)} \equiv \frac{1}{(1-\delta^2)} \frac{\pi_1}{\alpha^{(c)}\sigma_1^2} - \frac{\delta\sigma_2}{(1-\delta^2)\sigma_1} \frac{\pi_2}{\alpha^{(c)}\sigma_2^2} \quad (17)$$

$$\overline{\zeta}_2^{(c)} \equiv \frac{1}{(1-\delta^2)} \frac{\pi_2}{\alpha^{(c)}\sigma_2^2} - \frac{\delta\sigma_1}{(1-\delta^2)\sigma_2} \frac{\pi_1}{\alpha^{(c)}\sigma_1^2} \quad (18)$$

(where $\pi_i \equiv (g_i + d_i - r)$, $i = 1, 2$), are constant "long-run" or "equilibrium" components of chartist demands. The investors' demand functions (4), (5),

(15) and (16) generalise in a straightforward way to the case of two risky assets the ones derived for the one risky asset case in [5], the difference being the hedging demand components that depend on agents' beliefs about correlation of returns.

## 4 The Dynamical System

We assume that the market clearing function is performed by a *market maker*, who knows the fundamental price process in each market. For the sake of simplicity it is assumed that the (nominal) supply of shares of asset $i$, $i = 1, 2$, is constant at the level $N_i$, equal to investors' "long-run" demand, i.e. $N_i = \overline{\zeta}_i^{(f)} + \overline{\zeta}_i^{(c)}$, where $\overline{\zeta}_i^{(f)} \equiv h_i$ and $\overline{\zeta}_i^{(c)}$ are defined by (6), (7), (17), and (18). The market maker adjusts the prices in each market according to

$$P_{i,t+1} = P_{i,t} + g_i + \beta_i[\zeta_{i,t}^{(f)} + \zeta_{i,t}^{(c)} - N_i]$$

where $\zeta_{i,t}^{(f)} + \zeta_{i,t}^{(c)} - N_i$ represents the excess demand for asset $i$ at time $t$, and $\beta_i$ ($\beta_i > 0$) is the speed of reaction of the price of asset $i$. This means that the (relative) price change operated by the market maker is higher (lower) than the change in the underlying fundamental in case of positive (negative) excess demand. For each asset it is convenient to define the new variables $q_{i,t} = P_{i,t} - W_{i,t}$, $\xi_{i,t} = m_{i,t}^{(c)} - g_i$, the deviation of (log) price from (log) fundamental value (i.e. the log price/fundamental ratio), and the deviation of the chartist expected capital gain from the underlying trend, respectively. In terms of the new variables, the time evolution of prices and agents' beliefs about expected returns, variances and correlation is described by the iteration of the following 7-dimensional nonlinear map[5]

$$T : \begin{cases} q_1' = q_1 + \beta_1[-a_1 q_1 + b_2 q_2 + (\zeta_1^{(c)} - \overline{\zeta}_1^{(c)})] \\ q_2' = q_2 + \beta_2[-a_2 q_2 + b_1 q_1 + (\zeta_2^{(c)} - \overline{\zeta}_2^{(c)})] \\ \xi_i' = (1-c)\xi_i + c(q_i' - q_i) \qquad i = 1, 2 \\ v_i' = (1-c)v_i + c(1-c)(q_i' - q_i - \xi_i)^2 \qquad i = 1, 2 \\ K' = (1-c)K + c(1-c)(q_1' - q_1 - \xi_1)(q_2' - q_2 - \xi_2) \end{cases} \tag{19}$$

where the chartist demand functions $\zeta_1^{(c)}$, $\zeta_2^{(c)}$ are rewritten in terms of the state variables as

$$\zeta_1^{(c)} = \frac{(v_2 + \sigma_2^2)(\xi_1 + \pi_1) - (K + \delta\sigma_1\sigma_2)(\xi_2 + \pi_2)}{\alpha^{(c)}[(v_1 + \sigma_1^2)(v_2 + \sigma_2^2) - \delta^2\sigma_1^2\sigma_2^2 - K^2 - 2K\delta\sigma_1\sigma_2]} \tag{20}$$

$$\zeta_2^{(c)} = \frac{(v_1 + \sigma_1^2)(\xi_2 + \pi_2) - (K + \delta\sigma_1\sigma_2)(\xi_1 + \pi_1)}{\alpha^{(c)}[(v_1 + \sigma_1^2)(v_2 + \sigma_2^2) - \delta^2\sigma_1^2\sigma_2^2 - K^2 - 2K\delta\sigma_1\sigma_2]} \tag{21}$$

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[5] The symbol $'$ denotes the unit time advancement operator, i.e. if $x$ is the value of a state variable at time $t$, then $x'$ denotes the value of the same variable at $(t+1)$

and $\overline{\zeta}_1^{(c)}$ and $\overline{\zeta}_2^{(c)}$ are given by (17) and (18), respectively. With these changes of variables the unique "fundamental" steady state of the model, that we denote by $O$, is characterized by zero equilibrium levels of the state variables, $\bar{q}_i = \bar{\xi}_i = \overline{v}_i = 0$, $i = 1, 2$, $\overline{K} = 0$, which means that prices are equal to fundamentals and grow at the exogenously determined fundamental rates.

Throughout the rest of this paper we will focus on the simplified case of zero "long-run" correlation between returns ($\delta = 0$) where analytical results can be obtained about the local asymptotic stability and the bifurcations of the steady state. On the other hand, as shown in [6], many aspects of the dynamic behavior of the general model can be better understood within this particular case. In the case $\delta = 0$ the dividend yields of the two risky assets are not correlated in agents' beliefs, i.e. no correlation between returns is expected at the steady state, and the dynamic equations for $q_1$ and $q_2$ in (19) become

$$q_1' = q_1 + \beta_1[-a_1 q_1 + (\zeta_1^{(c)} - \overline{\zeta}_1^{(c)})]; \quad q_2' = q_2 + \beta_2[-a_2 q_2 + (\zeta_2^{(c)} - \overline{\zeta}_2^{(c)})] \tag{22}$$

where $a_i = \eta_i/(\alpha^{(f)}\sigma_i^2)$ represents the *strength of fundamentalist demand* for asset $i$, $i = 1, 2$, while chartist demands become

$$\zeta_1^{(c)} = \frac{(v_2 + \sigma_2^2)(\xi_1 + \pi_1) - K(\xi_2 + \pi_2)}{\alpha^{(c)}[(v_1 + \sigma_1^2)(v_2 + \sigma_2^2) - K^2]}; \quad \zeta_2^{(c)} = \frac{(v_1 + \sigma_1^2)(\xi_2 + \pi_2) - K(\xi_1 + \pi_1)}{\alpha^{(c)}[(v_1 + \sigma_1^2)(v_2 + \sigma_2^2) - K^2]} \tag{23}$$

with equilibrium levels $\overline{\zeta}_i^{(c)} = \pi_i/(\alpha^{(c)}\sigma_i^2)$, $i = 1, 2$. As one can also argue from eqs. (6), (7), (17) and (18), in the case $\delta = 0$ the steady state demand of each asset by each agent type is independent from the agent's beliefs about the other asset. On the contrary, eq. (23) says that when $K \neq 0$, and thus the system is not at the steady state, chartist demand for each asset does depend on beliefs about the other asset: what determines the "coupling" of the two markets is precisely the dynamic updating of the covariance [6].

## 5 Local Stability Analysis

It can be easily checked that the Jacobian matrix of the map $T$ computed at the steady state $O$ has the following "upper block triangular" structure

$$DT(O) = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & (1-c)\mathbf{I} \end{bmatrix}$$

where $\mathbf{0}$ is the null $(3 \times 4)$ matrix, $\mathbf{I}$ is the three-dimensional identity matrix and $\mathbf{A}$ is a four-dimensional matrix. This implies that three eigenvalues of

[6] Without the dynamic equation for $K$ in (19) (and assuming $K_t = 0$ for all $t$) the resulting 6-$D$ dynamical system would be made up of two uncoupled $3 - D$ systems in the state variables $(q_i, \xi_i, v_i)$, $i = 1, 2$, each describing the independent dynamics of a single asset

$DT(O)$ are of modulus smaller that one (being all equal to $(1-c)$), while the remaining eigenvalues are the characteristic roots of the submatrix $\mathbf{A}$. It follows that a sufficient condition for the local asymptotic stability is that the four eigenvalues of $\mathbf{A}$ lie inside the unit circle in the complex plane. Let us now focus on the particular case of zero "long-run" correlation, i.e. $\delta = 0$: in this case the matrix $\mathbf{A}$ is block diagonal, $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix}$, where the two-dimensional matrices $\mathbf{A}_i$, $i = 1, 2$, have identical structure, given by

$$\mathbf{A}_i = \begin{bmatrix} 1 - a_i\beta_i & \beta_i\theta_i \\ -c\beta_i a_i & 1 - c + c\beta_i\theta_i \end{bmatrix}$$

and the aggregate parameter $\theta_i \equiv \partial\zeta_i^{(c)}(O)/\partial\xi_i = 1/(\alpha^{(c)}\sigma_i^2)$ can be interpreted as the *strength of chartist demand* in the $i$th market (at the steady state). It follows that the four eigenvalues of $\mathbf{A}$ can be obtained separately as characteristic roots of $\mathbf{A}_1$ and $\mathbf{A}_2$. Notice that $\mathbf{A}_1$ does not depend on the parameters of market 2, and vice versa. This means that at the steady state the two markets are uncoupled from each other, and the behavior of asset 1 (asset 2) is uniquely associated with the eigenvalues of $\mathbf{A}_1$ ($\mathbf{A}_2$).

The region of the space of parameters $(\beta_i, \theta_i, c, a_i)$ where both eigenvalues associated with asset $i$ are in absolute value less than unity can be obtained from the following set of inequalities[7]: $\mathcal{P}_i(1) = 1 - Tr(\mathbf{A}_i) + Det(\mathbf{A}_i) > 0$; $\mathcal{P}_i(-1) = 1 + Tr(\mathbf{A}_i) + Det(\mathbf{A}_i) > 0$; $\mathcal{P}_i(0) = Det(\mathbf{A}_i) < 1$, where $\mathcal{P}_i(\lambda) = \lambda^2 - Tr(\mathbf{A}_i)\ \lambda + Det(\mathbf{A}_i)$ is the characteristic polynomial of $\mathbf{A}_i$. In terms of the (positive) parameters $\beta_i$, $\theta_i$, $c$, $a_i$ the above conditions are reduced to

$$a_i\beta_i(2-c) < 2(2-c) + 2c\beta_i\theta_i \qquad a_i\beta_i(1-c) > c\ [\beta_i\theta_i - 1] \tag{24}$$

It follows that a sufficient condition for the local asymptotic stability of the steady state $O$ is that (24) holds for both $i = 1$ and $i = 2$.

The conditions (24) are very similar to the ones obtained for the simpler two-dimensional, single risky asset model analyzed in [5]. *Fig. 1* is a qualitative picture, in the space of the parameters $(c, a_i)$, $0 < c < 1$, $a_i > 0$, of the region $S_i$ where the pair of eigenvalues associated with asset $i$ are of modulus smaller than one. The region $S_i$ of *Fig. 1* is bounded by the curves of equation $1 + Tr(\mathbf{A}_i) + Det(\mathbf{A}_i) = 0$ and $Det(\mathbf{A}_i) = 1$, and is obtained in a case where the chartist demand or the price reaction in the $i$-th market are sufficiently strong ($\theta_i\beta_i > 1$). On the bifurcation curve of equation $Det(\mathbf{A}_i) = 1$, denoted by "Hopf-curve" in *Fig. 1*, the matrix $\mathbf{A}_i$ has complex eigenvalues equal to one in modulus. When the Hopf-curve is crossed as shown by the arrow in *Fig. 1*, i.e. when the chartist extrapolation parameter $c$ becomes higher than the bifurcation value $c_{Hi} \equiv a_i\beta_i/[\beta_i(a_i + \theta_i) - 1]$, then the (complex) eigenvalues

[7] See e.g. [11]

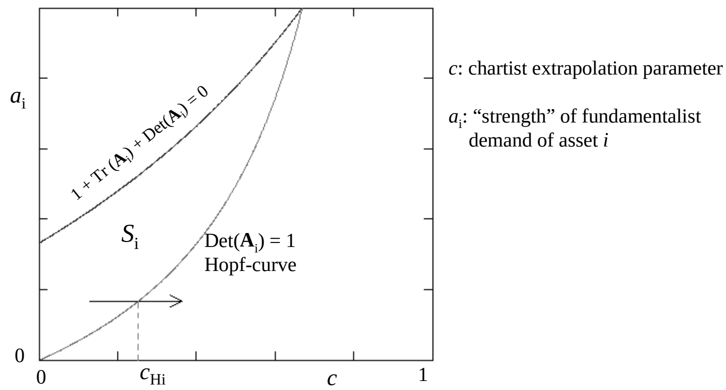


**Fig. 1.** Case $\delta = 0$: region in the $(c, a_i)$ parameter plane where the pair of eigenvalues associated with asset $i$ are less than unity in absolute value

associated with asset $i$ become of modulus higher than one[8]. As a consequence of the above analysis, the bifurcations of the steady state in the case $\delta = 0$ can be easily analyzed by combining the two regions $S_i$, $i = 1, 2$, each of the type represented in *Fig. 1*, associated with the two risky assets.

## 6 Hopf-Bifurcations and Coupled Price Fluctuations

With the help of the local analysis, we now take the chartist extrapolation rate $c$ as a bifurcation parameter in order to show how "coupled" fluctuations of the prices of the two assets may arise as a consequence of the interaction of heterogeneous agents. The analysis will be restricted to the case $\delta = 0$, where no correlation between returns is expected at the steady state: this will prove that the resulting interdependence of the two markets is endogenously determined by the interaction of time varying beliefs (especially chartists' updating rules) with the out-of-equilibrium adjustments of prices. Throughout this section it is assumed that asset 2 has higher expected risk premium and higher volatility than asset 1 in equilibrium ($\pi_2 > \pi_1$, $\sigma_2 > \sigma_1$) and therefore the strength of chartist demand is higher in market 1 ($\theta_1 \equiv 1/(\alpha^{(c)}\sigma_1^2)$) than in market 2 ($\theta_2 \equiv 1/(\alpha^{(c)}\sigma_2^2)$). We will also assume that chartists are less risk averse than fundamentalists ($\alpha^{(c)} < \alpha^{(f)}$). The parameters are fixed at the values reported in *Figs. 2, 3*. *Fig. 2* shows, for $i = 1, 2$, the region $S_i$ of the parameter plane $(c, a_i)$ where the two eigenvalues associated with market $i$ are of modulus smaller than 1 (the region $S_1$ is bounded by solid curves, $S_2$ by dashed curves). This means that for a given choice of $c$, $a_1$, $a_2$ the steady state $O$ is locally asymptotically stable when both $(c, a_1)$ lies inside $S_1$ and

[8]Note that higher values of the parameter $\theta_i$ (strength of chartist demand) cause the Hopf-curve to move to the left, thus shrinking the region $S_i$ of *Fig. 1*

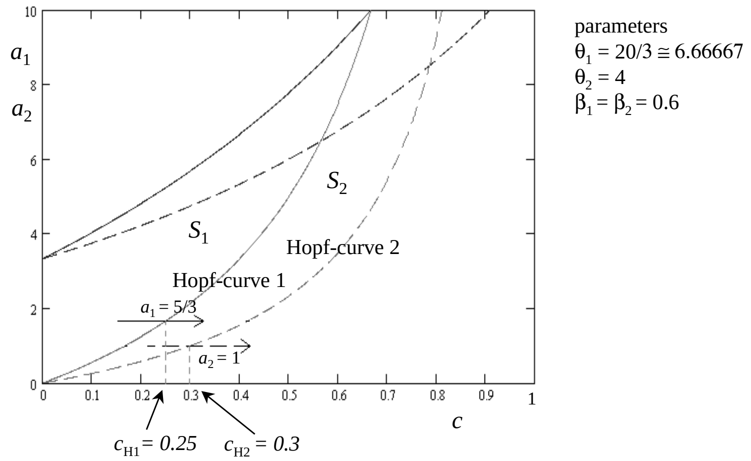


**Fig. 2.** Bifurcation curves in the case $\delta = 0$

$(c, a_2)$ lies inside $S_2$. We take $a_1 = 5/3 \cong 1.66667$, $a_2 = 1$, and $c$ as a varying parameter; the two arrows in *Fig. 2* help to follow the bifurcation path, and suggest the existence of 3 dynamic scenarios.

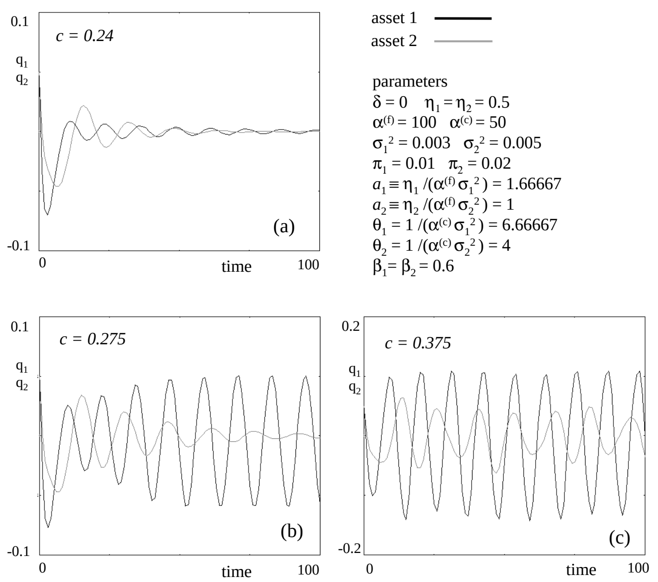


**Fig. 3.** Appearance of “coupled” fluctuations of prices in the two markets; time series of log price/fundamental ratios for increasing values of the chartist extrapolation parameter $c$

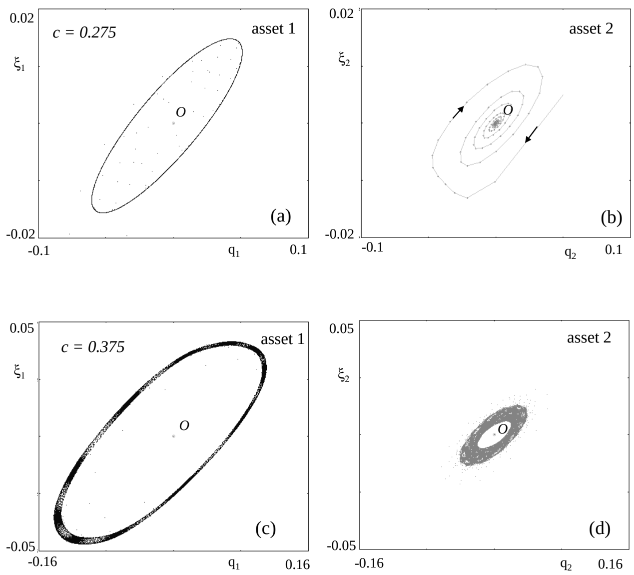


**Fig. 4.** Phase space transition for increasing values of the chartist parameter $c$: a limit cycle with fluctuations in market 1 (a), and market 2 "in equilibrium" (b), changes into a torus, characterized by long-run price fluctuations in both markets (c, d)

*(i)* $c < c_{H1}$ ($c_{H1} = 0.25$ in this example). In this case both pairs of eigenvalues are of modulus smaller than one; the steady state is locally stable. In the case represented in *Fig. 3a* ($c = 0.24$) the equilibrium is an attracting focus and both prices converge with dampened fluctuations to their fundamental values.

*(ii)* $c_{H1} < c < c_{H2}$ ($c_{H2} = 0.3$ in this example). When $c$ crosses the bifurcation value $c_{H1}$, the pair of (complex) eigenvalues associated with asset 1 exit the unit circle of the complex plane; a supercritical Hopf-bifurcation of the steady state occurs, which becomes a repelling focus, and an attracting closed curve exists for $c > c_{H1}$. The behavior of the prices is represented in *Fig 3b* ($c = 0.275$). It can be checked numerically that the existence of the limit cycle, at least for $c$ sufficiently close to $c_{H1}$, has no effect on the long-run behavior of the price of asset 2, which converges to its fundamental. *Figs. 4a,b* represent the projections of a trajectory in the $(q_1, \xi_1)$ and $(q_2, \xi_2)$-planes, respectively. On the limit cycle, the two blocks of variables $(q_1, \xi_1, v_1)$ and $(q_2, \xi_2, v_2)$ behave independently from each other (the former fluctuate regularly while the latter are fixed at their equilibrium values $(0, 0, 0)$), and agents' demand for asset 1 does not depend on the behavior of asset 2 and vice versa. The "uncoupled" dynamics of the two assets is due to the particular eigenvalue structure of the Jacobian matrix at the steady state, and to the

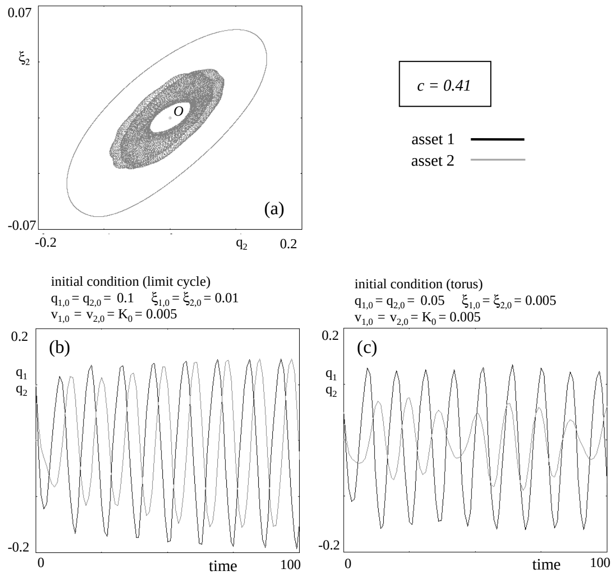


**Fig. 5.** Coexistence of a torus and a limit cycle; (a) projection of the coexisting attractors in the $(q_2, \xi_2)$-plane; (b, c) time series of log price/fundamental ratios, with two different initial conditions, converging to the limit cycle and to the torus, respectively

fact that for $c_{H1} < c < c_{H2}$ the pair of eigenvalues associated with market 1 are of modulus greater than 1, while the ones associated with market 2 are smaller than 1 in modulus.

*(iii)* $c > c_{H2}$. When $c$ crosses the bifurcation value $c_{H2}$ also the pair of (complex) eigenvalues associated with asset 2 exit the unit circle; strictly related to this second crossing, for higher values of $c$ a "secondary" Hopf-bifurcation occurs, taking place from the existing limit cycle. Its effect is to change the limit cycle into a *torus*, see *Fig. 3c* and *Figs. 4c,d* (where $c = 0.375$). This means that long-run price fluctuations appear also in market 2; the dynamics of the two prices on the torus are "coupled", i.e. interdependent, and the agents' demand for asset 1 also depends on the behavior of asset 2 and vice versa [9].

The interdependence between the two markets becomes stronger for higher values of $c$ when a "global" bifurcation occurs (not associated with the eigenvalue structure at the steady state), that creates a new attractor (a new limit cycle). For a range of values of $c$ the new attractor coexists with the previously

[9] Of course qualitatively different bifurcation paths will be possible for different choices of the fundamentalist parameters $a_1$ and $a_2$; for instance it could happen that $c_{H2} < c_{H1}$. The bifurcation analysis of this section can however be easily adapted to these qualitatively different cases

created torus (see *Fig. 5a*). The system converges to one or the other of the coexisting attractors depending on the initial condition, but the long-run dynamics are characterized in both cases by “coupled” fluctuations of the prices of the two assets (*Figs. 5b,c*).

## 7 Time Varying Correlation of Returns

This section performs a simple numerical experiment that shows how the interaction of agents' time varying beliefs, the market trading mechanism and simple noise processes may affect the correlation of realized returns of the two risky assets. As an example of external noise, i.i.d. random disturbances are added in each period to the log price/fundamental ratio of asset 2 ($q_2$), so that eq. (22) becomes $q_2' = q_2 + \beta_2[-a_2 q_2 + (\zeta_2^{(c)} - \overline{\zeta}_2^{(c)})] + \epsilon$, where the random variable $\epsilon$ is assumed uniformly distributed with zero mean. This means that the deterministic growth of the fundamental price of asset 2 (expressed by eq. (3)) is now affected by a multiplicative random shock.

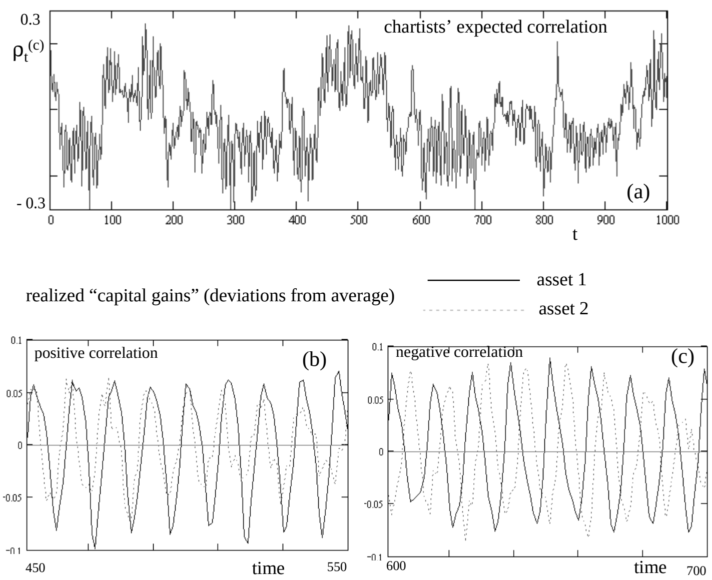


**Fig. 6.** (a) time varying chartists' expected correlation in a stochastic simulation of the dynamical system; (b, c) time series of realized capital gains (deviations from average) in periods of positive and negative expected correlation, respectively

A stochastic trajectory is obtained with the same parameters as in *Fig. 3c (*and *Figs. 4c,d*), with $\epsilon$ uniformly distributed in $[-0.02, 0.02]$. The unique attractor of the underlying deterministic system is a high dimensional torus, with fluctuations in both markets. *Fig. 6a* represents, at each point in time, the chartists' estimate of the correlation of returns over the next period (the quantity $\rho_t^{(c)}$ defined in (14), with $\delta = 0$ in this case). The path of the estimated correlation frequently switches from positive to negative values and again to positive over the 1000 time periods of the simulation, but wide subperiods can be identified, where agents' estimated correlation is significantly positive (e.g. the time interval $[450, 550]$) or negative (e.g. the interval $[600, 700]$). These cyclically varying beliefs are strongly related with the behavior of realized price changes. For instance, if we consider the time series of realized *capital gains*, restricted to the same time intervals, we report a remarkable positive correlation ($\overline{\rho} = 0.522$) in the period $[450, 550]$, and a negative correlation ($\overline{\rho} = -0.501$) in the period $[600, 700]$, as it is also suggested by *Figs. 6b,c*. In order to compare actual and expected correlation of *returns* (including the *dividend yields*) we assume that the realizations of the dividend yield processes are exactly as in agents' beliefs, with variances $\sigma_1^2 = 0.003$, $\sigma_2^2 = 0.005$, correlation $\delta = 0$ (and uncorrelated with the time series of capital gains). The resulting correlation of returns turns out to be $\widehat{\rho} = 0.153$ in the period $[450, 550]$ and $\widehat{\rho} = -0.179$ in the period $[600, 700]$: in both periods the sign and the magnitude of the correlation agree with agents' expectations represented in *Fig. 6a*. Similar phenomena can be observed also when noise is added to other dynamic variables, and under different regimes of parameters of the deterministic model. In all these cases periodic changes in the correlation of returns seem to occur, driven by external random events and agents' time varying beliefs.

## 8 Conclusions

We have set up a dynamic model of fundamentalists and chartists who interact in a financial market with two risky assets and a riskless asset, where price adjustments are operated by a market maker on the basis of the excess demand. Despite the high dimension of the resulting dynamical system, a complete local stability analysis of the "fundamental" steady state has been performed, which reveals how chartists can cause long-run, interdependent fluctuations of the prices of the risky assets, especially when the updating of chartists' beliefs is highly sensitive to the most recent price history. Numerical inspection of the global dynamics reveals, among other things, situations of co-existing attractors, which may be an important element in generating complex behaviour especially when background market noise is added to the model. Furthermore, when simple stochastic factors are introduced into the underlying nonlinear deterministic system, the random trajectories generated by the model show that the correlation of realized returns may periodically change, strictly linked to agents' time varying beliefs.

Although this has been a very preliminary study of the combined effect of agents' heterogeneity, time varying beliefs and portfolio diversification, it provides a useful framework in which to analyse how the capital asset pricing model relationships are modified in the expectations feedback heterogeneous agent framework. For instance the time varying beliefs about variances and correlation could provide a basis for better understanding some widely reported empirical facts, such as the time varying betas, that are not fully explained in the standard CAPM framework.

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