12 A Goodwin-type Model with a Piecewise Linear Investment Function

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12.1 Introduction

The model studied in the present Chapter is a variation of the one presented in Chapter 11, so for a background we refer to the introduction of that Chapter. The difference is that we now replace the cubic investment function by a five-piece linear one (see Fig.1). We shall see that the dynamic behavior of the model is different. In terms of economics we have the advantage that the constant pieces automatically prevent the investment function from cutting the horizontal axis more than once.

12.2 Center bifurcation of the fixed point ($a = 1$)

In Chapter 6 we have described the simplest version of the Hicksian business cycle model defined by a two-dimensional piecewise linear map, showing that the main bifurcation scenarios in the model is the center bifurcation of the fixed point resulting in periodic or quasiperiodic dynamics (see also Hommes (1991), Gallegati *et al*. (2003)). In the present chapter we shall see the emergence of more complex dynamics, such as multistability and chaos, in different version of the business cycle model also defined by a two-dimensional piecewise linear map.

We consider the dynamical system generated by a family of two - dimensional piecewise linear maps $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} cx + I(x-y) \\ x \end{pmatrix}, \tag{1}$$

where $I(z)$ is the piecewise linear investment function defined as

$$I(z) = \begin{cases} -h, & z \leq -\tau; \\ lz - a + l, & -\tau < z < -1; \\ az, & -1 \leq z \leq 1; \\ lz + a - l, & 1 < z \leq \tau; \\ h, & z \geq \tau; \end{cases} \tag{2}$$

where $\tau = (h - a + l)/l$ (see Fig.1). The map F depends on four real parameters a, c, l and h, such that

$$a > 0, 0 < c < 1, l < 0, 0 < h \leq a. \tag{3}$$

The investment function given in (2) is symmetric with respect to the origin (for an analogous business cycle model with symmetric cubic investment function see Puu and Sushko (2004)). It is not a generic case, and breaking of the symmetry can be introduced in several ways, but in this chapter, for simplicity, we limit our consideration to the symmetric case. It can be easily seen that the map F is also symmetric with respect to the origin, which immediately implies the following

Property. *Any invariant set A of the map F (i.e., such that $F(A) = A$) is either symmetric itself with respect to the origin, or there exists one more invariant set A' symmetric to A with respect to the origin.*

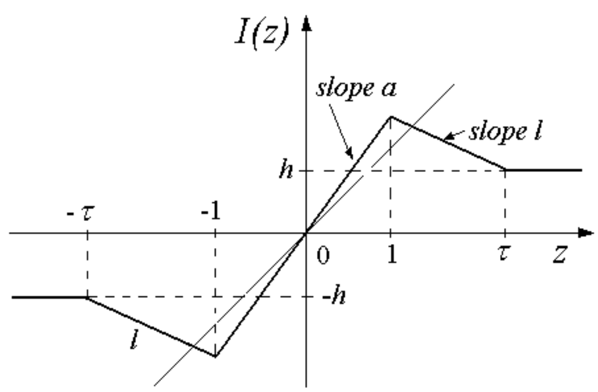

Figure 1: *The investment function.*

One of the aims of the present consideration is to illustrate the *center bifurcation* described in detail in Chapter 2, which occurs for piecewise linear maps when the fixed point loses stability with a pair of complex-conjugate eigenvalues on the unit circle (see also Sushko *et al*. (2003)). In a similarity to the Neimark-Sacker bifurcation (see Chapter 1, or Kuznetsov (1995)) it results in an attracting closed invariant curve $\mathcal{C}$ homeomorphic to a circle, and dynamics of F on $\mathcal{C}$ is reduced to a rotation with rational or irrational rotation number. We will present some examples of the bifurcation sequences leading through the breakdown of the curve $\mathcal{C}$ to complex dynamics.

The map F is given by five linear maps F_i, $i = 1, ..., 5$, defined, respectively, in five regions R_i of the phase plane:

$$\begin{aligned} F_1 \quad &: \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (c+a)x - ay \\ x \end{pmatrix}, \\ R_1 \quad &= \quad \{(x,y) : x - 1 \leq y \leq x + 1\}\,; \end{aligned} \tag{4}$$

$$\begin{aligned} F_{2,3} \quad &: \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (c+l)x - ly \pm (l-a) \\ x \end{pmatrix}, \\ R_2 \quad &= \quad \{(x,y) : x + 1 < y < x + \tau\}\,, \\ R_3 \quad &= \quad \{(x,y) : x - \tau < y < x - 1\}\,; \end{aligned} \tag{5}$$

$$\begin{aligned} F_{4,5} \quad &: \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} cx \mp h \\ x \end{pmatrix} \\ R_4 \quad &= \quad \{(x,y) : y \geq x + \tau\}\,, \\ R_5 \quad &= \quad \{(x,y) : y \leq x - \tau\}\,. \end{aligned} \tag{6}$$

So, in the phase space of the map F there are four straight lines on which F changes its definition:

$$\begin{aligned} LC_{-1},\ LC_{-1}^{s} \quad &: \quad y = x \pm 1, \\ LC'_{-1},\ LC'^{s}_{-1} \quad &: \quad y = x \pm \tau. \end{aligned}$$

Their images by F are called *critical lines*:

$$\begin{aligned} LC,\ LC^{s} \quad &: \quad y = (x \pm a)/c, \\ LC',\ LC'^{s} \quad &: \quad y = (x \pm h)/c. \end{aligned}$$

The i-th iteration by F of the critical line is a broken line, called critical line of rank i.

It can be easily verified that the map F is noninvertible. Fig.2 shows schematically the folding action of F : Any point of the phase plane on the right of LC or on the left of LC^s has zero preimages, any point between LC and LC', or between LC^s and LC'^s has two preimages, any point between LC' and LC'^s has one preimage and, finally, any point of the straight lines LC' and LC'^s has infinitely many preimages (given that the whole regions R_4 and R_5 are mapped, respectively, into the straight lines LC' and LC'^s). So, the map F has noninvertibility of $(Z_0 - Z_2 - Z_\infty - Z_1 - Z_\infty - Z_2 - Z_0)$ kind (as a reference to noninvertibility and theory of critical lines see Mira *et al*. (1996)).

An obvious result can be immediately reached: For the parameter range considered the dynamics of F are always bounded, namely, in the phase plane there exists an absorbing area bounded by the critical lines LC'_{-1}, LC'^s_{-1}, LC and LC^s.

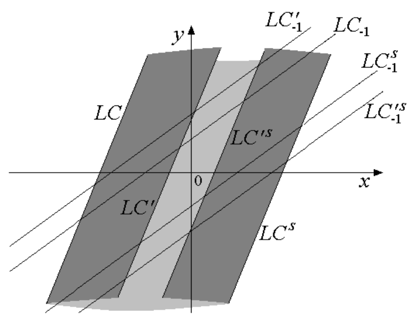

Figure 2: *The folding action of the map F.*

The unique fixed point of F is the fixed point of the map F_1 which is $(x^*, y^*) = (0, 0)$, given that the fixed points of other linear maps F_i, $i = 2, ..., 5$, are on the main diagonal belonging to R_1, and thus they are not fixed points of F. The eigenvalues of the Jacobian matrix of F_1 are

$$\lambda_{1,2} = (a + c \pm \sqrt{(a+c)^2 - 4a})/2, \tag{7}$$

so that for $a > 0$, $0 < c < 1$ the fixed point (x^*, y^*) is a node if $(c + a)^2 > 4a$, and a focus if $(c+a)^2 < 4a$, being attracting for $a < 1$ and repelling for $a > 1$. At $a = 1$ the fixed point undergoes the center bifurcation. Before the description of its results let us first derive some other properties of F.

The eigenvalues of the Jacobian matrix of the maps F_2 and F_3 (which differ only by a shift constant) are

$$\mu_{1,2} = (l + c \pm \sqrt{(l+c)^2 - 4l})/2. \tag{8}$$

Given that $l < 0$, we have that $\mu_{1,2}$ are always real and $0 < \mu_1 < 1$, while $-1 < \mu_2 < 0$ if $c > -2l - 1$, and $\mu_2 < -1$ if $c < -2l - 1$. The eigenvalues of the Jacobian matrix of the maps F_4 and F_5 (which also differ only by a shift constant) are $\nu_1 = c$ and $\nu_2 = 0$. We can state

Proposition 1. *If $a < 1$ and the ranges of c, l and h are as given in* (3), *then the fixed point (x^*, y^*) is the global attractor of the map F.*

Obviously, the map F cannot have divergent trajectories. To see that the proposition is true, we have to show that it cannot have other attractors. For $c > -2l-1$, when all the maps F_i, $i = 1, ..., 5$ are contractions, the statement is obvious. While in case $c < -2l - 1$, when $\mu_{1,2}$ are such that $0 < \mu_1 < 1$ and $\mu_2 < -1$, the statement can be proved by contradiction: Suppose, there exist another attractor. It necessarily must belong to an absorbing area, made up by images of the critical lines, but a simple geometrical reasoning shows that such an area shrinks by F to the origin.

Consider now the map F given in (1) exactly at the bifurcation value $a = 1$, when the fixed point (x^*, y^*) undergoes the center bifurcation. Using the results presented in Section 2.2 of Chapter 2, we can state that at $a = 1$ there exists an invariant region in the phase plane, and its structure depends only on the map F_1 and the critical lines LC_{-1}, LC^s_{-1} (and, thus, it depends only on the parameters a and c). Let us repeat that if at $a = 1$ the map F_1 is defined by the rotation matrix with some rational rotation number m/n, which holds for

$$c = c_{m/n} \stackrel{def}{=} 2\cos(2\pi m/n) - 1, \tag{9}$$

then any point from some neighborhood of the fixed point is periodic with rotation number m/n, and all points of the same periodic orbit are located on an invariant ellipse. The following proposition describes the dynamics of F in such a case:

Proposition 2. *Let $a = 1$, $c = c_{m/n}$, then in the phase plane of the map F there exists an invariant polygon P such that*

- *If n is even then P has n edges which are two generating segments $S_1 \subset LC_{-1}$, $S_1^s \subset LC^s_{-1}$ and their images $S_{i+1} = F_1(S_i) \subset LC_{i-1}$, $S^s_{i+1} = F_1(S^s_i) \subset LC^s_{i-1}$, $i = 1, ..., n/2 - 1$;*

- *If n is odd then P has $2n$ edges which are two generating segment $S_1 \subset LC_{-1}$, $S_1^s \subset LC_{-1}^s$ and their images $S_{i+1} = F_1(S_i) \subset LC_{i-1}$, $S_{i+1}^s = F_1(S_i^s) \subset LC_{i-1}^s$, $i = 1, ..., n-1$.*

Any initial point $(x_0, y_0) \in P$ is periodic with rotation number m/n, while any $(x_0, y_0) \notin P$ is mapped inside P in a finite number of iterations.

See Fig.3 with an example of the polygon P with 14 edges at $a = 1$, $c = c_{1/7}$.

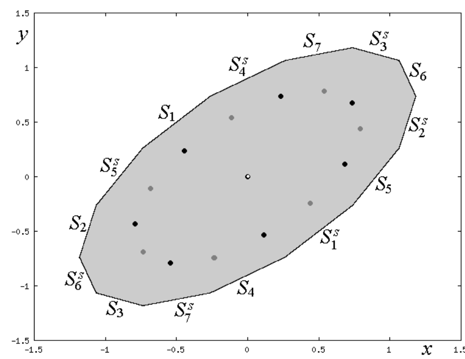

Figure 3: *The invariant polygon P of the map F at $a = 1$, $c = c_{1/7} = 2\cos(2\pi/7) - 1$. Any point of P is periodic with rotation number $1/7$; As an example, two such cycles are shown by black and gray circles.*

If at $a = 1$ the map F_1 is defined by the rotation matrix with an irrational rotation number ρ, which holds for

$$c = c_\rho \overset{def}{=} 2\cos(2\pi\rho) - 1, \tag{10}$$

then any point from some neighborhood of the fixed point is quasiperiodic, and all points of the same quasiperiodic orbit are dense on the corresponding invariant ellipse. In such a case the following proposition holds:

Proposition 3. *Let $a = 1$, $c = c_\rho$. Then in the phase space of the map F there exists an invariant region Q, bounded by an invariant ellipse $\mathcal{E}$ of the map F_1 tangent to both critical lines LC_{-1}, LC_{-1}^s (and to all their images). Any initial point $(x_0, y_0) \in Q$ belongs to a quasiperiodic orbit dense in the corresponding ellipse of F_1, while any $(x_0, y_0) \notin Q$ is mapped inside Q in a finite number of iterations.*

Note that Propositions 2 and 3 hold for any values of l and h. We already know that generally, increasing a, i.e., for $a = 1 + \varepsilon$, $\varepsilon > 0$, only the boundary of the region P or Q remains invariant for some sufficiently small ε, being an attracting closed invariant curve $\mathcal{C}$ on which the dynamics of F are periodic or quasiperiodic, respectively. Further increase of a leads to destruction of this curve and then to more complex dynamics.

12.3 Destruction of the curve $\mathcal{C}$ and routes to chaos ($a > 1$)

Before presenting some general description of the dynamics of F for $a > 1$, let us first consider a particular parameter value $a = h$. In such a case we have $\tau = 1$, $LC_{-1} = LC'_{-1}$ (and $LC^s_{-1} = LC'^s_{-1}$) so that the map F is given by three linear maps F_1, F_2 (which is equal to F_4) and F_3 (equals F_5), being noninvertible of $(Z_0 - Z_\infty - Z_1 - Z_\infty - Z_0)$ kind. This case corresponds to the version of the piecewise linear Hicksian business cycle model with both 'floor' and 'ceiling' (which are on the same distance from the origin in our case) incorporated in the investment function. Using the results obtained for piecewise linear maps of such a kind of noninvertibility (see Section 2.3 of Chapter 2), we can state

Proposition 4. *For* $a = h > 1$, $0 < c < 1$, *in the phase plane of the map* F *there exists an invariant closed attracting curve* $\mathcal{C}$*, homeomorphic to a circle, made up by a finite number of images of two generating segments, belonging, respectively, to* LC_{-1} *and* LC^s_{-1}*. The map* F *on* $\mathcal{C}$ *is reduced to a rotation with rational or irrational rotation number, and has, respectively, either periodic or quasiperiodic dynamics.*

We emphasize that $a = h$ is a particular case in which the curve $\mathcal{C}$ exists for any $a > 1$ and cannot be destroyed. Without going into details we just present an example of the curve $\mathcal{C}$ in a case of rotation number $1/7$, so that the map F has an attracting and a saddle cycle of period 7. Given that such cycles are not symmetric with respect to the origin, then, according to Property stated in the previous section, there must exist two more cycles of period 7, symmetric to the first two cycles, respectively: Fig.4 shows the phase portrait of the map F at $a = h = 2$, $c = 0.21$, with two coexisting attracting cycles of period 7 and their basins of attraction, bounded by the stable sets of two saddle cycles of period 7; The curve $\mathcal{C}$ is made up by 6 segments which are 3 images of the generating segments of LC_{-1} and LC^s_{-1}. The curve $\mathcal{C}$ is also formed by the closure of unstable sets of the saddle cycles of period 7.

Fig.5 presents a two-dimensional bifurcation diagram for the map F at $a = h$, where the regions corresponding to the attracting cycles of period

n, $n \leq 32$, are shown by different gray tonalities. We don't describe here the structure of this bifurcation diagram referring to Chapter 2 where a detailed description of a similar diagram is presented (we only remind that an

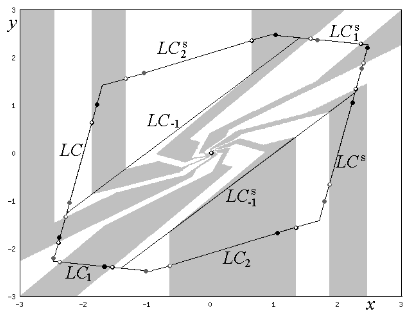

Figure 4: *The phase portrait of the map F at $a = h = 2$, $c = 0.21$: Two coexisting attracting 7-cycles (the black and gray circles) are shown together with their basins of attraction, bounded by the stable sets of two saddle 7-cycles.*

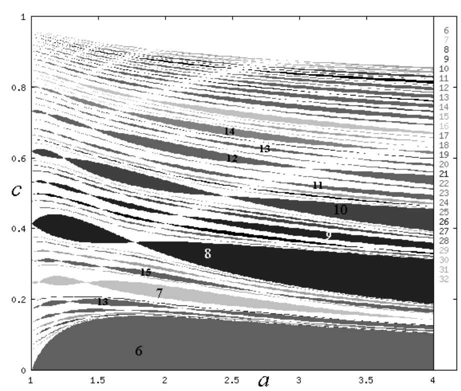

Figure 5: *The two-dimensional bifurcation diagram of the map F in the (a, c)-parameter plane in case $a = h$.*

odd periodicity region is related to two coexisting attracting cycles of the same period and, respectively, two saddle cycles). We would like to compare this diagram with two analogous bifurcation diagrams: One is related to the Hicksian business cycle model with the 'floor' in the investment function and the 'ceiling' in the income function (see Fig.3 of Chapter 6), and the other one is related to the Hicksian business cycle model with only one limit which is the 'floor' in the investment function (see Fig.4 of Chapter 2). All the three versions of the nonlinear Hicksian model possess either periodic or quasiperiodic dynamics and more complex dynamics cannot occur. The first two models never produce divergent trajectories. In the second model for sufficiently large a the main periodicity tongues mainly survive related to a particular trajectory which touches both upper and lower limits, while in the first model all periodicity tongues, born at $a = 1$, exist for any $a > 1$.

Consider now the case $a > h$. Given that the main interest of the present consideration is related to the bifurcation scenarios developing while a increases starting from $a = 1$, we consider $h < 1$. For example, let us fix $h = 0.5$. Regarding the value of the parameter l, note that for $l \to 0_-$ we have $\tau \to \infty$. Obviously, in such a case after the center bifurcation, that is for $a = 1 + \varepsilon$, $\varepsilon > 0$, an attractor of the map F belongs to the regions R_1, R_2, R_3 and has no points in the regions R_4 and R_5. Thus, only the maps F_1, F_2 and F_3 are involved in the asymptotic dynamics. In such a case we can apply the results presented in Section 2.6 of Chapter 2 and state that for sufficiently small $\varepsilon > 0$ in the phase space of the map F there exists an invariant closed attracting curve $\mathcal{C}$ homeomorphic to a circle, and dynamics of F on $\mathcal{C}$ are either periodic or quasiperiodic. The main difference of this case with respect to the case $a = h$ is that the curve $\mathcal{C}$ is not formed exactly by a finite number of proper segments of images the critical lines, but it is just a limit set for such images, and, as a consequence, the curve $\mathcal{C}$ has not finite but infinite number of segments and, thus, infinitely many corner points, which in case of a rational rotation are accumulating to the points of the related attracting cycle. The curve $\mathcal{C}$ can be destroyed by one of the mechanisms recalled in Section 2.6 of Chapter 2. In the example which we describe here this mechanism is related to transformation of the attracting cycle (which is born as a node) to a focus (see Fig.7a).

Fig.6 presents two-dimensional bifurcation diagrams of the map F in the (a, c)-parameter plane at fixed $h = 0.5$ and $l = -0.1$ (a), $l = -0.3$ (b). For the parameter ranges presented in these diagrams only the maps F_i, $i = 1, 2, 3$, are involved in the asymptotic dynamics (i.e., the attractor, or attractors, in case of multistability, are located in the regions R_i). It can be seen that the structure of both diagrams near the bifurcation value $a = 1$ is similar to the case $a = h$, namely, the parameter point $(a, c) = (1, c_{m/n})$ is the starting point for the periodicity tongue related to the attracting cycle with the rotation number m/n (or to the two attracting cycles when n is odd), but increasing the value of a such a periodicity tongue is destroyed. Comparing Fig.6a and Fig.6b, it can be seen that decreasing l such a destruction occurs for smaller values of a.

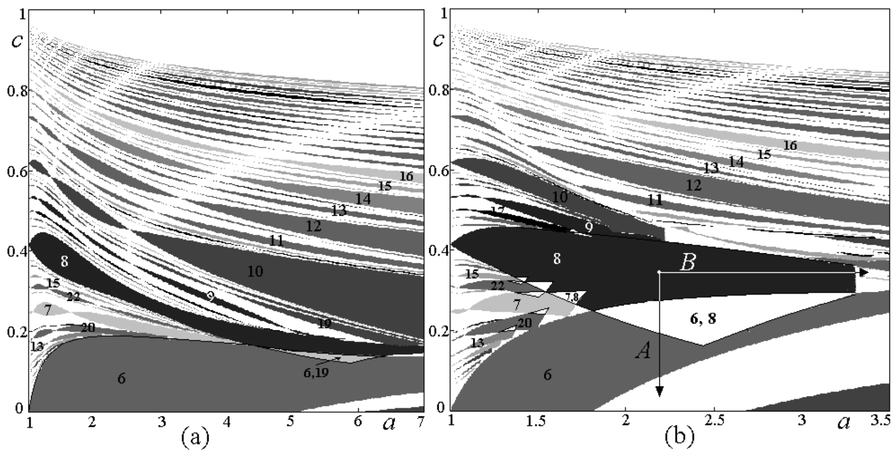

Figure 6: *Two-dimensional bifurcation diagrams of the map F in the (a, c)-parameter plane at $h = 0.5$, $l = -0.1$ (a) and $l = -0.3$ (b).*

Let us give some examples of bifurcation scenarios which can be realized if the (a, c)-parameter point moves inside a periodicity tongue. For this purpose we choose the 8-periodicity tongue and move the parameter point in the directions indicated in Fig.6b by the straight lines A and B. The starting point of the 8-periodicity tongue is $(a, c_{1/8}) = (1, \sqrt{2}-1)$. Increasing a, for example at $a = 1.005$, in the phase plane there exists the closed attracting invariant curve $\mathcal{C}$, made up by the closure of the unstable set of the saddle cycle of period 8, but already for $a = 1.01$ the attracting 8-cycle is a focus, so the curve $\mathcal{C}$ is no longer homeomorphic to a circle. Fig.7a presents an example of the saddle-focus connection at $a = 2.2$ and $c = 0.35$.

Starting from this parameter point we decrease the value of c following the parameter path A. Fig.7b presents the phase portrait of the map F at $a = 2.2$, $c = 0.33$: It can be seen that the unstable set of the saddle 8-cycle has selfintersections (which is impossible for invertible maps). At $c \approx 0.3085$ the saddle 8-cycle undergoes a homoclinic bifurcation (see Fig.7c with an enlarged part of the phase plane), after which the saddle-focus connection no longer exists. At $c \approx 0.28$ a border-collision bifurcation (Nusse and Yorke (1995)) occurs giving rise to an attracting and a saddle cycle of period 6 (an analog of the saddle-node bifurcation), so that the parameter point enters the bistability region. The enlarged part of the phase space related to this bifurcation is shown in Fig.7d. We emphasize that at the bifurcation the merging points of the attracting and saddle cycles are critical points. After

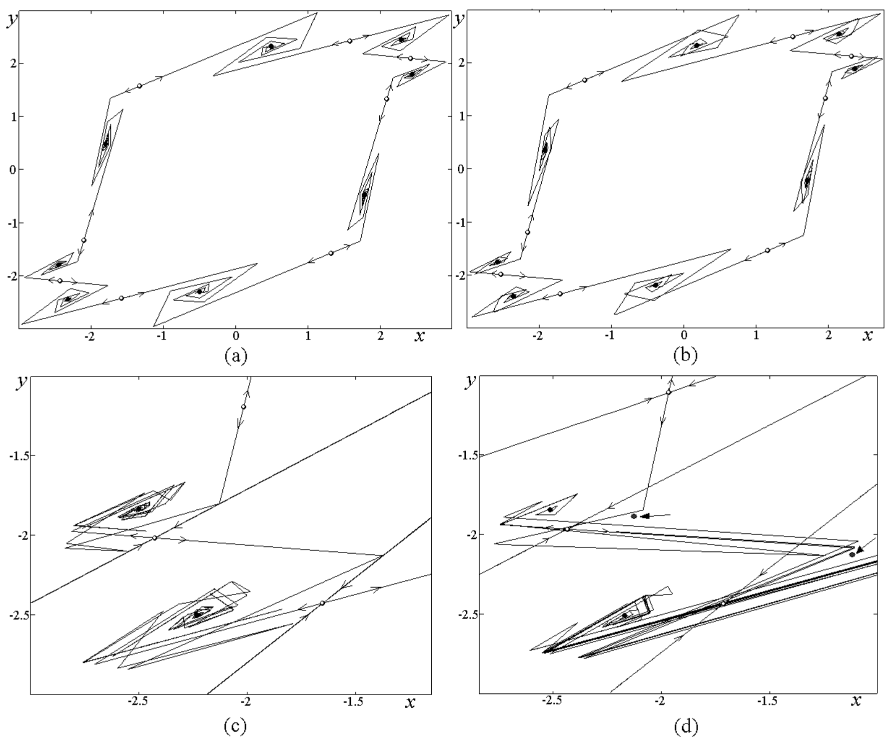

Figure 7: *The saddle-focus connection at* $a = 2.2$, $l = -0.3$, $h = 0.5$, $c = 0.35$ *(a) and* $c = 0.33$ *(b). The enlarged part of the phase plane at* $c = 0.3085$ *(c) (the homoclinic bifurcation of the saddle* 8*-cycle), and at* $c = 0.28$ *(d) (the 'saddle-node' border-collision bifurcation for an attracting and a saddle* 6*-cycle, two couples of merging points of which are shown by gray circles and indicated by the arrows).*

the bifurcation the basin of attraction of the attracting 6-cycle is bounded by the stable set of the saddle 6-cycle, while the basin of attraction of the attracting 8-cycle is bounded by the stable set of the saddle 8-cycle (see Fig.8). If we continue to decrease the value of c the basin of the attracting

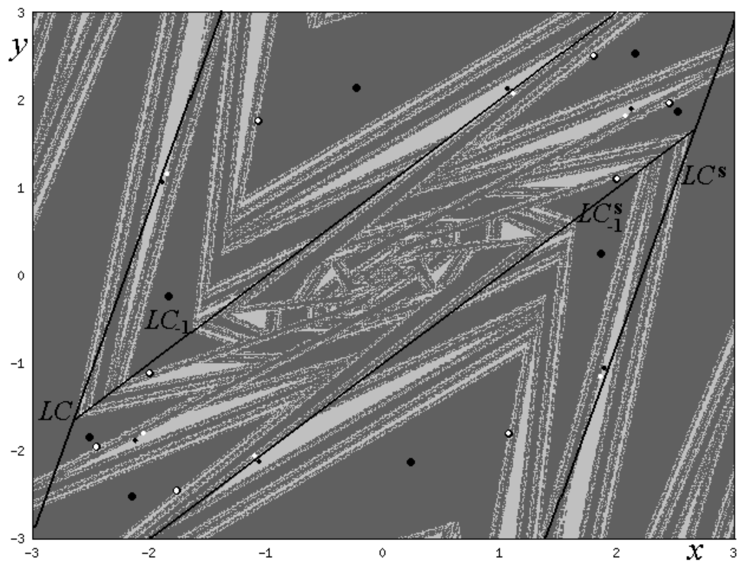

Figure 8: *Basins of attraction of coexisting attracting* 6*-cycle and* 8*-cycle.*

8 cycle decreases. At $c \approx 0.174$ the border-collision bifurcation occurs for the attracting and saddle cycles of period 8, when a pair of points of the attracting and the saddle cycles merges on LC_{-1} and another pair merges on LC^s_{-1}. On further decreasing of c the parameter point leaves the bistability region.

At $c \approx 0.085$ the attracting 6-cycle becomes a regular saddle: One of its eigenvalue passes through 1. Due to the piecewise linear definition of the map F, it is a particular kind of a pitchfork-like border-collision bifurcation: A k-cycle becomes a saddle and two coexisting k-cyclic chaotic attractors appear. Fig.9 shows schematically the phase portrait of the map F at the bifurcation value, where $[l_i, r_i]$, $i = 1, ..., 6$, denotes a segment of the eigenvector related to the eigenvalue 1, passing through the point p_i of the 6-cycle, and l_i, r_i are intersection points of the eigendirection with the related critical lines. Any point of the segment $[p_i, r_i]$ or $[l_i, p_i]$ is periodic of period 6. This bifurcation gives rise to two cyclic chaotic attractors of period 6 (see Fig.10 which shows such attractors at $a = 2.46$, $c = 0.081$). If we increase the value of a then at $a \approx 2.48$ the first homoclinic bifurcation of the saddle 6 cycle gives rise to the pairwise merging of the pieces of the attractors into a 6-pieces cyclic chaotic attractor, which then becomes a one-piece attractor due to the first homoclinic bifurcation of another saddle cycle of period 6, born due to the 'saddle-node' border-collision bifurcation. (For the de-

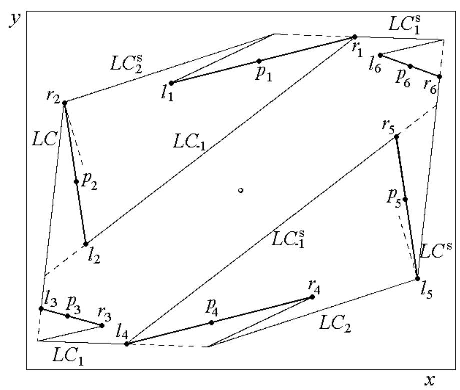

Figure 9: *The schematic view of the phase portrait of the map F at the bifurcation value related to the pichfolk bifurcation of the attracting* 6*-cycle.*

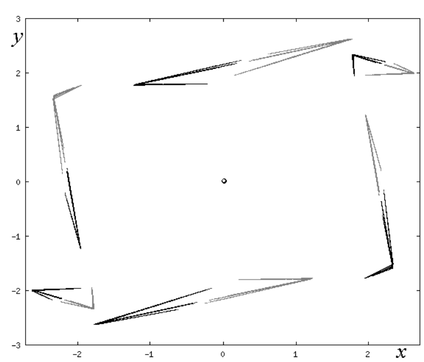

Figure 10: *Two cyclic chaotic attractors of period* 6 *are shown by different gray tonalities. Here* $a = 2.46$, $c = 0.081$, $l = -0.3$, $h = 0.5$.

scription of different mechanisms of reunion of pieces of a cyclical chaotic attractor in piecewise linear maps see Maistrenko *et al* (1998)).

Let us consider now the parameter path indicated in Fig.6b by B, increasing a from $a = 2.2$ at fixed $c = 0.35$. At $a \approx 3.33333$ the attracting 8-cycle undergoes the center bifurcation: At the bifurcation value there exist in the phase space 8 invariant regions (because of numerical precision, we cannot say precisely if the invariant region is bounded by an ellipse, or it is a polygon with a high number of edges). After the bifurcation the map F has 8-cyclic attracting rings (see Fig.11a which shows the attractor at $a = 3.33334$). Further increasing of the value a leads to the merging of 8 pieces of the attractor into one-piece attractor due to the homoclinic bifurcation of the saddle 8 cycle (see Fig.11b which shows the attractor soon after the merging, at $a = 3.335$).

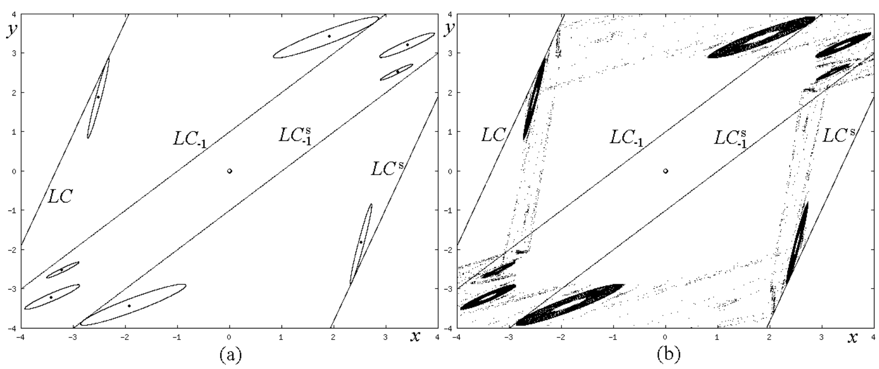

Figure 11: *The attractor of the map F at $c = 0.35$, $l = -0.3$, $h = 0.5$, $a = 3.33334$ (a) and $a = 3.335$ (b).*

All the examples presented above are related to the case in which only the maps F_1, F_2 and F_3 are involved in asymptotic dynamics, that is the constant branches of the investment function $I(z)$ (see Fig.1) play a role only for transient, but not asymptotic dynamics, which seems to be more reasonable from the economic point of view. To give an example of the case in which all five linear maps F_i, $i = 1, ..., 5$ are involved in asymptotic dynamics, we show two bifurcation diagrams in the (l, c)-parameter plane in Fig.12, where $h = 0.5$, $a = 1.1$ (a) and $a = 3$ (b). We have numerically checked whether the trajectory (after some transient) has points in the regions R_4 and R_5. The related parameter values are indicated by black points in Fig.12b (the corresponding attracting set is a one-piece chaotic attractor), while in

Fig.12a for the whole range of l the limit trajectory has no points in these regions.

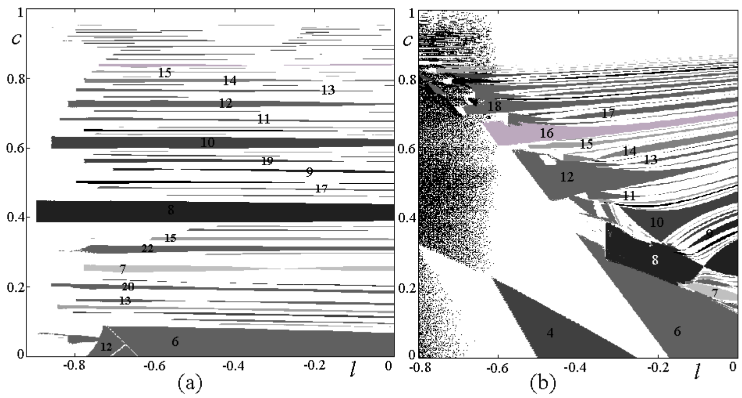

Figure 12: *Two-dimensional bifurcation diagram of the map F in the (l, c)-parameter plane at $h = 0.5$, $a = 1.1$ (a) and $a = 3$ (b).*

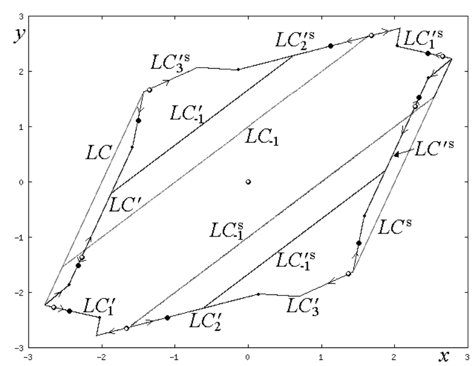

Figure 13: *The attracting closed invariant curve C at $a = 2$, $c = 0.35$, $l = -0.3$ and $h = 1.8$.*

To get a closed invariant attracting curve located in all five regions of the phase space we take $h > 1$: Fig.13 shows an example of such a curve at $a = 2$, $c = 0.35$, $l = -0.3$ and $h = 1.8$. Peculiarity of such a case is related to the fact that now we have a composition of the linear maps with zero and nonzero eigenvalues, and it influences the bifurcation scenarios, which can be realized varying the parameters. The curve $\mathcal{C}$, presented in Fig.13, is made up by 4 images of the generating segment of LC'_{-1} and 4 images of the generating segment of LC'^{s}_{-1}, while the images of the critical lines LC_{-1} and LC^{s}_{-1} bound an absorbing area (or, more precisely, and absorbing ring) including the curve $\mathcal{C}$. The map F is reduced on $\mathcal{C}$ to a rotation with the rotation number $1/8$ and has two cycles of period 8, one attracting and one saddle. Increasing a the number of segments of the curve $\mathcal{C}$ increases (see Fig.14a), and then coinciding segments of the unstable set of the saddle 8-cycle appear (see Fig.14b). It would be interesting to give a detailed analysis of possible scenarios of the destruction of a closed invariant curve in such a case, but we leave it as a subject for a future study.

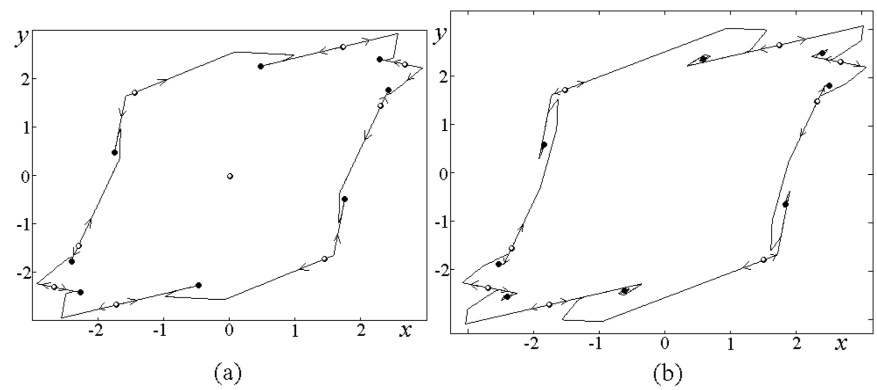

Figure 14: *The attracting closed invariant curve* C *at* $c = 0.35$, $l = -0.3$, $h = 1.8$, $a = 2.16$ *(a) and* $a = 2.3$ *(b).*

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