6 The Hicksian Model with Investment Floor and Income Ceiling

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6.1 Introduction

As we saw in Chapter 3, Hicks (1950) modified the Samuelson (1939) linear multiplier-accelerator model through introducing two constraints. The linear multiplier-accelerator model itself only has two options: Exponentially explosive or damped motion. According to Hicks, only the explosive case is interesting, as only this produces persistent motion endogenously, but it had to be limited through two (linear) constraints for which Hicks gave factual explanations.

When the cycle is in its depression phase it may happen that income decreases so fast that more capital can be dispensed with than what disappears through depreciation, i.e., natural wear and aging. As a result, the linear accelerator would predict not only negative investments (disinvestments), but to an extent that implies active destruction of capital. To avoid this, Hicks introduced his floor to disinvestment at the natural depreciation level.

When the cycle is in its prosperity phase, then it may happen that income would grow at a pace which does not fit available resources. Hicks has a discussion about what then happens, in terms of inflation, but he contended himself with stating that (real) income could not grow faster than available resources which put a ceiling on the income variable.

Hicks never formulated his final model with floor and ceiling mathematically, it seems that this was eventually done by Rau (1974), where the accelerator-generated investments were limited downwards through the natural depreciation floor, and where the income is limited upwards through the ceiling, determined by available resources. Formally:

$$I_t = \max(a(Y_{t-1} - Y_{t-2}), -I^f);$$

$$C_t = cY_{t-1};$$
$$Y_t = \min(C_t + I_t, Y^c).$$

Eliminating C_t and I_t, one has the single recurrence equation:

$$Y_t = \min(cY_{t-1} + \max(a(Y_{t-1} - Y_{t-2}), -I^f), Y^c). \tag{1}$$

It remains to say that Hicks's original discussion included an exponential growth in autonomous expenditures, combined with the bounds I^f and Y^c growing at the same pace, but taking the bounds as constant and deleting the autonomous expenditures, gives a more clear-cut version. It was this that was originally analyzed in detail by Hommes (1991), and the notation above comes from Hommes. However, there were some pieces missing in his discussion, such as two-dimensional bifurcation diagrams, which makes it motivated to make a new attack on this model.

6.2 Description of the map

Let $x_t := Y_t$, $y_t := Y_{t-1}$, $d := I^f$ and $r := Y^c$. Then the model given in (1) can be rewritten as a two-dimensional piecewise linear continuous map $F : \mathbb{R}^2 \to \mathbb{R}^2$:

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \min(cx + \max(a(x-y), -d), r) \\ x \end{pmatrix}, \tag{2}$$

which depends on four real parameters: $a > 0$, $0 < c < 1$, $d > 0$, $r > 0$.

The map F is given by three linear maps F_i, $i = 1, 2, 3$, defined, respectively, in three regions R_i of the phase plane:

$$\begin{aligned} F_1 \quad &: \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (c+a)x - ay \\ x \end{pmatrix}; \\ R_1 \quad &= \quad \{(x,y) : (1 + c/a)x - r/a \le y \le x + d/a\}; \end{aligned} \tag{3}$$

$$\begin{aligned} F_2 \quad &: \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} cx - d \\ x \end{pmatrix}; \\ R_2 \quad &= \quad \{(x,y) : y > x + d/a, x \le (d+r)/c\}; \end{aligned} \tag{4}$$

$$\begin{aligned} F_3 \quad &: \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} r \\ x \end{pmatrix}; \\ R_3 \quad &= \quad \mathbb{R}^2/R_1/R_2. \end{aligned} \tag{5}$$

Three half lines denoted LC_{-1}, LC'_{-1} and LC''_{-1} are boundaries of the regions R_i:

$$\begin{aligned} LC_{-1} &: \; y = x + d/a, \; x \le (r+d)/c, \\ LC'_{-1} &: \; y = (1 + c/a)x - r/a, \; x < (r+d)/c, \\ LC''_{-1} &: \; x = (r+d)/c, \; y > (r+d)/c + d/a. \end{aligned}$$

Their images by F are called critical lines:

$$\begin{aligned} LC_0 &: \; y = (x+d)/c, \; x \le r, \\ LC'_0 &: \; x = r, \; y < (r+d)/c. \end{aligned}$$

The image of LC''_{-1} by F is a point $(r, (r+d)/c)$. A qualitative view of the phase plane of the map F for $a > 1$, $d < r$ and $a > c^2/(1-c)$ is shown in Fig.1 (the last inequality indicates that the intersection point of LC'_{-1} and LC_0 is in the negative quadrant).

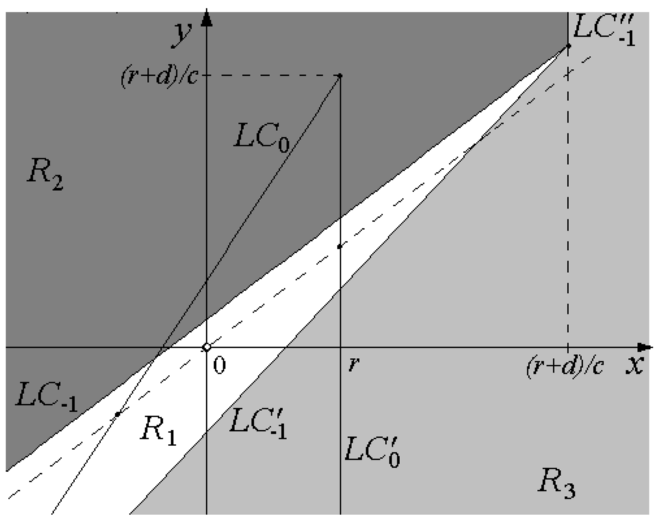

Figure 1: *Critical lines of the map F for $a > 1$, $d < r$, $a > c^2/(1-c)$.*

As it was mentioned in the introduction, an analogous model has been studied by Hommes (1991). Main conclusions of this paper hold also for the map F, namely, for $a > 1$ the map F has an attracting set $\mathcal{C}$ homeomorphic to a circle and all the trajectories of F (except for the fixed point) are attracted to this set. It was also proved that the dynamics of the map F on $\mathcal{C}$ are either periodic or quasiperiodic. In our consideration we show how the set $\mathcal{C}$ appears relating this to the center bifurcation described in detail in Chapter 2. We also discuss the structure of the two-dimensional bifurcation diagram in the (a, c)-parameter plane.

First note that the maps F_2 and F_3 have simple dynamics: The eigenvalues of F_2 are $\mu_1 = c$, $0 < c < 1$, $\mu_2 = 0$, so that any initial point $(x_0, y_0) \in R_2$ is mapped into a point of LC_0, while the map F_3 has two zero eigenvalues, and any $(x_0, y_0) \in R_3$ is mapped into a point of the straight line $x = r$. In such a way the whole phase plane is mapped in one step to the straight line $x = r$ and a cone $D = \{(x, y) : y \leq (x + d)/c,\ x \leq r\}$ (see Fig.1). Thus, the map F is a noninvertible map of so-called $(Z_\infty - Z_1 - Z_0)$ type: Any point belonging to the critical lines or to the half line $x = r$, $y > (r + d)/c$, has infinitely many preimages, any inner point of D has one preimage and any other point of the plane has no preimages.

The map F has a unique fixed point $(x^*, y^*) = (0, 0)$ which is the fixed point of the map F_1 (while the fixed points of the maps F_2 and F_3 belong to R_1, thus, they are not fixed points for the map F). The eigenvalues of the Jacobian matrix of F_1 are

$$\lambda_{1,2} = (a + c \pm \sqrt{(a + c)^2 - 4a})/2, \tag{6}$$

so that for the parameter range considered the fixed point (x^*, y^*) is a node if $(c+a)^2 > 4a$, and a focus if $(c+a)^2 < 4a$, being attracting for $a < 1$ and repelling for $a > 1$. Thus, for $a < 1$ the fixed point (x^*, y^*) is the unique global attractor of the map F (given that F_2 and F_3 are contractions).

6.3 Center bifurcation $(a = 1)$

At $a = 1$ the fixed point (x^*, y^*) loses stability with a pair of complex-conjugate eigenvalues crossing the unit circle, that is the center bifurcation occurs. First we describe the phase portrait of the map F exactly at the bifurcation value $a = 1$. Analogous description is presented in Section 2.2 of Chapter 2 for a two-dimensional piecewise linear map defined by two linear maps, which for the particular parameter value $b = 0$ are the maps F_1 and F_2 given in (3) and (4). It is proved that for the parameter values corresponding to the center bifurcation there exists an invariant region in the phase plane, which either is a polygon bounded by a finite number of images of a proper segment of the critical line, or the invariant region is bounded by an ellipse and all the images of the critical line are tangent to this ellipse (see Propositions 1 and 2 of Chapter 2). In the following we use these results for the considered map F specifying which critical lines are involved in the construction of the invariant region.

The map F_1 at $a = 1$ is defined by a rotation matrix. Moreover, if

$$c = c_{m/n} \overset{def}{=} 2\cos(2\pi m/n) - 1, \tag{7}$$

then the fixed point (x^*, y^*) is locally a center with rotation number m/n, so that any point in some neighborhood of (x^*, y^*) is periodic with rotation number m/n, and all points of the same periodic orbit are located on an invariant ellipse of the map F_1. Note that from $c > 0$ it follows that $m/n < 1/6$. Denote

$$c = c^* \overset{def}{=} 1 - (d/r)^2. \tag{8}$$

Proposition 1. *Let $a = 1$, $c = c_{m/n}$, then in the phase space of the map F there exists an invariant polygon P such that*

- *if $c_{m/n} < c^*$ then P has n edges which are the generating segment $S_1 \subset LC_{-1}$ and its $n-1$ images $S_{i+1} = F_1(S_i) \subset LC_{i-1}$, $i = 1, ..., n-1$;*
- *if $c_{m/n} > c^*$ then P has n edges which are the generating segment $S'_1 \subset LC'_{-1}$ and its $n-1$ images $S'_{i+1} = F_1(S'_i) \subset LC'_{i-1}$;*
- *if $c_{m/n} = c^*$ then P has $2n$ edges which are the segments S_i and S'_i, $i = 1, ..., n$.*

Any initial point $(x_0, y_0) \in P$ is periodic with rotation number m/n, while any $(x_0, y_0) \notin P$ is mapped in a finite number of steps into the boundary of P.

The proof of the proposition is similar to the one presented in Section 2.2 of Chapter 2. The value c^* is obtained from the condition of an invariant ellipse of F_1 to be tangent to both critical lines LC_{-1} and LC'_{-1}. It can be shown that for $c_{m/n} < c^*$ only the upper boundary LC_{-1} is involved in the construction of the invariant region, while if $c_{m/n} > c^*$ we have to iterate the generating segment of the lower boundary LC'_{-1} to get the boundary of the invariant region. An example of the invariant polygon P in the case $c_{m/n} = c^*$ is presented in Fig.2, where $a = 1$, $d = 10$, $r = 10/\sqrt{2-\sqrt{2}}$, $c = c_{1/8} = c^* = \sqrt{2} - 1$. For such parameter values the polygon P has 16 edges, which are the segments $S_i \subset LC_{i-2}$ and $S'_i \subset LC'_{i-2}$, $i = 1, ..., 8$.

Any point of P is periodic with rotation number $1/8$ (in Fig.2 the points of two such cycles belonging to the boundary of P are shown by black and gray circles), while any point $(x_0, y_0) \notin P$ is mapped to the boundary of P.

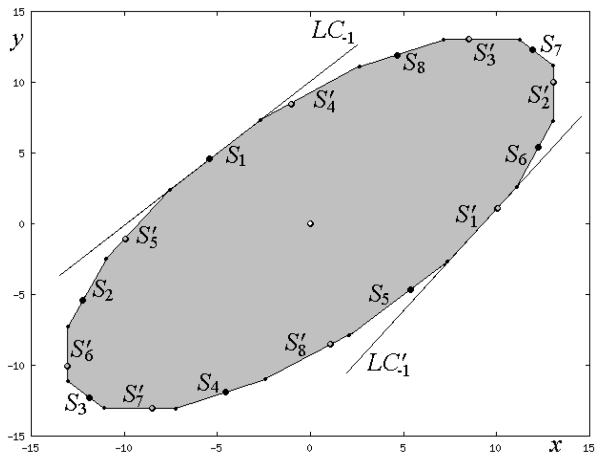

Figure 2: *The invariant polygon* P *with* 16 *edges at* $a = 1$, $c = c_{1/8} = \sqrt{2} - 1 = c^*$, $d = 10$, $r = 10/\sqrt{2 - \sqrt{2}}$.

Consider now the case in which the map F_1 is defined by the rotation matrix with an irrational rotation number ρ, which holds if

$$c = c_\rho \stackrel{def}{=} 2\cos(2\pi\rho) - 1, \tag{9}$$

where $\rho < 1/6$. Then any point in some neighborhood of the fixed point (x^*, y^*) is quasiperiodic, and all points of the same quasiperiodic orbit are dense on the corresponding invariant ellipse of the map F_1. Using the Proposition 2 of Chapter 2 and the values c^* given in (8) we can state the following

Proposition 2. *Let* $a = 1$, $c = c_\rho$. *Then in the phase space of the map* F *there exists an invariant region* Q, *bounded by an invariant ellipse* $\mathcal{E}$ *of the map* F_1 *which is tangent to* LC_{-1} *(and to all its images) if* $c < c^*$, *to* LC'_{-1} *if* $c > c^*$, *and to both critical lines* LC_{-1} *and* LC'_{-1} *if* $c = c^*$. *Any initial point* $(x_0, y_0) \in Q$ *belongs to a quasiperiodic orbit dense in the corresponding invariant ellipse of* F_1, *while any initial point* $(x_0, y_0) \notin Q$ *is mapped to* $\mathcal{E}$.

Note that from (8) it follows that if $d > r$ then the inequality $c^* < 0$ holds, thus, given $c > 0$, for $d > r$ only the lower boundary LC'_{-1} is involved in the construction of the invariant region of the map F at $a = 1$.

6.4 Bifurcation structure of the (a, c)-parameter plane

In this section we describe the dynamics of the map F after the center bifurcation, that is for $a > 1$. In short, an initial point (x_0, y_0) from some neighborhood of the unstable fixed point (x^*, y^*) moves away from it under the map F_1 and in a finite number k of iterations it necessarily enters either the region R_2, or R_3 (in the case in which (x^*, y^*) is a focus the statement is obvious, while if (x^*, y^*) is a repelling node this can be easy verified using the eigenvalues $\lambda_{1,2}$ given in (6) and the corresponding eigenvectors). If $(x_k, y_k) \in R_2$, then the map F_2 is applied: $F_2(x_k, y_k) = (x_{k+1}, y_{k+1}) \in LC_0$. All consequent iterations by F_2 give points on LC_0 approaching the attracting fixed point of F_2 (which belongs to R_1), until the trajectory enters R_1 where the map F_1 is applied again. If $(x_k, y_k) \in R_3$, then the map F_3 is applied: $F_3(x_k, y_k) = (x_{k+1}, y_{k+1}) \in LC'_0$. We have that either $(x_{k+1}, y_{k+1}) \in R_1$, or $(x_{k+1}, y_{k+1}) \in R_3$ and one more application of F_3 gives its fixed point $(r, r) \in R_1$, so, the map F_1 is applied to this point. In such a way the dynamics appear to be bounded.

Indeed, it was proved in Hommes (1991), that for $a > 1$ any trajectory of F rotates with the same rotation number around the unstable fixed point, and it is attracted to a closed invariant curve $\mathcal{C}$ homeomorphic to a circle. It was also proved that the dynamics of F on $\mathcal{C}$, depending on the parameters, are either periodic or quasiperiodic. We can state that such a curve $\mathcal{C}$ is born due to the center bifurcation of the fixed point, described in the previous section: Namely, the bounded region P (or Q), which is invariant for $a = 1$, exists also for $a > 1$, but only its boundary remains invariant, and this boundary is the curve $\mathcal{C}$.

We refer as well to Chapter 2 in which it is shown that also in a more generic case of a two-dimensional piecewise linear map, defined by two linear maps, the center bifurcation can give rise to the appearance of a closed invariant attracting curve $\mathcal{C}$, on which the map is reduced to a rotation with rational or irrational rotation number. Recall that in the case of a rational rotation number m/n the map has an attracting and a saddle m/n-cycle on $\mathcal{C}$, so that the curve $\mathcal{C}$ is formed by the unstable set of the saddle cycle, approaching the points of the attracting cycle. While in the case of an irrational rotation number the map has quasiperiodic orbits on $\mathcal{C}$. In Section 2.3 of Chapter 2 the curve $\mathcal{C}$ is described in detail for the map defined by the linear maps F_1 and F_2 given in (3) and (4). So, we can use these results for the considered map F if the curve $\mathcal{C}$ belongs to the regions R_1, R_2 and has no intersection with the region R_3, thus, only the maps F_1 and F_2 are involved in the asymptotic dynamics. Obviously, we have a qualitatively similar case if the curve $\mathcal{C}$ has no intersection with the region R_2 and, thus, only the maps F_1 and F_3 are applied to the points on $\mathcal{C}$. One more possibility is the case in which the curve $\mathcal{C}$ belongs to all the three regions R_i, $i = 1, 2, 3$. We can state that in the first and second cases the curve $\mathcal{C}$ can be obtained by iterating the generating segment of LC_{-1} and LC'_{-1}, respectively, while in the third case both generating segments can be used to get the curve $\mathcal{C}$.

To see which parameter values correspond to the cases described above we present in Fig.3 a two-dimensional bifurcation diagram in the (a, c)-parameter plane for fixed values $d = 10$, $r = 30$. Different gray tonalities indicate regions corresponding to attracting cycles of different periods $n \leq 41$ (note that regions related to the attracting cycles of the same period n, but with different rotation numbers are shown by the same gray tonality). The white region in Fig.3 is related either to periodic orbits of period $n > 41$, or to quasiperiodic orbits.

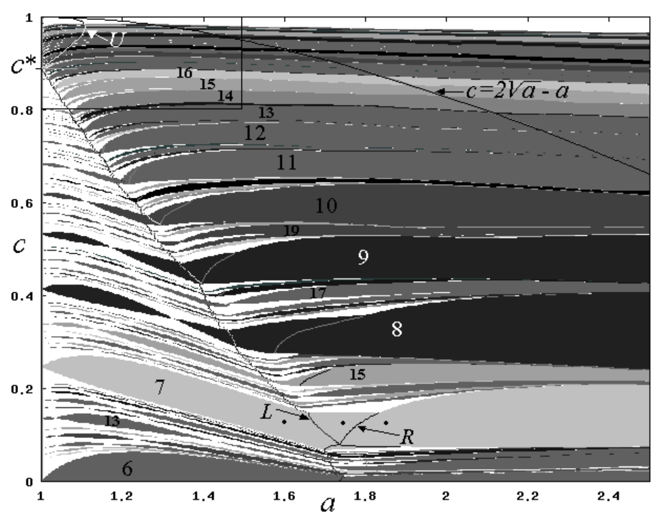

Figure 3: *Two-dimensional bifurcation diagram of the map F in the (a, c)-parameter plane at $d = 10$, $r = 30$. Regions corresponding to attracting cycles of different periods $n \leq 41$ are shown by various gray tonalities.*

Let us first comment on some particular parameter values of the bifurcation line $a = 1$. As described in the previous section, at $a = 1$, $c = c_{m/n}$ given in (7), in the phase plane of F there exists an invariant polygon P such that any point of P is periodic with the rotation number m/n. So, the points $a = 1$, $c = c_{m/n}$, for different $m/n < 1/6$, are starting points for the corresponding periodicity tongues. For example, $a = 1$, $c = c_{1/8} = \sqrt{2} - 1$ is the point from which the 8-periodicity tongue starts, corresponding to the attracting cycle with the rotation number $1/8$. Recall that according to the summation rule (see Hao and Zheng (1998)), between any two rotation numbers m_1/n_1 and m_2/n_2 there is also the rotation number $m'/n' = (m_1+m_2)/(n_1+n_2)$, so that $a = 1$, $c = c_{m'/n'}$ is the starting point for the corresponding periodicity region. If the (a, c)-parameter point is taken inside the periodicity region, then the map F has the attracting and saddle cycles with corresponding rotation number, and the unstable set of the saddle cycle form the closed invariant attracting curve $\mathcal{C}$. Note, that in the case in which both constrains are involved in the asymptotic dynamics, the map F may have two attracting cycles and two saddles of the same period coexisting on the invariant curve (as it occurs, for example, inside the 7-periodicity tongue at $a = 2.9$, $c = 0.136$, $d = 10$, $r = 30$). While if the (a, c)-parameter point belongs to the boundary of the periodicity region, then the border-collision bifurcation occurs (see Nusse and Yorke (1995)) for the attracting and saddle cycles, giving rise to their merging and disappearance (see Chapter 2).

The parameter points $a = 1$, $c = c_\rho$ given in (9), for different irrational numbers $\rho < 1/6$ correspond to the case in which any point of the invariant region Q is quasiperiodic. Such parameter points are starting points for the curves related to quasiperiodic orbits of the map F.

At $a = 1$, $c = c^* = 8/9$, (which is the value c^* given in (8) at $d = 10$ and $r = 30$) there exists an invariant ellipse of F_1 tangent to both critical lines LC_{-1} and LC'_{-1}, so that for $c < c^*$ the boundary of the invariant region can be obtained by iterating the generating segment of LC_{-1}, while for $c > c^*$ we can iterate the segment of LC'_{-1}. Thus, after the center bifurcation for $c < c^*$ at first only LC_{-1} is involved in the asymptotic dynamics, and then increasing a there is a contact of the curve $\mathcal{C}$ with the lower boundary LC'_{-1}. And vice versa for $c > c^*$. For example, the curve denoted by L inside the 7-periodicity region in Fig.3 indicates a collision of the curve $\mathcal{C}$ with the lower boundary LC'_{-1}. The curves related to similar collision are shown also inside some other periodicity regions. Before this collision the dynamics of F on $\mathcal{C}$ is as described in Proposition 3 of Chapter 2, while after both boundaries LC_{-1} and LC'_{-1} are involved in the asymptotic dynamics.

One more curve shown inside the periodicity regions (for example, the one denoted by R inside the 7-periodicity region) indicates that the point $(x, y) = (r, r)$ becomes a point of the corresponding attracting cycle.

To clarify, let us present examples of the phase portrait of the map F corresponding to three different parameter points inside the 7-periodicity region, indicated in Fig.3. Fig.4 shows the closed invariant attracting curve $\mathcal{C}$ at $a = 1.6$, $c = 0.125$, when $\mathcal{C}$ has no intersection with the region R_3, being made up by 7 segments of the images of the generating segment of LC_{-1}.

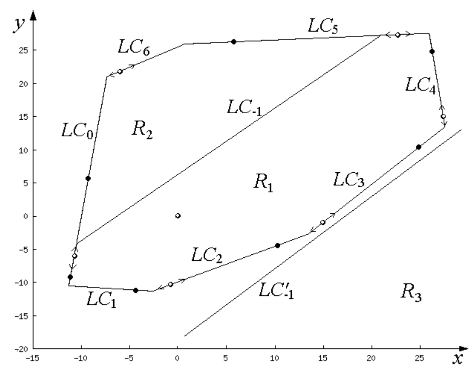

Figure 4: *The attracting closed invariant curve* $\mathcal{C}$ *with the attracting and saddle cycles of period* 7 *at* $a = 1.6$, $c = 0.125$, $d = 10$, $r = 30$.

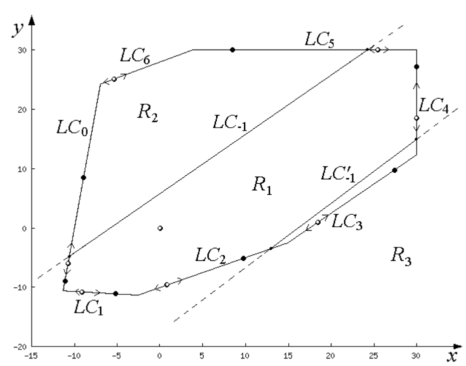

Figure 5: *The attracting closed invariant curve* $\mathcal{C}$ *with the attracting and saddle cycles of period* 7 *at* $a = 1.75$, $c = 0.125$, $d = 10$, $r = 30$.

The closed invariant curve $\mathcal{C}$ corresponding to the parameter values $a = 1.75$, $c = 0.125$, is shown in Fig.5. In such a case both boundaries LC_{-1} and LC'_{-1} are involved in the dynamics. It can be easily seen that images of the generating segments of LC_{-1} and LC'_{-1} form the same set, so it does not matter which segment is iterating to get the curve $\mathcal{C}$.

Fig.6 presents an example of $\mathcal{C}$ at $a = 1.85$, $c = 0.125$, when two consequent points of the attracting cycle belong to the region R_3, so that $(x, y) = (r, r)$ is a point of the attracting cycle.

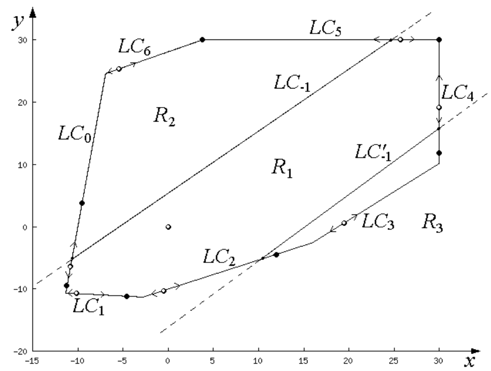

Figure 6: *The attracting closed invariant curve $\mathcal{C}$ at* $a = 1.85$, $c = 0.125$, $d = 10$, $r = 30$.

In Fig.7 we show the enlarged window of the bifurcation diagram presented in Fig.3 in order to indicate the (a, c)-parameter region corresponding to the case in which only the lower boundary is involved in the asymptotic dynamics. The curve denoted by U indicates the contact of the trajectory with the upper boundary LC_{-1}, so that just after the center bifurcation, for $c > c^*$ at first only the lower boundary LC'_{-1} is involved in the asymptotic dynamics (see Fig.8 with an example of the attracting closed invariant curve $\mathcal{C}$ at $a = 1.05$, $c = 0.94$). Then, increasing a the trajectory has a contact also with the upper boundary LC_{-1}. Note that for the main periodicity tongues (those related to the rotation number $1/n$) just after the center bifurcation the point (r, r) immediately becomes a point of the attracting cycle, because after the bifurcation two points of the attracting cycle must be in the region R_3, but we know that two successive applications of F_3 give the point (r, r). In Hommes (1991) it was proved that if and only if the attracting set $\mathcal{C}$ contains the point (r, r), then the restriction of the map F to $\mathcal{C}$ is topologically conjugate to a piecewise linear nondecreasing circle map f, and there exists a unique circle arc I on which f is constant being strictly increasing on the complement of I. From this statement it follows that in such a case the map F cannot have quasiperiodic trajectory, but only periodic ones.

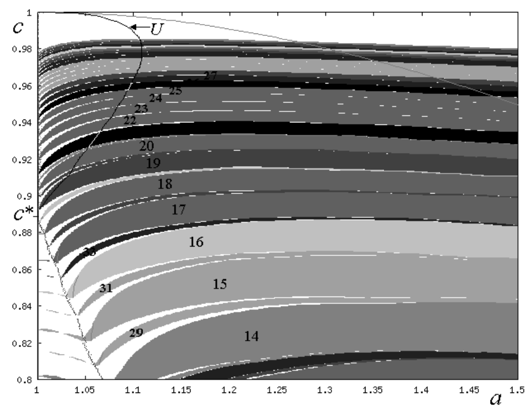

Figure 7: *Enlarged window of the bifurcation diagram of the map F shown in Fig.3.*

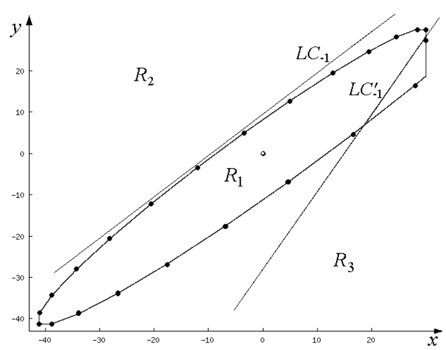

Figure 8: *The attracting closed invariant curve $\mathcal{C}$ with the attracting* 23*-cycle on it at* $a = 1.05$, $c = 0.94$, $d = 10$, $r = 30$.

Summarizing, we state that for $c < c^*$ (or $c > c^*$) given in (7), increasing the values of a from $a = 1$, the closed invariant attracting curve $\mathcal{C}$ at first is made up by a finite number of images of LC_{-1} (or LC'_{-1}, respectively), then a contact with LC'_{-1} (or LC_{-1}) occurs after which to get the curve $\mathcal{C}$ we can iterate the generating segment either LC_{-1} or LC'_{-1}. As long as the curve $\mathcal{C}$ does not contain the point (r, r), the dynamics of F on $\mathcal{C}$ are either periodic, or quasiperiodic, while if (r, r) belongs to $\mathcal{C}$ then dynamics are only periodic.

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