Chapter 2

NONINVERTIBLE MAPS

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Abstract

In this chapter we briefly reconsider some of the main properties of noninvertible maps which characterize the related dynamics. We shall recall how the effects of noninvertibility may be important in understanding global bifurcations both in the structure of the invariant attracting sets and in the structure of the basins of attraction.

2.1 Introduction

The object of the present chapter is to describe some properties on nonlinear dynamics in discrete systems, associated with noninvertible functions. Let us consider a dynamic model which is described by an iterative process, i.e. a map: The state of the system changes under the action of some function T. The state x may be a scalar or a vector of state variables. The state (or phase) space is a set $X \subseteq R^m$ where m is an integer denoting the dimension of the vector state variable x, $m \in \{1, 2, 3...\}$, and $T : X \to X$. A discrete dynamical system (DDS for short) is represented by the standard notation

$$x_{n+1} = T(x_n) \quad or \quad x' = T(x) \tag{2.1}$$

The object of the theory of DDS is that to understand which kind of values will be obtained asymptotically, and this depending on the initial value (or initial condition) x_0 in the phase

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space. Also important will be the *bifurcations*, which are responsible of the changes in the qualitative behaviors of the trajectories of the iterative process. To this scope we recall that the bifurcations are studied in the parameter space, which includes all the parameters which are considered in a model under study. Whenever the parameters have a fixed value we have a dynamic system whose invariant sets in the phase space of interest are our object of investigation, as well as the description of the dynamic behavior associated with the points in the phase space (are the trajectories converging to the same set ? are some of them uninteresting for us because associated to divergent dynamics ? and so on). Then, as the parameters are varied, things may change smoothly (as under a deformation, we shall say "via an homeomorphism", which is a continuous invertible function) or not smoothly (and often some drastic change may occur), in which case we say that a bifurcation takes place. Roughly speaking, we say that a bifurcation takes place at some specific parameters setting when the dynamics occurring "before" and "after" (when the condition is not fulfilled) cannot be obtained one from the other by a smooth change (via an homeomorphism).

To study DDS it is important to introduce first a few definitions and terms. Let us consider a map $x' = T(x)$, T is defined from X into itself. The point x' is called the *rank-1 image* of x. A point x such that $T(x) = x'$ is called a *rank-1 preimage* of x'. The point $x(n) = T^n(x)$, $n \in \mathbb{N}$, is called image of rank-n of the point x, where T^0 is identified with the identity map and $T^n(\cdot) = T \circ T^{n-1}(\cdot) = T(T^{n-1}(\cdot))$. A point x such that $T^n(x) = y$ is called rank-n preimage of y.

Let $A \subset X$ be a such that $T(A) \subseteq A$, then A is called *trapping set.* We have two kinds of trapping set: either (a) $T(A) = A$, then A is called *invariant set,* or (b) $T(A) \subset A$ than A is strictly mapped into itself, and in this case $T^{n+1}(A) \subseteq T^n(A)$ for any $n > 0$. When A is a compact set then the intersection of the nested sequence of sets is a closed nonempty invariant set, say $B = \cap_{n>0} T^n(A)$, then $T(B) = B$ (note that the number of iterations necessary to get the invariant set B may be finite or infinite). It is important to stress the properties of an invariant set $A \subseteq X$. As by definition any point of $T(A)$ is the image of at least one point of A, we have that for an invariant set $A \subseteq X$, for which $T(A) = A$, this properly holds for any point in A, that is:

Property 1. If T is invariant on A then any point of A has at least one rank-1 preimage in A, and iteratively: any point of A has an infinite sequence of preimages in A.

The behavior of points in a neighborhood of an invariant set A depends on the local dynamics (A may be attracting, repelling, or neither of the two).

An *attracting set* is a closed invariant set A which possess a *trapping neighborhood,* that is, a neighborhood U, with $A \subset Int(U)$, such that $A = \cap_{n\geq 0} T^n(U)$ (as in the case of the set B constructed above). In other words, if A is an attracting set for T, then a neighborhood U of A exists such that the iterates $T^n(x)$ tend to A for any $x \in U$ (and not necessarily enter A). An *attractor* is an attracting set with a dense orbit.

The *basin of attraction* of an attracting set A, $\mathcal{D}(A)$, it the set of all the points whose trajectory has the limit set in A (roughly speaking, whose trajectory tends to A).

$$\mathcal{D}(A) = \{x | T^n(x) \to A \quad \text{as } n \to +\infty\}. \tag{2.2}$$

As the attracting set possesses a neighborhood U of points having this property, then the basin is made up of all the possible preimages of U: $\mathcal{D}(A) = \cup_{n\geq 0} T^{-n}(U)$. Sometimes it is useful to consider as neighborhood U the *immediate basin*, which is the largest connected component of the basin which contains the attracting set A.

A *repelling set* is a compact invariant set K which possesses a neighborhood U such that for any point $x_0 \in U\backslash K$, the trajectory $x_0 \to x_1 \to ...$ must satisfy $x_n \notin U$ for at least one value of $n \geq 0$ (but such a trajectory may also come back again in U, as it occurs when homoclinic trajectories exist). A *repellor* is a repelling set with a dense orbit.

It is worth noticing that this definition is a very strict one (as we shall see below, by using this definition a saddle cycle cannot be called repelling, but only unstable). Some authors use "expanding" in its place, keeping a more soft definition for a repelling set saying that a closed invariant set K which is not attracting is called *repelling if however close to K there are points whose trajectories goes away from K*. In this chapter we adopt this second (more soft) definition.

Regarding the *invariant sets*, the simplest case is that of "*fixed point*". We say that x^* is a fixed point (or equilibrium point) of the DDS if it satisfies

$$x^* = f(x^*)$$

That is: starting in that point the system never changes. Then, given that it is very difficult to be exactly in a fixed point, it is important to understand when (i.e. under which conditions) starting from a different state and iterating the process we are approaching the equilibrium, and when this occurs for all the points in a suitable neighborhood, we call it *attracting:* The definition given above is fulfilled. When for some points, also very close to an equilibrium, the process will lead the state far away from it, then it is unstable.

A map T is said to be noninvertible (or "many-to-one", see Fig. 2.1), if distinct points $x \neq y$ exist which have the same image, $T(x) = T(y) = x'$. This can be equivalently stated by saying that points exist which have several rank-1 preimages, i.e. the inverse relation $x = T^{-1}(x')$ may be multi-valued. Geometrically, the action of a noninvertible map T can be described by saying that it "folds and pleats" the space, so that two distinct points are mapped into the same point. Equivalently, we could also say that several inverses are defined, and these inverses "unfold" the space.

For a continuous noninvertible map T, the space $\mathbb{R}^m$ can be subdivided into regions Z_k, $k \geq 0$, whose points have k distinct rank-1 preimages. Generally, as the point x' varies in $\mathbb{R}^m$, pairs of preimages appear or disappear as this point crosses the boundaries which separate different regions. Hence, such boundaries are characterized by the presence of at least two coincident (or merging) preimages. This leads to the definition of the critical sets, one of the distinguishing features of noninvertible maps ([1], [2], [3], [4], [5], [6]): The *critical set* CS of a continuous map T is defined as the locus of points having at least two coincident $rank-1$ preimages, located on a set CS_{-1} called *set of merging preimages*. The critical set CS is the n-dimensional generalization of the notion of critical value (when it is a local minimum or maximum value) of a one-dimensional map[1], and of the notion of *critical curve* LC of a noninvertible two-dimensional map (from the French "Ligne Critique"). The set CS_{-1} is the generalization of the notion of critical point (when it is a local extremum point) of a one-dimensional map, and of the fold curve LC_{-1} of a

[1]This terminology, and notation, originates from the notion of critical points as it is used in the classical works of Julia and Fatou.

two-dimensional noninvertible map (for piecewise smooth continuous maps). The critical set CS is generally formed by $(n-1)$-dimensional hypersurfaces of $\mathbb{R}^m$, and portions of CS separate regions Z_k of the phase space characterized by a different number of $rank-1$ preimages, for example Z_k and Z_{k+2} (this is the standard occurrence).

2.2 One-dimensional maps

Let us consider first the case of a one-dimensional phase space, as all the main properties of dynamical systems and chaotic behaviors can be well introduced in this space. As a very simple example consider the Myrberg's map (Myrberg was the first author who studied in details the bifurcations of such non-invertible one-dimensional maps, still in 1963, and via a linear homeomorphism this map is topologically conjugated with the standard logistic map):

$$x' = f(x) \quad : \quad f(x) = x^2 + c \tag{2.3}$$

For $c \in [-2, 0]$ we have $f : I \to I,\ I = [q^*_{-1}, q^*]$ where q^* is the repelling positive fixed point. At $c = 0$ the slope at the stable fixed point p^* is zero (also called superstable), and then, decreasing c, the slope from positive becomes negative, reaching the value -1 and a flip bifurcation takes place, leading to the appearance of a stable cycle of period 2.

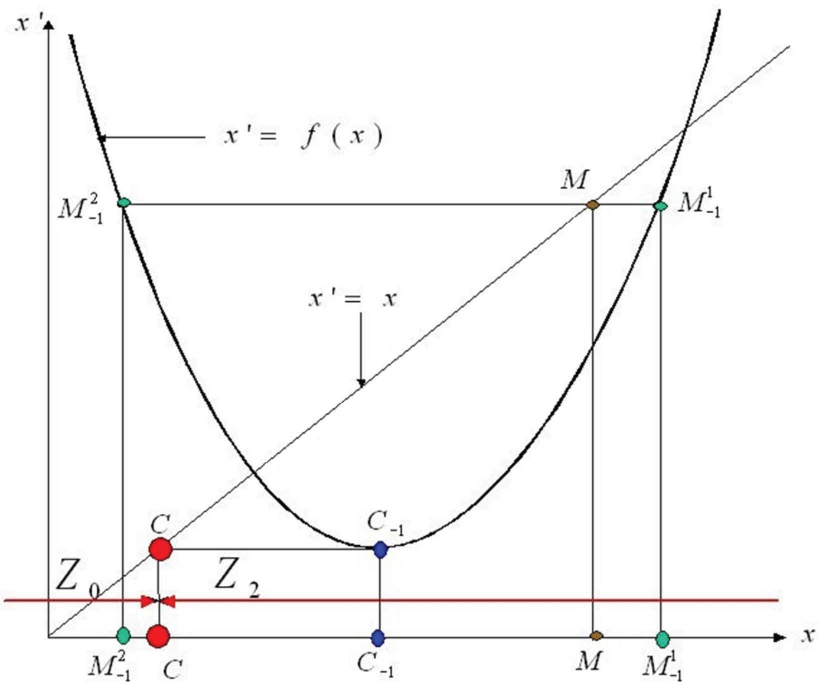

Figure 2.1: Myrberg's map.

We recall that for a nonlinear function the stability/instability of a fixed point x^* is a local property, which may be investigated by the first order approximation of the function in the fixed point, that is, by using the first derivative in the fixed point, also called eigenvalue, $\lambda = f'(x^*)$. We can summarize as follows:

$$\begin{aligned} -1 &< \lambda = f'(x^*) < 1 \;:\; locally\; asymptotically\; stable\; fixed\, point \\ \lambda &= +1\, bifurcation\; (fold,\; transcritical\; or\; pitchfork) \\ \lambda &= -1\; flip\; bifurcation \end{aligned}$$

In general, a $k-$cycle ($k \geq 1$) is a sequence of k consecutive distinct points x_i, $i = 1, 2, ..., k$ visited iteratively by the map, and such that $f^k(x_i) = x_i$ for any point x_i. That is, stated in other words, each of the periodic points is a fixed point of the map $f^k = f \circ f \circ ... \circ f$. The stability/instability of a cycle is determined by the stability/instability condition of a fixed point of the map f^k and from the chain rule we have, for each point x_i of the cycle,

$$\lambda = \frac{d}{dx}(f^k(x))|_{x_i} = \prod_{j=1}^{k} f'(x_j) \tag{2.4}$$

Summarizing, if we consider a one–dimensional map $x_{n+1} = f(x_n)$ and a $k-$cycle of points $\{x_1, ..., x_k\}$, $k \geq 1$ (for $k = 1$ we have a fixed point), the condition $|\lambda| < 1$ (resp. > 1) is a sufficient condition to conclude that the $k-$cycle is an attractor (resp. repellor), as λ is the slope, or eigenvalue, in any point x_i of the map f^k.

We have not considered the bifurcation cases in which $|\lambda| = 1$, because the behavior depends on the kind of bifurcation. This can be found in several textbooks ([7], [8], [9], [10], [11]), and we simply recall that the bifurcations associated with $\lambda = -1$ are related to the period-doubling of the cycle, and it is frequently called *flip bifurcation*. That is, crossing this bifurcation value, when suitable transversality conditions are satisfied, then a stable $k-$cycle becomes unstable and a stable $2k-$cycle (of double period) appears around it. While the bifurcations associated with $\lambda = +1$ may be of three different kinds: (i) either related to a *fold bifurcation*, giving rise to a pair of $k-$cycles, one attracting and one repelling, (ii) or to a *change of stability* (also called *transcritical*), a pair of stable/unstable cycles merge after which they exchange their stability, i.e. become unstable/stable respectively, (iii) or a *pitchfork bifurcation* occurs at which a stable $k-$cycle becomes unstable and two new $k-$cycles appear around it, both stable.

As observed several years ago by the pioneers of such studies ([3], [12], [13], [14], [15], [16]) still in the one-dimensional case we can see that once that the monotonicity (i.e. the invertibility property) is lost, then very complicated paths may occur, which may be predictable or not (although the model is completely deterministic). As a standard example let us consider the simple Myrberg's map (whose graph is a parabola), which for $c < 0$ has the positive fixed point always unstable and the negative fixed point which may be stable or unstable, depending on the slope (or eigenvalue) in that point. This map has a unique critical point C ($x = c$), which separates the real line into the two subsets (see Fig. 2.1): $Z_0 = (-\infty, C)$, where no inverses are defined, and $Z_2 = (C, +\infty)$, whose points have two rank-1 distinct preimages. These preimages can be computed by the two inverses

$$x_1 = f_1^{-1}(x') = \sqrt{x' - c} \;\; ; \;\; x_2 = f_2^{-1}(x') = -\sqrt{x' - c}. \tag{2.5}$$

If $x' \in Z_2$, its two rank-1 preimages, computed according to (2.5), are located symmetrically with respect to the point $C_{-1}(x = 0) = f_1^{-1}(C) = f_2^{-1}(C)$. Hence, $C_{-1}(x = 0)$ is the point where the two merging preimages of C are located. The map f folds the real line (Fig. 2.1), while the two inverses unfold it. As the map (2.3) is differentiable, at C_{-1} the first derivative vanishes. However, note that in general a critical point may even be a point where the map is not differentiable. This happens for continuous piecewise differentiable maps such as the well known tent map or other piecewise linear maps. In these maps critical points are located at the kinks where two branches with slopes of opposite sign join and local maxima and minima are located.

Similarly we can reason with a one-dimensional bimodal map, an example of which is shown in Fig. 2.2.

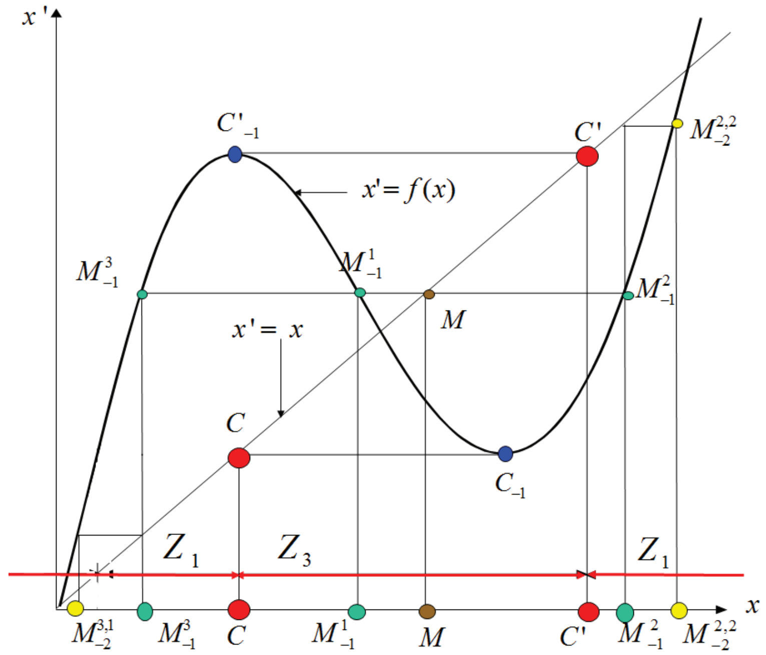

Figure 2.2: Bimodal map.

Here we have two critical points: C the points of local maximum, and C', the point of local minimum. These points separate the real line in three regions, called $Z_1 - Z_3 - Z_1$, where $Z_3 =]C, C'[$ as any point $M \in Z_3$ has three distinct rank-1 preimages in R (see Fig. 2.2).

2.3 Two-dimensional maps

In a generic two-dimensional map T, and in analogy with the one-dimensional case, the set LC_{-1} is included in the set where $\det J_T(x, y)$ changes sign, since T is locally an orientation preserving map near points (x, y) such that $\det J_T(x, y) > 0$ and orientation reversing if $\det J_T(x, y) < 0$. When the map is continuously differentiable the points of LC_{-1} necessarily belong to the set where the Jacobian determinant vanishes, and $LC = T(LC_{-1})$ belongs to boundaries which separate regions Z_k characterized by a different number of preimages. In order to give a geometrical interpretation of the action of a multi-valued inverse relation T^{-1}, it is useful to consider a region Z_k as the superposition of k sheets, each associated with a different inverse. Such a representation is equivalent to a *Riemann foliation* of the plane (see e.g. Mira et al., [4]). Different sheets are connected by folds joining two sheets, and the projections of such folds on the phase plane are arcs of LC.

We can easily extend the definition given above to the m-dimensional case. It is clear that the relation $CS = T(CS_{-1})$ holds, and the points of CS_{-1}, in which the map is continuously differentiable, are necessarily points where the Jacobian determinant vanishes. In fact, in any neighborhood of a point of CS_{-1} there are at least two distinct points which are mapped by T in the same point. Accordingly, the map is not locally invertible in points of CS_{-1}.

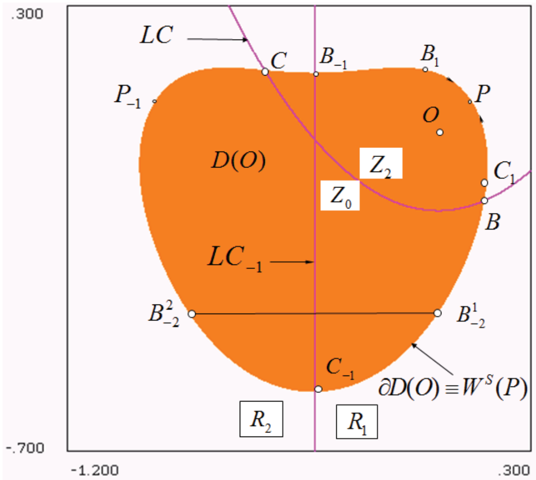

Figure 2.3:

The true extension of the properties of the Myrberg map can be analyzed in a two-dimensional non-invertible map. As a simple example let us consider the map T defined by

$$T : \begin{cases} x' = y \\ y' = bx + x^2 + y^2 \end{cases} \tag{2.6}$$

which was considered in [6]. The points which are the analogue of the critical points of a one-dimensional map are now associated with the vanishing of the Jacobian determinant. Here we have

$$J_T(x,y) = \begin{bmatrix} 0 & 1 \\ b+2x & 2y \end{bmatrix} , \quad \det J_T(x,y) = -2x - b \tag{2.7}$$

then the set defined by $\det J_T(x,y) = 0$, here $x = -\frac{b}{2}$, represents the so called critical line LC_{-1}, and its image, $LC = T(LC_{-1})$ here the hyperbola of equation $y = x^2 - 2b$, is a set which separates the phase plane into two regions: Z_0 and Z_2. Each point belonging to Z_0 has no rank-1 preimage, while each point belonging to Z_2 has two distinct rank-1 preimages, located one on the right and one on the left of LC_{-1} (the regions denoted as R_1 and R_2 in Fig. 2.3).

In Fig. 2.3 we show the phase plane at $b = 0.8$, when the origin $O = (0,0)$ is an attracting fixed point. His basin of attraction $\mathcal{D}(O)$ is also shown, which is bounded by a smooth frontier, which is made up by the stable set of an unstable fixed point of the map: a saddle P.

We recall that while in *invertible maps* the basin of attraction of some attracting set A is an invariant set, for *noninvertible maps* this property is not always satisfied. In fact, whenever there is a region Z_0 and a basin has a portion in it, then the image by the map of the basin is a set strictly included into itself. So in Fig. 2.3 we have $T(\mathcal{D}(A)) \subset \mathcal{D}(A)$, and the generic properties are as follows:

$$T(\mathcal{D}(A)) \subseteq \mathcal{D}(A) \ \ , \ \ T^{-1}(\mathcal{D}(A)) = \mathcal{D}(A) \tag{2.8}$$

However, one of the main distinguishing features of noninvertible maps is in the structure of the basins of attraction. In fact, while in *invertible maps* the basin of attraction of some attracting set A is a simply connected set, in *noninvertible maps* we may have basins of attraction which are connected but not simply (i.e. with so-called *holes*) or also non connected (i.e. disconnected, with so-called *island*). Important global bifurcations of the structure of the basins of attraction, leading a simply connected basin to a connected basin but with holes or to a non connected basin, are due to contact bifurcations between the basins and the critical curves of a map.

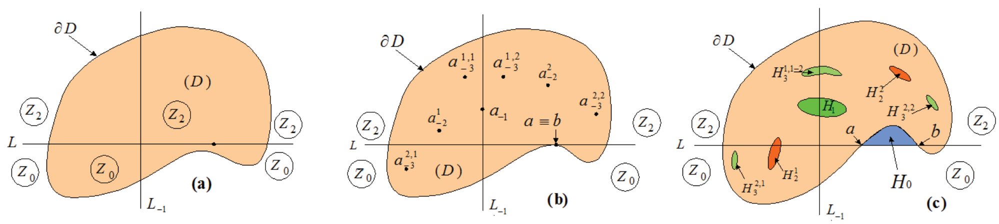

Figure 2.4:

In Fig. 2.4 we schematically show this contact bifurcation: a portion of the frontier has a contact with the critical curve LC, here denoted L, and crosses it, leaving a region H_0 whose points have a different fate (they do not belong to the considered basin). But this portion H_0 belongs to the region Z_2 and thus has two distinct rank-1 preimages which form together a hole H_1 inside the old basin. Then this area H_1 has an infinite sequence of preimages inside the old basin, and all such areas are holes of points having a different fate. So a simply connected basin is transformed into a multiply-connected basin. Note that if the points outside are converging to a different attractor, and thus belong to a different basin, for it the same bifurcation has created new portions, islands, inside a different area, and thus it has been transformed from connected to non connected.

Another import properly, due to the noninvertibility, is the boundary of absorbing areas made by portions of critical curves. As it occurs in one-dimensional maps, where absorbing intervals are bounded by the images of the critical point, also now the images of the critical curve, called critical curves of higher rank, are used to bound absorbing areas as well as chaotic areas. Consider the following map T:

$$T \; : \left\{ \begin{array}{l} x' = ax + y \\ y' = b + x^2 \end{array} \right. \tag{2.9}$$

(which was extensively considered in [4] and [5]). Here we have

$$J_T(x,y) = \left[\begin{array}{cc} a & 1 \\ 2x & 0 \end{array} \right] \quad , \quad \det J_T(x,y) = -2x \tag{2.10}$$

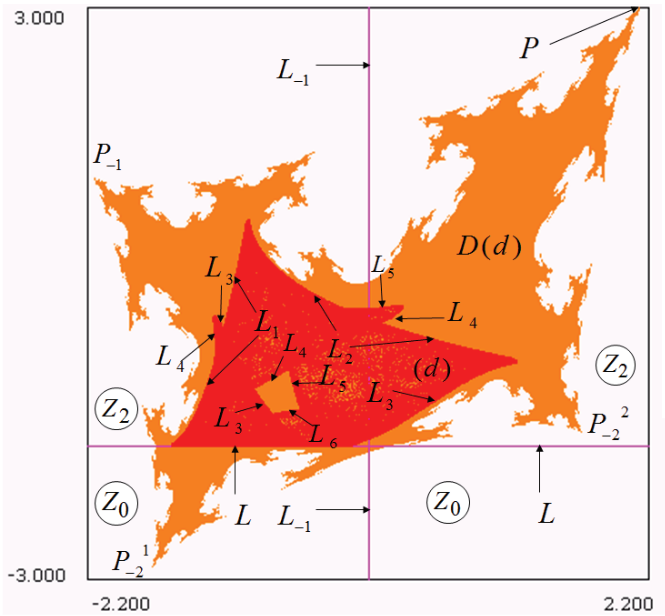

Figure 2.5:

then the set defined by $\det J_T(x,y) = 0$, here $x = 0$, represents the critical line LC_{-1}, and its image, $LC = T(LC_{-1})$, here the line of equation $y = b$, is a set which separates the phase plane into two regions: Z_0 and Z_2. An example of chaotic area is shown in Fig. 2.5, at $a = -0.42$ and $b = -1.6$. *In [4] it is proved that the boundary of the chaotic area is given by portions of critical curves belonging to the images of the segment (called generating arc g) of LC_{-1} belonging to the area itself.*

The bifurcations leading to changes in the structure of the basins (connected, multiply connected or disconnected) are called *contact bifurcations* (see in [4], [17]) because they are due to the contact of the frontier of the basin with the critical set LC. While bifurcations leading to changes in the structure of the chaotic areas (reunion of chaotic pieces, explosion to a wide area, final bifurcation, etc.) are also called *contact bifurcations* but due to the contact of two (at least) *different invariant sets*.

The chaotic area, bounded by portions of critical arcs of increasing ranks L_r, $r = 0, 1, ..., 5$, $L_0 = L$ (the critical curve), is shown in Fig. 2.5 inside its basin of attraction, whose frontier has a fractal structure. Also the process leading from a smooth basin boundary to a fractal basin boundary, is explained via sequence of contact bifurcation involving the critical curves. Let us show this process via a guiding example, by using the map (2.9), at the parameter at $a = -0.42$ and decreasing the value of the parameter b.

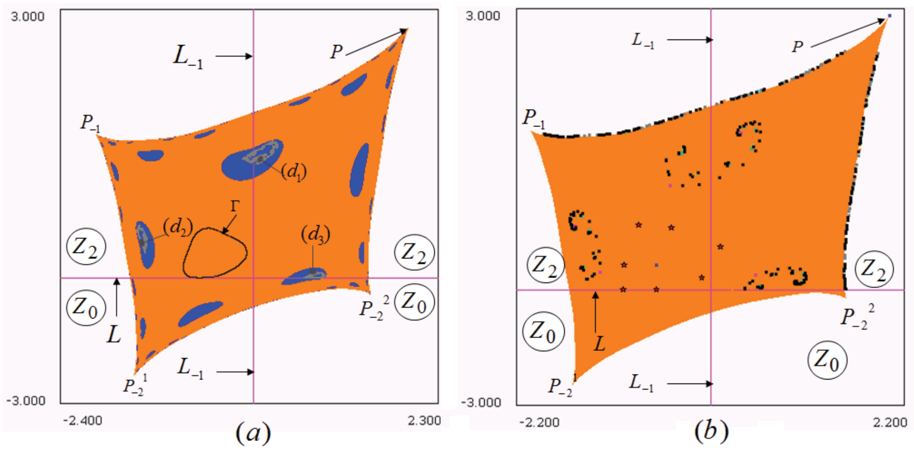

Figure 2.6:

In Fig. 2.6a, at $b = -1.09$ we see the coexistence of two different attractors: of a closed invariant curve Γ with a multiply-connected basin, and a chaotic attractor (chaotic area) made up of three cyclical pieces (d_i), $i = 1, 2, 3$, whose basin is non connected. This basin consists of the three immediate basins of the three areas (d_i), $i = 1, 2, 3$, and related preimages of any rank. The external boundary, separating the basins of the two attractors from the basin of divergent trajectories, includes a repelling node P and its preimages. In Fig. 2.6b, at $b = -1.25$ the chaotic attractor (d_i), $i = 1, 2, 3$, has been destroyed (via a contact bifurcation with the immediate basin boundary) leaving a chaotic repellor, or *strange repellor* (SR) (although not visible, in the figure, appearing as inside the basin of the unique attracting set, now an attracting cycle of period 7). That is, at $b = -1.25$ the unique attracting set is now an attracting cycle of period 7 (star points), and its basin boundary is made up of an "external" part, including the repelling fixed point P, its preimages, and infinitely repelling cycles of increasing period belonging to another strange repellor (SR') (black points on the quadrangular piecewise smooth line), and an "internal" part made up of the fractal set (SR) (some of its repelling cycles being the black points inside the brown region). The basin boundary is thus fractal.

In Fig. 2.7a, at $b = -1.35$ we see the effect of the contact bifurcation between the external frontier and the critical curve LC. The attracting set is still a 7-cycle, and its basin is multiply connected. Infinitely many preimages of the region H_0 give areas (holes) inside the basin, which are here made up of points having divergent trajectories, and now the infinitely many holes are accumulating on the strange repellors (SR) and (SR'), thus a new fractal structure has been created, that related to the basin with its boundary. The crossing of the preimage of the repelling node P denoted as P^2_{-2} trough the critical curve LC gives as result the fractal basin shown in Fig. 2.5.

We also note that in Fig. 2.5 the attractor is a chaotic area quite close to the frontier of its basin. A contact of this chaotic attractor with the frontier of its basin will transform the chaotic attractor into a chaotic repellor. An example is shown in Fig. 2.7b: almost all the points have divergent trajectories, however the transient part in the old chaotic area shows the old shape as a kind of ghost. This gives rise to a chaotic transient toward infinity.

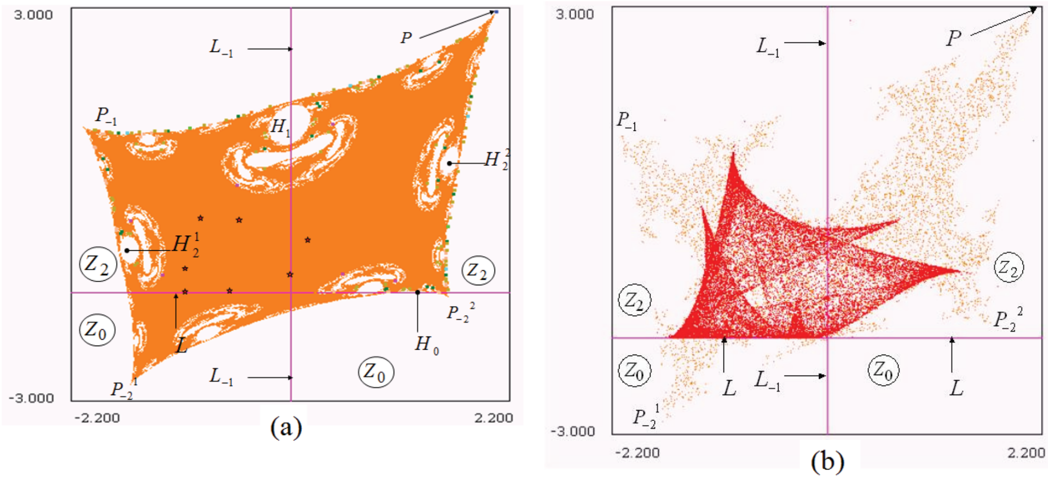

Figure 2.7:

2.4 Conclusion

To conclude our short summary, it is worth to note the importance of the global structures and of the global bifurcations, in the interpretation of the dynamics of maps in the applied context. As in fact we may have simple attractors with a very small basin of attraction, or with a fractal shape, and coexistent with several other attractors (as it occurs for example in the case of Cournot duopoly models) which limit the importance of the attractivity property, or at least must be followed by other kind of considerations.

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