11 Hotelling Type Duopoly and Oligopoly

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1 Introduction

Hotelling's seminal contribution of 1929 was one of several successful attempts to give a precise interpretation to Bertrand's sweeping criticism of Cournot's duopoly model of 1838. Chamberlin's of 1932 was another. The common point was that if the commodity were homogenous, and the competitors were quantity adjusters, as originally assumed by Cournot, then any one competitor could, by undercutting the other competitor's price, however slightly, recover the entire market as its share. This would lead to a price war ending first when marginal costs were barely covered. Both Hotelling and Chamberlin assumed price, not quantity, to be the decision variable. Chamberlin assumed the commodity to be perceived as *heterogeneous* by the consumers, so that they would prefer one brand or dealer among similar ones, and only gradually desert their favourite when price differences grew too much adverse. Hotelling, in contrast, assumed the commodity to be perceived as *perfectly homogenous* by the consumers, but incorporated space, location, and transportation costs, which provided each competitor with a local monopoly area, with competition only at the fringes.

In this way Hotelling's variant included a location problem. Hotelling assumed demand to be *completely inelastic*. This made the location choice unstable. In order to maximise market shares, both competitors would eventually crowd in the same point. So, what Hotelling showed was that his suggested solution was no solution. To judge from passing comments, Hotelling understood how the case would work out with elastic demand: The competitors would tend to gravitate closer than placing themselves in the centres of their respective markets, but would no longer cluster in the same point as with inelastic demand.

However, Hotelling did not analyse the case of elastic demand formally, and this set the standard for following re-treatments. Hotelling's theory became a theory of location, and it was almost forgotten that it primarily aimed at being a theory of duopoly pricing.

Therefore, we think it rewarding to pin down the facts of Hotelling's model with elastic demand. Lerner and Singer, in their ingenious graphic analysis of 1937, took the first step. They assumed a given reservation price. Whenever the actual price was lower, the customers would buy a fixed quantity, quite as Hotelling assumed, whenever it was higher, they would, however, buy nothing. Later, in 1941, Smithies replaced this step function by a linear decreasing demand function, and presented an insightful verbal analysis for it, though he still considered the problem "*too complex to be treated by rigorous methods*".

However, the neat formal analysis of monopoly pricing by Beckmann 1968 and 1976, can easily be extended also to duopoly. With linear demand, profits become cubic in prices and quadratic in locations even in 1D, so inner profit maxima for both location and pricing do exist for a wide range of parameter values and can even be obtained in closed form.

The case hence evades all problems pointed out by d'Aspremont *et. al*. in their note 1979 on the original Hotelling case, where profits become quadratic in price and linear in location. Linear demand is a much more interesting case to study than the artificial case of transportation costs which increase with the square of the distance, as suggested by d'Aspremont *et. al*.

2 Local Oligopoly Conditions

Suppose a firm is located along a 1D line at point x_i. To the left there is another firm at point x_{i-1}, to the right of it another at point x_{i+1}. The firms charge "mill" prices p_{i-1}, p_i, p_{i+1}, so the consumers pay for transportation and the good becomes more and more expensive farther away from the producing firm.

Unlike supply, which is concentrated to a discrete set of locations, demand is continuously distributed. At any point *x*, provided the good is transported from the firm located at x_i, the density of demand is given by the local demand function:

$$q = f(z) \qquad \text{with} \qquad z = p_i + k\left|x - x_i\right| \tag{1}$$

where k denotes the (constant) transportation cost per unit distance. Now take the linear specification

$$f(z) = \alpha - \beta z \tag{2}$$

defined for $z \geq 0$, and with $f(z) = 0$ if $z \geq \alpha / \beta$. Total demand for the *i:th* firm then becomes

$$Q_i = \int_{a_i}^{b_i} f\left(p_i + k|x - x_i|\right)dx \tag{3}$$

For the linear function (2) we get (3) as the closed form integral:

$$Q_i = (\alpha - \beta p_i)(b_i - a_i) - \frac{\beta k}{2}\left((a_i - x_i)^2 + (b_i - x_i)^2\right) \tag{4}$$

Here a_i, b_i denote the boundary points for the market interval. Given the firm is located between competitors, these points are not fixed, but determined by the conditions that prices from different suppliers be equal in boundary points:

$$p_i + k|a_i - x_i| = p_{i-1} + k|a_i - x_{i-1}| \tag{5}$$

$$p_i + k|b_i - x_i| = p_{i+1} + k|b_i - x_{i+1}| \tag{6}$$

As $x_{i-1} < a_i < x_i < b_i < x_{i+1}$, the conditions (5)-(6) yield:

$$a_i = \frac{x_{i-1} + x_i}{2} + \frac{p_i - p_{i-1}}{2k} \qquad b_i = \frac{x_i + x_{i+1}}{2} + \frac{p_{i+1} - p_i}{2k} \tag{7}$$

This implies that always $b_i = a_{i+1}$. The geometry of the "price landscape" of this case is illustrated in Fig. 1.

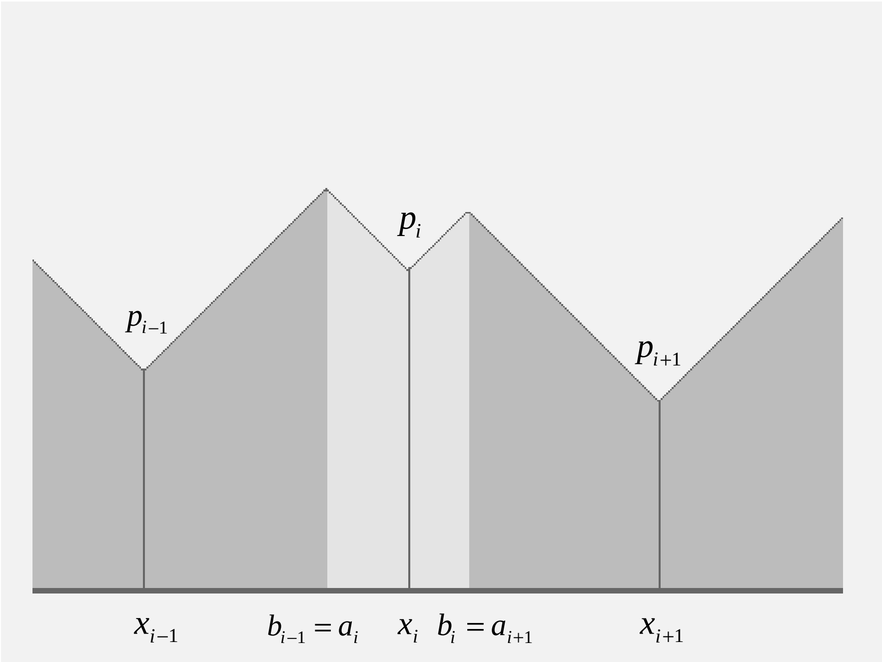

Fig. 1. Local price variation over the region around the i:th firm.

Substituting from (7) in the quantity integral (4) it now becomes:

$$Q_i = (\alpha - \beta p_i)\frac{1}{2k}\left(k(x_{i+1} - x_{i-1}) + p_{i+1} + p_{i-1} - 2p_i\right) \\ - \frac{\beta}{8k}\left((kx_{i-1} - kx_i + p_i - p_{i-1})^2 + (kx_{i+1} - kx_i + p_{i+1} - p_i)^2\right) \tag{8}$$

The i:th firm will maximize its profits: $\Pi_i = p_i Q_i - C_i(Q_i)$, where we also take a linear production cost function $C_i(Q_i) = c_i Q_i$. Thus:

$$\Pi_i = (p_i - c_i)Q_i \tag{9}$$

Each firm maximizes Π_i , by choosing its price p_i *and* its location x_i .

Suppose we first want to find the optimal location. Taking $\partial \Pi_i / \partial x_i = 0$, or even $\partial Q_i / \partial x_i = 0$, as the multiplicative factor $(p_i - c_i)$ of (9) does not contain location x_i, and solving we get:

$$x_i = \frac{x_{i-1} + x_{i+1}}{2} + \frac{p_{i+1} - p_{i-1}}{2k} \tag{10}$$

This makes sense: If the competitors' prices are equal, the firm locates midway in between, whereas a higher price of the competitor left or right drags the firm in that direction. The second order condition presently becomes $\partial^2 \Pi_i / \partial x_i^2 = -\frac{1}{2}\beta k(p_i - c_i) < 0$, so as price has to exceed marginal cost, i.e. $p_i > c_i$, the second order condition is negative, and the location choice indeed yields maximum profit. This is quite nice, because in the original Hotelling model, where $\beta = 0$, there were no inner solutions.

Given the firm has chosen an optimal location, we can substitute for x_i from (10) back in the expression (8) for Q_i and hence in (9) $\Pi_i = (p_i - c_i)Q_i$. In order to attain a concise formula, define a new compound variable:

$$\lambda_i = p_{i-1} + p_{i+1} + k(x_{i+1} - x_{i-1}) \tag{11}$$

We can also write $\lambda_i = p_{i-1} + k(x_i - x_{i-1}) + p_{i+1} + k(x_{i+1} - x_i)$, so λ_i denotes the *sum* of the prices of the commodity, including transportation, if transported to x_i from the left neighbour x_{i-1} *and* from the right neighbour x_{i+1}. The formula (11) is, however, more useful for stressing that λ_i only depends on x_{i-1} and x_{i+1}, but not on x_i.

With the new variable defined, the profits of the *i*:th firm become:

$$\Pi_i = \frac{\beta}{16k}(p_i - c_i)(\lambda_i - 2p_i)\left(8\frac{\alpha}{\beta} - \lambda_i - 6p_i\right) \tag{12}$$

Next, differentiating (12), a cubic in p_i, with respect to p_i and solving for the variable, we get two solutions:

$$p_i = \frac{4}{9}\frac{\alpha}{\beta} + \frac{1}{3}c_i + \frac{1}{9}\lambda_i$$
$$\pm \frac{1}{18}\sqrt{36\left(\frac{\alpha}{\beta} - c_i\right)^2 - 12\left(\frac{\alpha}{\beta} - c_i\right)\left(2\frac{\alpha}{\beta} - \lambda_i\right) + 13\left(2\frac{\alpha}{\beta} - \lambda_i\right)^2} \tag{13}$$

We have the second derivatives:

$$\frac{\partial^2 \Pi_i}{\partial p_i^2} = \pm \frac{1}{4k}\sqrt{36\left(\frac{\alpha}{\beta} - c_i\right)^2 - 12\left(\frac{\alpha}{\beta} - c_i\right)\left(2\frac{\alpha}{\beta} - \lambda_i\right) + 13\left(2\frac{\alpha}{\beta} - \lambda_i\right)^2} \tag{14}$$

so the second solution is the maximum. From (13)-(14) the profit minimising price is the higher, and from (12) we find that profits increase without limit after that minimum, but never mind. Such prices do not count because they exceed the maximum admissible value α / β.

3 Disjoint Monopolies

The linear demand function has certain discontinuity problems where it cuts the axes. Above the price $p_i = \alpha / \beta$, demand drops to zero and remains zero. Of course, prices cannot be negative either. They must even be higher than the positive unit production costs $p_i > c_i$. What was said above about maximum price must hold for *price plus transportation cost*. Hence, the highest total prices for the consumers, at the boundary points a_i, b_i, must not exceed α / β, i.e.:

$$\frac{\alpha}{\beta} \geq p_i + k\left|a_i - x_i\right| \qquad \frac{\alpha}{\beta} \geq p_i + k\left|b_i - x_i\right| \tag{15}$$

Both actually boil down to the same condition once we consider how a_i, b_i and x_i were determined in the above formulas (7) and (10):

$$\frac{\alpha}{\beta} \geq \frac{1}{4}\left(p_{i-1} + 2p_i + p_{i+1}\right) - \frac{k}{4}\left(x_{i+1} - x_{i-1}\right) \tag{16}$$

If we want to avoid discontinuities due to the linear demand function, we must check that condition (16) holds all the time.

Should the condition not be satisfied, then market diameter $2R_i$ becomes less than the interval available. Demand drops to zero at $\alpha - \beta p_i - \beta k R_i = 0$, which yields

$$R_i = \frac{1}{k}\left(\frac{\alpha}{\beta} - p_i\right) \tag{17}$$

Then the quantity integral (4) becomes:

$$Q_i = \int_{x_i - R_i}^{x_i + R_i} f\left(p_i + k\left|x - x_i\right|\right)dx = 2\left(\alpha - \beta p_i\right)R_i - \beta k R_i^2 \tag{18}$$

or, with substitution for the market radius from (17),

$$Q_i = \frac{\beta}{k}\left(\frac{\alpha}{\beta} - p_i\right)^2 \tag{19}$$

Profits then become:

$$\Pi_i = \frac{\beta}{k}\left(p_i - c_i\right)\left(\frac{\alpha}{\beta} - p_i\right)^2 \tag{20}$$

Differentiating this cubic with respect to p_i, equating the derivative to zero, and solving, we get two solutions:

$$p_i = \frac{\alpha}{\beta} \qquad \text{and} \qquad p_i = \frac{1}{3}\frac{\alpha}{\beta} + \frac{2}{3}c_i \tag{21}$$

The second derivatives of (20) for the two solutions (21) are respectively

$$\frac{\partial^2 \Pi_i}{\partial p_i^2} = \pm \frac{2\beta}{k}\left(\frac{\alpha}{\beta} - c_i\right) \tag{22}$$

so, again, given marginal cost c_i does not exceed maximum price α / β, the first solution is a minimum and the second a maximum. The second solution (21) in fact is the well known solution to a spatial monopoly (mill) pricing problem in 1D with a linear demand function.

We can also easily calculate maximum monopoly profits by substituting from the second expression of (21) into (20)

$$\Pi_i = \frac{4\beta}{27k}\left(\frac{\alpha}{\beta} - c_i\right)^3 \tag{23}$$

This solution is relevant when demand drops to zero at a distance before the prices with accumulated transportation costs break even for the competitors. The result then is that the market areas of neighbouring firms no longer touch, but are isolated, possibly with intervals in between which are not served by any firm - the price would simply be too high for anybody to buy the commodity. Whether this occurs seems to be a question of how many firms crowd on a given distance, what the maximum price is, and what the marginal costs are. The present case represents a number of *non-competing monopolies*. It is first when the firms are squeezed closer together that oligopoly arises.

To see that disjoint monopolies seamlessly go over into oligopoly, take (16) as an equality, and substitute into (11). Then we obtain:

$$\lambda_i = 4\frac{\alpha}{\beta} - 2p_i \tag{24}$$

which is further substituted into equation (12) and after simplification yields (20) quite as above, obtained for the case of monopoly. Further, substituting from (24) into the solution for oligopoly price (13), rearranging, taking squares to get rid of the root sign, and factoring, we get:

$$\left(\frac{\alpha}{\beta}-p_i\right)\left(\frac{1}{3}\frac{\alpha}{\beta}+\frac{2}{3}c_i-p_i\right)=0 \tag{25}$$

which gives the monopoly solutions (21) back.

We can transform formula (17), by expressing market radius in terms of marginal cost, which is a parameter, in stead of in terms of price, which is an endogenous variable. Just substitute from the second expression (21) in (17):

$$R_i=\frac{2}{3}\frac{1}{k}\left(\frac{\alpha}{\beta}-c_i\right) \tag{26}$$

The total space $\tilde{L}$ occupied by n touching monopolies is hence:

$$\tilde{L}=\sum_{i=1}^{n}2R_i=\frac{4}{3}\frac{1}{k}\sum_{i=1}^{n}\left(\frac{\alpha}{\beta}-c_i\right) \tag{27}$$

It is clear that the total length L of the space available must be less than $\tilde{L}$ in order that oligopolistic competition should develop, i.e.

$$L<\frac{4}{3}\frac{1}{k}\sum_{i=1}^{n}\left(\frac{\alpha}{\beta}-c_i\right) \tag{28}$$

Note that a market radius can be defined not only in the case of monopoly, but also in the case of oligopoly, provided we deal with a firm having competitors to both sides. The choice rule for optimal location (10) implies that the firm always places itself in the *midpoint of its market interval* $[a_i, b_i]$. To see this, substitute from (10) into (7), and form the differences:

$$x_i-a_i=\frac{x_{i+1}-x_{i-1}}{4}+\frac{p_{i-1}-2p_i+p_{i+1}}{4k} \tag{29}$$

$$b_i-x_i=\frac{x_{i+1}-x_{i-1}}{4}+\frac{p_{i-1}-2p_i+p_{i+1}}{4k} \tag{30}$$

The expressions being equal, we can define market radius:

$$R_i = \frac{x_{i+1} - x_{i-1}}{4} + \frac{p_{i-1} - 2p_i + p_{i+1}}{4k} \tag{31}$$

The result was based on the fact that there were other firms to the right and left of each firm, so what if a_i or b_i is fixed, i.e. that the firm is the leftmost or rightmost in a fixed interval? Then things become very different, as we will see. It is no longer possible to define a market radius, because the firms do not locate in the centres of their market areas. They may even crowd in the same point, as in the original Hotelling case.

4 Space Dimensions and Boundary Conditions

The above discussion started out from the original Hotelling case, firms located on a line in 1D, though we formulated the matter more generally by considering one firm in relation to its neighbours. As we know, Hotelling considered *two* firms on a *fixed interval*, and then each firm only has a neighbour to one side, the boundary point to the other being fixed from the outset. We will consider this case at some length. It is, however, interesting to consider also different types of similar models.

First, we may note that it makes a profound difference whether we have fixed boundaries or not. We already mentioned Hotelling's conjecture about a remaining tendency to gravitate towards the midpoint even when demand is elastic. This will in fact be shown to be true in the sequel. However, consider instead three oligopolists on a circle, hence without any boundary points. Then we already know from the above discussion that each firm always locates in the midpoint between the competitors, so there is no gravitation towards the middle. The Hotelling clustering phenomena hence depend on the presence of fixed endpoints.

Second, we should take note that most interesting phenomena in geographical space occur in 2D, the linear case only being a way of making things more manageable in analysis. In 2D areas have shape in addition to size, and there hence arise many more geometrical configurations than just beads on a string.

In unbounded space the regular tessellations first come into mind, triangles, squares, and hexagons with the firms located in the midpoints. Given all mill prices are equal, things become simple enough, but, as we know from the Launhardt funnel construction, any price difference sets up fourth order curves for market boundaries, which make closed form integration for demand or even for market area almost impossible to perform.

At least this is the case when we relate transportation costs to an Euclidean distance metric. This, however, is not the only choice. Given a chessboard lattice of locations, where each firm has neighbours to four sides, we can combine it with a "Manhattan" North-South, East-West transportation cost metric of the form $|x+y|+|x-y|$, where x and y are the Euclidean space coordinates. Then the loci for constant delivered price from each dealer become squares, quite in conformity with the location pattern. Likewise we can consider a triangular lattice of locations, combined with the metric $\left|x+\sqrt{3}y\right|+\left|x-\sqrt{3}y\right|+|2x|$. This case arises when there are streets in three directions intersecting at angles of 60 degrees, quite as there are streets in two directions intersecting at angles of 90 degrees in the Manhattan case. Each firm is now surrounded by six neighbours, and the curves of constant delivered price in conformity become hexagonal. These cases seem to be the simplest candidates for treatment in 2D, though we will not attempt these at present.

As for *bounded* space, it is not even obvious which case is the natural counterpart to Hotelling's duopolists on a line segment, maybe it is three competitors in an equilateral triangle, and there seems not to be even an obvious guess about whether the crowding phenomenon carries over to 2D.

We should also note that in 2D the powers for integrals of total sales and profits are raised by one. To see this, we can just check the trivial cases of independent monopoly market areas.

Given there are no close competitors, each firm would have a circular market area of radius R_i, determined by the vanishing of demand on the boundary circle, i.e. through the condition : $\alpha-\beta(p_i+kR_i)=0$. From this we find:

$$R_i=\frac{1}{k}\left(\frac{\alpha}{\beta}-p_i\right) \tag{32}$$

Given an Euclidean transport cost metric, the total sales for the firm are

$$Q_i=\int_0^{2\pi}\int_0^{R_i}\left(\alpha-\beta\left(p_i+kr\right)\right)rdrd\theta \tag{33}$$

which simplifies to

$$Q_i = 2\pi(\alpha - \beta p_i)\frac{R_i^2}{2} + 2\pi\beta k\frac{R_i^3}{3} \tag{34}$$

or, substituting from (32), to

$$Q_i = \frac{\pi\beta}{3k^2}\left(\frac{\alpha}{\beta} - p_i\right)^3 \tag{35}$$

The corresponding profits then are:

$$\Pi_i = \frac{\pi\beta}{3k^2}(p - c_i)\left(\frac{\alpha}{\beta} - p_i\right)^3 \tag{36}$$

It is now appropriate to maximise (36) with respect to p_i, which yields the profit maximising solution:

$$p_i = \frac{1}{4}\frac{\alpha}{\beta} + \frac{3}{4}c_i \tag{37}$$

This is as well known as (21) in the 1D case, and similar, only the coefficients are different. In both cases the mill price chosen is a weighted average of maximum price and marginal production cost.

Equations (35)-(36) illustrate the raising of powers referred to. The independent monopoly case is possible only if the space available allows circular market areas to be packed together without deforming the circular boundaries. If not, we have to consider other possibilities. There are intermediate cases discussed in the literature (for instance hexagons with rounded corners), and, of course, the three regular tessellations of the plane: triangles, squares, and hexagons. No matter how interesting, we, however, now leave the 2D topics.

5 The Hotelling Case: Equilibrium

5.1 Between Monopoly and Crowding. It is now time to deal, not only with a firm in its relation to the closest competitors, but the global setup. Suppose we just have two competitors. Further, suppose $a_1 = -1$ and $b_2 = 1$ are fixed. This corresponds to the original Hotelling model. From (7):

$$a_2 = b_1 = \frac{x_1 + x_2}{2} + \frac{p_2 - p_1}{2k} \tag{38}$$

From (4) then:

$$Q_1 = (\alpha - \beta p_1)(1 + b_1) - \frac{\beta k}{2}\left((1 + x_1)^2 + (b_1 - x_1)^2\right) \tag{39}$$

$$Q_2 = (\alpha - \beta p_2)(1 - a_2) - \frac{\beta k}{2}(a_2 - x_2)^2 + \left((1 - x_2)^2\right) \tag{40}$$

The optimal locations can be found by differentiating (39)-(40) with respect to x_1, x_2 respectively, as location does not enter the multiplicative factor for profits. Thus we obtain:

$$x_1 = \frac{x_2 - 4}{5} + \frac{p_2 - 3p_1}{5k} + \frac{2\alpha}{5\beta k} \tag{41}$$

$$x_2 = \frac{x_1 + 4}{5} + \frac{3p_2 - p_1}{5k} - \frac{2\alpha}{5\beta k} \tag{42}$$

Note that (41)-(42) are different from (10), as we now have one boundary point fixed for each firm. Also note that if we solve the location choice equations (41)-(42) *as a simultaneous system*, we get $x_1 + x_2 = (p_2 - p_1)/k$, and so from (38):

$$a_2 = b_1 = \frac{p_2 - p_1}{k} \tag{43}$$

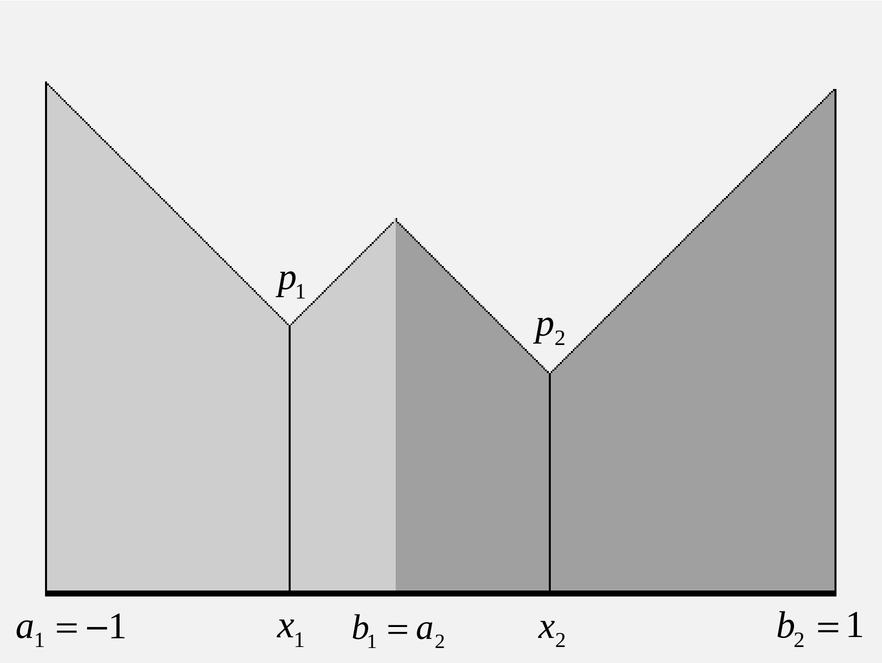

Fig. 2. Global price variation with two firms on the interval [-1, 1].

Hence *in equilibrium* the inner market boundary point only depends on the price difference divided by transportation cost. The geometry of the Hotelling case is displayed in Fig. 2.

In disequilibrium, the inner market boundary point, resulting from location choice by the first firm, taking the location of the second as given, is obtained by substituting (41) into (38):

$$b_1 = \frac{3x_2 - 2}{5} + \frac{3p_2 - 4p_1}{5k} + \frac{\alpha}{5\beta k} \tag{44}$$

Similarly, by substituting from (42) in (38), the choice of the second firm results in:

$$a_2 = \frac{3x_1 + 2}{5} + \frac{4p_2 - 3p_1}{5k} - \frac{\alpha}{5\beta k} \tag{45}$$

Expressions (41)-(42) and (44)-(45) substituted in (39)-(40), result in one single formula:

$$Q_i = \frac{\beta}{10k}\left(6\left(\frac{\alpha}{\beta} - p_i\right)^2 - 4\left(\frac{\alpha}{\beta} - p_i\right)\left(\frac{\alpha}{\beta} - \lambda_i\right) - \left(\frac{\alpha}{\beta} - \lambda_i\right)^2\right) \tag{46}$$

provided we define

$$\lambda_1 = p_2 + k(1 + x_2) \quad \text{and} \quad \lambda_2 = p_1 + k(1 - x_1) \tag{47}$$

Note that λ_i have the following interpretation: They are the prices accumulated by transportation costs if the commodity is transported from the firm to the right all the way to the left endpoint, or from the left firm all the way to the right endpoint.

From (46) we have profits:

$$\Pi_i = \frac{\beta}{10k}(p_i - c)\left(6\left(\frac{\alpha}{\beta} - p_i\right)^2 - 4\left(\frac{\alpha}{\beta} - p_i\right)\left(\frac{\alpha}{\beta} - \lambda_i\right) - \left(\frac{\alpha}{\beta} - \lambda_i\right)^2\right) \tag{48}$$

which looks more complex than (12). This again is due to the fixed endpoints.

Optimising (48) with respect to p_i we obtain:

$$p_i = \frac{4}{9}\frac{\alpha}{\beta} + \frac{3}{9}c_i + \frac{2}{9}\lambda_i \pm \frac{1}{18}\sqrt{36\left(\frac{\alpha}{\beta} - c_i\right)^2 - 24\left(\frac{\alpha}{\beta} - c_i\right)\left(\frac{\alpha}{\beta} - \lambda_i\right) + 34\left(\frac{\alpha}{\beta} - \lambda_i\right)^2} \tag{49}$$

Again the second derivative has the sign of the root term of (49), i.e.

$$\frac{\partial^2 \Pi_i}{\partial p_i^2} = \pm 2\sqrt{36\left(\frac{\alpha}{\beta} - c_i\right)^2 - 24\left(\frac{\alpha}{\beta} - c_i\right)\left(\frac{\alpha}{\beta} - \lambda_i\right) + 34\left(\frac{\alpha}{\beta} - \lambda_i\right)^2} \qquad (50)$$

so the root with the minus sign corresponds to the maximum.

Let us check out equilibrium. Then equations (41)-(42), (47), and (49) hold as a simultaneous system. It is nonlinear and not so obvious for a closed form solution, unless the firms are identical, i.e. $c_i = c$ for i = 1, 2, so for simplicity we assume this to be the case. A good guess is that they then in equilibrium also charge the same mill prices, i.e. $p_i = p$ for i = 1,2. From (41)-(42) we obtain, substituting $p_i = p$

$$x_1 = -\frac{2}{3} + \frac{1}{3k}\left(\frac{\alpha}{\beta} - p\right) \quad \text{and} \quad x_2 = \frac{2}{3} - \frac{1}{3k}\left(\frac{\alpha}{\beta} - p\right) \qquad (51)$$

As we see, $x_1 + x_2 = 0$, so *the firms locate symmetrically around the zero point*, the midpoint of the whole interval [-1, 1], though not as a rule in the midpoints, -0.5 and 0.5, of their respective market areas. From (43) we already saw that in equilibrium with equal prices the markets are separated by the zero point.

Try to substitute zero in the left hand sides of (51), and note that both firms *locate at the same point*, the Hotelling case, if

$$\frac{1}{k}\left(\frac{\alpha}{\beta} - p\right) = 2 \qquad (52)$$

Also note that the firms locate exactly in the middle of their intervals $x_1 = -0.5$, $x_2 = 0.5$, *the case of disjoint monopolies*, if

$$\frac{1}{k}\left(\frac{\alpha}{\beta} - p\right) = \frac{1}{2} \qquad (53)$$

Normally they tend to locate closer together. With infallible intuition Hotelling noted that with elastic demand "*the tendency ... to establish business ex-*

cessively close ... will be less marked", but the competitors would "*not go as far ... as public welfare would require*", i.e. locating in the midpoints of their markets, due to the "*tempting intermediate market*". As a matter of fact, they would, but only in the extreme case of adjacent monopolies. However, monopoly pricing would not correspond to the demands of public welfare either, for a different but quite obvious reason. In Hotelling's original case it was possible that once the firms located in the same point, competition would take the form of price cutting even until the marginal cost was reached (the classical competitive solution), but then the location choice would imply loss of public welfare due to excessive transportation costs.

In order to avoid both the extreme cases, monopoly and crowding in the centre, we would require $k/2 \le \alpha/\beta - p \le 2k$, i.e. maximum price must overshoot the equilibrium price by between half and double the transportation cost rate. Once we established equilibrium price we will again be able to translate this condition in terms of production cost.

Note that Hotelling's crowding phenomenon has to do with the global setup of the model: How many competitors there are, and also with the boundary conditions. In the present case we deal with a region that is a fixed interval. But if we in stead consider three firms on a circle periphery, then, as we see from the rule (10), each firm locates halfway between its neighbours. In equilibrium the firms will be equally spaced, in the centres of their respective subintervals, and crowding would never occur.

Next, substitute (51) in (47):

$$\lambda_1 = \lambda_2 = -\frac{\alpha}{3\beta} + \frac{4}{3}p + \frac{5}{3}k \tag{54}$$

So the auxiliary variables become equal for the two identical competitors, and given also that marginal costs are equal, we get by substituting from (54) and $c_i = c$ in (49):

$$p = \frac{2}{5}\frac{\alpha}{\beta} + \frac{3}{5}c + \frac{8}{5}k - \frac{3}{10}\sqrt{4\left(\frac{\alpha}{\beta} - c\right)^2 - 8\left(\frac{\alpha}{\beta} - c\right)k + 34k^2} \tag{55}$$

Note that after substitution from (54) in (49), p is substituted under the root sign, so we have to rearrange, take squares and solve for p anew in order to get (55). Also note that we took the smaller root right away because we already know it to correspond to maximum. We can now obtain *equilibrium price p* by substituting for the three *parameters* of the model: maximum price α / β according to the demand function, the unit production cost *c*, and the transportation cost rate *k* in (55). Then, substituting the equilibrium price *p*, the maximum price α / β, and the transportation rate *k* into (51) we also obtain the equilibrium locations x_1, x_2.

Before continuing note that we can use (55) for replacing the endogenous variable *p* by the parameter *c* in equations (52)-(53). The condition for crowding and for noncompetitive monopolies thus read:

$$\frac{1}{k}\left(\frac{\alpha}{\beta}-c\right)=\frac{11}{4} \qquad \text{and} \qquad \frac{1}{k}\left(\frac{\alpha}{\beta}-c\right)=\frac{3}{4} \tag{56}$$

The parameter interval between monopoly and crowding is hence $0.75 < (\alpha / \beta - c) / k < 2.75$.

5.2 Price Cutting. We have disregarded one complication. As we saw, the Hotelling identical duopolists on a fixed interval may locate in the same point, if $(\alpha / \beta - c) / k = 2.75$, even when demand is elastic. They may then start a price cutting war until the level of marginal costs is reached, or they may form a collusive monopoly, if law permits.

It is notable that this competitive situation may occur even if the firms do *not* locate in the same point *but just sufficiently close*. Each firm can then, by undercutting the competitor's price *by the transportation cost over the distance between the locations* take the whole market. This, of course, is feasible only if the undercut price still exceeds marginal cost for production. But a more interesting criterion is when the maximum duopoly profits, from each firm taking half the total interval, break even with undercutting the competitor's price and taking the whole market. As we will see it is possible to establish an exact condition for this to happen.

First, calculate the profits, when each firm takes half the market:

$$\begin{aligned}\Pi_1 &= (p-c)\int_{-1}^{0}\left(\alpha-\beta\left(p+k\left|x-x_1\right|\right)\right)dx \\ &= (p-c)\left(\alpha-\beta p-\frac{\beta k}{2}\left(x_1^2+\left(1+x_1\right)^2\right)\right)\end{aligned} \tag{57}$$

$$\Pi_2 = (p-c)\int_0^1 \left(\alpha - \beta\left(p + k\left|x - x_2\right|\right)\right)dx \tag{58}$$
$$= (p-c)\left(\alpha - \beta p - \frac{\beta k}{2}\left(x_2^2 + (1-x_2)^2\right)\right)$$

Note that these profits are equal because from (51) the firms locate symmetrically around the origin, i.e. $x_2 = -x_1$. Further, prices and marginal costs are equal for the firms, as indicated by dropping the indices. Substituting for price and location from (55) and (51) in (57)-(58), and using:

$$\kappa = \frac{1}{k}\left(\frac{\alpha}{\beta} - c\right) \tag{59}$$

we obtain *maximum* profits for the duopolists (equal for both firms):

$$\Pi_1 = \Pi_2 = \tag{60}$$
$$\frac{\beta k^2}{250}\left(\left(2\kappa^2 - 44\kappa + 377\right)\sqrt{4\kappa^2 - 8\kappa + 34} + 4\kappa^3 - 92\kappa^2 + 482\kappa - 2194\right)$$

Note that maximum duopoly profit, apart from the multiplicative factor βk^2, only depends on the compound parameter κ, as defined in (59). Note from (23) that, using (59), we can also put maximum *monopoly* profit in the same form: $\Pi_i = (4/27)\beta k^2 \kappa^3$. So if we want to find at which point monopoly profit breaks even with duopoly profit we would equate this expression to (60) and could find the break even point in terms of the compound parameter κ alone. Further from (56) we find that the criteria not only for monopoly, but also for clustering are in terms of this parameter alone. It so turns out that κ, the difference between maximum price and production cost divided by the transportation cost rate, is crucial for most of the topics we discuss.

Now, consider that if any of the firms undercuts the equilibrium price *p*, as given by (55), with the amount $k(x_2 - x_1) = -2kx_1 = 2kx_2$, which is the transportation cost between the duopoly locations of the firms, it will take the whole market as its share. However, this takes for granted that the undercutting firm does *not* relocate, as assumed by d'Aspremont *et. al*. in their discussion for the case of inelastic demand. Putting things in this way we get a *too restrictive estimate* of when price undercutting becomes profitable.

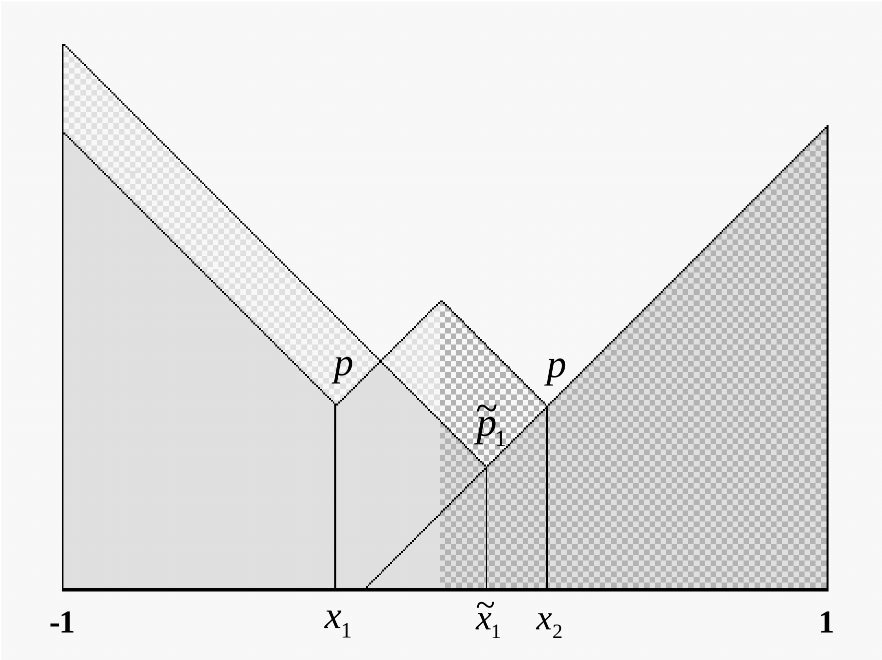

Fig. 3. Price undercutting and relocation by the first firm.

As a rule the firm trying price undercutting would *not remain where it is located*. It would relocate and choose the perfect location in combination with price, which always yields a higher profit than if it were constrained to remain where it is.

There are constraints for the choice of location and price for the firms:

$$\tilde{p}_1 = p - k(x_2 - \tilde{x}_1) \quad \text{and} \quad \tilde{p}_2 = p - k(\tilde{x}_2 - x_1) \tag{61}$$

Here p denotes the original duopoly price. It has no index as it is the equilibrium price for both duopolists to which the other firm is assumed to continue to adhere. So p is given according to (55), as before, and so are x_1, x_2, by (51). But the firms can choose $\tilde{x}_1$, $\tilde{p}_1$ and $\tilde{x}_2$, $\tilde{p}_2$ as they wish provided (61) is fulfilled. Note that the constraints result in undercutting without change of location as indicated above if we take $\tilde{x}_1 = -x_2$ or $\tilde{x}_2 = -x_1$. The undercutting situation is illustrated in Fig. 3. Note also that given the way we have drawn the picture, unchanged location x_1 is no choice at all, because the first firm could then not even charge a positive undercutting price, so there would be no profit even if production costs were zero.

Now the undercutting profits for the first firm are:

$$\Pi_1 = (\tilde{p}_1 - c)\int_{-1}^{1}\left(\alpha - \beta\left(\tilde{p}_1 + k\left|x - \tilde{x}_1\right|\right)\right)dx \tag{62}$$

$$= (\tilde{p}_1 - c)\left(2(\alpha - \beta\tilde{p}_1) - \frac{\beta k}{2}\left((1+\tilde{x}_1)^2 + (1-\tilde{x}_1)^2\right)\right)$$

and for the second firm:

$$\Pi_2 = (\tilde{p}_2 - c)\int_{-1}^{1}\left(\alpha - \beta\left(\tilde{p}_2 + k\left|x - \tilde{x}_2\right|\right)\right)dx \tag{63}$$

$$= (\tilde{p}_2 - c)\left(2(\alpha - \beta\tilde{p}_2) - \frac{\beta k}{2}\left((1-\tilde{x}_2)^2 + (1+\tilde{x}_2)^2\right)\right)$$

We now have just one free optimization variable for each firm, either price or location, the other is dependent so as to fit on the relevant linear constraint of (61). Suppose we choose location for optimising. Then we substitute for $\tilde{p}_1$ from (61) in (62) and optimise with respect to $\tilde{x}_1$. Likewise we substitute for $\tilde{p}_2$ from (61) in (63) and optimise with respect to $\tilde{x}_2$. The solutions for location become more messy than before, as we now solve quadratic equations:

$$\tilde{x}_1 = \frac{1}{3}(x_2 + 1) - \frac{1}{3}\frac{p-c}{k} - 1 + \frac{1}{3k}\sqrt{(c - p + k(1+x_2))^2 + 6k\left(\frac{\alpha}{\beta} - p + kx_2\right)} \tag{64}$$

$$\tilde{x}_2 = \frac{1}{3}(x_1 - 1) + \frac{1}{3}\frac{p-c}{k} + 1 - \frac{1}{3k}\sqrt{(c - p + k(1-x_1))^2 + 6k\left(\frac{\alpha}{\beta} - p - kx_1\right)} \tag{65}$$

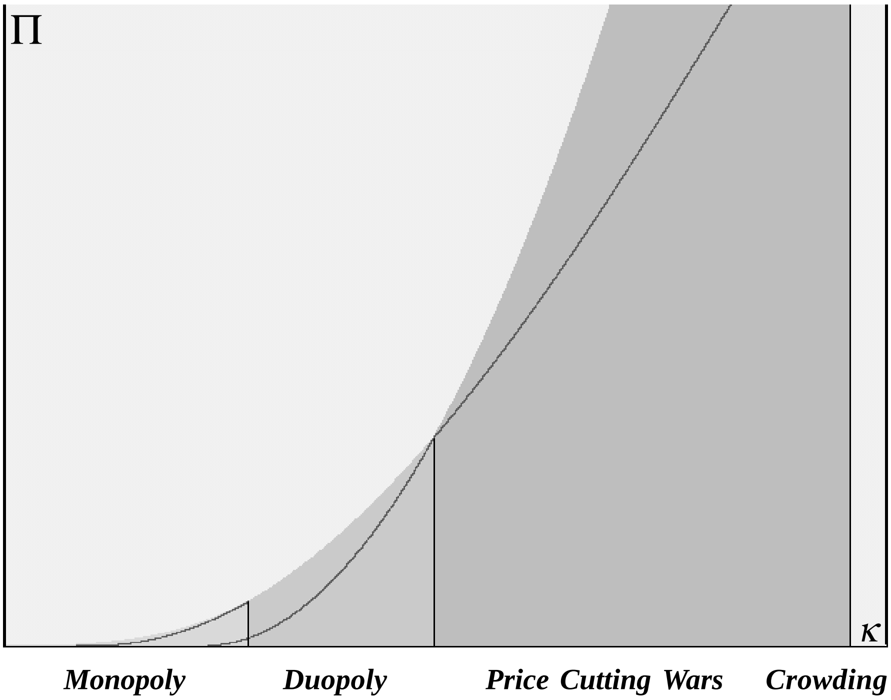

Fig. 4. Profit curves and competition regimes.

Given $x_2 = -x_1$ we hence also get $\tilde{x}_1 = -\tilde{x}_2$ which is reassuring. However, note that undercutting is a strategy under the condition that the competitor adheres to previous duopolistic behaviour. The case $x_2 = -x_1$ is a conceivable equilibrium, but $\tilde{x}_1 = -\tilde{x}_2$ is not. If both try to undercut at the same time both get disappointed. Undercutting is hence like the von Stackelberg 1938 case, rather than like the Cournot 1838 case.

But, supposing that only one firm undercuts, we can now first substitute the location according to (64)-(65) in (61) to obtain the corresponding undercutting prices, and then substitute for both in the profit expressions (62)-(63). Further, in either case, we again substitute for duopoly equilibrium price p (assumed to be retained by the other firm) from (55), and for the location of the other firm x_2 or x_1 from (51). Finally, again using the compound parameter from (59), profits become:

$$\Pi_1 = \Pi_2 = \frac{2\beta k^2}{3375}\left(13\kappa^2 + 34\kappa - 17 - 6(\kappa - 6)\sqrt{4\kappa^2 - 8\kappa + 34}\right)^{\frac{3}{2}} \tag{66}$$

$$+\frac{2\beta k^2}{3375}\left((31\kappa^2 + 28\kappa - 29)\sqrt{4\kappa^2 - 8\kappa + 34} - 63\kappa^3 + 369\kappa^2 - 819\kappa - 2187\right)$$

Note that (66) states that undercutting profits are equal for both firms, provided one firm undercuts but the other does not. As the undercutting firm takes the whole market, the profits of the other firm become zero. If both try undercutting at the same time, we have a new situation which is worth exploring in order to make explicit the dynamics of economic warfare.

It is yet interesting to establish criteria for when such warfare does *not* occur, i.e. the firms find it most profitable to adhere to duopoly behaviour. Equating (66), price cutting profit, to (60), duopoly profit, cancelling the equal factors βk^2, and rearranging to get rid of the roots, we finally obtain the nasty looking tenth order polynomial equation:

$$\begin{aligned}&3584\kappa^{10} - 216832\kappa^9 + 4696592\kappa^8 - 50448000\kappa^7 \\ &+294183456\kappa^6 - 1142701440\kappa^5 + 3361000744\kappa^4 \\ &-6631985696\kappa^3 + 9946507384\kappa^2 - 8248430736\kappa + 2165411397 = 0\end{aligned} \tag{67}$$

It has three pairs of complex roots, and four real ones. As always, repeated squaring produces lots of nonsense roots. In fact only one real root makes sense. It is approximately equal to $\kappa \approx 1.2146$. So it seems that *we should have* $0.75 < \kappa < 1.2146$ *to ensure that true oligopolistic competition be established, without monopoly, and without price cutting competition.* The various regimes: monopoly, duopoly, and price cutting competition, are illustrated in Fig. 4 by their respective profit functions, and we see that their intersections define sharp points in which one regime becomes more profitable than another.

6 The Hotelling Case: Dynamics

6.1 Immediate Location Adjustment. We can now set up the Hotelling model as a dynamical system from (41)-(42) and (49):

$$x_1' = \frac{x_2 - 4}{5} + \frac{p_2 - 3p_1}{5k} + \frac{2\alpha}{5\beta k} \tag{68}$$

$$x_2' = \frac{x_1 + 4}{5} + \frac{3p_2 - p_1}{5k} - \frac{2\alpha}{5\beta k} \tag{69}$$

$$p_1' = \frac{4}{9}\frac{\alpha}{\beta} + \frac{3}{9}c_1 + \frac{2}{9}\lambda_1 - \frac{1}{18}\sqrt{\Delta_1} \tag{70}$$

$$p_2' = \frac{4}{9}\frac{\alpha}{\beta} + \frac{3}{9}c_2 + \frac{2}{9}\lambda_2 - \frac{1}{18}\sqrt{\Delta_2} \tag{71}$$

where

$$\lambda_1 = p_2 + k(1 + x_2) \tag{72}$$

$$\lambda_2 = p_1 + k(1 - x_1) \tag{73}$$

as stated in equation (47), and

$$\Delta_i = 36\left(\frac{\alpha}{\beta} - c_i\right)^2 - 24\left(\frac{\alpha}{\beta} - c_i\right)\left(\frac{\alpha}{\beta} - \lambda_i\right) + 34\left(\frac{\alpha}{\beta} - \lambda_i\right)^2 \tag{74}$$

for $i = 1, 2$. The dash indicates advancing the map one period, from t to $t + 1$. We are interested in the dynamic behaviour of the system (68)-(71). As we shall see the system is a contraction, so any initial condition gives rise to a trajectory converging to the attracting fixed point, which exists and is necessarily unique.

To see this, consider the Jacobian matrix of the system (68)-(71):

$$J = \frac{\partial\left(x_1', x_2', p_1', p_2'\right)}{\partial\left(x_1, x_2, p_1, p_2\right)} = \begin{bmatrix} 0 & \frac{1}{5} & -\frac{3}{5k} & \frac{1}{5k} \\ \frac{1}{5} & 0 & -\frac{1}{5k} & \frac{3}{5k} \\ 0 & k\Phi_1 & 0 & \Phi_1 \\ -k\Phi_2 & 0 & \Phi_2 & 0 \end{bmatrix} \tag{75}$$

where

$$\Phi_i = \frac{\partial p_i'}{\partial x_j} = \frac{2}{9} - \frac{1}{9\sqrt{\Delta_i}}\left(6\left(\frac{\alpha}{\beta} - c_i\right) - 17\left(\frac{\alpha}{\beta} - \lambda_i\right)\right) \tag{76}$$

for $i, j = 1, 2$ and $i \neq j$. As a preliminary lemma, let us show that the values of these partial derivatives are always bounded (whatever the parameter values and the points in phase space).

Lemma. *The following inequalities hold*:

$$-\frac{1}{4} < \Phi_i < \frac{3}{4} \tag{77}$$

for i = 1, 2.

Proof: We have $\Phi_i > -1/4$ iff

$$\frac{24}{17}\left(\frac{\alpha}{\beta} - c_i\right) - 4\left(\frac{\alpha}{\beta} - \lambda_i\right) < \sqrt{\Delta_i} \tag{78}$$

The inequality is certainly true when the left hand side is negative, otherwise we solve:

$$\left(\frac{24}{17}\left(\frac{\alpha}{\beta} - c_i\right) - 4\left(\frac{\alpha}{\beta} - \lambda_i\right)\right)^2 < \Delta_i \tag{79}$$

which gives:

$$18\left(\frac{\alpha}{\beta}-\lambda_i\right)^2-12.706\left(\frac{\alpha}{\beta}-\lambda_i\right)\left(\frac{\alpha}{\beta}-c_i\right)+34.5052\left(\frac{\alpha}{\beta}-c_2\right)^2>0 \qquad (80)$$

As the discriminant of the left hand side is negative, this inequality is always satisfied.

On the other hand, $\Phi_i < 3/4$ iff

$$-\frac{24}{19}\left(\frac{\alpha}{\beta}-c_i\right)+4\frac{17}{19}\left(\frac{\alpha}{\beta}-\lambda_i\right)<\sqrt{\Delta_i} \qquad (81)$$

Again, this inequality is certainly true if the left hand side is negative. If not we solve for

$$\left(-\frac{24}{19}\left(\frac{\alpha}{\beta}-c_i\right)+4\frac{17}{19}\left(\frac{\alpha}{\beta}-\lambda_i\right)\right)^2<\Delta_i \qquad (82)$$

which gives

$$24.19\left(\frac{\alpha}{\beta}-\lambda_i\right)^2-14.958\left(\frac{\alpha}{\beta}-\lambda_i\right)\left(\frac{\alpha}{\beta}-c_i\right)+34.4\left(\frac{\alpha}{\beta}-c_2\right)^2>0 \qquad (83)$$

Again, the discriminant of the left hand side is negative, so the inequality is always satisfied. ■

We hence prove the following:

Theorem. *The map defined in* (68)-(71) *is a contraction.*

Proof: In order to show that the map is a contraction it is enough to prove that all the eigenvalues, say *z*, of the matrix *J* as stated in (75) are less than 1 in absolute value at all the points in the phase space. To this end we solve the characteristic polynomial equation:

$$\mathrm{Det}(J)=z^4+C_1z^3+C_2z^2+C_3z+C_4=0 \qquad (84)$$

where from the Jacobian (75)

$$C_1 = 0 \tag{85}$$

$$C_2 = -\left(\frac{1}{5} - \Phi_1\right)\left(\frac{1}{5} - \Phi_2\right) \tag{86}$$

$$C_3 = \frac{3}{25}\left(\Phi_1 + \Phi_2 - 10\Phi_1\Phi_2\right) \tag{87}$$

$$C_4 = -\frac{9}{25}\Phi_1\Phi_2 \tag{88}$$

It is known from Fairbrother (1973) that the necessary and sufficient conditions for (84) to have all roots less than 1 in absolute value are:

$$1 - C_4 > 0 \tag{89}$$

$$3\left(1 + C_4\right) - C_2 > 0 \tag{90}$$

$$1 + C_2 + C_4 - \left|C_1 + C_3\right| > 0 \tag{91}$$

$$\left(1 - C_4\right)^2\left(1 + C_4 - C_2\right) + \left(C_1 - C_3\right)\left(C_3 - C_1C_4\right) > 0 \tag{92}$$

which upon substitution from (85)-(88) become:

$$1 + \frac{9}{25}\Phi_1\Phi_2 > 0 \tag{93}$$

$$3\left(1 - \frac{9}{25}\Phi_1\Phi_2\right) + \left(\frac{1}{5} - \Phi_1\right)\left(\frac{1}{5} - \Phi_2\right) > 0 \tag{94}$$

$$1 - \left(\frac{1}{5} - \Phi_1\right)\left(\frac{1}{5} - \Phi_2\right) - \frac{9}{25}\Phi_1\Phi_2 - \left|\frac{3}{25}\left(\Phi_1 + \Phi_2 - 10\Phi_1\Phi_2\right)\right| > 0 \tag{95}$$

$$\left(1+\frac{9}{25}\Phi_1\Phi_2\right)^2\left(1-\frac{9}{25}\Phi_1\Phi_2+\left(\frac{1}{5}-\Phi_1\right)\left(\frac{1}{5}-\Phi_2\right)\right) \\ -\left(\frac{3}{25}\left(\Phi_1+\Phi_2-10\Phi_1\Phi_2\right)\right)^2>0 \tag{96}$$

Given the result of the above Lemma, stating that $-\frac{1}{4}<\Phi_i<\frac{3}{4}$ for $i = 1, 2$, it is easy to see that conditions (93)-(96) are always satisfied.

(*i*) The inferior limit value for the expression on the left in (93) is $\left(1+\frac{9}{25}\left(-\frac{1}{4}\right)\frac{3}{4}\right)$ which is positive.

(*ii*) Next, rearranging the condition stated on the left in (94) we get $\frac{1}{25}\left(76-2\Phi_1\Phi_2-5\left(\Phi_1+\Phi_2\right)\right)>\frac{1}{25}\left(76-2\left(\frac{3}{4}\right)^2-5\left(\frac{3}{2}\right)\right)>0$.

(*iii*) As for (95) we consider two cases:

(*a*) Suppose $\left(\Phi_1+\Phi_2-10\Phi_1\Phi_2\right)>0$. Rearranging expression (95), we get $\frac{2}{25}\left(12-2\Phi_1\Phi_2+\Phi_1+\Phi_2\right)=\frac{2}{25}\left(12+8\Phi_1\Phi_2+\Phi_1+\Phi_2-10\Phi_1\Phi_2\right)$ $>\frac{2}{25}\left(12+8\Phi_1\Phi_2\right)>\frac{2}{25}\left(12+8\left(-\frac{1}{4}\right)\frac{3}{4}\right)>0$.

(*b*) Next, suppose $\left(\Phi_1+\Phi_2-10\Phi_1\Phi_2\right)<0$. Then, rearranging (95), its left hand side equals $\frac{8}{25}\left(3-8\Phi_1\Phi_2+\Phi_1+\Phi_2\right)$, the inferior limit value of which is zero (obtained for $\Phi_1=\Phi_2=\frac{3}{4}$).

(*iv*) As for (96) we use the maximum value of $\left|\Phi_1+\Phi_2-10\Phi_1\Phi_2\right|$, which is 33/8 (obtained for $\Phi_1=\Phi_2=\frac{3}{4}$). Rearranging the left hand side of (96): $\frac{1}{25}\left(1+\frac{9}{25}\Phi_1\Phi_2\right)^2\left(26+16\Phi_1\Phi_2-5\left(\Phi_1+\Phi_2\right)\right)-\left(\frac{3}{25}\right)^2\left(\Phi_1+\Phi_2-10\Phi_1\Phi_2\right)^2>$ $\frac{1}{25}\left(1+\frac{9}{25}\left(-\frac{1}{4}\right)\frac{3}{4}\right)^2\left(26+16\left(-\frac{1}{4}\right)\frac{3}{4}-5\left(\frac{3}{2}\right)\right)-\left(\frac{3}{25}\right)^2\left(\frac{33}{8}\right)^2>0$. ■

6.2 Delayed or Adaptive Location Adjustment. Let us now generalize the model (68)-(71) as follows:

$$x_1' = (1-\sigma_1)x_1 + \sigma_1\left(\frac{x_2 - 4}{5} + \frac{p_2 - 3p_1}{5k} + \frac{2\alpha}{5\beta k}\right) \tag{97}$$

$$x_2' = (1-\sigma_2)x_2 + \sigma_2\left(\frac{x_1 + 4}{5} + \frac{3p_2 - p_1}{5k} - \frac{2\alpha}{5\beta k}\right) \tag{98}$$

$$p_1' = \frac{4}{9}\frac{\alpha}{\beta} + \frac{3}{9}c_1 + \frac{2}{9}\lambda_1 - \frac{1}{18}\sqrt{\Delta_1} \tag{99}$$

$$p_2' = \frac{4}{9}\frac{\alpha}{\beta} + \frac{3}{9}c_2 + \frac{2}{9}\lambda_2 - \frac{1}{18}\sqrt{\Delta_2} \tag{100}$$

where $\sigma_i \in [0,1]$ while the λ_i and Δ_i are as given in (72)-(73) and (74) respectively.

The difference to the previous model is that we now allow for a slower location adjustment than price adjustment by assuming an adaptive process. The firms need not immediately jump to the new optimum locations but may experience a certain inertia due to the considerable cost associated with relocation. In reality, the firms would relocate only now and then, whenever the potential gains from relocation offset the substantial relocation costs, but then take full relocation steps.

In a model it makes no harm to represent this conservatism to relocation in terms of smaller steps in stead. Over a longer period the outcomes are equivalent. The coefficients σ_i can represent any speed of reaction, from total inertia when σ_i = 0, to extreme agility when σ_i = 1.

We note that the system (97)-(100) in fact becomes identical to (68)-(71) for σ_i = 1, and that the equilibrium of the system (97)-(100) is the same as for (68)-(71), which we already know to exist and to be unique. The study of its stability, however, is now more complex. In order to investigate the properties of (97)-(100), we make use of the Jacobian matrix which now is:

$$J=\frac{\partial(x_1',x_2',p_1',p_2')}{\partial(x_1,x_2,p_1,p_2)}=\begin{bmatrix}(1-\sigma_1) & \frac{\sigma_1}{5} & -\frac{3\sigma_1}{5k} & \frac{\sigma_1}{5k}\\ \frac{\sigma_2}{5} & (1-\sigma_2) & -\frac{\sigma_2}{5k} & \frac{3\sigma_2}{5k}\\ 0 & k\Phi_1 & 0 & \Phi_1\\ -k\Phi_2 & 0 & \Phi_2 & 0\end{bmatrix} \tag{101}$$

where the Φ_i are the derivatives defined in (76), with upper and lower bounds still as stated in the above Lemma. The characteristic polynomial of the Jacobian matrix (101) is

$$\text{Det}(J)=P(z)=z^4+C_1z^3+C_2z^2+C_3z+C_4 \tag{102}$$

where now

$$C_1=-(1-\sigma_1)-(1-\sigma_2) \tag{103}$$

$$C_2=(1-\sigma_1)(1-\sigma_2)-\left(\frac{\sigma_1}{5}-\Phi_1\right)\left(\frac{\sigma_2}{5}-\Phi_2\right) \tag{104}$$

$$\begin{aligned}C_3&=\left(2-8\frac{\sigma_1}{5}-8\frac{\sigma_2}{5}\right)\Phi_1\Phi_2\\&+\left(3\frac{\sigma_1}{5}\frac{\sigma_2}{5}-\frac{\sigma_2}{5}(1-\sigma_1)\right)\Phi_1+\left(3\frac{\sigma_1}{5}\frac{\sigma_2}{5}-\frac{\sigma_1}{5}(1-\sigma_2)\right)\Phi_2\end{aligned} \tag{105}$$

$$C_4=-\left(9\frac{\sigma_1}{5}\frac{\sigma_2}{5}-3\frac{\sigma_1}{5}(1-\sigma_2)-3\frac{\sigma_2}{5}(1-\sigma_1)+(1-\sigma_1)(1-\sigma_2)\right)\Phi_1\Phi_2 \tag{106}$$

It is clear that in order to have a contraction map the conditions can be obtained by introducing the above coefficients in (89)-(92), but the analysis is now more complex, and even if numerical experiments suggest that the system is still always stable, we are not able to prove this rigorously, except for the limiting cases $\sigma_i = 1$, dealt with above, and $\sigma_i = 0$, dealt with below.

However, we can prove that the equilibrium is always stable in the symmetric case. Considering identical firms ($c_1 = c_2 = c$) we already stated the explicit expressions for Nash equilibrium, whence $p_1 = p_2 = p$, as given in (55), and $\Phi_1 = \Phi_2 = \Phi$. Further assume $\sigma_1 = \sigma_2 = \sigma$ for the sake of complete symmetry. Evaluated in the equilibrium point, (103)-(106) become:

$$C_1 = -2(1-\sigma) \tag{107}$$

$$C_2 = (1-\sigma)^2 - \left(\frac{\sigma}{5} - \Phi\right)^2 \tag{108}$$

$$C_3 = \left(4\frac{\sigma}{5}\Phi - 2\Phi^2\right)\left(8\frac{\sigma}{5} - 1\right) \tag{109}$$

$$C_4 = -\left(8\frac{\sigma}{5} - 1\right)^2 \Phi^2 \tag{110}$$

With these coefficients the characteristic polynomial factorizes:

$$P(z) = P_1(z)P_2(z) \tag{111}$$

where

$$P_1(z) = z^2 + \left(4\frac{\sigma}{5} + \Phi - 1\right)z + \left(8\frac{\sigma}{5} - 1\right)\Phi \tag{112}$$

$$P_2(z) = z^2 + \left(6\frac{\sigma}{5} - \Phi - 1\right)z - \left(8\frac{\sigma}{5} - 1\right)\Phi \tag{113}$$

so that it is possible to get the expressions for the four eigenvalues explicitly. To prove stability we can employ the usual necessary and sufficient conditions to have the two eigenvalues of a quadratic polynomial in modulus less than 1:

$$P_1(1) > 0, \quad P_1(-1) > 0, \quad P_1(0) < 1 \tag{114}$$

$$P_2(1) > 0, \quad P_2(-1) > 0, \quad P_2(0) < 1 \tag{115}$$

From (107)-(110) and (112)-(113) we have:

$$P_1(1) = 8\frac{\sigma}{5}\Phi + 4\frac{\sigma}{5} > 0 \quad \text{iff} \quad \Phi > -\frac{1}{2} \tag{116}$$

$$P_1(-1) = \frac{10-4\sigma}{5} - \frac{10-8\sigma}{5}\Phi > 0 \quad \text{iff} \quad \Phi < \frac{5-2\sigma}{5-4\sigma} \tag{117}$$

$$P_1(0) = \left(8\frac{\sigma}{5} - 1\right)\Phi < 1 \quad \text{iff} \quad \Phi < (>) \frac{5}{8\sigma - 5} \quad \text{for} \quad \sigma > (<) \frac{5}{8} \tag{118}$$

$$P_2(1) = -8\frac{\sigma}{5}\Phi + 6\frac{\sigma}{5} > 0 \quad \text{iff} \quad \Phi < \frac{3}{4} \tag{119}$$

$$P_2(-1) = \frac{10-6\sigma}{5} + \frac{10-8\sigma}{5}\Phi > 0 \quad \text{iff} \quad \Phi > -\frac{5-3\sigma}{5-4\sigma} \tag{120}$$

$$P_2(0) = -\left(8\frac{\sigma}{5} - 1\right)\Phi < 1 \quad \text{iff} \quad \Phi > (<) -\frac{5}{8\sigma - 5} \quad \text{for} \quad \sigma > (<) \frac{5}{8} \tag{121}$$

As $-\frac{1}{4} < \Phi < \frac{3}{4}$ it is easy to see that the six conditions (116)-(121) are always satisfied for any $\sigma \in (0,1]$. Thus the Nash equilibrium is locally stable, whatever the values of the parameters. Hence a bifurcation can never occur.

The case $\sigma = 0$ is degenerate. Given any choice of locations x_1, x_2 these are never changed, and there is a pure price dynamics given by the two equations:

$$p_1' = \frac{4}{9}\frac{\alpha}{\beta} + \frac{3}{9}c_1 + \frac{2}{9}\lambda_1 - \frac{1}{18}\sqrt{\Delta_1} \tag{122}$$

$$p_2' = \frac{4}{9}\frac{\alpha}{\beta} + \frac{3}{9}c_2 + \frac{2}{9}\lambda_2 - \frac{1}{18}\sqrt{\Delta_2} \tag{123}$$

This two-dimensional system is a contraction. The Jacobian matrix is:

$$J = \begin{bmatrix} 0 & \Phi_1 \\ \Phi_2 & 0 \end{bmatrix} \tag{124}$$

with $-\frac{1}{4} < \Phi_i < \frac{3}{4}$ for i = 1, 2. The two eigenvalues (real or complex conjugates), always less than 1 in absolute value, are $z = \pm\sqrt{\Phi_1 \Phi_2}$.

Throughout the discussion we kept the coefficients α and β from the demand function, though nowhere in the reaction functions did these parameters enter except in terms of their ratio α / β, so we could just have used one symbol for their ratio. Further, inspecting the reaction formulas it is obvious that if we redefine all value variables (prices, production costs, transportation costs) as ratios to this maximum price α / β, then nothing at all is changed. Accordingly, we could even put $\alpha / \beta = 1$ without any loss of generality. We will profit from this possibility in Section 8 below.

7 Triopoly with an Intermediate Firm

Despite the seeming stability inherent in the model of two firms on a fixed interval, the introduction of a third intermediate firm introduces some manifest instability. The intermediate firm will, as we have seen, locate between the competitors, in the centre of its market area. On the other hand the latter will still tend to gravitate towards the centre. At a certain point it may hence occur that the intermediate firm becomes so squeezed by its neighbours that it will find larger market areas by moving out to the left or right on one or the other side of the competitor there. As a rule, in equilibrium, the intermediate firm will also have to charge a lower price than the competitors, even when all three are identical in terms of equal production costs. This enhances the tendency for the intermediate firm to move out, and so introduces a locational instability in the process.

Let us now reconsider the Hotelling case, but suppose that there is an intermediate firm, i.e. three firms on a fixed interval. As we will see the case has some unexpected surprises. Again assume the interval is [-1,1]. Now there are three firms located at $x_1, x_2, x_3 \in [-1, 1]$. The demand quantities can be recovered immediately from (39)-(40) for the extremal firms, and from (4) for the intermediate one:

$$Q_1 = (\alpha - \beta p_1)(1 + b_1) - \frac{\beta k}{2}\left((1 + x_1)^2 + (b_1 - x_1)^2\right) \tag{125}$$

$$Q_2 = (\alpha - \beta p_2)(b_2 - a_2) - \frac{\beta k}{2}\left((a_2 - x_2)^2 + (b_2 - x_2)^2\right) \tag{126}$$

$$Q_3 = (\alpha - \beta p_3)(1 - a_3) - \frac{\beta k}{2}\left((a_3 - x_3)^2 + (1 - x_3)^2\right) \tag{127}$$

where we used $a_1 = -1$ and $b_3 = 1$. We can recover the two remaining market boundary points $a_2 = b_1$ and $a_3 = b_2$ directly from (7) above.

So, substitute for these, and maximise sales according to (125)-(127) with respect to the choice of location. As we recall, the multiplicative factor (price minus unit cost) in profits does not influence location. Hence, differentiating (125)-(127) and solving for location, we obtain:

$$x_1 = \frac{x_2 - 4}{5} + \frac{p_2 - 3p_1}{5k} + \frac{2\alpha}{5\beta k} \tag{128}$$

$$x_2 = \frac{x_1 + x_3}{2} + \frac{p_3 - p_1}{2k} \tag{129}$$

$$x_3 = \frac{x_2 + 4}{5} + \frac{3p_3 - p_2}{5k} - \frac{2\alpha}{5\beta k} \tag{130}$$

As we see, (128) and (130) are quite similar to (41)-(42), whereas (129) resembles (10) above.

Adding (128) and (130) and substituting from (129) we obtain the handy expression:

$$x_2 = x_1 + x_3 = \frac{p_3 - p_1}{k} \tag{131}$$

Considering equilibrium with three identical firms (equal marginal production costs), we could conjecture that, (125) and (127) being similar, as well as (128) and (130), we would have equal prices at least for the extremal firms. So, substituting $p_1 = p_3$ in (131) we find that: i) *the extremal firms locate symmetrically around the origin*; ii) *the intermediate firm locates midway between the competitors*. The conjecture about prices of the extremally located firms in fact turns out to be true, though the intermediate firm will as a rule have to charge a lower price.

The intermediate firm always locates in the middle of its market area $[a_2, b_2]$ - whether in equilibrium or not! From (7) and (129) we find that for $i = 2$, equations (29)-(30) always hold. This is, however not true for the extremal firms, which have a location bias towards the centre.

Consider for a moment the case of disjoint identical monopoly firms (i.e. having equal production costs). Then all three firms charge equal prices. Another case of price equality is if all three cluster in the centre. Substituting $p_1 = p_2 = p_3 = p$ in (128)-(130), we obtain:

$$x_1 = -\frac{4}{5} + \frac{2}{5}\frac{1}{k}\left(\frac{\alpha}{\beta} - p\right) \tag{132}$$

$$x_2 = 0 \tag{133}$$

$$x_3 = \frac{4}{5} - \frac{2}{5}\frac{1}{k}\left(\frac{\alpha}{\beta} - p\right) \tag{134}$$

First try the case of clustering by substituting zeros in all left hand sides. Then we easily obtain: $\alpha / \beta - p = 2k$ quite as in the case of two competitors, so if maximum price overshoots equilibrium price by twice the transportation cost, then all firms crowd in the centre, not only two, but even three.

Next try the case when the firms locate in the midpoints of their respective markets. The central one always is at the origin according to (133), but the outer ones would have to locate in the points $\mp 2/3$. Substituting in (132) and (134) we then have $\alpha/\beta - p = k/3$. It is easy to check that this corresponds to the case of disjoint monopolies. The denominator in the condition is different from the duopoly case, but the firms are now three in stead of two, so a shorter segment is available for each.

For the general case of duopoly equilibrium we have to consider one special problem. As the extremal firms tend to locate close it may happen that the intermediate firm finds that it will obtain a larger market interval by leaving its central position and moving left or right of their competitors. (We suggested that the extremal firms may also charge higher prices, so a larger market with a higher potential price might result in higher profit.) If so there will be set up a location instability, and we have to check the exact point at which this happens.

We now have to settle the issue of pricing. Assume the firms are identical, i.e. that all the marginal costs are equal to c. From (125)-(127) we then get output, which we multiply by $(p_i - c)$ to get profits and then substitute for locations x_i from (128)-(130). Again define auxiliary variables to simplify the formulas. Let:

$$\lambda_1 = p_2 + k(1 + x_2) \quad \lambda_2 = p_1 + p_3 + k(x_3 - x_1) \quad \lambda_3 = p_2 + k(1 - x_2) \tag{135}$$

Note that the first and last resemble (47), whereas the middle one resembles (11). Using (135) we get profits:

$$\Pi_1 = \frac{\beta}{10k}(p_1 - c)\left(6\left(\frac{\alpha}{\beta} - p_1\right)^2 - 4\left(\frac{\alpha}{\beta} - p_1\right)\left(\frac{\alpha}{\beta} - \lambda_1\right) - \left(\frac{\alpha}{\beta} - \lambda_1\right)^2\right) \tag{136}$$

$$\Pi_2 = \frac{\beta}{16k}(p_2 - c)(\lambda_2 - 2p_2)\left(8\frac{\alpha}{\beta} - \lambda_2 - 6p_2\right) \tag{137}$$

$$\Pi_3 = \frac{\beta}{10k}(p_3 - c)\left(6\left(\frac{\alpha}{\beta} - p_3\right)^2 - 4\left(\frac{\alpha}{\beta} - p_3\right)\left(\frac{\alpha}{\beta} - \lambda_3\right) - \left(\frac{\alpha}{\beta} - \lambda_3\right)^2\right) \tag{138}$$

Note that (136) and (138) look similar, and indeed become identical in equilibrium when $x_2 = 0$ according to (131), given equal prices for the extremal firms and hence $\lambda_1 = \lambda_3$ from (135). Note also the similarity of (136) and (138) to equation (48), as well as of (137) to (12).

Next differentiate with respect to p_i, equate to zero, and solve. In order to make the formulas more concise let us define the two expressions:

$$A_i = 36\left(\frac{\alpha}{\beta} - c\right)^2 - 24\left(\frac{\alpha}{\beta} - c\right)\left(\frac{\alpha}{\beta} - \lambda_i\right) + 34\left(\frac{\alpha}{\beta} - \lambda_i\right)^2 \qquad (139)$$

$$B_i = 36\left(\frac{\alpha}{\beta} - c\right)^2 - 12\left(\frac{\alpha}{\beta} - c\right)\left(2\frac{\alpha}{\beta} - \lambda_i\right) + 13\left(2\frac{\alpha}{\beta} - \lambda_i\right)^2 \qquad (140)$$

Using (139)-(140) we can write:

$$p_1 = \frac{4}{9}\frac{\alpha}{\beta} + \frac{3}{9}c + \frac{2}{9}\lambda_1 - \frac{1}{18}\sqrt{A_1} \qquad (141)$$

$$p_2 = \frac{4}{9}\frac{\alpha}{\beta} + \frac{3}{9}c + \frac{1}{9}\lambda_2 - \frac{1}{18}\sqrt{B_2} \qquad (142)$$

$$p_3 = \frac{4}{9}\frac{\alpha}{\beta} + \frac{3}{9}c + \frac{2}{9}\lambda_3 - \frac{1}{18}\sqrt{A_3} \qquad (143)$$

Again we note the similarity of (141) and (143) to (49), and of (142) to (13). In equilibrium with identical firms we have $p_1 = p_3$ but, as a rule, $p_2 < p_1 = p_3$, except in the obvious case of disjoint monopolies.

We can now easily state the maximum profits of the various firms when a profit maximising price policy is chosen according to (141)-(143). Substituting from (141)-(143) in (136)-(138) we obtain:

$$\Pi_1 = \frac{\beta}{2430k}\left(\frac{1}{2}(A_1)^{\frac{3}{2}} + \left(2\frac{\alpha}{\beta} - 3c + \lambda_1\right)\left(A_1 - 120\left(\frac{\alpha}{\beta} - \lambda_1\right)^2\right)\right) \tag{144}$$

$$\Pi_2 = \frac{\beta}{3888k}\left((B_2)^{\frac{3}{2}} - \left(4\frac{\alpha}{\beta} - 6c + \lambda_2\right)\left(16\frac{\alpha}{\beta} - 6c - 5\lambda_2\right)\left(8\frac{\alpha}{\beta} + 6c - 7\lambda_2\right)\right) \tag{145}$$

$$\Pi_3 = \frac{\beta}{2430k}\left(\frac{1}{2}(A_3)^{\frac{3}{2}} + \left(2\frac{\alpha}{\beta} - 3c + \lambda_3\right)\left(A_3 - 120\left(\frac{\alpha}{\beta} - \lambda_3\right)^2\right)\right) \tag{146}$$

As we consider identical firms in equilibrium (135) apply for the auxiliary variables, further with $p_1 = p_3$.

Consider now the case when the middle firm moves out from its central position either right or left of one of the extremal firms. Suppose it moves right, to the position $\vec{x}_2$. Note that all the symbols associated with a right move are identified by the arrow pointing right. The market boundary points then become $\vec{a}_2$, to be obtained from (7), and $\vec{b}_2 = 1$, as prescribed by the fixed interval. From (7) we immediately get:

$$\vec{a}_2 = \frac{x_3 + \vec{x}_2}{2} + \frac{\vec{p}_2 - p_3}{2k} \tag{147}$$

where $\vec{p}_2$ denotes the yet undetermined price of the previously middle firm in its new rightmost position. Similarly, for a left move to $\overleftarrow{x}_2$ (indicated by arrows pointing left) we get $\overleftarrow{a}_2 = -1$ in the new position and $\overleftarrow{b}_2$, likewise obtained from (7):

$$\overleftarrow{b}_2 = \frac{x_1 + \overleftarrow{x}_2}{2} + \frac{p_1 - \overleftarrow{p}_2}{2k} \tag{148}$$

Next, from (4) we get demand for each case:

$$\vec{Q}_2 = (\alpha - \beta\, \vec{p}_2)(1 - \vec{a}_2) - \frac{\beta k}{2}\left((\vec{a}_2 - \vec{x}_2)^2 + (1 - \vec{x}_2)^2\right) \tag{149}$$

$$\overleftarrow{Q}_2 = (\alpha - \beta\, \overleftarrow{p}_2)\left(1 + \overleftarrow{b}_2\right) - \frac{\beta k}{2}\left((1 + \overleftarrow{x}_2)^2 + \left(\overleftarrow{b}_2 - \overleftarrow{x}_2\right)^2\right) \tag{150}$$

We can substitute for $\vec{a}_2$ from (147) in (149) and then as usual differentiate with respect to $\vec{x}_2$, put equal to zero, and solve for the optimum location in the new interval. In the same way substitute for $\overleftarrow{b}_2$ from (148) in (150) and do the same with respect to $\overleftarrow{x}_2$. We get:

$$\vec{x}_2 = \frac{x_3 + 4}{5} + \frac{3\vec{p}_2 - p_3}{5k} - \frac{2\alpha}{5\beta k} \tag{151}$$

$$\overleftarrow{x}_2 = \frac{x_1 - 4}{5} - \frac{3\overleftarrow{p}_2 - p_1}{5k} + \frac{2\alpha}{5\beta k} \tag{152}$$

where $\partial^2 \vec{Q}_2 / \partial \vec{x}_2{}^2 = \partial^2 \overleftarrow{Q}_2 / \partial \overleftarrow{x}_2{}^2 = -5\beta k / 4 < 0$ guarantee the fulfilment of the second order conditions for profit maxima in both cases.

Substituting for $\vec{x}_2$ from (151) in (149) and for $\overleftarrow{x}_2$ from (152) in (150) we obtain total sales in the new optimal locations:

$$\vec{Q}_2 = \frac{\beta}{10k}\left(6\left(\frac{\alpha}{\beta} - \vec{p}_2\right)^2 - 4\left(\frac{\alpha}{\beta} - \vec{p}_2\right)\left(\frac{\alpha}{\beta} - \vec{\lambda}_2\right) - \left(\frac{\alpha}{\beta} - \vec{\lambda}_2\right)^2\right) \tag{153}$$

$$\overleftarrow{Q}_2 = \frac{\beta}{10k}\left(6\left(\frac{\alpha}{\beta} - \overleftarrow{p}_2\right)^2 - 4\left(\frac{\alpha}{\beta} - \overleftarrow{p}_2\right)\left(\frac{\alpha}{\beta} - \overleftarrow{\lambda}_2\right) - \left(\frac{\alpha}{\beta} - \overleftarrow{\lambda}_2\right)^2\right) \tag{154}$$

respectively, where we have defined new auxiliary variables

$$\vec{\lambda}_2 = p_3 + k(1 - x_3) \qquad \overleftarrow{\lambda}_2 = p_1 + k(1 + x_1) \tag{155}$$

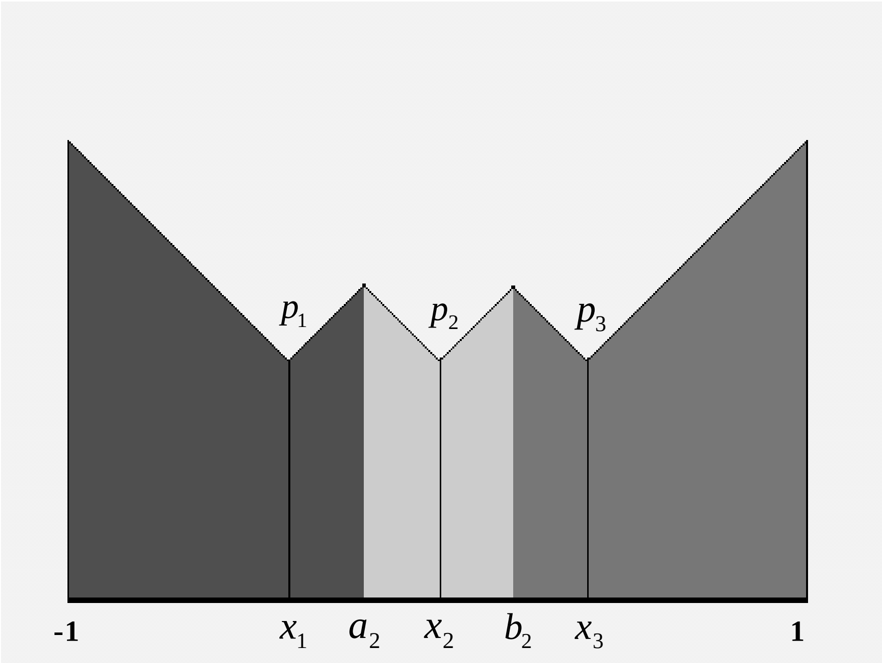

Fig. 5. Price landscape for three competitors on a line segment.

different from the middle formula (135) as the firm is now in an extremal position.

From (153)-(154) we get profits:

$$\vec{\Pi}_2 = \frac{\beta}{10k}\left(\vec{p}_2 - c\right)\left(6\left(\frac{\alpha}{\beta} - \vec{p}_2\right)^2 - 4\left(\frac{\alpha}{\beta} - \vec{p}_2\right)\left(\frac{\alpha}{\beta} - \vec{\lambda}_2\right) - \left(\frac{\alpha}{\beta} - \vec{\lambda}_2\right)^2\right) \quad (156)$$

$$\overleftarrow{\Pi}_2 = \frac{\beta}{10k}\left(\overleftarrow{p}_2 - c\right)\left(6\left(\frac{\alpha}{\beta} - \overleftarrow{p}_2\right)^2 - 4\left(\frac{\alpha}{\beta} - \overleftarrow{p}_2\right)\left(\frac{\alpha}{\beta} - \overleftarrow{\lambda}_2\right) - \left(\frac{\alpha}{\beta} - \overleftarrow{\lambda}_2\right)^2\right) \quad (157)$$

which we maximise with respect to $\vec{p}_2$ or $\overleftarrow{p}_2$, depending on the direction of jump, always using the corresponding $\vec{\lambda}_2$ or $\overleftarrow{\lambda}_2$ from (155), and solve for the variable. In order to obtain a concise expression we again define some abbreviations:

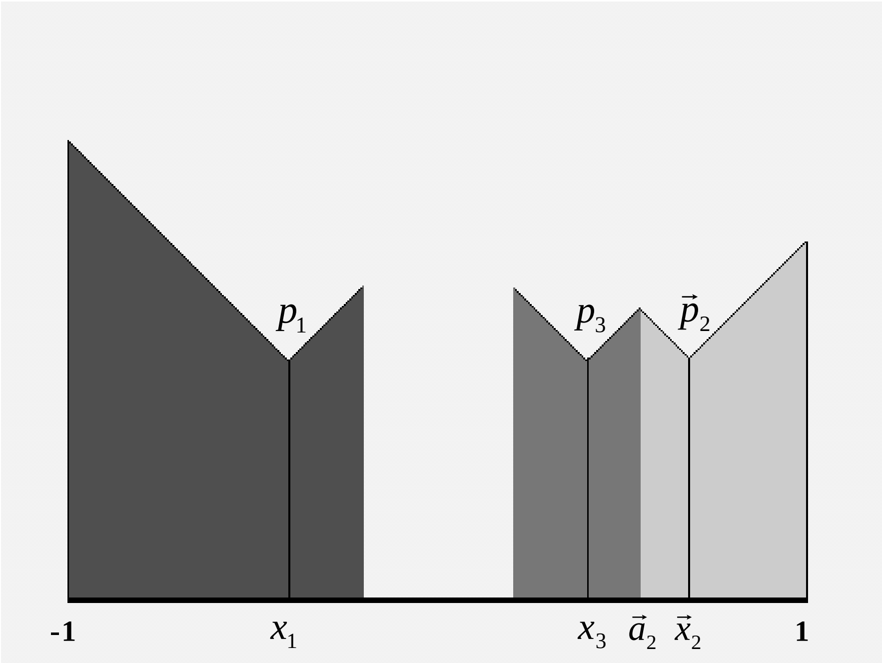

Fig. 6. Price landscape with the squeezed middle firm jumping right.

$$\vec{A}_2 = 36\left(\frac{\alpha}{\beta} - c\right)^2 - 24\left(\frac{\alpha}{\beta} - c\right)\left(\frac{\alpha}{\beta} - \vec{\lambda}_2\right) + 34\left(\frac{\alpha}{\beta} - \vec{\lambda}_2\right)^2 \tag{158}$$

$$\overleftarrow{A}_2 = 36\left(\frac{\alpha}{\beta} - c\right)^2 - 24\left(\frac{\alpha}{\beta} - c\right)\left(\frac{\alpha}{\beta} - \overleftarrow{\lambda}_2\right) + 34\left(\frac{\alpha}{\beta} - \overleftarrow{\lambda}_2\right)^2 \tag{159}$$

Using this we obtain relocation prices:

$$\vec{p}_2 = \frac{4}{9}\frac{\alpha}{\beta} + \frac{3}{9}c + \frac{2}{9}\vec{\lambda}_2 - \frac{1}{18}\sqrt{\vec{A}_2} \qquad \overleftarrow{p}_2 = \frac{4}{9}\frac{\alpha}{\beta} + \frac{3}{9}c + \frac{2}{9}\overleftarrow{\lambda}_2 - \frac{1}{18}\sqrt{\overleftarrow{A}_2} \tag{160}$$

Finally, substitute from (160) in (156)-(157) to obtain maximum profits in this new location:

$$\vec{\Pi}_2 = \frac{\beta}{2430k}\left(\frac{1}{2}\left(\vec{A}_2\right)^{\frac{3}{2}} + \left(2\frac{\alpha}{\beta} - 3c + \vec{\lambda}_2\right)\left(\vec{A}_2 - 120\left(\frac{\alpha}{\beta} - \vec{\lambda}_2\right)^2\right)\right) \tag{161}$$

$$\overleftarrow{\Pi}_2 = \frac{\beta}{2430k}\left(\frac{1}{2}\left(\overleftarrow{A}_2\right)^{\frac{3}{2}} + \left(2\frac{\alpha}{\beta} - 3c + \overleftarrow{\lambda}_2\right)\left(\overleftarrow{A}_1 - 120\left(\frac{\alpha}{\beta} - \overleftarrow{\lambda}_2\right)^2\right)\right) \tag{162}$$

These expressions (161)-(162) for profits in case of moving out, are what we have to compare to (145) in order to establish if jumping out is profitable or not. Such comparisons do not result in manageable closed form criteria, but are better carried out by the computer for numerical cases.

The geometry of the original middle location and a jump is illustrated in Figs. 5-6.

Even though *only the middle firm* may have reason to jump *from an equilibrium position* due to the fact that it is squeezed by the outer firms, it is perfectly possible that, in any disequilibrium state in the dynamic process, an outer firm may in stead consider jumping in between the competitors. So, if we want to set up an explicit dynamical process we have to consider jumping possibilities for all three firms, even the extremal ones which may want to jump in into the middle or even to the other end. Only computer programs are conceivable for checking the dynamics of such processes. Preliminary simulations indicate the possibility of never ending reshuffling of sequences of firms.

8 Different forms of the Demand Function

If one considers the absence of complex dynamics a problem, a linear demand function may seem to be one, at least in 1D models, because, as we have seen in Section 6 above, the derivatives of the reaction functions are too low for the Nash equilibrium to be ever destabilised.

However, linear demand is not the only choice. As mentioned, Lerner and Singer first tried a step function, before Smithies tried the linear. We might want to check the whole class of demand functions $p^{\rho} + q^{\rho} = 1$, with the parameter r between unity (the linear), and infinity (the step function). This function is of the Minkowski metric type, or as economists might prefer to say, CES type, even if the curvature would be in the wrong direction compared to normal isoquants. Note that for the linear case we now put $\alpha / \beta = 1$, as we noted that we may without loss of generality. The problem with such a family is that we have to integrate expressions such as (3) over space, and easily end up with expressions of a monstrous complexity right at the outset.

However, we can easily find these things out with a broken line demand function in the unit square with a kink point along the positive diagonal:

$$q = \begin{cases} 1 - \frac{1-\mu}{\mu}\left(p_i + k\left|x - x_i\right|\right) & p_i + k\left|x - x_i\right| \le \mu \\ \mu - \frac{\mu}{1-\mu}\left(p_i + k\left|x - x_i\right| - \mu\right) & p_i + k\left|x - x_i\right| > \mu \end{cases} \tag{163}$$

If $\mu = 1/2$ we have the linear case back, if $\mu \to 1$, we approach the Lerner and Singer case.

From (163) we see that the kink point is located at $p_i + k\left|x - x_i\right| = \mu$. For the integration of total demand we have to locate the points in space where this kink occurs, i.e. $p_i + k\left(x_i - \xi_i\right) = p_i + k\left(\eta_i - x_i\right) = \mu$. Solving we get:

$$\xi_i = x_i - \frac{p_i - \mu}{k} \quad \text{and} \quad \eta_i = x_i + \frac{p_i - \mu}{k} \tag{164}$$

so we can easily calculate the total demand for the general firm:

$$\begin{aligned} Q_i = & \int_{a_i}^{\xi_i} \left(1 - \frac{1-\mu}{\mu}\left(p_i + k\left|x - x_i\right|\right)\right) dx \\ & + \int_{\xi_i}^{\eta_i} \left(\mu - \frac{\mu}{1-\mu}\left(p_i + k\left|x - x_i\right| - \mu\right)\right) dx \\ & + \int_{\eta_i}^{b_i} \left(1 - \frac{1-\mu}{\mu}\left(p_i + k\left|x - x_i\right|\right)\right) dx \end{aligned} \tag{165}$$

Quite as above, a_i, b_i denote the boundary points for the market interval, whereas ξ_i, η_i are defined in (164) above. Breaking up the integral in three terms makes integration easy. We get the closed form expression:

$$Q_i = \frac{\mu}{1-\mu}\left((1-p_i)(b_i - a_i) - \frac{k}{2}\left((a_i - x_i)^2 + (b_i - x_i)^2\right)\right) - \frac{\mu(2\mu-1)(\mu-p_i)^2}{k(1-\mu)\mu^2} \tag{166}$$

This is quite similar to (4), only note that we have simplified by putting α and β equal to unity. Further, recall that the linear demand function case we dealt with has $\mu = 1/2$, so then the coefficient or the first term becomes unitary whereas the second term vanishes altogether. This is indeed reassuring. We note that the effect of the kink in the demand function is to scale up the first term and subtract an additional term.

We can now easily obtain all the relevant closed form expressions for the Hotelling duopolists on a line. Just recall that we have $a_1 = -1$, $b_2 = 1$, and quite as in (10) $a_2 = b_1 = (x_1 + x_2)/2 + (p_2 - p_1)/2k$. Optimising (166) with respect to location we note that *i*) location does not enter the additional term and *ii*) the multiplicative factor makes no difference for the optimum location condition. Accordingly, (41)-(42) still hold, as restated here:

$$x_1 = \frac{x_2 - 4}{5} + \frac{p_2 - 3p_1 + 2}{5k} \qquad x_2 = \frac{x_1 + 4}{5} + \frac{3p_2 - p_1 - 2}{5k} \tag{167}$$

Again there is a slight simplification due to our choice of α and β equal to unity.

The rest of the procedure follows as above. Define the auxiliary variables:

$$\lambda_1 = p_2 + k(1 + x_2) \qquad \lambda_2 = p_1 + k(1 - x_1) \tag{168}$$

and we can write the total demand for the Hotelling duopolists:

$$Q_i = \frac{\mu}{1-\mu}\frac{1}{10k}\left(6(1-p_i)^2 - 4(1-p_i)(1-\lambda_i) - (1-\lambda_i)^2\right) - \frac{\mu(2\mu-1)(\mu-p_i)^2}{k(1-\mu)\mu^2} \tag{169}$$

Profits are next obtained by multiplying the quantity expressions through by the factor $(p_i - c_i)$, which still makes profits cubic in prices. Optimization with respect to prices is straightforward, though the occurrence of the kink makes the expressions slightly more complicated.

The important thing, however, is that the derivatives of the reaction functions still seem to retain the property of being bounded in a range which makes the Nash equilibrium stable.

By conclusion we should stress that we left many loose ends worth further study, such as phenomena in 2D, processes where a third competitor in a fixed interval may feel squeezed and set up a series of location order changes, or phenomena in unbounded space, such as three firms on a circle in 1D, or a sphere in 2D.

9 Selected References

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