6 Duopoly with Piecewise Linear Discontinuous Reaction Functions

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1 Introduction

In the previous Chapter (Section 5) we have considered a variant of Palander's model of duopoly with kinked demand, and decreasing marginal costs. In the original model, with constant, or even zero, marginal costs no more interesting dynamics than periodic orbits can occur. This is because the resulting slopes of the reaction functions are too low. A way to raise the slopes is assuming the marginal cost line to be downsloping.

Of course, a globally downsloping marginal cost line does not make any sense at all, because it will eventually (once it reaches negative values) result in production costs decreasing with increasing production, and further even in negative production costs.

However, a piecewise linear marginal cost function could approximate the traditional U-shaped textbook case, which itself is awkward to work with as even in the simplest case we have to consider roots of cubic equations (see Section 4 of the previous Chapter). We could then focus the study on just what happens in one downsloping section of the approximating train of straight line segments, and take care to check that we only study processes which are confined to this section. We already found indications of chaos, and will now proceed to a more detailed and more formal study of these cases. In particular we focus on the changes in the dynamics as we change two of the parameters, more precisely the slopes of the two pieces in the kinked demand function.

2 Description of the General Model

Consider the general duopoly model with a piecewise linear continuous demand function of the form

$$p = f(x+y) = \begin{cases} \alpha_1 - \beta_1(x+y); & 0 \le x+y \le q; \\ \alpha_2 - \beta_2(x+y); & q < x+y \le q_1; \\ 0; & x+y > q_1; \end{cases} \quad (1)$$

where

$$q = \frac{\alpha_1 - \alpha_2}{\beta_1 - \beta_2}; \quad q_1 = \frac{\alpha_2}{\beta_2}$$

are kink points; $\alpha_i > 0; \beta_i > 0; \ i = 1; 2;$ (see fig.1).

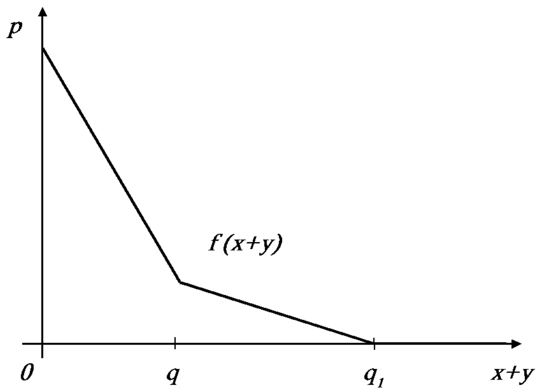

Figure 1: *Demand function.*

Let $\alpha_1 > \alpha_2$; then $\beta_1 > \beta_2$, $q > 0$, $f(q) = (\alpha_2\beta_1 - \alpha_1\beta_2)/(\beta_1 - \beta_2) > 0$.

Let the cost function of the first competitor be

$$C_x(x) = a_x x - b_x x^2; \quad 0 < x < \frac{a_x}{2b_x}; \quad (2)$$

and of the second

$$C_y(y) = a_y y - b_y y^2; \quad 0 < y < \frac{a_y}{2b_y}. \quad (3)$$

We define the function $C_x(x)$ in the interval $0 < x < a_x/2b_x$ because only when the marginal cost is positive does the model make sense, and positive marginal cost corresponds to the increasing branch of the cost function. For the same reason the function $C_y(y)$ is defined for $0 < y < a_y/2b_y$.

Note that if $a_x = 0$ and $b_x > 0$, then $C_x(x) < 0$, so that this case is not allowed. We shall consider $a_x > 0$, $b_x \geq 0$ as well as $a_y > 0$, $b_y \geq 0$.

Let $0 \leq x + y \leq q_1$. The marginal revenue $MR_x(x,y)$ and the marginal cost $MC_x(x)$ of the first firm are:

$$\begin{aligned} MR_x(x,y) &= (f(x+y)x)'_x = \begin{cases} \alpha_1 - 2\beta_1 x - \beta_1 y, & 0 \leq x+y \leq q, \\ \alpha_2 - 2\beta_2 x - \beta_2 y, & q < x+y < q_1, \end{cases} \\ MC_x(x) &= a_x - 2b_x x \end{aligned}$$

and of the second one are:

$$\begin{aligned} MR_y(x,y) &= (f(x+y)y)'_y = \begin{cases} \alpha_1 - 2\beta_1 y - \beta_1 x, & 0 \leq x+y \leq q, \\ \alpha_2 - 2\beta_2 y - \beta_2 x, & q < x+y < q_1, \end{cases} \\ MC_y(y) &= a_y - 2b_y y \end{aligned}$$

The profit functions of the two competitors are respectively

$$\begin{aligned} \Pi_x(x,y) &= f(x+y)x - C_x(x) \\ &= \begin{cases} (\alpha_1 - a_x - \beta_1 y)x - (\beta_1 - b_x)x^2, & 0 \leq x+y \leq q, \\ (\alpha_2 - a_x - \beta_2 y)x - (\beta_2 - b_x)x^2, & q < x+y < q_1, \end{cases} \end{aligned} \tag{4}$$

and

$$\begin{aligned} \Pi_y(x,y) &= f(x+y)y - C_y(y) \\ &= \begin{cases} (\alpha_1 - a_y - \beta_1 x)y - (\beta_1 - b_y)y^2, & 0 \leq x+y \leq q, \\ (\alpha_2 - a_x - \beta_2 x)y - (\beta_2 - b_y)y^2, & q < x+y < q_1. \end{cases} \end{aligned}$$

In order to obtain local maximal profits of the first competitor we have to solve the equation

$$\frac{\partial}{\partial x}(\Pi_x(x,y)) = 0 \tag{5}$$

with respect to x (or, in other words, we solve the equation $MR_x(x,y) = MC_x(x)$ with respect to x). Note that the solution of (5) is indeed a profit maximum if the second derivative of $\Pi_x(x,y)$ is negative, i.e.

$$\frac{\partial^2}{\partial x^2}(\Pi_x(x,y)) = -2(\beta_i - b_x) < 0,$$

which is true if

$$\beta_i > b_x, \quad i = 1, 2.$$

Using the same arguments for the second competitor we get the following conditions:

$$\beta_i > b_y, \quad i = 1, 2.$$

Solving (5) with respect to x we get the following two functions denoted by $\varphi_1(y)$ and $\varphi_2(y)$ (associated with the two functions defining $\Pi_x(x, y)$ in (4)):

$$x = \varphi_1(y) = \frac{(\alpha_1 - \beta_1 y - a_x)}{2(\beta_1 - b_x)}, \tag{6}$$

$$x = \varphi_2(y) = \frac{(\alpha_2 - \beta_2 y - a_x)}{2(\beta_2 - b_x)}. \tag{7}$$

The function $\varphi_1(y)$ is defined for $0 \le x + y \le q$, i.e. $0 \le \varphi_1(y) + y \le q$, from which it follows that if $\beta_1 > 2b_x$ then $\varphi_1(y)$ is given for

$$0 \le y \le y_1,$$

where

$$y_1 = \frac{2q(b_x - \beta_1) + \alpha_1 - a_x}{2b_x - \beta_1} = \frac{f(q) - a_x}{2b_x - \beta_1} + q,$$

while if $\beta_1 < 2b_x$ then $\varphi_1(y)$ is defined for

$$y_1 \le y \le y_2,$$

where

$$y_2 = \frac{\alpha_1 - a_x}{2b_x - \beta_1}.$$

The condition $x = \varphi_1(y) \ge 0$ also has to be fulfilled and it holds for

$$y \le y_{0,1} = \frac{\alpha_1 - a_x}{\beta_1}.$$

Note that from $y \geq 0$ it follows that $y_{0,1} \geq 0$, i.e. the following condition has to be satisfied:

$$\alpha_1 \geq a_x.$$

The function $\varphi_2(y)$ is defined for $q < x + y < q_1$, i.e. $q < \varphi_2(y) + y < q_1$, from which it follows that if $\beta_2 > 2b_x$ then $\varphi_2(y)$ is given for

$$y_3 < y < y_4,$$

where

$$y_3 = \frac{2q(b_x - \beta_2) + \alpha_2 - a_x}{2b_x - \beta_2} = \frac{f(q) - a_x}{2b_x - \beta_2} + q,$$

$$y_4 = q_1 - \frac{a_x}{2b_x - \beta_2},$$

while if $\beta_2 < 2b_x$ then $\varphi_2(y)$ is defined for

$$y_4 < y < y_3.$$

From $x = \varphi_2(y) \geq 0$ we get the inequality

$$y \leq y_{0,2} = \frac{\alpha_2 - a_x}{\beta_2}$$

which together with the condition $y \geq 0$ gives the following condition

$$\alpha_2 \geq a_x.$$

In order to have the cost function defined in the suitable range (given in (2)) we substitute $x = \varphi_1(y)$ and $x = \varphi_2(y)$ into the marginal cost expression and ask for positivity:

$$MC_x(\varphi_i(y)) = a_x - 2b_x \varphi_i(y) > 0,$$

which, when $b_x \neq 0$, holds if

$$y > \frac{\alpha_i}{\beta_i} - \frac{a_x}{b_x}, \quad i = 1, 2.$$

A sufficient condition for the inequality above to be fulfilled is

$$\frac{\alpha_i}{\beta_i} < \frac{a_x}{b_x}; \quad i = 1;2:$$

Summarizing, it can be shown that two qualitatively different cases occur for the range of definition of the functions $\varphi_1(y)$ and $\varphi_2(y)$ depending on $a_x \gtrless f(q)$:

1) If $a_x \leq f(q)$; then $y_{0;1} \leq y_{0;2}$. Moreover, the function $\varphi_1(y)$ is defined only for $\beta_1 > 2b_x$ (as $y_1 > y_{0;1}$ for $\beta_1 < 2b_x$) :

$$x = \varphi_1(y) \ \text{for } 0 \leq y \leq y_1; \ \beta_1 > 2b_x:$$

The function $\varphi_2(y)$ is given as follows:

$$x = \varphi_2(y) \left\{ \begin{array}{l} \text{for } y_3 < y < y_{0;2} \ \text{if } \beta_2 > 2b_x; \\ \text{for } 0 < y < y_{0;2} \ \text{if } \beta_2 < 2b_x: \end{array} \right.$$

In the last inequality the function $\varphi_2(y)$ is defined for $0 < y < y_{0;2}$ because when $\beta_2 < 2b_x$ then $y_4 < 0$:

2) If $a_x > f(q)$; then $y_{0;1} > y_{0;2}$. The functions $\varphi_1(y)$ and $\varphi_2(y)$ are defined as follows:

$$x = \varphi_1(y) \left\{ \begin{array}{l} \text{for } 0 \leq y \leq y_{0;1} \ \text{if } \beta_1 > 2b_x; \\ \text{for } y_1 \leq y < y_{0;1} \ \text{if } \beta_1 < 2b_x; \end{array} \right.$$

$$x = \varphi_2(y) \quad \text{for } 0 < y < y_3; \ \beta_2 < 2b_x;$$

(as $y_3 > y_{0;2}$ for $\beta_2 > 2b_x$):

Using the same arguments for the second competitor we can repeat all the computations performed above putting x instead of y and vice versa, denoting the two solutions of the equation

$$\frac{\partial}{\partial y}(\Pi_y(x;y)) = 0$$

by $\psi_i(x)$ (instead of $\varphi_i(y)$); $i = 1;2$:

Let us collect all the parameter conditions which are to be fulfilled in order to have a meaningful duopoly model with the demand function (1) and the cost functions (2) and (3):

$$\begin{aligned}
&\alpha_i > 0; \ \beta_i > 0;\\
&\alpha_1 > \alpha_2; \ \beta_1 > \beta_2; \ (\alpha_2\beta_1 - \alpha_1\beta_2) > 0;\\
&a_x > 0; \ b_x \geq 0; \ a_y > 0; \ b_y \geq 0;\\
&a_x \leq \alpha_i; \ b_x < \beta_i; \ a_y \leq \alpha_i; \ b_y < \beta_i;\\
&\alpha_i/\beta_i < a_x/b_x \ \text{ for } b_x \neq 0;\\
&\alpha_i/\beta_i < a_y/b_y \ \text{ for } b_y \neq 0.
\end{aligned} \tag{8}$$

The functions $\varphi_1(y)$ and $\varphi_2(y)$ are the solutions of the local optimization problem for the first competitor. In order to get a global maximal profit (denote it by $\Pi_1(y)$) we have to substitute the functions $\varphi_1(y)$ and $\varphi_2(y)$ in the profit expression (4), compare two profits and choose the function that corresponds to the maximum. In this way we obtain a reaction function of the first competitor x defined (at each y) by the function $\varphi_i(y)$; $i = 1, 2$; which corresponds to the global maximal profit.

Denote $\Pi_x(\varphi_1(y), y)$ by $\Pi_{x,1}(y)$ and $\Pi_x(\varphi_2(y), y)$ by $\Pi_{x,2}(y)$:

$$\Pi_{x,1}(y) = \frac{(\alpha_1 - \beta_1 y - a_x)^2}{4(\beta_1 - b_x)}; \quad \Pi_{x,2}(y) = \frac{(\alpha_2 - \beta_2 y - a_x)^2}{4(\beta_2 - b_x)};$$

the graphs of these functions are convex parabolas which intersect the vertical axis in

$$\Pi_{x,1}(0) = \frac{(\alpha_1 - a_x)^2}{4(\beta_1 - b_x)}; \quad \Pi_{x,2}(0) = \frac{(\alpha_2 - a_x)^2}{4(\beta_2 - b_x)};$$

and have minimum value (zero) at the following points:

$$\Pi_{x,1}(y) = 0 \text{ at } y = y_{0,1}; \quad \Pi_{x,2}(y) = 0 \text{ at } y = y_{0,2}.$$

It follows from the conditions $\varphi_1(y) \geq 0$ that the function $\Pi_{x,1}(y)$ is defined for $0 \leq y \leq y_{0,1}$; from the condition $\varphi_2(y) \geq 0$ we get that $\Pi_{x,2}(y)$ is defined for $0 \leq y \leq y_{0,2}$. In order to have the function $\Pi_1(y)$ defined for all positive y, let $\Pi_1(y)$ be given by

$$\Pi_1(y) = \begin{cases} \max\{\Pi_{x,1}(y), \Pi_{x,2}(y)\}, & 0 \leq y \leq \max\{y_{0,1}, y_{0,2}\}, \\ 0, & y > \max\{y_{0,1}, y_{0,2}\}. \end{cases}$$

It is easy to get that $\Pi_{x,1}(y) \cap \Pi_{x,2}(y) = \{\bar{y}, \bar{y}_1\}$, where

$$\bar{y} = \frac{(\alpha_1 - a_x)\sqrt{(\beta_2 - b_x)} - (\alpha_2 - a_x)\sqrt{(\beta_1 - b_x)}}{\beta_1\sqrt{(\beta_2 - b_x)} - \beta_2\sqrt{(\beta_1 - b_x)}};$$

$$\bar{y}_1 = \frac{(\alpha_1 - a_x)\sqrt{(\beta_2 - b_x)} + (\alpha_2 - a_x)\sqrt{(\beta_1 - b_x)}}{\beta_1\sqrt{(\beta_2 - b_x)} + \beta_2\sqrt{(\beta_1 - b_x)}}.$$

Note that in order to avoid division by zero in the expression for $\bar{y}$, the following condition should be checked:

$$b_x \neq \frac{\beta_1\beta_2}{\beta_1 + \beta_2}.$$

Proposition 1 *Let* $a_x < f(q)$, $\beta_1 > 2b_x$, $\bar{y} > 0$ *and*

$$b_x < \frac{\beta_1\beta_2}{\beta_1 + \beta_2}. \tag{9}$$

Then the reaction function of the first competitor is

$$x = \begin{cases} \varphi_1(y), & 0 \le y < \bar{y}; \\ \varphi_2(y), & \bar{y} \le y \le y_{0,2}; \\ 0, & y > y_{0,2}. \end{cases} \tag{10}$$

Proof. In fact, it follows from the assumptions on the parameters that $y_{0,1} < y_{0,2}$, $\bar{y} < \bar{y}_1$, $\bar{y} < y_{0,1}$, $y_{0,1} < \bar{y}_1 < y_{0,2}$, that is the point $\bar{y}_1$ belongs to the interval where only the profit function $\Pi_{x,2}(y)$ is defined, thus only the point $\bar{y}$ is related to a jump from one function to the other.

The function $\varphi_1(y)$ is defined for $0 < y \le y_1$, $\beta_1 > 2b_x$. It can be shown that $\bar{y} < y_1 < y_{0,1}$.

If $\beta_2 > 2b_x$ then the function $\varphi_2(y)$ is defined for $y_3 < y < y_{0,2}$. It follows from the assumptions that $y_3 < \bar{y}$; if $\beta_2 < 2b_x$ then $\varphi_2(y)$ is defined for $0 < y < y_{0,2}$.

Thus, the global maximal profit for the first competitor is given by

$$\Pi_1(y) = \begin{cases} \Pi_{x,1}(y), & 0 \le y < \bar{y}; \\ \Pi_{x,2}(y), & \bar{y} \le y \le y_{0,2}; \\ 0, & y > y_{0,2}; \end{cases}$$

(see fig.2 a), from which it follows that the reaction function is given by (10)

Proposition 2 *Let* $a_x < f(q);\ \beta_1 > 2b_x;\ \bar{y} < 0$ *and*

$$b_x < \frac{\beta_1 \beta_2}{\beta_1 + \beta_2}.$$

Then the reaction function of the first competitor is

$$x = \begin{cases} \varphi_2(y), & 0 \le y \le y_{0,2}, \\ 0, & y > y_{0,2}. \end{cases} \tag{11}$$

Proof. It is easy to see that if $\bar{y} < 0$ then the global maximal profit is

$$\Pi_1(y) = \begin{cases} \Pi_{x,2}(y), & 0 \le y \le y_{0,2}, \\ 0, & y > y_{0,2}, \end{cases}$$

and the reaction function of the first competitor in this case is given by (11).

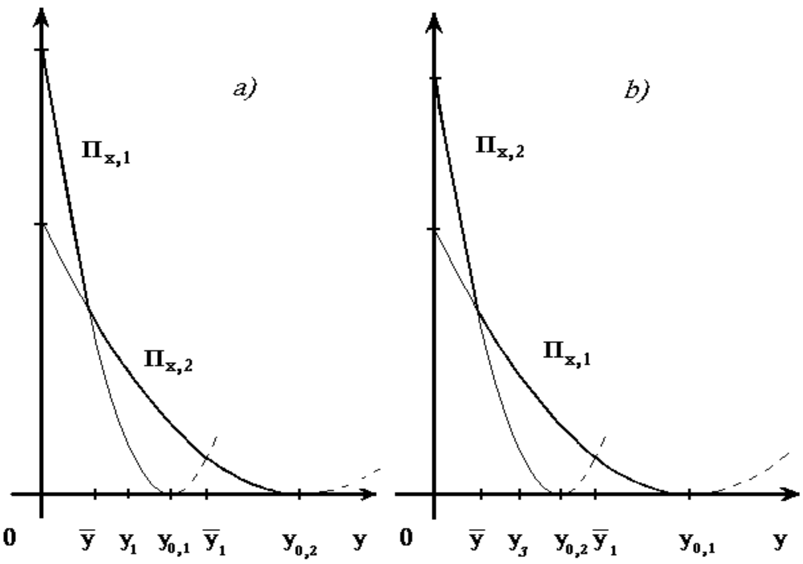

Figure 2: *The profit functions* $\Pi_{x,1}$ *and* $\Pi_{x,2}$ *corresponding to the branches* $\varphi_1(y)$ *and* $\varphi_2(y)$ *of the reaction function of the first competitor in the case of parameter assumptions of Proposition 1 (a) and Proposition 3 (b).*

Corollary 1 *If the assumptions* $a_x < f(q);\ \beta_1 > 2b_x$ *are maintained but the inequality (9) is not satisfied (which can occur only if* $\beta_2 < 2b_x$*) then* $\bar{y} > \bar{y}_1$*, i.e.* $\bar{y}$ *belongs to the interval where the profit function is equal to zero. Thus the reaction function of the first competitor is defined by (11).*

Proposition 3 *Let* $a_x > f(q);\ \beta_2 < 2b_x;\ \bar{y} > 0$ *and*

$$b_x > \frac{\beta_1\beta_2}{\beta_1+\beta_2}. \tag{12}$$

Then the reaction function of the first competitor is

$$x = \begin{cases} \varphi_2(y); & 0 \le y < \bar{y}; \\ \varphi_1(y); & \bar{y} \le y \le y_{0;1}; \\ 0; & y > y_{0;1}. \end{cases} \tag{13}$$

Proof. It follows from the assumptions on the parameters that $y_{0;1} > y_{0;2};\ \bar{y} < \bar{y}_1;\ \bar{y} < y_{0;2};\ y_{0;2} < \bar{y}_1 < y_{0;1}$; that is the point $\bar{y}_1$ belongs to the interval where only the profit function $\Pi_{x;1}(y)$ is defined. Thus only the point $\bar{y}$ is related to a jump from one function to the other.

The function $\varphi_2(y)$ is defined for $0 < y \le y_3;\ \beta_2 < 2b_x$. It can be shown that $\bar{y} < y_3 < y_{0;2}$.

If $\beta_1 < 2b_x$ then the function $\varphi_1(y)$ is defined for $y_1 \le y < y_{0;1}$. It follows from the assumptions that $y_1 < y_3;\ y_1 < y_{0;1}$ and $y_1 < \bar{y}$; if $\beta_1 > 2b_x$ then $\varphi_1(y)$ is defined for $0 < y < y_{0;1}$.

Thus, the global maximal profit of the first competitor is

$$\Pi_1(y) = \begin{cases} \Pi_{x;2}(y); & 0 \le y < \bar{y}; \\ \Pi_{x;1}(y); & \bar{y} \le y \le y_{0;1}; \\ 0; & y > y_{0;1}; \end{cases}$$

(see fig.2 b), and the reaction function is given by (13).

Proposition 4 *Let* $a_x > f(q);\ \beta_2 < 2b_x;\ \bar{y} < 0$ *and*

$$b_x > \frac{\beta_1\beta_2}{\beta_1+\beta_2}.$$

Then the reaction function of the first competitor is

$$x = \left\{ \begin{array}{ll} \varphi_1(y); & 0 \le y \le y_{0;1}; \\ 0; & y > y_{0;1}. \end{array} \right. \tag{14}$$

Proof. It can be easily shown that if $\bar{y} < 0$ then the global maximal profit is

$$\pi_1(y) = \begin{cases} \pi_{x,1}(y), & 0 \le y \le y_{0,1}, \\ 0, & y > y_{0,1}, \end{cases}$$

and the reaction function of the first competitor is as (14).

Corollary 2 *If the inequality (12) is not satisfied (which can occur only if* $\beta_1 > 2b_x$*) but the assumptions* $a_x > f(q), \beta_2 < 2b_x$ *hold, then* $\bar{y} > \bar{y}_1$*, i.e.* $\bar{y}$ *belongs to the interval where the profit function is zero. It can be shown that in this case the reaction function is defined by (14).*

Corollary 3 *If* $a_x = f(q)$ *then* $y_{0,1} = y_{0,2} = q$. *If the inequality (9) holds then the reaction function of the first competitor is given by (14), while if the inequality (12) is satisfied then the reaction function is given by (13).*

Using the same arguments for the second competitor one can get similar results for his reaction function putting x instead of y and vice versa, and $\psi_i(x)$ instead of $\varphi_i(y)$, $i = 1, 2$.

3 An Example of the Duopoly Model with Piecewise Linear Demand and Nonlinear Cost

In this section we shall propose a two-parameter family of models defined by the demand function (1) assuming quadratic cost functions, (2) and (3), in order to investigate the dynamics of the piecewise linear models which come out from the process described in the previous section. We shall consider an economic example which generalizes a model proposed by Palander (1936, 1939). The original Palander's example was already considered in Section 5 of the previous Chapter, both in the case of zero cost functions and in the case of a linear cost function. The model we propose here has a demand function given by

$$p = f(x+y) = \begin{cases} 150 - \beta_1(x+y), & 0 \le x+y \le q, \\ 64 - \beta_2(x+y), & q < x+y \le q_1, \\ 0, & x+y > q_1, \end{cases}$$

where

$$q = 86/(\beta_1 - \beta_2), \quad q_1 = 64/\beta_2,$$

so that our linear components of the demand function depend on the parameters β_1 and β_2 which model the slopes of the linear pieces. We shall let these parameters vary in the region denoted by $\mathcal{P}$:

$$\mathcal{P} = \{(\beta_1, \beta_2) : 0.043 < \beta_1 < 0.065,\ 0.0022 < \beta_2 < 0.0026\},$$

all giving examples close to the one proposed by Palander.

The cost functions are assumed of quadratic shape, and we shall keep them fixed as follows:

$$\begin{aligned} C_x(x) &= 53x - 0.00128x^2, \\ C_y(x) &= 53y - 0.00125y^2. \end{aligned}$$

In this way we have fixed the following parameters:

$$\begin{aligned} a_1 &= 150,\ a_2 = 64, \\ a_x &= 53,\ b_x = 0.00128, \\ a_y &= 53,\ b_y = 0.00125, \end{aligned}$$

and shall vary the parameters β_1 and β_2 in the region $\mathcal{P}$. We shall see that in spite of the narrow intervals in which these parameters are let to vary, the model reveals a very rich variety of dunamic behaviours. It is easy to verify that whichever are the values of β_1 and β_2 in the given ranges, the conditions given in (8) are fulfilled, and that the following also hold:

$$\begin{aligned} &\beta_1 > 2b_x,\ \beta_2 < 2b_x, \\ &\beta_1 > 2b_y,\ \beta_2 < 2b_y, \\ &a_x < f(q),\ a_y < f(q), \\ &b_x < \beta_1\beta_2/(\beta_1 + \beta_2), \\ &b_y < \beta_1\beta_2/(\beta_1 + \beta_2), \\ &\overline{y} > 0,\ \overline{x} > 0. \end{aligned} \qquad (15)$$

Thus, from Proposition 1 it follows that the reaction function of the first competitor is

$$x = \varphi(y) = \begin{cases} \varphi_1(y) = s_1 y + d_1, & 0 \le y < \overline{y}, \\ \varphi_2(y) = s_2 y + d_2, & \overline{y} \le y \le y_{0,2} = \frac{97}{2}, \\ 0, & y > y_{0,2} = \frac{97}{2}, \end{cases} \qquad (16)$$

where

$$s_1 = -\frac{\beta_1}{2(\beta_1 - 0.00128)}; \quad d_1 = \frac{97}{2(\beta_1 - 0.00128)};$$
$$s_2 = -\frac{\beta_2}{2(\beta_2 - 0.00128)}; \quad d_2 = \frac{11}{2(\beta_2 - 0.00128)};$$

$$\bar{y} = \frac{3\left(97\beta_2 - 11\beta_1\right)}{\sqrt{(\beta_2 - 0.00128)}\sqrt{(\beta_1 - 0.00128)} - 0.00128\,(\beta_1 - \beta_2)} + q;$$

while the reaction function of the second competitor is given by

$$y = \psi(x) = \begin{cases} \psi_1(x) = m_1 y + l_1; & 0 \le x < \bar{x}; \\ \psi_2(x) = m_2 y + l_2; & \bar{x} \le x \le x_{0;2} = \frac{11}{2}; \\ 0; & x > x_{0;2} = \frac{11}{2}. \end{cases} \quad (17)$$

where

$$m_1 = -\frac{\beta_1}{2(\beta_1 - 0.00125)}; \quad l_1 = \frac{97}{2(\beta_1 - 0.00125)};$$
$$m_2 = -\frac{\beta_2}{2(\beta_2 - 0.00125)}; \quad l_2 = \frac{11}{2(\beta_2 - 0.00125)};$$

$$\bar{x} = \frac{3\left(97\beta_2 - 11\beta_1\right)}{\sqrt{(\beta_2 - 0.00125)}\sqrt{(\beta_1 - 0.00125)} - 0.00125\,(\beta_1 - \beta_2)} + q.$$

4 Dynamic Behavior of the Model

As usual, a dynamic model is get by introducing a time lag in the variables x and y, that is

$$\begin{cases} x_{t+1} = \varphi(y_t) \\ y_{t+1} = \psi(x_t) \end{cases}$$

so that the time evolution of the model is described by the iterations of *the two-dimensional piecewise linear discontinuous map* $T : R^2_+ \to R^2_+$, $R^2_+ = [0, +\infty) \times [0, +\infty)$, of the form

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \varphi(y) \\ \psi(x) \end{pmatrix}, \qquad (18)$$

where the functions $\varphi(y)$ and $\psi(x)$ are given in (16) and (17) respectively.

The study of a two dimensional duopoly model has already been performed in many papers by several authors. Let us recall the paper by Bischi, Mammana and Gardini (2000) which puts in evidence some general results for the class of maps (18), also recalled in Chapter 3. In our case the dynamics of the model can be described by means of the properties of the piecewise linear one-dimensional discontinuous maps $F : R^+ \to R^+$ and $G : R^+ \to R^+$ given by

$$\begin{aligned} F &: x \mapsto F(x) = (\varphi \circ \psi)(x), \\ G &: y \mapsto G(y) = (\psi \circ \varphi)(y). \end{aligned}$$

and in order to understand the dynamics of the map T, we can consider only the map F from which the properties of the map G can be deduced (for example, if x_i is a periodic point of F then $y_i = \psi(x_i)$ is a periodic point of G).

The map F is given by the following linear functions:

$$x \mapsto F(x) = \begin{cases} (\varphi_2 \circ \psi_1)(x), & 0 \le x < \overline{x},\ \overline{y} \le \psi_1(x) \le y_{0,2}; \\ (\varphi_1 \circ \psi_1)(x), & 0 \le x < \overline{x},\ 0 \le \psi_1(x) < \overline{y}; \\ (\varphi_2 \circ \psi_2)(x), & \overline{x} \le x \le x_{0,2},\ \overline{y} \le \psi_2(x) \le y_{0,2}; \\ (\varphi_1 \circ \psi_2)(x), & \overline{x} \le x \le x_{0,2},\ 0 \le \psi_2(x) < \overline{y}; \\ 0, & 0 \le x < \overline{x},\ \psi_1(x) > y_{0,2}; \\ 0, & \overline{x} \le x \le x_{0,2},\ \psi_2(x) > y_{0,2}; \\ 0, & x > x_{0,2}. \end{cases}$$

It is a discontinuous map with break points $\overline{x}$, $x1 = \psi_1^{-1}(\overline{y})$, $x2 = \psi_2^{-1}(\overline{y})$, $x3 = \psi_1^{-1}(y_{0,2})$, $x4 = \psi_2^{-1}(y_{0,2})$ and $x_{0,2}$ (if the values $\psi_1^{-1}(\overline{y})$, $\psi_2^{-1}(\overline{y})$, $\psi_1^{-1}(y_{0,2})$ and $\psi_2^{-1}(y_{0,2})$ are defined), an example is shown in fig.3.

Depending on the parameters $(\beta_1, \beta_2) \in P$ the map F may be made up by a different numbers of linear pieces.

Let L_1 denote the curve in the $(\beta_1;\beta_2)-$ parameter plane of equation $\varphi_1(\overline{x}) = \overline{y}$ (see fig.4). It can be shown that if the parameter point $(\beta_1;\beta_2) \in P$ is above the curve L_1 (i.e. the condition $\varphi_1(\overline{x}) < \overline{y}$ is satisfied) then the function $F(x)$ consists of 5 linear pieces (see fig.3):

$$x \mapsto F(x) = \begin{cases} s_2m_1x + d_2 - s_2l_1; & 0 < x \le x1; \\ s_1m_1x + d_1 - s_1l_1; & x1 < x < \overline{x}; \\ s_2m_2x + d_2 - s_2l_2; & \overline{x} \le x \le x2; \\ s_1m_2x + d_1 - s_1l_2; & x2 < x \le x_{0;2}; \\ 0; & x > x_{0;2}. \end{cases}$$

The map F in this case has a fixed point $x_1^* = (d_1 - s_1l_1)/(1 - s_1m_1) \in (x1;\overline{x})$ which is attracting in the considered parameter range (being $0 < s_1m_1 < 1$).

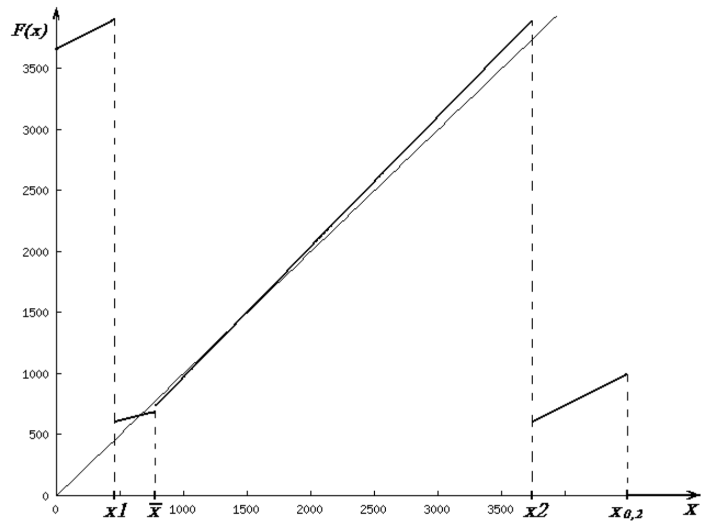

Figure 3: *An example of the map* F *given by* 5 *linear functions with break points* $x1; \overline{x}; x2$ *and* $x_{0;2}$ $(\beta_1 = 0{:}045;\ \beta_2 = 0{:}00245)$:

The map F has one more fixed point $x_2^* = (d_2 - s_2l_2)/(1 - s_2m_2) \in (\overline{x}; x2)$ if $F(\overline{x}) < \overline{x}$ and $F(x2) > x2$, or if $F(\overline{x}) > \overline{x}$ and $F(x2) < x2$: Let L_3 denote the curve in the $(\beta_1;\beta_2)-$ parameter plane given by the equation $F(\overline{x}) = \overline{x}$ and L_5 denote the curve of equation $F(x2) = x2$ (see fig.4): The fixed point x_2^* exists and is repelling if the parameter point $(\beta_1;\beta_2)$ is below

L_3 and is attracting if (β_1, β_2) is above L_5. The fixed point x_2^* does not exist if (β_1, β_2) is below L_5 and above L_3.

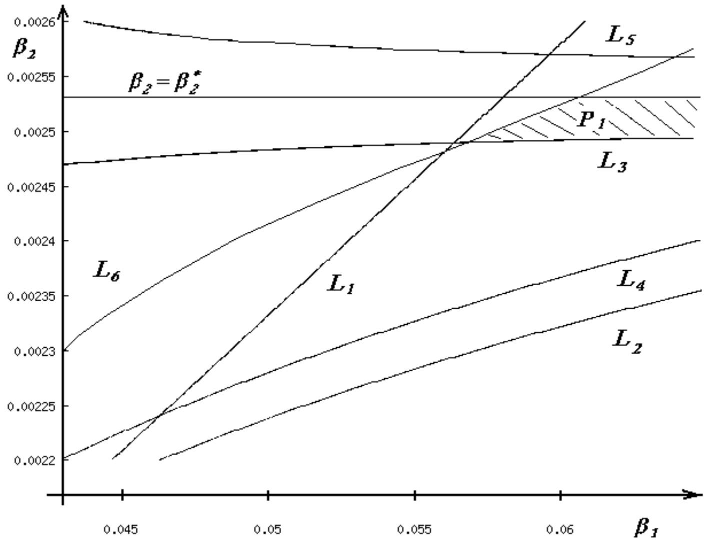

Figure 4: $(\beta_1, \beta_2)-$ *parameter plane where the curves* L_i, $i = 1, ..., 6$ *and the region* P_1 *are represented.*

Let L_2 denote the curve in the (β_1, β_2) parameter plane given by $\psi_2(\overline{x}) = y_{0,2}$. If the parameter point $(\beta_1, \beta_2) \in P$ is below the curve L_2 (i.e. $\psi_2(\overline{x}) > y_{0,2}$) then the map F is given by

$$x \mapsto F(x) = \begin{cases} s_2 m_1 x + d_2 - s_2 l, & 0 < x < \overline{x}; \\ 0, & \overline{x} \le x < x3; \\ s_2 m_2 x + d_2 - s_2 l_2, & x3 \le x \le x2; \\ s_1 m_2 x + d_1 - s_1 l_2, & x2 < x \le x_{0,2}; \\ 0, & x > x_{0,2}. \end{cases}$$

(see fig.5). We do not consider further this case in the present paper, however we note that it can raise intersting issues from the economical point of view because the presence of a zero branch for $\overline{x} \le x < x3$ can give rise to attracting cycles with points on the axes. This means that we may have zero quantity produced by one competitor and non-zero by the other in some periods and the opposite situation in other periods, i.e. shifting monopoly.

If the parameter point $(\beta_1, \beta_2) \in P$ is below the curve L_1 but above the curve L_2 (i.e. $\psi_1(\overline{x}) > \overline{y}$ and $\psi_2(\overline{x}) < y_{0,2}$) then the map F is given by

$$x \mapsto F(x) = \begin{cases} f_1(x) = s_2 m_1 x + d_2 - s_2 l; & 0 < x < \overline{x}; \\ f_2(x) = s_2 m_2 x + d_2 - s_2 l_2; & \overline{x} \leq x \leq x2; \\ f_3(x) = s_1 m_2 x + d_1 - s_1 l_2; & x2 < x \leq x_{0,2}; \\ 0; & x > x_{0,2}; \end{cases} \tag{19}$$

A trajectory of the map F will never visit the zero branch if the parameter point (β_1, β_2) is above the curve denoted by L_4 (see fig.4) of equation $f_2(x2) = x_{0,2}$ (i.e., if $f_2(x2) < x_{0,2}$ then the interval $(0, x_{0,2})$ is absorbing).

When the inequalities $f_2(x2) > x2$ and $f_2(\overline{x}) > \overline{x}$ are both satisfied (i.e. (β_1, β_2) is below the curve L_5 and above L_3) then the map F has no fixed points (an example is shown in fig.6).

It can be verified that the slopes $s_2 m_1$ and $s_1 m_2$ of the functions $f_1(x)$ and $f_3(x)$ are positive and less then 1. Thus the map F is a contraction on the intervals $(0, \overline{x})$ and $(x2, x_{0,2})$. F is expanding on the interval $(\overline{x}, x2)$ if $s_2 m_2 > 1$, which holds if

$$\beta_2 < \beta_2^* = \left((b_x + b_y) + \sqrt{(b_x + b_y)^2 - 3 b_x b_y}\right)/3.$$

In the family we are considering we have $\beta_2^* \approx 0.00253$, the line $\beta_2 = \beta_2^*$ is shown in fig.4.

In order to define a particular region, denoted by P_1 in fig.4, let us introduce in the parameter plane another curve, L_6, of equation $f_2(\overline{x}) = f_3(x2)$ (with $f_2(\overline{x}) < f_3(x2)$ below L_6). The three curves L_3, L_6 and $\beta_2 = \beta_2^*$ bound a region P_1 :

$$P_1 = \{(\beta_1, \beta_2) \in P : \overline{x} < f_2(\overline{x}) < f_3(x2),\ \beta_2 < \beta_2^*\}$$

such that for $(\beta_1, \beta_2) \in P_1$ the resulting map F has an invariant absorbing interval $I = I_1 \cup I_2 = [f_3(x2), x2] \cup [x2, f_2(x2)]$. That is, $F(I) = I$ (so that $F(x) \in I\ \forall x \in I$) and the trajectory of any point $x_0 \in [0, +\infty)$ is mapped into I in a finite number of iterations. A qualitative picture of the graph of $F(x)$ in this case is shown in fig.7.

Being F expanding in I_1 any trajectory must have at least one point in the interval I_2, and any point $x_0 \in I_2$ maps into the interval I_1, i.e. $x_1 = f_3(x_0) \in I_1$, then it is necessary to make some iterations by f_2 in order to get the first return in I_2 (i.e, again a point in the interval I_2). Let k be the least integer such that $f_2^k(x_1) \in I_2$, then this number is obtained taking the

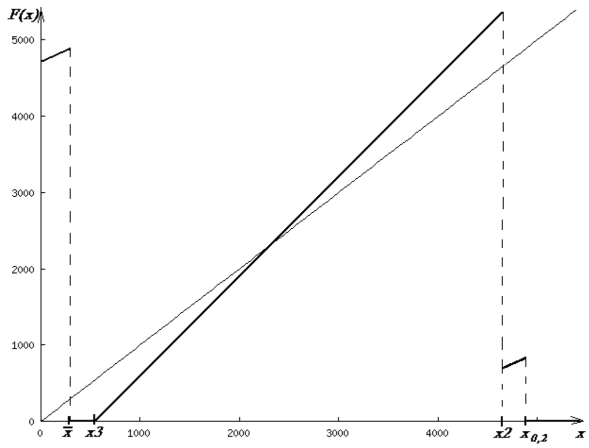

Figure 5: *An example of the function* $F(x)$ *with zero branches where* ¯$_1$ = 0:055; ¯$_2$ = 0:00225.

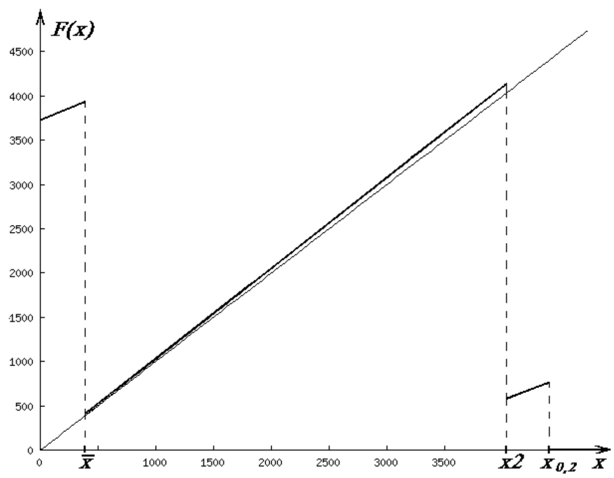

Figure 6: *An example of the function* $F(x)$ *given by* 4 *linear pieces with break points* $\overline{x}$; $x2$ *and* $x_{0;2}$ *where* ¯$_1$ = 0:06; ¯$_2$ = 0:0025.

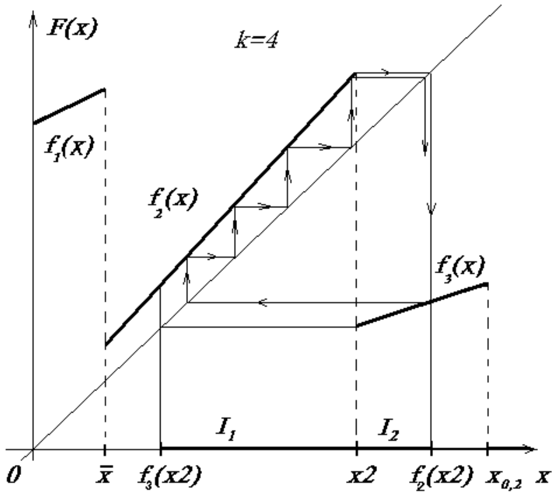

Figure 7: *Qualitative picture of the function* $F(x)$ *at* $(\beta_1; \beta_2) \in P_1$.

trajectory with the initial point $x_0^* = f_2(x2) \in I_2$. That is, k is get as the least integer for which $f_2^k(f_3(x_0^*)) \in I_2$, i.e. $x2 \leq f_2^k(f_3(x_0^*)) \leq f_2(x2)$ (see fig.7). We then prove the following

Proposition 5 *Let* $(\beta_1; \beta_2) \in P_1$. *If*

$$k > n^* = -\frac{\ln s_1 m_2}{\ln s_2 m_2},$$

then the map F *is chaotic in* $I = I_1 \cup I_2 = [f_3(x2); x2] \cup [x2; f_2(x2)]$; *and the map* T *is chaotic in the cartesian product* $I \times J$ *being* $J = \psi(I)$.

Proof. It was shown that the interval I is invariant for $(\beta_1; \beta_2) \in P_1$. In order to prove the proposition it is enough to show that when $k > n^*$ then the map F can have only repelling cycles in I. Let us consider a periodic trajectory, then it must have at least one point in I_2; let $x_0 \in I_2$ and $x_1 = f_3(x_0) \in I_1$; then let n_1 be the integer giving its first return in I_2, i.e. $f_2^{n_1}(x_1) \in I_2$. It follows that the eigenvalue of such a cycle has a factor $\lambda_1 = s_1 m_2 (s_2 m_2)^{n_1}$; and the eigenvalue of the cycle is given by the product of similar factors: $\lambda_1 \lambda_2 ... \lambda_r$ for some suitable integer $r \geq 1$ and

$\mu_i = s_1 m_2 (s_2 m_2)^{n_i}$ for $i = 1, ..., r$. It is enough to have $\mu_i > 1\ \forall i$ to state that the cycle is repelling, and the inequality $\mu_i > 1$ is satisfied if

$$n_i > n^* = -\frac{\ln s_1 m_2}{\ln s_2 m_2}.$$

As for any point of I_2 we have $n_i \geq k$ (by definition of k); when $k > n^*$ then $n_i > n^*\ \forall i$ from which it follows that any cycle of the map F has the eigenvalue higher than 1; i.e. F has only repelling cycles.

Thus, any trajectory with initial point $x_0 \in I$ is either asymptotically periodic to a repelling cycle, or aperiodic, and F is chaotic in I. A similar reasoning apply to the map G, and due to the relation between F and G we have that G is chaotic in $J = \psi(I)$ so that the two-dimensional map T is chaotic in the cartesian product $I \times J$. $\square$

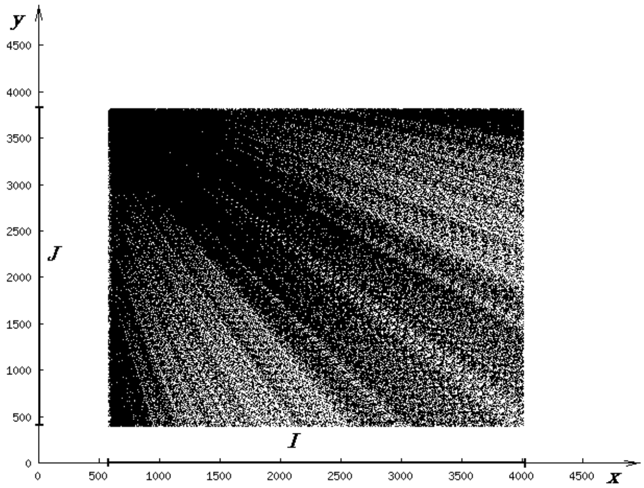

Figure 8: *An example of chaotic attractors of the map* T *at* $\beta_1 = 0.06$; $\beta_2 = 0.0025$.

As an example let us fix $\beta_1 = 0.06$; $\beta_2 = 0.0025$. For such parameter values $n^* \approx 27.6455$; $k = 78$; so that the assumptions of Proposition 5 are satisfied and the map T has a chaotic attractor, the rectangle $I \times J$ which is shown in fig.8.

The two-dimensional bifurcation diagram of the map F in the (β_1, β_2) - parameter plane is shown in fig.9 where the regions of regular and chaotic behavior are indicated. It is recalled in Chapter 3 that one of the characteristic properties of the class of maps (18) is the multistability, i.e. the coexistence of distinct attractors that may be stable cycles or attracting cyclic chaotic sets. In fig.9 the regions which correspond to coexisting attracting cycles

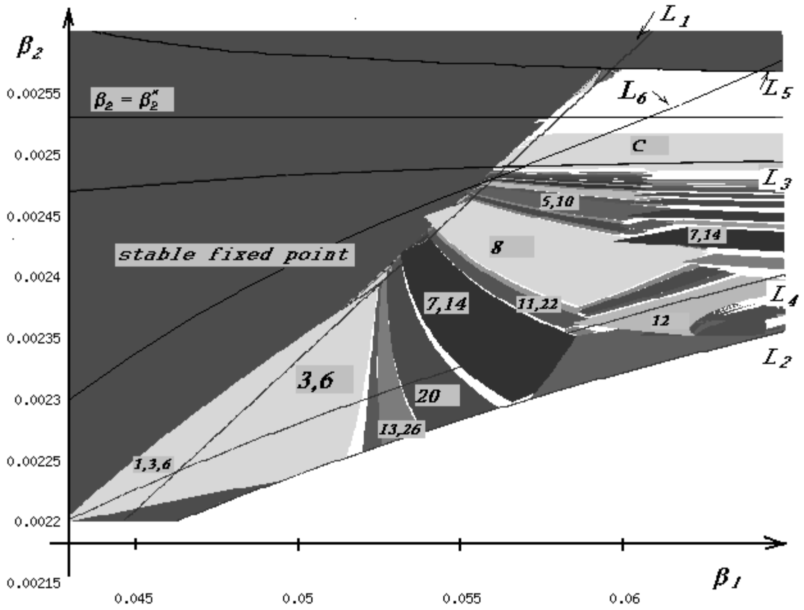

Figure 9: *Two-dimensional bifurcation diagram in the* (¯$_1$ ¯$_2$) *parameter plane for the map* F *where regions corresponding to attracting cycles of the map* T *are indicated by numbers*;*region which corresponds to chaotic behavior is indicated by* C:

are indicated by numbers (which denote the periods of the cycles of the two-dimensional map T). From the properties of the maps of type (18) we know that in order to get all the attractors of the map T it is enough to consider first all the attractors of the one-dimensional map F (which, on they turn, are determined by considering the trajectories with initial points equal to the two values associated with all the break points of F), and then, by using the results of Bischi, Mammana, Gardini (2000), we can get all the attractors of T: For instance, the region indicated by 7;14 corresponds to the parameter values such that F has an attracting cycle of period 7, so that the map T has *one* attracting cycle of period 7 and *three* coexisting attracting cycles of period 14. In fig.10 we show the basins of attraction in the phase plane (x, y) of the *four* distinct cycles of T for an example in that parameter region, with $\beta_1 = 0.055$ and $\beta_2 = 0.00234$.

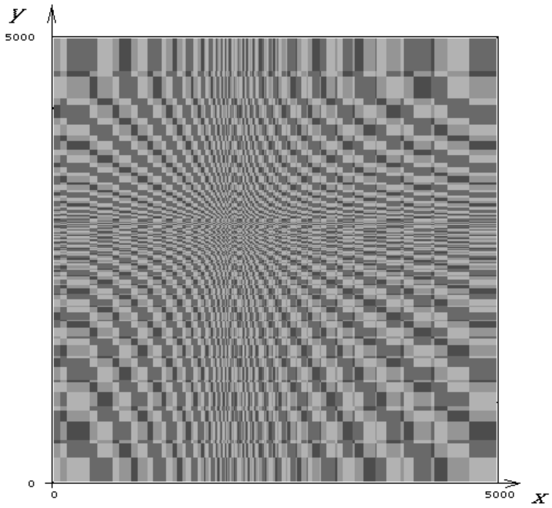

Figure 10: *An example of basins of attraction of one cycle of period* 7 *and three cycles of period* 14.

Also the structure of the basins is typical for this class of maps (18), as we know that the basin boundaries always belong to vertical and horizontal lines.

The region indicated by 20 corresponds to the parameter values such that the map F has an attracting cycle of period 10, so that T has *five* distinct attracting cycles of period 20.

It can be shown that the transition from one region to another is due to a border-collision bifurcation, occurring when some point of an attracting cycle of F coincides with a break point (see, for instance, Nusse and Yorke (1995) where such a bifurcation is descibed for piecewise smooth continuous one-dimensional maps).

If the parameter point (β_1, β_2) belongs to the region denoted by C then the map T has a chaotic attracting set. The region C was obtained by calculating the Lyapunov exponent which is positive for $(\beta_1, \beta_2) \in C$.

An example of a n-piece chaotic attractor is presented in fig.11 where $\beta_1 = 0.065;\ \beta_2 = 0.002488.$ Note that in this case $(\beta_1; \beta_2) \in C$ is below the curve L_3 (and thus out of the region called P_1).

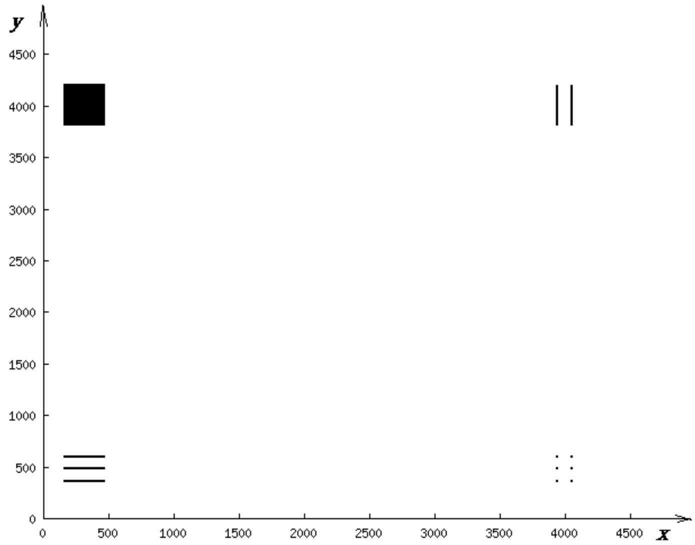

Figure 11: *An example of* n*-pieces chaotic attractor of the map* T *at* $\beta_1 = 0.065;\ \beta_2 = 0.002488.$

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