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Homoclinic tangles associated with closed invariant curves in families of 2D maps

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Abstract

In this paper we describe some sequences of global bifurcations of attracting and repelling closed invariant curves of two-dimensional maps that have a fixed point which may lose stability both via a supercritical Neimark bifurcation and a supercritical flip bifurcation. These bifurcations, characterized by the creation of heteroclinic and homoclinic connections or homoclinic tangles, are first described through qualitative phase diagrams and then by numerical examples.

1 Introduction

The local and global bifurcations associated with closed invariant curves have been one of the main interesting subjects of the last decades in the field of Dynamical Systems. Clearly the local bifurcations are mainly those related with the supercritical Neimark bifurcation of fixed points in two-dimensional maps, and the related structure of the bifurcation diagram, in a two-dimensional parameter plane, given by the so called "Arnold's tongues" associated with a rational rotation number p/q (see [11], [10], [16], [18], [6], [12]). Other properties of the Arnold tongues have been investigated by several authors, see e.g. [4], [7], [8], [5], [13], [14], [17], to cite a few. In particular, some global bifurcations associated with the appearance/disappearance of closed invariant curves, which may be related with homoclinic tangles of saddles, has been recently emphasized in [1], [2], [3]. In this paper we focus on some

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possible mechanisms, which are associated with saddle-focus cycles and global bifurcations related to homoclinic tangles which cause the appearance/disappearance or modifications of closed invariant curves, attracting or repelling.

It is well known that a saddle and an attracting cycle may give rise to a closed attracting curve, which is called an *heteroclinic connection*, moreover the attracting cycle may be a node or a focus (giving rise to a closed curve homeomorphic to a circle, or not homeomorphic, respectively). Similar closed invariant curves, but repelling, may be associated with a saddle and a repelling cycle, node or focus. In the first case the attracting closed curve is made up by the unstable set of the saddle, while in the second case the repelling closed curve is made up by the stable set of the saddle.

The bifurcation mechanisms described in this paper may be associated with a saddle-connection, also called *homoclinic loop,* defined as a closed invariant curve formed by the merging of a branch of the stable set of a periodic point of a saddle cycle with the unstable branch of another periodic point of the same saddle, thus forming a closed connection among the periodic points of the saddle. This is a structurally unstable situation, which causes a bifurcation between two qualitatively different dynamic behaviors. This kind of bifurcation cannot be predicted by a local investigation, so that it can be classified as a *global bifurcation*. In particular, the unstable branch of the saddle involved in the bifurcation exhibits different dynamic behaviors before and after the bifurcation, because it reaches two different attracting sets. A similar property holds for the stable branch of the saddle involved in the bifurcation: before and after the bifurcation the preimages of the local stable set come from different invariant repelling sets. Such homoclinic loops of saddle are known to occur in the resonant cases of the Neimark bifurcation (see [12], [9]), and recently they have been observed in some families of maps in relation with a subcritical Neimark bifurcation (see [2]). However, we shall see that its occurrence is quite common also far from the Neimark bifurcation, and is related with several bifurcations of closed invariant curves. As we shall see in the examples shown in this paper, when dealing with maps this structurally unstable situation is often replaced by the following sequence of bifurcations: first, a cyclical heteroclinic tangency (or homoclinic tangent bifurcation of non simple type), followed by a parameter range of transverse crossing between the stable and unstable set of the saddle cycle, that gives rise to cyclical heteroclinic points (or cyclical homoclinic connections), followed by a second cyclical heteroclinic or homoclinic tangency. In other words, the simple homoclinic loop of a saddle, quite common in continuous flows, is often replaced, in maps, by a range of parameters that give rise to an *homoclinic tangle* of the saddle (and related complex dynamics).

The paper is organized as follows. In Sec.2 we give a qualitative description of the bifurcations which lead to the appearance of closed invariant curves, attracting and repelling, and their further qualitative changes. In Sec.3 these sequences of bifurcations are shown through numerical explorations by using a family of symmetric maps.

2 Qualitative description of the bifurcations

Let us consider a two-dimensional map T with a fixed point O which can lose stability via a supercritical Neimark bifurcation, as well as via a supercritical flip bifurcation. In other words, we assume that the local stability analysis of O has a stability region in a two-dimensional parameter plane, which is bounded by a supercritical Neimark bifurcation curve, say $\mathbf{N}$, and a supercritical flip bifurcation curve, say $\mathbf{F}$.

If the parameters are outside the stability region and close to the flip bifurcation curve $\mathbf{F}$, then the fixed point O is a saddle and an attracting cycle of period 2 exists, say with points Q_1 and Q_2. Instead, if the parameters are beyond the Neimark bifurcation curve $\mathbf{N}$, then the fixed point O is a repelling focus and a closed invariant attracting curve exists around it. Thus, if we vary the parameters following a bifurcation path connecting the two regions described above, outside the stability region, the dynamic scenario must change from the former situation to the latter one, and some global bifurcation must occur leading to the creation of the closed invariant curve.

Before starting the qualitative description of some global bifurcations, let us remind that the dynamics of the restriction of a map to a closed invariant curve (attracting or repelling) is either quasiperiodic (i.e. the limit set of any trajectory on the closed invariant curve is the curve itself), or periodic (i.e. the curve is formed by a saddle-node or saddle-focus connection, as explained in Sec.1). However, generally the dynamics on the closed curve is periodic, but of very high period, so that it is numerically indistinguishable from a quasiperiodic one.

The appearance of a closed invariant curve far from a Neimark bifurcation is necessarily something which is related to some global bifurcation. Often a pair of closed invariant curves appear simultaneously, one attracting and one repelling, and this can be considered as a kind of "saddle-node" bifurcation for closed invariant curves (see e.g. [12]). However we remark that the true mechanism is probably related with some pair of cycles, born by saddle-node bifurcation, followed by a saddle-connection, i.e. an homoclinic connection. This is qualitatively illustrated in Fig.1 by using cycles of period 4. Fig.1a shows a pair of cycles, and the stable manifold of the saddle is not forming a closed invariant curve, however the unstable branch issuing from one periodic point of the saddle is approaching the stable branch

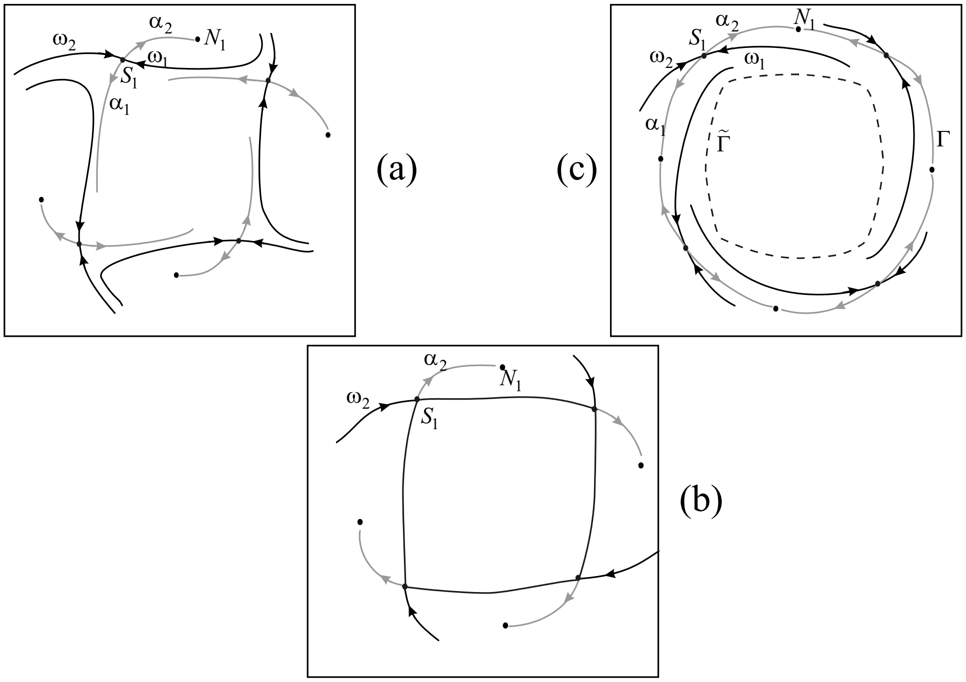

Figure 1: *Qualitative picture of the mechanism causing the appearance of two invariant closed curves, one attracting and one repelling*

issuing from another periodic point of the same saddle, and so on cyclically.

The bifurcation situation is shown in Fig.1b: Two branches merge cyclically and a closed invariant curve is formed connecting only the points of the saddle cycle. This structurally unstable situation gives rise to the stable one, shown in Fig.1c, in which two closed invariant curves exist: one repelling (denoted $\widetilde{\Gamma}$) and one attracting (denoted Γ). The attracting one is formed by the saddle-node connection, that is, it is made up by the unstable set of the saddle, and the closed invariant curve includes the two cycles (a saddle and a node). Clearly the qualitative description of the bifurcation shown in Fig.1 may work also with an attracting focus instead of an attracting node. This is shown in Fig.2. Fig.2a illustrates a pair of 4-cycles, one saddle and one attracting focus, which are not forming a closed connection. In the region inside we have an attractor, which is here considered a closed invariant curve. In fact, the sequence of bifurcations described in Figs.2-3 may be associated with

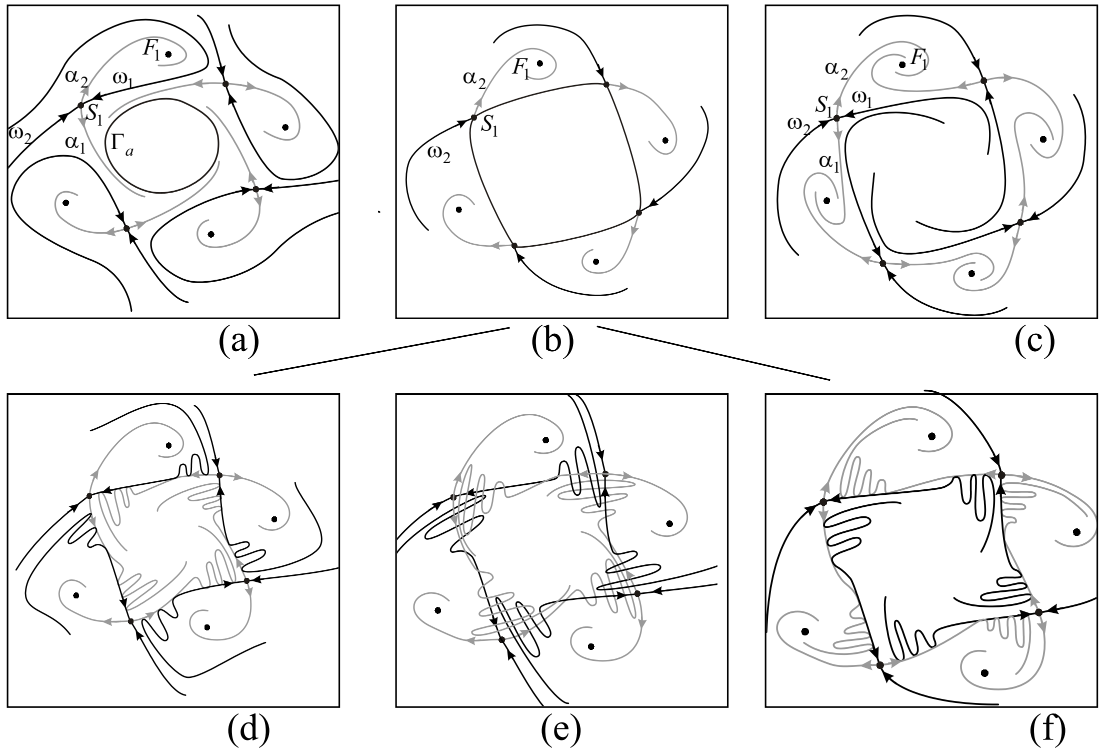

Figure 2: *Qualitative picture of the mechanism transforming an attracting closed curve Γ_a into a saddle-focus connection. The bifurcation is associated with a pair of cycles located "outside" the closed curve.*

the transition of a pair of cycles from "outside" a closed invariant curve (Fig.2a) to "inside" (Fig.3c). In Fig.2a the pair of cycles, saddle and attracting focus, exists outside an attracting closed curve, denoted Γ_a. One branch (denoted $\cup\alpha_{2,i}$) of the unstable set of the saddle tends to the stable focus cycle, while the other branch of the saddle (denoted $\cup\alpha_{1,i}$) goes towards the attracting closed curve. As the closed curve increases in size, it may merge with the unstable branches of the saddle and the stable ones. That is, at the bifurcation, the unstable branches $\cup\alpha_{1,i}$ and the stable ones $\cup\omega_{1,i}$, $i = 1, ...4$, may form a homoclinic loop, as shown in Fig.2b, thus creating a closed connection, while the invariant curve Γ_a no longer exists. After the bifurcation, a saddle-focus connection between the 4-cycles exists (Fig.2c), thus an attracting closed invariant curve still exists, given by the heteroclinic connection formed by the unstable set of the saddle winding around the attracting focus. The bifurcation situation described in Fig.2b, which is the one frequently observed in flows, is rarely observed in discrete models. In fact, in maps Fig.2b is more frequently substituted by a homoclinic tangle, as qualitatively described in Figs.2d,e,f. That is, a first tangency occurs between the unstable branches $\cup\alpha_{1,i}$ and the stable ones $\cup\omega_{1,i}$ (Fig.2d), followed by their transverse crossing (Fig.2e) and by a second tangency between the same manifolds $\cup\alpha_{1,i}$ and $\cup\omega_{1,i}$ (Fig.2f). Clearly, the homoclinic tangle is associated with complex dynamics, as we know from the homoclinic theorem for saddles (see e.g. [10], [16], [18], [6], [12]), in this situation a chaotic repellor exists, made up of infinitely many (countable) repelling cycles, and uncountable aperiodic trajectories.

The situation reached in Fig.2c is the starting point of a second bifurcation, as shown in Fig.3a. The outer branches of the saddle, $\cup\alpha_{2,i}$ and $\cup\omega_{2,i}$, approach each other and they ultimately merge, causing a bifurcation of the saddle-focus connection, as qualitatively shown in Fig.3b. After the bifurcation a closed invariant curve (denoted Γ_b) surrounds the pair of cycles, and the saddle-focus connection no longer exists (Fig.3c). As before, Fig.3b is more frequently substituted by an homoclinic tangle, as qualitatively shown in Figs.3d,e,f: A first tangency occurs between the unstable branches $\cup\alpha_{2,i}$ and the stable ones $\cup\omega_{2,i}$ (Fig.3d), followed by their transverse crossing (Fig.3e) and a second tangency between the same manifolds $\cup\alpha_{2,i}$ and $\cup\omega_{2,i}$ (Fig.3f) leads to the disappearance of all the homoclinic points. In the example described in the next section we shall see that the sequences of bifurcations described above may occur several times.

3 A family of symmetric maps

The family of two-dimensional maps that we consider is the one already introduced in [3], but at a different parameter regime. We consider

$$T : \begin{cases} x' = ax + y \\ y' = bx + cy + d \arctan y \end{cases} \tag{1}$$

with $a < 0$ and $d < 0$. The map (1) has a fixed point in the origin O whose local stability is obtained from the analysis of the eigenvalues of the Jacobian matrix evaluated in O :

$$DT(O) = \begin{bmatrix} a & 1 \\ b & c+d \end{bmatrix}. \tag{2}$$

Let $\mathcal{P}(z) = z^2 - Trz + Det$ be the characteristic polynomial, where Tr and Det denote the trace and the determinant of $DT(O)$, respectively. Then the stability region of O is determined by the following conditions: $\mathcal{P}(1) = 1 - Tr + Det > 0$,

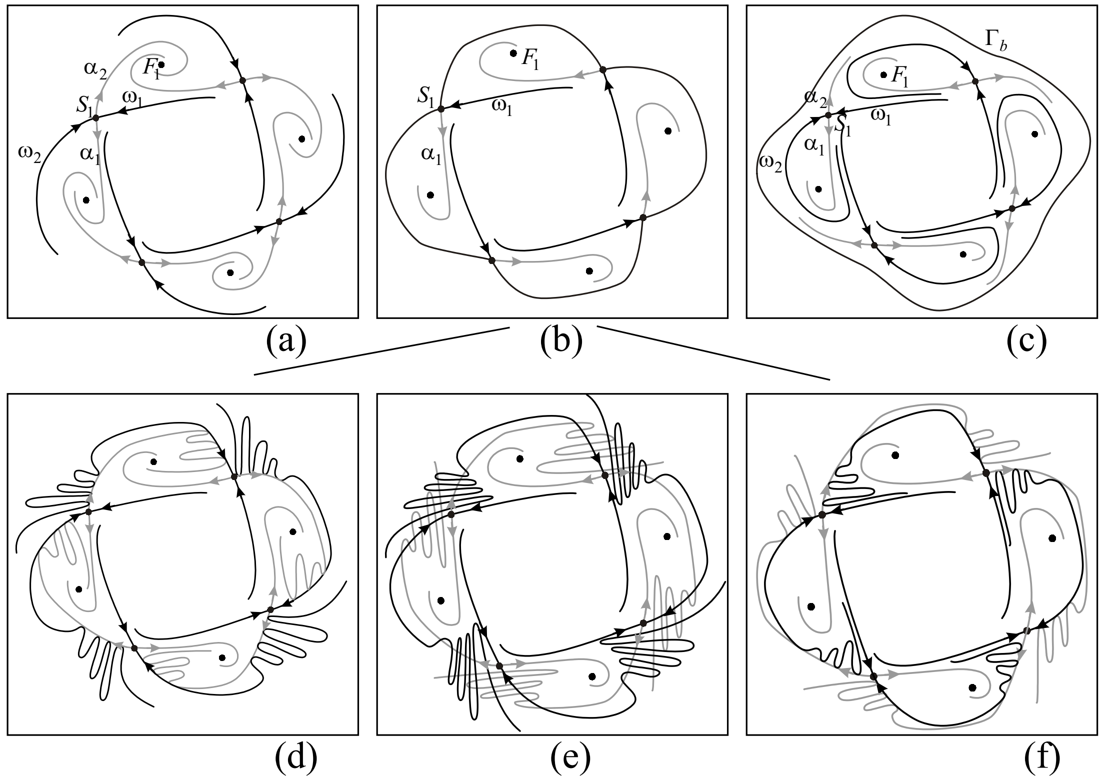

Figure 3: *Qualitative picture of the mechanism transforming a saddle-focus connection into an attracting closed curve* Γ_b*. At the end, the pair of cycle involved in the bifurcation is "inside" the closed curve*

$\mathcal{P}(-1) = 1 + Tr + Det > 0$, $Det < 1$ (see e.g. [11], p. 159, or [15], p.52], or any standard book on discrete dynamical systems). This region is represented by the grey-shaded triangle in Fig.4a. Simple computations show that the pitchfork bifurcation curve **P**, defined by $\mathcal{P}(1) = 0$, has equation $b = (1-a)(1-c-d)$, i.e. a straight line in the (c, b) parameter plane, assuming a and d as fixed values, the line **P** in Fig.4a. The flip bifurcation curve, defined by $\mathcal{P}(-1) = 0$, has equation $b = (1+a)(1+c+d)$, again a straight line in the (c, b) parameter plane, denoted by **F** in Fig.4a. The Neimark bifurcation curve, defined by $Det = 1$, corresponds to $b = a(c+d) - 1$, another straight line in the (c, b) parameter plane, denoted by **N**.

The map (1) is symmetric with respect to the origin, because $T(-x,-y) = -T(x,y)$. This implies that any invariant set of T is either symmetric with respect to O, or it admits a symmetric invariant set. In particular, this holds for the cycles of T. Thus any cycle of T of odd period necessarily coexists with a symmetric one having the same characteristics. For the same reason, all the basins of attraction are either symmetric with respect to O. We also notice that the map T can be

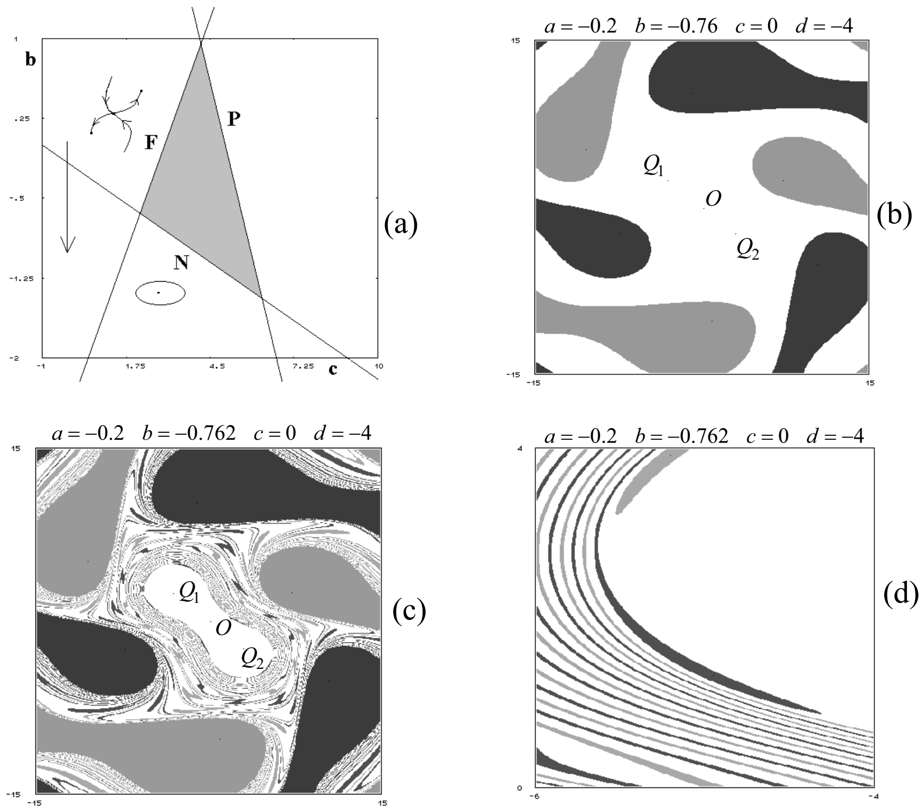

Figure 4: (a) *Stability region of the fixed point* O. (b) *After the flip bifurcation an attracting cycle of period* 2 *exists, as well as two attracting cycles of period* 3*, appeared via saddle-node bifurcation together with two saddle cycles. The basins of attraction of the attracting* 3*-cycles are represented in different grey tonalities.* (c) *At a lower value of* b*, two attracting cycles of period* 5 *appear via saddle-node bifurcation.* (d) *An enlargement of the previous figure shows that the basin boundaries of the two cycles of period* 5 *are winding very closely.*

invertible or noninvertible, according to the set of parameters considered. However, the parameter values that we shall use in the examples of the following section always belong to the region in which T is uniquely invertible.

3.1 Appearance of cycles and two closed invariant curves

In the following we consider some sequences of numerical simulations obtained with the fixed values $a = -0.2$, $c = 0$ and $d = -4$, while we gradually decrease the parameter b, along the bifurcation path in Fig.4a. Let us start with a value of the parameter b outside the stability region, not far from the flip bifurcation curve **F**, so that the fixed point O is a saddle, and the 2-cycle $Q_1 - Q_2$ is stable, being the only attractor at such a value of b.

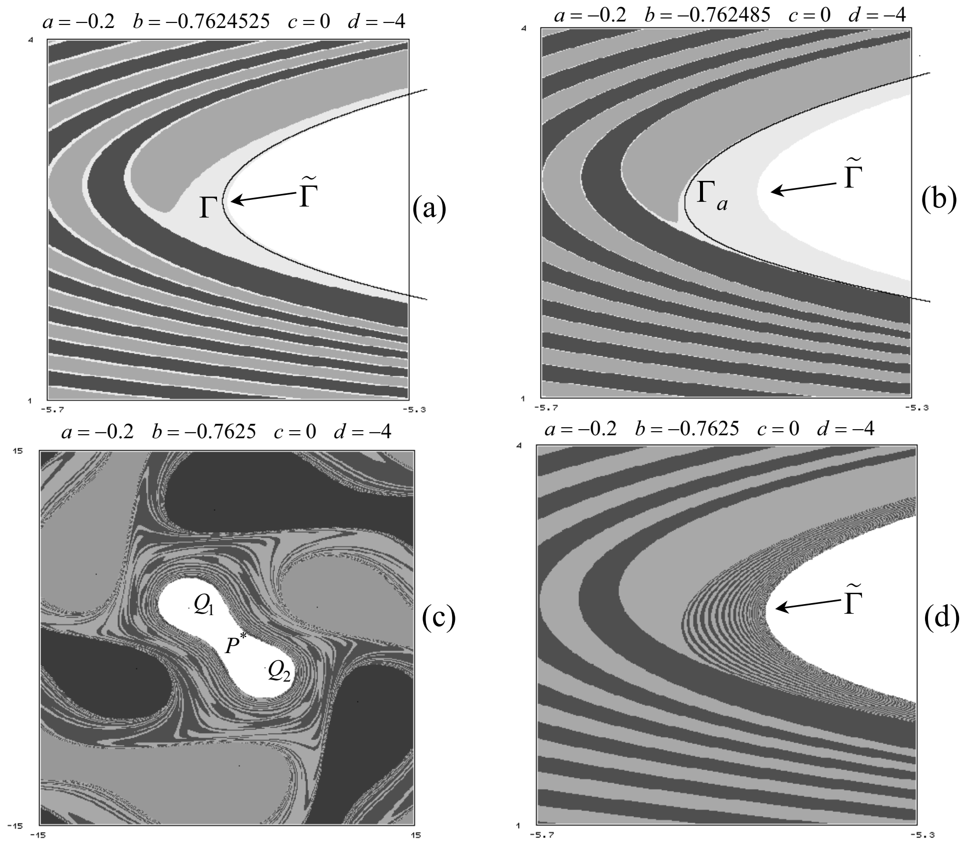

Figure 5: (a) *Two invariant closed curves have appeared: A repelling closed invariant curve* $\widetilde{\Gamma}$ *is bounding, on one side, the basin of attraction of the stable* 2*-cycle, while, on the other side, it is the boundary of the basin of a new attractor, a closed invariant curve* Γ. (b) *An enlargement at a different value of* b *shows that the basins of attraction of the two* 5*-cycles are approaching each other and the attracting closed curve, now denoted by* Γ_a. (c) *After the global bifurcation qualitatively described in Fig.*2*, the curve* Γ_a *disappears.* (d) *An enlargement of the previous figure.*

As the value of b is decreased, a pair of cycles of period 3 appears by saddle-node bifurcation, and being of odd period also the symmetric ones exists. The stable node 3-cycles then turn into foci, and at $b = -0.76$ the basins of the three attractors in the phase space are shown in Fig.4b. As the value of b is further decreased, another pair of cycles of period 5 appears via saddle-node bifurcation, and being of odd period also the symmetric ones exists. The stable node 5-cycles then turn into foci, and at $b = -0.762$ the basins of the five coexisting attractors in the phase space are shown in Fig.4c. The enlargement in Fig.4d shows that the basin boundaries of the 5-cycles are winding very closely. A bifurcation is going to occur. In fact, at a lower value of b we can see, in the enlargement in Fig.5a, that a pair of closed invariant curves have appeared: A repelling closed invariant curve $\widetilde{\Gamma}$ is bounding, on one side, the basin of attraction of the stable 2-cycle, while, on the other side, it is the boundary of the basin of a new attractor, a closed invariant curve Γ. Thus, in this situation there are several coexisting attractors: a 2-cycle, a pair of 3-cycles, a pair of 5-cycles and Γ.

3.2 Transitions between two closed invariant curves

The enlargement in Fig.5b shows that the basins of the 5-cycles are approaching each other, as well as the closed invariant curve, now denoted Γ_a. The 5-cycles existing outside this closed curve Γ_a in Fig.5b are going to move inside, as shown in Fig.6c,d, following the sequence of bifurcations described in Figs.2,3. The first bifurcation, due to the merging of the inner branches of the saddles of period 5, causes the disappearance of the closed curve Γ_a, leaving a new closed curve connecting the 5-cycles. That is, the two attracting cycles of period 5 together with the two saddles of period 5, form an attracting closed invariant curve: a saddle-focus connection (Fig.5c and its enlargement in Fig.5d). As b is further decreased the second bifurcation (qualitatively shown in Fig.3) occurs. In the enlargement of Fig.6a we can see that the outer branches of the 5-saddles are approaching each other and Figs.6b,c, after the bifurcation, show that a closed attracting curve Γ_b exists surrounding the 5-cycles and the saddle-focus connection no longer exists.

It is quite difficult to see, in the above sequence of bifurcations, whether the bifurcations in Fig.2b and Fig.3b occur as saddle-connections or with the homoclinic tangles. However, as b is further decreased, another sequence of bifurcations similar to the previous one occurs, causing the transitions of the cycles of period 3 from outside a closed curve Γ_a to inside a closed curve Γ_b, and now the homoclinic tangles are clearly visible. This example is shown in Fig.7. The closed attracting curve Γ_a approaches the saddle cycles of period 3, as shown in Fig.7a, and the first bifurcation that we have qualitatively described in Fig.2 is going to occur. The merging of Γ_a

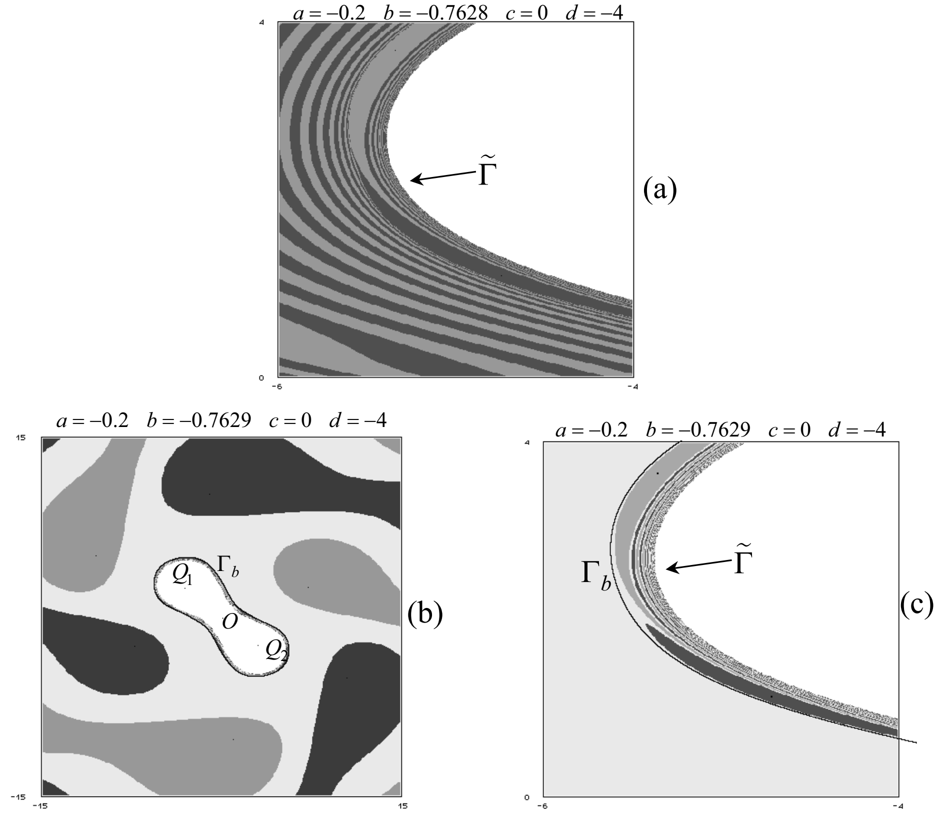

Figure 6: (a) *An enlargement of the basins of the two cycles of period* 5 *shows that a bifurcation is coming.* (b) *After the bifurcation qualitatively described in Fig.*3*, the saddle-focus connection no longer exists and an attracting closed curve* Γ_b *exists.* (c) *An enlargement of the previous figure shows that the cycle of period* 5 *are "inside"* Γ_b.

with the inner branches of the saddle 3-cycles, via an homoclinic tangle, causes the disappearance of the attracting set Γ_a, leaving a saddle-focus connection between the 3-cycles. This is the situation in Fig.7d, and the homoclinic tangle occurring during the bifurcation is shown in Fig.7b and its enlargement in Fig.7c.

On further decreasing the parameter b the outer branches of the 3-cycles saddle approach each other, as shown in Fig.7e, and the second bifurcation qualitatively described in Fig.3 (via homoclinic tangle) is going to occur, which will cause the disappearance of the saddle-focus connection. The situation after the bifurcation is shown in Fig.7f: A wider closed attracting curve exists, Γ_b, surrounding the cycles of period 3, and a pair of cycles of period 10 (a saddle and a stable focus) already exists outside, and as b decreases the bifurcation mechanisms start again.

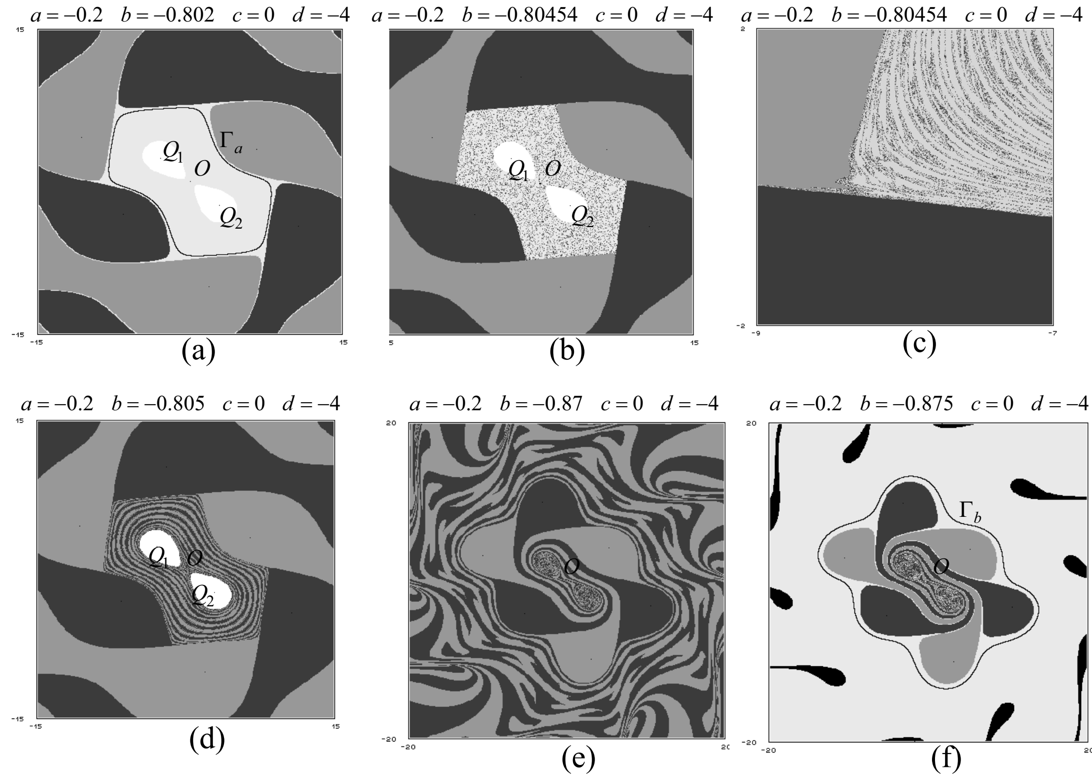

Figure 7: (a) *The attracting closed curve* Γ_a *approaches the saddle cycles of period* 3. (b) *The intermingled portion of the basin denotes the existence of a chaotic repellor, due to a homoclinic tangle.* (c) *An enlargement of the previous figure shows the oscillations of the stable set of the saddle cycles.* (d) *At the end of the homoclinic tangle the closed curve* Γ_a *disappears, leaving a saddle-focus connection between the cycles of period* 3. (e) *The outer branches of the stable sets of the saddle cycles approach each other and a new bifurcation is going to occur.* (f) *After the bifurcation an attracting closed curve* Γ_b *surrounds the cycles of period* 3. *A pair of cycles of period* 10 *(saddle and node) already exists outside and as* b *decreases the bifurcation mechanisms starts again.*

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