

# A stability analysis of the perfect foresight map in nonlinear models of monetary dynamics

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## Abstract

We undertake an analysis of the dynamic behaviour of a discrete time nonlinear monetary dynamics model under perfect foresight expectations. The model derives its interest from the fact that it is a basic mechanism in a broad class of descriptive macrodynamic models. Our analysis makes transparent the multivalued nature of the nonlinear perfect foresight map governing price dynamics. This results in a number of possible discontinuous maps, all of whose dynamics we analyze in response to an unanticipated monetary shock.

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## 1. Introduction

An analysis of descriptive dynamic economic models under the assumption of perfect foresight (rational expectations in a stochastic setting) has been a feature of the economics literature for three decades. A common feature of many such models is a saddle point structure which posed to early researchers the conundrum of how the economy could end up on a non-divergent path, given that most initial conditions would otherwise place it on a divergent path. This conundrum was resolved by the widespread adoption of the jump-variable technique, which relied upon the presumed full-knowledge by agents of their economic environment. It was assumed that, armed with this knowledge and realizing that the given initial values placed them on a divergent path, the agents would calculate the required change in initial values that would place the economy on the stable branch of the saddle point, from where it would move towards the equilibrium point. If there were some unanticipated change in some underlying economic parameter (e.g. change in the money supply) that moved the economy to a new equilibrium point, then the agents would calculate the new jump required in order to arrive on the stable branch of the new saddle point. Despite some misgivings about the conceptual basis of this technique, see e.g. [1,2], it became standard practice in descriptive dynamic economic modelling. A good recent exposition of the jump-variable technique is contained in [3].

The jump-variable technique originally arose in the context of the monetary growth dynamics model studied by Sargent and Wallace [4] and further elaborated by Burmeister in [5]. The antecedent of this model is probably the work of Cagan [6]. Whilst now-a-days the jump-variable technique is more likely to be encountered in models of exchange rate dynamics (see e.g. [7]), it still warrants study in the context of the models of monetary growth as there the saddle point structure implied by the perfect foresight assumption occurs in the lowest possible number of dimensions. This

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low dimensionality facilitates a study of the type of maps that arise in nonlinear perfect foresight models, which is the aim of this paper. Conceptually one would expect that the economic behaviour we discover for the nonlinear monetary dynamics model would carry over to nonlinear versions of other models, such as models of exchange rate dynamics. However we must leave analysis of such models for future research for which the analytical framework of this paper provides a conceptual basis.

In order to avoid the somewhat arbitrary nature of the initial jump imposed on prices by the jump-variable technique, Chiarella [8] proposed a nonlinear money demand function based on simple portfolio considerations. In this setting when expected inflation becomes large positive (negative) the agents move towards holding all wealth in real physical goods (money). If such a nonlinear money demand function is combined with sluggish price adjustment then, in a continuous time setting, the economy is stabilized, with prices tending either to the steady-state point or to a relaxation cycle. The nonlinear monetary dynamics model with perfect foresight being viewed as the limit of adaptive expectations was analyzed fairly extensively in a continuous time setting in [9]. The analysis was taken further by Flaschel and Sethi [10] who proposed a mechanism for overcoming an indeterminacy problem that can arise under certain parameter constellations.

Since many economic applications involve use of a discrete time framework it is important to also fully understand perfect foresight dynamics in this framework. In [9,11], Chiarella gave a very preliminary and incomplete analysis of the adaptive expectations and perfect foresight mechanisms in discrete time. The discrete time framework poses a number of problems compared to the continuous time one. Firstly in the discrete time framework it is not possible to proceed via a continuous limiting process from adaptive expectations to perfect foresight, a procedure that helped considerably to clarify the perfect foresight dynamics in continuous time. Secondly, in discrete time under perfect foresight the map relating price at one time to price at the previous time may become multivalued. Economic agents thus face a number of choices when calculating their expectation of the rate of inflation over the next period, any one of which results in a discontinuous map for the price dynamics.

The aim of this paper is to undertake a complete analysis of the nonlinear monetary dynamics model under perfect foresight expectations in discrete time. We discuss all of the discontinuous maps driving the price dynamics when the aforementioned multi-valuedness occurs, and the different price dynamics that they imply. We also offer some rationale for a choice between these various maps. In a companion paper [12], we analyze the corresponding discrete time adaptive expectations mechanism. Whilst that framework does not pose the problem of multi-valuedness and discontinuous maps, its main technical difficulty lies in the fact that the maps governing the price dynamics are multi-dimensional.

A better understanding of the dynamics of perfect foresight maps is relevant to the recent efforts of Chiarella and Flaschel in [13] and Asada et al. in [14] to develop integrated disequilibrium macrodynamic models, in both open and closed economics. The expectations processes contained in the models of these works have as their limiting behaviour the perfect foresight maps studied in this paper. As such their properties may be one source of the rich dynamic behaviour that the integrated disequilibrium macrodynamic models exhibit.

The structure of the paper is as follows. In Section 2 we review the model of monetary dynamics and set it up in discrete time under perfect foresight expectations. We analyze the perfect foresight map that governs the price dynamics. In Section 3 we discuss briefly a generalization of our analysis to the case of partial perfect foresight, which may be considered as one way to capture bounded rationality. Section 4 concludes and makes suggestions for future research.

## 2. The nonlinear monetary dynamics model under perfect foresight

The continuous time version of the nonlinear model of monetary dynamics consists of the price adjustment equation

$$\dot{p} = \alpha[m - p - f(\pi)], \quad (1)$$

where  $p$  is the logarithm of the price level,  $m$  is the logarithm of the money supply (here assumed constant) and  $\pi$  is the expected rate of inflation i.e.

$$\pi(t) = E_t[\dot{p}(t)]. \quad (2)$$

According to Eq. (1), the rate of change of price in the goods market depends on the excess demand for real balances. The function  $f$  is the logarithm of the demand for real money balances, which on the basis of portfolio considerations explained in [9], is assumed to have the nonlinear form shown in Fig. 1.

If expectations are formed adaptively they would evolve according to

$$\tau \dot{\pi} = \dot{p} - \pi \quad (\tau \geq 0). \quad (3)$$

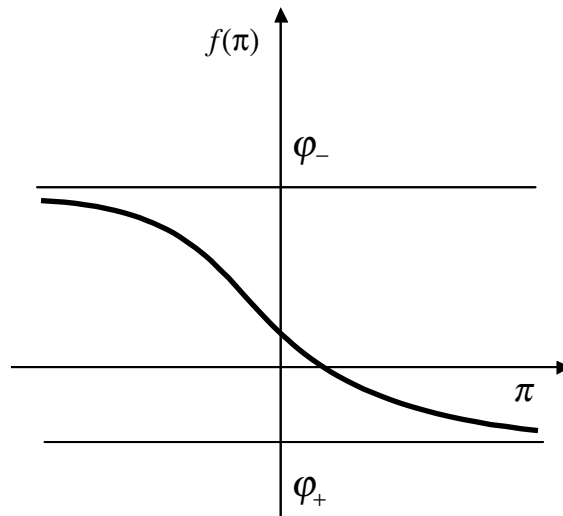


Fig. 1. Nonlinear money demand function.

In the special case  $\tau = 0$ , Eq. (3) reduces to myopic perfect foresight, according to which

$$\pi = \dot{p}. \tag{4}$$

Chiarella investigated the continuous time system (1) and (3), showing in particular that the cases  $\tau = 0$  and  $\tau = 0^+$  yield qualitatively similar dynamics. An advantage of viewing perfect foresight as the limiting case of adaptive expectations in this way is that it makes clear how the perfect foresight dynamics become a relaxation cycle in the phase plane of  $(\pi, p)$ . An analysis starting from the lower dimensional perfect foresight model does not easily allow one to see this feature. Chiarella also gave a brief analysis of a particular discrete time version of the perfect foresight case i.e. (1) and (4). However a complete analysis of the discrete versions of (1) and (3) has never been undertaken, at least to our knowledge. Such an analysis is the aim of this work.

In a certain sense the discrete time analysis of the nonlinear monetary dynamics model (1) and (4) is richer than the continuous time analysis because a number of discrete time maps may be obtained depending on what we assume about the information set of agents. In the analysis below we consider a family of maps that can arise in the discrete time setting.

The discrete time version of the price evolution equation (1) is given by

$$p_{t+1} = \alpha m + (1 - \alpha)p_t - \alpha f(\pi_{t,t+1}), \tag{5}$$

where now  $p_t$  denotes the logarithm of the price level at time  $t$ . In Eq. (5) the money demand function at time  $t$  depends on the rate of inflation expected for the next period. More precisely, let  $I_{t+1} = p_{t+1} - p_t$  be the inflation rate at time  $t + 1$ . At time  $t$  the agents do not know the exact value of  $I_{t+1}$  so that they have to form some expected value of it, which we shall denote  $\pi_{t,t+1}$ , in order to emphasize that it is formed at time  $t$  for the next time period  $t + 1$ . Thus

$$\pi_{t,t+1} = I_{t+1}^{(e)} = E_t(p_{t+1} - p_t). \tag{6}$$

To close the model given in (5) we shall here consider the *perfect foresight* case, in which agents are assumed to be able to forecast at time  $t$  the exact value of the future inflation rate  $I_{t+1} = (p_{t+1} - p_t)$ , i.e.

$$\pi_{t,t+1} = I_{t+1}. \tag{7}$$

The perfect foresight hypothesis leads to a one-dimensional map in *implicit* form for  $p_{t+1}$ , namely

$$p_{t+1} = \alpha m + (1 - \alpha)p_t - \alpha f(p_{t+1} - p_t), \tag{8}$$

obtained by substituting (7) in (5).

The *perfect foresight map* (8) will be studied in this section. Moreover we will show that the model (8) can be also obtained by assuming that agents calculate  $\pi_{t,t+1}$  from their knowledge of the mechanism of price formation. They are assumed to know Eq. (5) from which they form

$$E_t(p_{t+1} - p_t) = \alpha m - \alpha p_t - \alpha f(\pi_{t,t+1}),$$

i.e. they calculate  $\pi_{t,t+1}$  according to

$$\pi_{t,t+1} = \alpha m - \alpha p_t - \alpha f(\pi_{t,t+1}). \quad (9)$$

Solving (9) for  $\pi_{t,t+1}$  as a function of  $p_t$  the agents can then determine the price evolution equation by substituting (9) into (5). The full details of this procedure are discussed in Section 2.1.

In passing we note that the framework of this section could be extended to allow a consideration of a type of bounded rationality. In particular it could be assumed that the expectations are given by a weighted average of the perfect foresight prediction and the previous expectation, i.e.

$$\pi_{t,t+1} = \omega I_{t+1} + (1 - \omega)\pi_{t-1,t} \quad (10)$$

$$= \omega(p_{t+1} - p_t) + (1 - \omega)\pi_{t-1,t}, \quad (11)$$

where  $\omega$  is a positive constant,  $0 \leq \omega \leq 1$ . This scheme could be interpreted as assuming that the agents are not fully confident about their knowledge of the model, as they are in the perfect foresight case. Thus  $\omega$  could be interpreted as the degree of uncertainty in the perfect predictor, so that  $\omega = 1$  means they are fully certain (giving the perfect foresight model), whilst  $\omega = 0$  indicates a complete lack of confidence, in which case they remain with the previous prediction.

Eq. (10) in association with the price evolution (5) leads to the dynamic system for price and expectations given by

$$\begin{cases} p_{t+1} = p_t + \alpha(m - p_t - f(\pi_{t,t+1})), \\ \pi_{t,t+1} = \omega(p_{t+1} - p_t) + (1 - \omega)\pi_{t-1,t}, \end{cases} \quad (12)$$

which is a two-dimensional map in *implicit* form for  $p_{t+1}$  and  $\pi_{t,t+1}$ . We shall refer to (12) as the *partial perfect foresight model*. This map will be discussed further in Section 3, though its full analysis is left for future research.

### 2.1. The perfect foresight map

If we assume the perfect foresight hypothesis then, as we have seen, we obtain the implicit model given in (8) for the price dynamics, which we rewrite for convenience, as

$$p_{t+1} = \alpha m + (1 - \alpha)p_t - \alpha f(p_{t+1} - p_t). \quad (13)$$

Rearranging (13) in order to try to make explicit the variable  $p_{t+1}$  we obtain

$$(p_{t+1} - p_t) + \alpha f(p_{t+1} - p_t) = \alpha(m - p_t)$$

and defining the function

$$V(x) = x + \alpha f(x) \quad (14)$$

we can write

$$p_{t+1} = p_t + V^{-1}[\alpha(m - p_t)] \equiv F(p_t). \quad (15)$$

In (15)  $V^{-1}$  denotes the inverse function of  $V$  when  $V$  is invertible, or a suitable function defined by using one of the inverses of  $V$  when  $V$  is not uniquely invertible.

It is also possible to arrive at the map (15) by assuming that agents calculate  $\pi_{t,t+1}$  by using their knowledge of the price formation mechanism, i.e. according to Eq. (9), namely

$$\pi_{t,t+1} = \alpha m - \alpha p_t - \alpha f(\pi_{t,t+1}),$$

from which we obtain

$$\pi_{t,t+1} + \alpha f(\pi_{t,t+1}) = \alpha m - \alpha p_t,$$

which may be written

$$V(\pi_{t,t+1}) = \alpha m - \alpha p_t, \quad (16)$$

where  $V$  is the function defined in (14). Making explicit  $\pi_{t,t+1}$  in (16) and substituting that expression in the law of price dynamics (5), we obtain the one-dimensional map

$$p_{t+1} = \alpha m + (1 - \alpha)p_t - \alpha f(V^{-1}(\alpha m - \alpha p_t)). \quad (17)$$

The model (17) is equivalent to the one (15). In fact, adding and subtracting to the left side of (17) the quantity  $V^{-1}(\alpha m - \alpha p_t)$  we have

$$\begin{aligned} p_{t+1} &= \alpha m + (1 - \alpha)p_t + V^{-1}(\alpha m - \alpha p_t) - (V^{-1}(\alpha m - \alpha p_t) + \alpha f(V^{-1}(\alpha m - \alpha p_t))) \\ &= \alpha m + (1 - \alpha)p_t + V^{-1}(\alpha m - \alpha p_t) - V(V^{-1}(\alpha m - \alpha p_t)) = p_t + V^{-1}(\alpha m - \alpha p_t), \end{aligned}$$

which is exactly the map  $F$  of Eq. (15).

We shall refer to the one-dimensional map  $F$  in (15) as the *perfect foresight map*. In this section we study the dynamical properties of the one-dimensional map (15), which depends on the shape of the function  $F$ , which in turn depends on the invertibility of the function  $V$ . Some comments on the related dynamics shall also be given.

### 2.2. Graph of $V$ and invertibility condition

It is clear from (15) that the properties of the map  $F$  will be very much determined by the properties of the function  $V$ , and in particular on its inverse. So we turn first to a study of this issue.

In order to study the graph of the function  $V$  defined in (14), let us denote by  $\varphi_+$  and  $\varphi_-$  the horizontal asymptotes of the function  $f$ , whose qualitative graph is shown in Fig. 1, i.e.

$$\varphi_+ = \lim_{\pi \rightarrow +\infty} f(\pi); \quad \varphi_- = \lim_{\pi \rightarrow -\infty} f(\pi), \tag{18}$$

where we have  $\varphi_+ < \varphi_-$ , and let  $a$  be the absolute value of the minimum value of the first derivative of  $f$ , i.e.

$$\min_{\pi} f'(\pi) = -a. \tag{19}$$

Regarding the function  $V$ , it is simple to see that the two parallel straight lines

$$\begin{aligned} y &= \pi + \alpha\varphi_- \quad \text{as } \pi \rightarrow -\infty, \\ y &= \pi + \alpha\varphi_+ \quad \text{as } \pi \rightarrow +\infty \end{aligned}$$

are its asymptotes.

The first derivative of  $V$  is  $1 + \alpha f'(\pi)$ , so that when

$$\alpha \leq \frac{1}{a} \tag{20}$$

the function  $V$  is an increasing function, and thus uniquely invertible. Its inverse  $V^{-1}$  is an increasing function with asymptotes

$$\begin{aligned} y &= \pi - \alpha\varphi_- \quad \text{as } \pi \rightarrow -\infty, \\ y &= \pi - \alpha\varphi_+ \quad \text{as } \pi \rightarrow +\infty. \end{aligned}$$

A qualitative graph of  $V$  and its inverse when condition (20) is satisfied is shown in Fig. 2.

If condition (20) does not hold, the function  $V$  is no longer uniquely invertible and we have to decide how to choose the branches of the inverses in order to have a well defined function. The general form of  $V$  in this case is shown in Fig. 3, in which we have a maximum  $V_M$  at  $\pi_2 (V_M = V(\pi_2))$  and a minimum  $v_m$  at  $\pi_1 (v_m = V(\pi_1))$ . In the same figure we have also indicated (to be used below) the point  $\pi_2^e$  which is the other preimage of  $V(\pi_2)$  and  $\pi_1^e$ , the other preimage of  $V(\pi_1)$ . Furthermore we note that  $\pi_1^e < \pi_2 < \pi_1 < \pi_2^e$ . It is thus evident that  $V^{-1}(y)$  includes three distinct points when  $v_m < y < V_M$  while it only includes one point for  $y < v_m$  and  $y > V_M$ . For this reason we say, using the language of [15], that  $V$  is a function of type  $Z_1 - Z_3 - Z_1$ ,<sup>1</sup> whose inverse function is made up of one-three-one branches, as qualitatively shown in Fig. 3. In order to obtain a well defined model we have to decide how to choose  $V^{-1}(y)$  for  $v_m < y < V_M$ . If we choose the lower branch we would obtain for  $V^{-1}$  an increasing function,

<sup>1</sup> Recall that the notation  $Z_i$  ( $i = 1, 3$ ) denotes intervals whose points have the same number  $i$  of distinct rank-1 preimages; such intervals are separated by the critical points  $v_m$  and  $V_M$ .

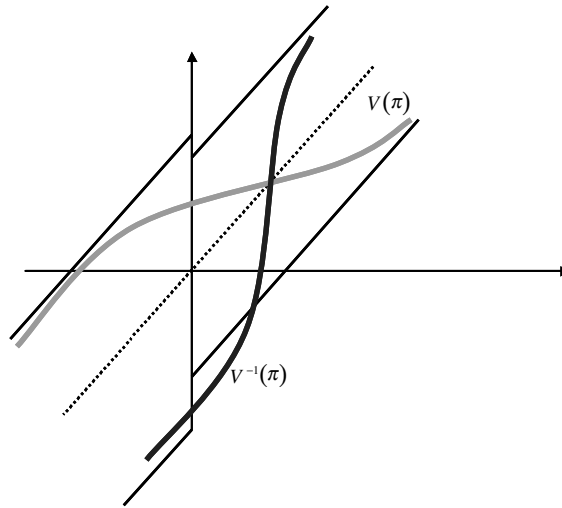


Fig. 2. The function  $V$  and its inverse in the case  $\alpha < 1/a$ .

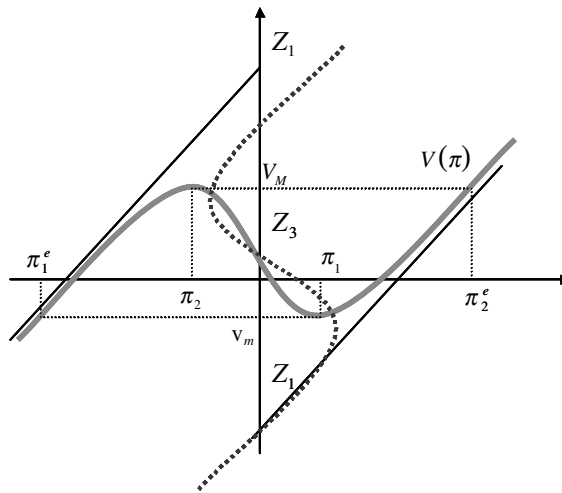


Fig. 3. The graph of  $V$ , when noninvertible. The intervals  $(-\infty, v_m)$  and  $(V_M, +\infty)$  are the  $Z_1$  regions, whereas the interval  $(v_m, V_M)$  is the  $Z_3$  one. The dotted curve is the inverse transformation of  $V$ .

discontinuous at  $V_M$ , with a jump of width  $(\pi_2^e - \pi_2)$ , as shown in the qualitative picture in Fig. 4a. If we choose the upper branch we would obtain for  $V^{-1}$  an increasing function, discontinuous at  $v_m$ , with a jump of width  $(\pi_1 - \pi_1^e)$  as shown in the qualitative picture in Fig. 4b. If we choose the middle branch we would obtain for  $V^{-1}$  a non-monotone function, with two points of discontinuity, one at  $v_m$  with a jump  $(\pi_1 - \pi_1^e)$ , and the other at  $V_M$  with a jump  $(\pi_2^e - \pi_2)$ , as shown in the qualitative picture in Fig. 4c.

### 2.3. The map $F(p)$

Having established the properties of the function  $V$  we now turn to a study of the map  $F$ . Recall that the function  $F$  is defined in (15) according to

$$F(p) = p + V^{-1}(\alpha(m - p)).$$

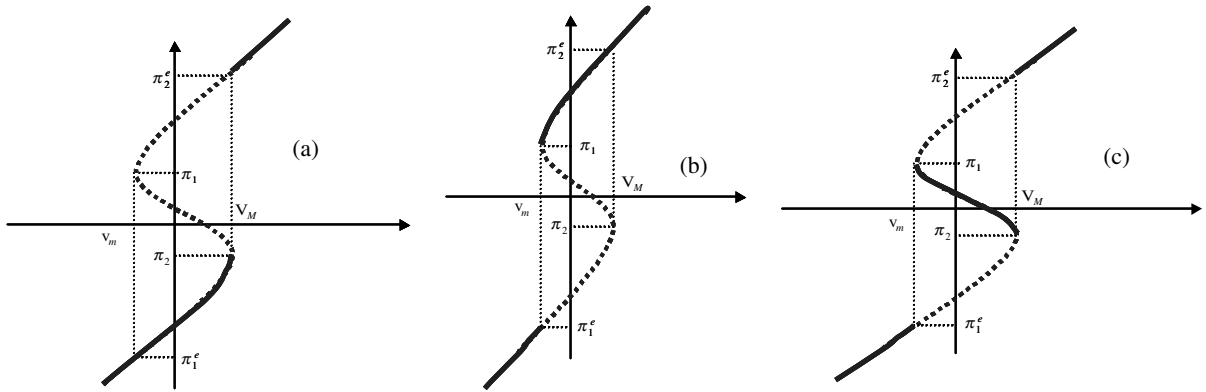


Fig. 4. (a)  $V^{-1}$  as an increasing function discontinuous at  $V_M$ . (b)  $V^{-1}$  as an increasing function discontinuous at  $v_m$ . (c)  $V^{-1}$  as a non-monotone function with two points of discontinuity.

Thus  $F$  is obtained by adding  $p$  to the composition of  $V^{-1}$  with the decreasing function  $\alpha(m - p)$ . The properties of  $F$  can be deduced from those of  $V^{-1}$  considered in the previous subsection, so that  $F(p)$  is a continuous function when  $V$  is invertible, while it is a discontinuous function when  $V$  is not uniquely invertible.

It is easy to see that

$$\lim_{p \rightarrow \pm\infty} F(p) = \begin{cases} \pm\infty & \text{if } \alpha < 1, \\ \mp\infty & \text{if } \alpha > 1 \end{cases}$$

and that  $F$  has the straight lines

$$\begin{aligned} y &= (1 - \alpha)p + \alpha(m - \varphi_-), \\ y &= (1 - \alpha)p + \alpha(m - \varphi_+) \end{aligned} \tag{21}$$

as asymptotes, for  $p \rightarrow +\infty$  and  $p \rightarrow -\infty$  respectively. These asymptotic properties hold independently of whether  $F$  is continuous or discontinuous.

When the function  $F$  is continuous it has only one fixed point, given by

$$p^* = m - f(0), \tag{22}$$

since  $F(p) = p$  holds iff  $V^{-1}(\alpha(m - p)) = 0$ , that is iff  $\alpha(m - p) = V(0)$ , or alternatively iff  $\alpha(m - p) = \alpha f(0)$ . When  $F$  has some points of discontinuity, then the fixed point may not exist. However, when it exists it is given by (22).

Moreover, by using the properties of inverse functions, we obtain, in any point  $p$  in which  $F$  is continuous,

$$F'(p) = 1 - \frac{\alpha}{1 + \alpha f'(V^{-1}(\alpha(m - p)))}, \tag{23}$$

and in particular at the fixed point we have

$$F'(p^*) = 1 - \frac{\alpha}{1 + \alpha f'(0)}.$$

**Proposition 1.** *If  $\alpha < \frac{1}{1+a}$  then the function  $F(p)$  is strictly increasing.*

*If  $1 < \alpha < \frac{1}{a}$  then the function  $F(p)$  is strictly decreasing.*

**Proof.** We note that  $F'(p) > 0$  iff  $f'(V^{-1}(\alpha(m - p))) > 1 - \frac{1}{\alpha}$  or  $f'(V^{-1}(\alpha(m - p))) < -\frac{1}{\alpha}$ . Since  $f'(\pi) \geq -a$  for any  $\pi$ , it follows that a sufficient condition to have  $F'(p) > 0$  for any  $p$  is  $-a > (1 - \frac{1}{\alpha})$ , that is,  $\alpha < \frac{1}{1+a}$ .

On the other hand  $F'(p) < 0$  iff  $-\frac{1}{\alpha} < f'(V^{-1}(\alpha(m - p))) < 1 - \frac{1}{\alpha}$  and a sufficient condition to have  $F'(p) < 0$  for any  $p$  is to have  $(1 - \frac{1}{\alpha}) > 0$ , that is,  $\alpha > 1$  and  $-\frac{1}{\alpha} < -a$ , that is,  $\alpha < \frac{1}{a}$ .  $\square$

While Proposition 1 gives sufficient conditions to obtain a strict monotone map  $F$ , the following proposition enunciates conditions under which the fixed point is stable or unstable, and the global dynamics also stable or divergent.

**Proposition 2.** *If  $\alpha \leq \frac{1}{a}$  then*

- (a)  $F(p)$  is a continuous function with  $F'(p) < 1$  for any  $p$ ;
- (b) its unique fixed point,  $p^* = m - f(0)$ , is globally stable for  $\alpha < \hat{\alpha}$ , where  $\hat{\alpha} = \frac{2}{1-2f'(0)}$ ;
- (c) if  $\alpha > 2$  then  $F'(p) < -1$  for any  $p$ , and the dynamics are divergent.

**Proof.** We recall from Section 2.2 that  $\alpha \leq \frac{1}{a}$  implies that  $V$  is invertible, moreover the same condition  $\alpha \leq \frac{1}{a}$  implies that  $0 \leq 1 + \alpha f'(\pi) < 1$  holds for any  $\pi$ , which implies that  $F'(p) < 1$  for any  $p$ , and statement (a) follows. In particular the condition  $F'(p) < 1$  is satisfied at the fixed point, thus the stability condition given in (b) follows from  $F'(p^*) > -1$ , which holds iff

$$\alpha < \hat{\alpha}, \quad \text{where } \hat{\alpha} = \frac{2}{1-2f'(0)} \tag{24}$$

so that  $\hat{\alpha}$  denotes a flip bifurcation value for the fixed point. Statement (c) is obtained noticing that a sufficient condition to have  $F'(p) < -1$  for any  $p$  is  $\alpha > 2$ , in which case every trajectory of the one-dimensional map  $p' = F(p)$  is divergent, except for the unstable fixed point.  $\square$

The following proposition is proved in Appendix A.

**Proposition 3.** *At  $\alpha = \hat{\alpha} = \frac{2}{1-2f'(0)}$  the map  $F(p)$  undergoes a supercritical flip bifurcation.*

#### 2.4. Dynamic analysis of the map $F(p)$

Having clarified in the previous subsection the properties of the map  $F$ , we now turn to study the behaviour that is implied for the price dynamics. We need to distinguish several cases.

##### 2.4.1. $F(p)$ continuous

Let us first consider the case in which  $F(p)$  is continuous, occurring when  $V$  is invertible, i.e. when the condition  $\alpha \leq \frac{1}{a}$  is satisfied.

For  $a \leq 1$  we may distinguish several intervals for the parameter  $\alpha$  in which the qualitative shape of the function  $F(p)$  is known, and thus also its possible dynamic behaviour:

- $1 < \alpha \leq \frac{1}{a}$ ; then  $F'(p) < 0$  for any  $p$  and  $F$  is a decreasing function (see Fig. 5a), thus the only possible attractors are a fixed point, a 2-cycle, or a 2-cycle at infinity, that is divergence.  
The fixed point  $p^* = m - f(0)$  is stable if  $\alpha < \hat{\alpha}$ . The flip bifurcation value  $\hat{\alpha}$  belongs to the interval  $(1, \frac{1}{a})$  for  $-\frac{1}{2} < f'(0) < \frac{1}{2} - a$ . For example when  $0 < a < \frac{1}{2}$  then  $1 < \hat{\alpha} < \frac{1}{a}$ , while for  $\frac{1}{2} < a < 1$  then  $\hat{\alpha} > \frac{1}{1+a}$  may be in any interval. When  $p^*$  is unstable ( $\alpha > \hat{\alpha}$ ) the attracting set is a cycle of period two as long as a 2-cycle exists at finite distance, otherwise the dynamics are divergent. For  $\alpha > 2$  we have  $F'(p) < -1$  for any  $p$ , so that all the trajectories are divergent, except for the unstable fixed point.
- $\alpha = 1$ ; then  $F$  is a decreasing function (see Fig. 5b). If  $\hat{\alpha} > 1$  (which occurs if  $f'(0) > -\frac{1}{2}$ ) then the fixed point is globally stable. If  $\hat{\alpha} < 1$  (which occurs if  $f'(0) < -\frac{1}{2}$ ) then the fixed point is unstable and the attracting set is a 2-cycle (globally attracting, except for the fixed point).
- $\frac{1}{1+a} < \alpha < 1$ ; then  $F$  is no longer a monotone function and its shape is illustrated in Fig. 5c. If  $\hat{\alpha} < 1$  (which occurs if  $f'(0) < -\frac{1}{2}$ ) then the fixed point may be stable or unstable. When it is unstable the more persistent attracting set is a 2-cycle, globally attracting, except for the fixed point. This is due to the fact that, if after the flip bifurcation the usual period-doubling sequence of bifurcations occurs, also a reverse sequence of bifurcations, ending with a stable 2-cycle, must take place. In fact, quite far from the flip bifurcation value, one periodic point of the 2-cycle becomes smaller than the local maximum of  $F$  and the other periodic point higher than the local minimum of  $F$ , and the 2-cycle is stable, because in these intervals we have the derivative  $0 < F'(p) < 1$ .
- $\alpha < \frac{1}{1+a}$ ; then  $F'(p) > 0$  for any  $p$  and  $F$  is an increasing function (see Fig. 5d), and since  $F'(p) < 1$  the fixed point is globally stable.

In the case  $a > 1$  we have that  $\frac{1}{a} < 1$ , and the invertibility of  $V$  occurs only in two intervals for  $\alpha$ , in which the same properties described above hold, that is:



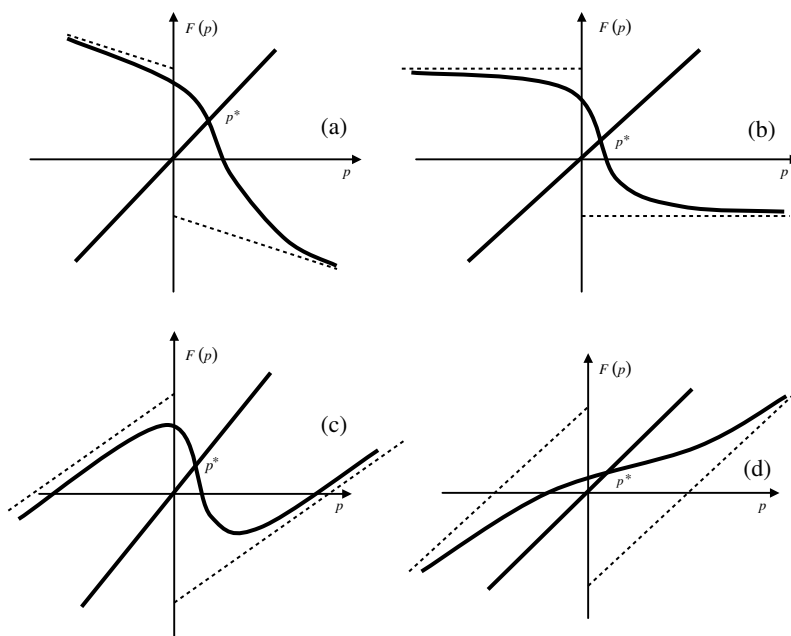


Fig. 5. The continuous map  $F$ . (a)  $\alpha > 1$ , (b)  $\alpha = 1$ , (c)  $1/(a + 1) < \alpha < 1$ , (d)  $\alpha < 1/(a + 1)$ .

- $\frac{1}{1+a} < \alpha \leq \frac{1}{a}$ ;  $F$  is not a monotone function and its shape is illustrated in Fig. 5c. Then the fixed point may be stable or unstable. When it is unstable then the more persistent attracting set is a 2-cycle, globally attracting, except for the fixed point.
- $\alpha < \frac{1}{1+a}$ ;  $F'(p) > 0$  for any  $p$  and  $F$  is an increasing function (see Fig. 5d), with a globally attracting fixed point.

We can thus state the following:

**Proposition 4.** Let  $\alpha \leq \frac{1}{a}$  then the dynamics of the continuous map  $p_{t+1} = F(p_t)$  are as follows:

- for  $\alpha < \hat{\alpha}$  every trajectory converges to the fixed point;
- for  $\hat{\alpha} < \alpha < 2$  every trajectory, except for the fixed point, converges to a 2-cycle, or to a more complex attractor (cycle of period  $2^i$ ,  $i > 2$ , or chaotic set), but this occurs in a small range of  $\alpha$  values;
- for  $\alpha > 2$  every trajectory, except for the fixed point, is divergent.

#### 2.4.2. $F(p)$ discontinuous

Let us now consider the case in which  $V$  is not invertible, that is  $\alpha > \frac{1}{a}$ . Then the map  $F$  is no longer continuous and the points of discontinuity may be one or two (as occurs for the function  $V^{-1}$ ). Now the fixed point may also not exist, in which case the attracting set is something more complex, we may have a cycle of any period, or also cyclical chaotic intervals. When the fixed point exists, it may be stable or unstable (via a flip bifurcation), and now when it is unstable the attracting set may be a 2-cycle, or a cycle of different period, or also cyclical chaotic intervals. Moreover, differently from what occurs when  $V$  is invertible, now it is possible to have two coexisting attractors.

Let us consider first the case in which the function  $V^{-1}$  is defined as an increasing function with a discontinuity. Here we study the case in which the discontinuity is in  $V_M = V(\pi_2)$ , the point where  $V$  attains its maximum, but analogous results can be obtained in the case of discontinuity in the point where  $V$  attains its minimum,  $v_m$ . In such a case we obtain

$$\lim_{p \rightarrow \left(m - \frac{V(\pi_2)}{\alpha}\right)^+} F(p) = m - \frac{V(\pi_2)}{\alpha} + \pi_2,$$

$$\lim_{p \rightarrow \left(m - \frac{V(\pi_2)}{\alpha}\right)^-} F(p) = m - \frac{V(\pi_2)}{\alpha} + \pi_2^e$$

and  $F$  has asymptotes given in (21). Moreover we can observe that

$$\begin{aligned} \lim_{p \rightarrow (m - \frac{V(\pi_2)}{\alpha})^+} F'(p) &= \lim_{p \rightarrow (m - \frac{V(\pi_2)}{\alpha})^+} \left[ 1 - \frac{\alpha}{1 + \alpha f'(V^{-1}(\alpha(m - p)))} \right] = \lim_{y \rightarrow V(\pi_2)^-} \left[ 1 - \frac{\alpha}{1 + \alpha f'(V^{-1}(y))} \right] \\ &= \lim_{y \rightarrow V(\pi_2)^-} \left[ 1 - \frac{\alpha}{V'(V^{-1}(y))} \right] = -\infty \end{aligned}$$

so that the map  $F$  cannot be an increasing function. We distinguish two cases.

- $\alpha > 1$ ; then the function  $F$  is decreasing, since its derivative is always negative, see Fig. 6. If  $p^*$  exists and is unstable, then for  $\alpha > 2$  every trajectory is divergent while for  $1 < \alpha < 2$  the  $\omega$ -limit set of the map is a stable 2-cycle which persists also when  $p^*$  disappears as  $\alpha$  varies. This is due to the fact that after the flip bifurcation, the periodic points of the 2-cycle belong to intervals where  $-1 < F'(p) < 0$ .
- $\frac{1}{a} < \alpha < 1$ ; then the function  $F$  is no longer monotone, having in a right neighborhood of  $m - \frac{V(\pi_2)}{\alpha}$  a minimum point, as shown in Fig. 7. On the left of the discontinuity point, the function  $F$  can be increasing or not, depending on the value of  $f'(\pi_2^c)$ . If  $p^*$  exists and is unstable, then the more persistent  $\omega$ -limit set of the map is a stable 2-cycle which persists also if  $p^*$  disappears. This is due to the fact that, if after the flip bifurcation the usual period-doubling sequence of bifurcations occurs, also a reverse sequence of bifurcations, ending with a stable 2-cycle, must take place. In fact, quite far from the flip bifurcation value, one periodic point of the 2-cycle becomes smaller than the local maximum of  $F$  and the other periodic point higher than the local minimum of  $F$ , and the 2-cycle is stable, because

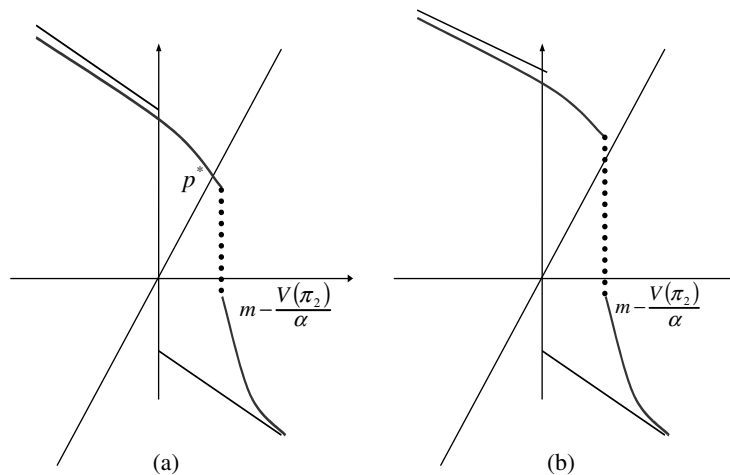


Fig. 6. The map  $F$  as a discontinuous decreasing function. (a) The fixed point  $p^*$  exists. (b) The fixed point does not exist.

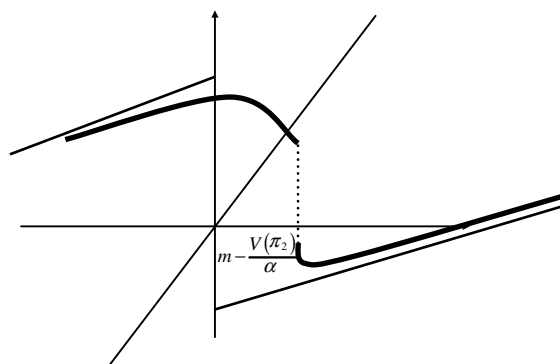


Fig. 7. The map  $F$  as a non-monotone, discontinuous function.

in these intervals we have the derivative  $0 < F'(p) < (1 - \alpha) < 1$ . While if  $p^*$  disappears when it is stable then cycles of higher period may appear, or cyclical chaotic intervals, and also two coexisting attractors may occur.

Let us now consider the case in which  $V^{-1}$  is defined as a non-monotone function, so that  $F$  has two points of discontinuity in  $p_1 = m - \frac{V(\pi_1)}{\alpha}$  and  $p_2 = m - \frac{V(\pi_2)}{\alpha}$  with

$$\lim_{p \rightarrow p_1^+} F'(p) = +\infty,$$

$$\lim_{p \rightarrow p_2^-} F'(p) = +\infty.$$

For  $p_1 < p < p_2$  we always have  $F'(p) > 0$  (i.e. an increasing branch). In the external intervals  $F$  may be decreasing (when  $\alpha > 1$ , since the slope of the asymptotes is  $(1 - \alpha) < 0$ ) (see Fig. 8), while for  $\alpha < 1$ , since the slope of the asymptotes is  $(1 - \alpha) > 0$  we may have increasing branches or non-monotone branches (see Figs. 9 and 10).

Regarding the dynamic behaviour, as in the previous case, when the fixed point is unstable or does not exist we may have several different attractors. In addition two coexisting attractors may exist, depending upon the value of  $\alpha$ .

As an example, let us consider the case in which the function  $f$  is defined as in Appendix B, and we fix the following values of the parameters:

$$m = 1, \quad b = 0.07, \quad g = 5, \quad \mu = 2$$

while the parameter  $\alpha$  is varied.

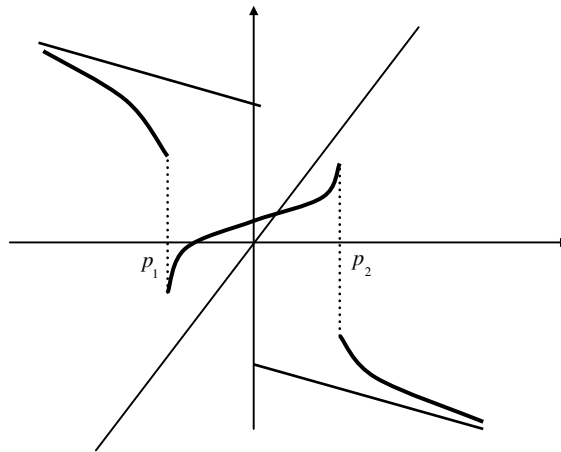


Fig. 8. The map  $F$  with three monotone branches: two decreasing and one increasing.

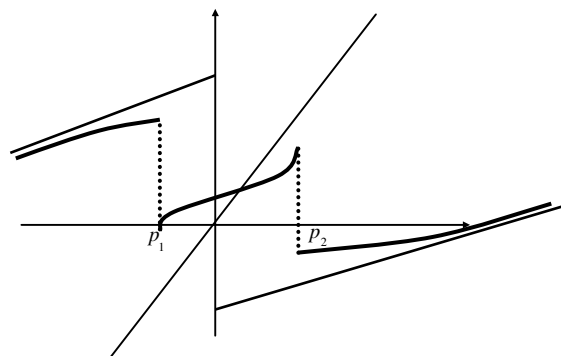


Fig. 9. The map  $F$  with three increasing branches.

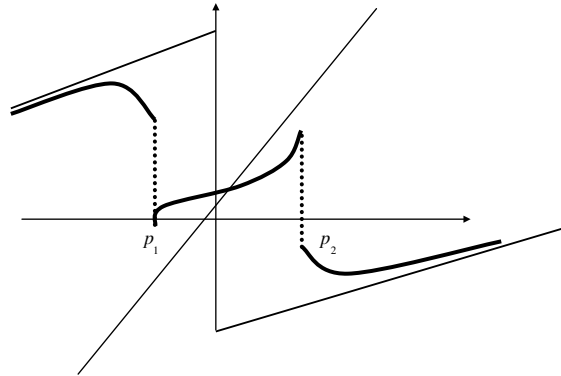


Fig. 10. The map  $F$  with three branches, two of them non-monotone.

For  $\alpha = 0.35$  the graph of  $F(p)$  is qualitatively similar to that of Fig. 10, and we have the fixed point which is globally attracting (see Fig. 11a). As  $\alpha$  increases the fixed point will disappear (when  $p^*$  merges with the discontinuity point  $p_2$ ) after which we may have several kinds of attractors. A stable cycle of period 6 is shown in Fig. 11b for  $\alpha = 0.39$ , a stable cycle of period 11, as shown in Fig. 11c for  $\alpha = 0.392$ , a cycle of high period (or chaotic intervals) as shown in Fig. 11d, a cycle of period 5 and so on.

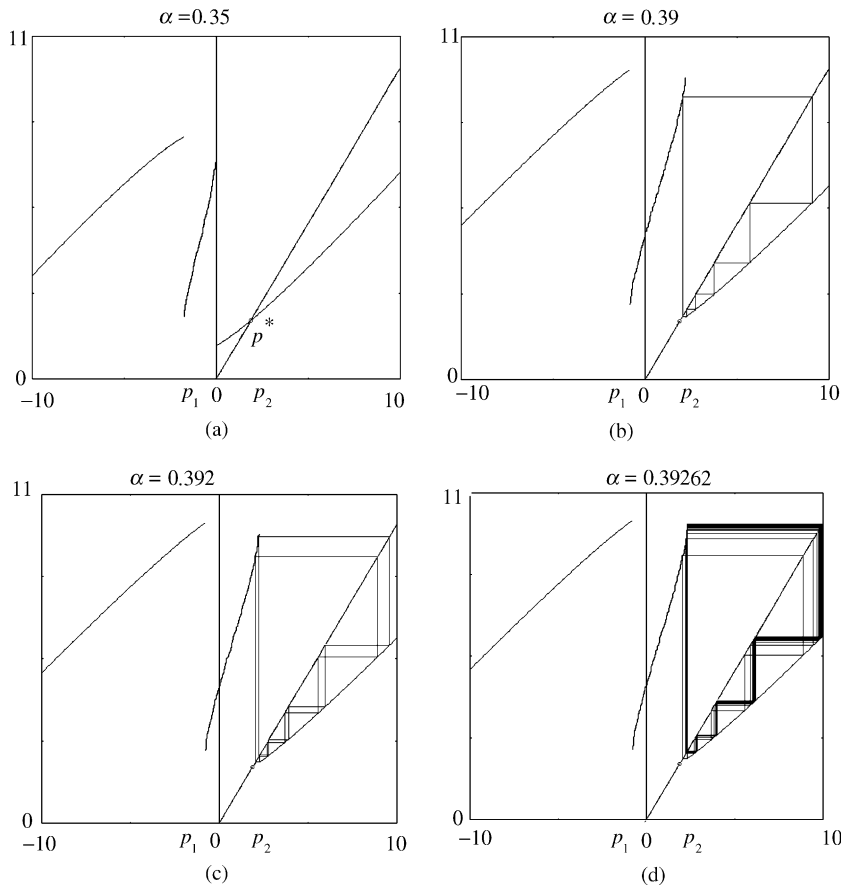


Fig. 11. Some attracting cycles occurring in the case of two discontinuity points. We fix  $m = 1$ ,  $b = 0.07$ ,  $g = 5$ ,  $\mu = 2$  and let  $\alpha$  vary. (a) Globally attracting fixed point. (b) An attracting cycle of period 6. (c) A stable cycle of period 11. (d) A cycle of high period (or chaotic intervals).

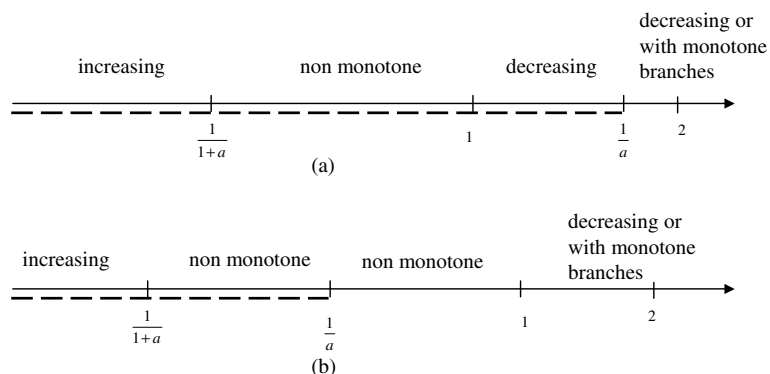


Fig. 12. Summary of the different shapes of  $F$ , The dotted bold line denotes the interval in which the map  $F$  is uniquely defined. (a)  $a < 1$ , (b)  $a > 1$ .

Summarizing, the map  $F(p)$  can assume different shapes, depending on the value of  $\alpha$ : Fig. 12 gives an illustration of the possibilities that may arise. Its dynamics are generally quite simple, either a fixed point or a 2-cycle. However, as we have seen above, more complex dynamical behaviour may arise when the map is discontinuous.

### 3. A partial perfect foresight model

Let us consider the partial perfect foresight model (12), which we rewrite using the simpler notation  $\pi_t$  instead of  $\pi_{t,t+1}$  as

$$\begin{cases} p_{t+1} = p_t + \alpha(m - p_t - f(\pi_t)), \\ \pi_t = \omega(p_{t+1} - p_t) + (1 - \omega)\pi_{t-1}. \end{cases} \tag{25}$$

This is a two-dimensional map, in fact if we assume that at time  $t$  the price  $p_t$  and the expected inflation rate  $\pi_{t-1}$  are given by the model (25) we obtain the price  $p_{t+1}$  of the next period and the expectation  $\pi_t$ .

Substituting the second equation of (25) into the first one we get

$$p_{t+1} = p_t + \alpha(m - p_t - f(\omega(p_{t+1} - p_t) + (1 - \omega)\pi_{t-1})), \tag{26}$$

which is an implicit equation from which we obtain  $p_{t+1}$  as a function of  $p_t$  and  $\pi_{t-1}$ ; with this value of  $p_{t+1}$  we then obtain  $\pi_t$ .

Alternatively: substituting the first equation of (25) into the second one we get

$$\pi_t = \omega\alpha m - \omega\alpha p_t - \omega\alpha f(\pi_t) + (1 - \omega)\pi_{t-1}$$

from which

$$\pi_t + \omega\alpha f(\pi_t) = \omega(\alpha m - \alpha p_t) + (1 - \omega)\pi_{t-1} \tag{27}$$

that is

$$V_\omega(\pi_t) = g_\omega(p_t, \pi_{t-1}),$$

where  $V_\omega(\pi_t) = \pi_t + \omega\alpha f(\pi_t)$  and  $g_\omega(p_t, \pi_{t-1}) = \omega(\alpha m - \alpha p_t) + (1 - \omega)\pi_{t-1}$ . Observe that the function  $g_\omega$  is linear in  $p_t$  and  $\pi_{t-1}$ . Now, if  $V_\omega(\pi_t)$  is invertible, then from (27) we can write  $\pi_t$  as

$$\pi_t = V_\omega^{-1}(g_\omega(p_t, \pi_{t-1})) = V_\omega^{-1}(\omega(\alpha m - \alpha p_t) + (1 - \omega)\pi_{t-1})$$

which, coupled with the equation for the price  $p_{t+1}$  obtained indifferently by the first or the second equation of (25), yields the explicit two-dimensional map, i.e.

$$\begin{cases} \pi_t = V_\omega^{-1}(\omega(\alpha m - \alpha p_t) + (1 - \omega)\pi_{t-1}), \\ p_{t+1} = \alpha m + (1 - \alpha)p_t - \alpha f(V_\omega^{-1}(\omega(\alpha m - \alpha p_t) + (1 - \omega)\pi_{t-1})) \end{cases}$$

or

$$\begin{cases} \pi_t = V_\omega^{-1}(\omega(\alpha m - \alpha p_t) + (1 - \omega)\pi_{t-1}), \\ p_{t+1} = p_t - \frac{1-\omega}{\omega} \pi_{t-1} + \frac{1}{\omega} V_\omega^{-1}(\omega(\alpha m - \alpha p_t) + (1 - \omega)\pi_{t-1}). \end{cases} \quad (28)$$

In particular, we consider the case  $\omega = 1$ . Since  $V_1(\pi_t) = \pi_t + \alpha f(\pi_t) = V(\pi_t)$  of Section 3, from (28) we can immediately see that the model reduces to the map

$$\begin{cases} \pi_t = V^{-1}(\alpha m - \alpha p_t), \\ p_{t+1} = p_t + V^{-1}(\alpha m - \alpha p_t) \end{cases} \quad (29)$$

which is the perfect foresight model. In fact the second equation of (29) is exactly the one-dimensional model (15) and the first one the corresponding value of the expected inflation rate. The map (28) provides us with one way to study price dynamics with boundedly rational expectations (at least in the sense discussed in this paper). It also provides us with a higher dimensional map from which the one-dimensional perfect foresight map may be obtained via a continuous limiting process, namely  $\omega \rightarrow 1$ . It may be that an understanding of this high dimensional map and its  $\omega \rightarrow 1$  limiting behaviour may give us some insights into how to choose between the various perfect foresight maps in the discontinuous case. This task we leave to future research. We recall that in continuous time, the consideration of perfect foresight as the limiting case of a higher dimensional model led to insights that were not possible in the lower dimensional perfect foresight model.

#### 4. Conclusion

We have set up a discrete time version of the nonlinear monetary dynamics model of Chiarella [8,9], and Flaschel and Sethi [10] under perfect foresight expectations. We have made explicit the map driving the perfect foresight dynamics and studied in detail its local and global dynamic features. Depending upon parameter constellations a fixed point or a 2-cycle is generally the outcome. We have also proposed a more general boundedly rational perfect foresight model that is more in keeping with the view that the limited information and computational ability of agents need to be taken into account. This framework would be of particular use in incorporating bounded rationality into the open economy integrated disequilibrium macrodynamic models in [14]. This task we leave to future research.

#### Acknowledgements

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#### Appendix A

**Proof of Proposition 3.** Looking for the existence of the 2-cycles of the map  $F(p)$ , let us denote the periodic points by  $p_1$  and  $p_2$ , then, by using the definition of the function, these points must satisfy the equations

$$\begin{aligned} p_2 &= p_1 + V^{-1}(\alpha(m - p_1)), \\ p_1 &= p_2 + V^{-1}(\alpha(m - p_2)) \end{aligned} \quad (A.1)$$

and thus also

$$V^{-1}(\alpha(m - p_1)) + V^{-1}(\alpha(m - p_2)) = 0.$$

In order to simplify the analysis, let us assume that  $V^{-1}$  is a symmetric function, satisfying  $V^{-1}(-z) = -V^{-1}(z)$ . In this case the condition given above is satisfied when

$$V^{-1}(\alpha(m - p_1)) = V^{-1}(-\alpha(m - p_2)),$$

that is, when  $2m = p_1 + p_2$  and substituting into (A.1) we have that the periodic points must satisfy the conditions:

$$p_2 = 2m - p_1,$$

$$2(m - p_1) = V^{-1}(\alpha(m - p_1)).$$

We note that we may rewrite the second condition as

$$\frac{2}{\alpha}\alpha(m - p_1) = V^{-1}(\alpha(m - p_1))$$

or

$$\frac{2}{\alpha}z = V^{-1}(z). \tag{A.2}$$

A 2-cycle exists when Eq. (A.2) has three solutions (the two periodic points and the fixed point). The graph of the function  $V^{-1}(z)$  intersects the straight line from the origin with slope  $\frac{2}{\alpha}$  in two more points besides the origin only when this slope is higher than the slope of the asymptotes of the function, and lower than the slope of the tangent in the origin, that is when  $\alpha$  satisfies

$$1 < \frac{2}{\alpha} < \frac{1}{1 + \alpha f'(0)}. \tag{A.3}$$

The conditions (A.3) correspond to the flip bifurcation condition (i.e. the two-cycle exists after the flip bifurcation of the fixed point),  $\alpha > \frac{2}{1 - 2f'(0)} = \hat{\alpha}$  and to the non-divergence condition  $\alpha < 2$ .

In the cases in which the function  $V^{-1}(z)$  is not symmetric, the computations are more complex, however similar reasoning hold and similar conditions exist, in fact the conditions for the existence of a 2-cycle are found by looking for the solutions different from zero of the equation

$$\frac{2 - \alpha}{\alpha}y = f(-y) - f(y).$$

In any case, whichever is the function  $V^{-1}(z)$  we note that

$$\hat{\alpha} < \frac{1}{a} \quad \text{iff } f'(0) < \frac{1}{2} - a,$$

$$\hat{\alpha} > 1 \quad \text{iff } f'(0) > -\frac{1}{2}.$$

### Appendix B

In this appendix we describe a possible choice for the function  $f(\pi)$ , whose qualitative graph is shown in Fig. 1, which is the function used in our numerical simulations. The function  $f(\pi)$  we consider is defined as

$$f(\pi) = 1 - \frac{g}{b + e^{-\mu\pi}}, \tag{B.1}$$

where the parameters  $g, b, \mu$  are positive. This  $f$  is defined in  $\mathbb{R}$  and it is always positive if  $g/b < 1$ , and its two horizontal asymptotes are

$$\varphi_- = 1 \quad \text{if } x \rightarrow -\infty,$$

$$\varphi_+ = 1 - \frac{g}{b} \quad \text{if } x \rightarrow +\infty.$$

The function  $f$  is decreasing, since its derivative is given by

$$f'(\pi) = -\frac{g\mu e^{-\mu\pi}}{(b + e^{-\mu\pi})^2}.$$

Furthermore  $f$  has an inflection point at

$$\pi_f = -\frac{\log b}{\mu}.$$

Then the minimum value of  $f'(\pi)$  in absolute value is

$$a = |f'(\pi_f)| = \frac{g\mu}{4b},$$

and the absolute value of the derivative at 0 is

$$|f'(0)| = \frac{g\mu}{(b+1)^2}.$$

Moreover we can observe that *the function  $f(\pi)$  is symmetric with respect to the point  $(0, f(0))$  iff  $b = 1$* ; in fact the symmetry condition  $f(\pi) - f(0) = f(0) - f(-\pi)$  for any  $\pi$  reduces to

$$(b-1)(e^{\mu\pi} + e^{-\mu\pi} - 2) = 0,$$

which is true for any  $\pi$  iff  $b = 1$ .

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