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A re-evaluation of adaptive expectations in light of global nonlinear dynamic analysis

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Abstract

We undertake an analysis of the dynamic behaviour of a discrete time nonlinear monetary dynamics model with adaptive expectation that is a basic mechanism in a broad class of descriptive macrodynamic models. We consider in particular a variety of ways in which the adaptive expectations mechanism may be formulated. These differ in the degree of rationality of the economic agents that inhabit the model. We study in detail the local and global properties of the maps determining the price dynamics.

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1. Introduction

Until the time of the rational expectations revolution in the early 1970s adaptive expectations was a common mechanism for modelling expectations in many dynamic

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macroeconomic models. The change in viewpoint on expectations modelling sparked by Sargent and Wallace (1973) and many subsequent authors caused the profession to regard adaptive expectations as a totally unsatisfactory explanation of expectations formation. It might even be fair to say that the concept itself was "excommunicated" from the economic order.

The main reason for the strong rejection of adaptive expectations had much to do with the linear framework in which it was applied. In that framework the dynamic models have unique steady states that are either stable or unstable (except for borderline cases of measure zero). Since instability in linear models can only lead to explosive outcomes, such cases were usually excluded by assumption on the range of model parameters under consideration. In the stable case economic agents would find themselves on either monotonic or damped fluctuating paths moving towards the steady state. The argument of the Rational Expectations School was that on such paths agents would realize that, even in the presence of some background noise, their expectations were consistently wrong and would therefore change their expectations formation scheme (of course to rational expectations).

What the arguments against adaptive expectations overlooked was the fact that the linear models frequently used in economics are usually an approximation to some underlying nonlinear model. If parameter sets for which the steady state is locally unstable are admitted, then as the paths move sufficiently far from the steady state the linear approximation breaks down and one needs to consider the nonlinear nature of the economic mechanism in order to obtain a true picture of the dynamics. When one takes into account the fact that nonlinear dynamical systems can produce dynamic paths that are not so regular and predictable, one of the major arguments against adaptive expectations does not seem so strong. Points along this line have been raised by George and Oxley (1985), Chiarella (1986, 1990), Oxley and George (1994) and Flaschel and Sethi (1999).

Arguing from a different perspective Burmeister (1980) pointed to a number of conceptual difficulties raised by the rational expectations approach. It is worth noting also that there is some evidence from experimental economics that economic agents do often adapt in the lagged fashion suggested by adaptive expectations. See in particular Hommes et al. (2000) who carried out experiments in the context of the cobweb model.

Advances in our understanding of nonlinear dynamic phenomena over the last decade have made even more pertinent the critique against the too rapid rejection of adaptive expectations by the earlier cited authors. These authors typically had in mind the possibility of chaotic dynamics as the reason that agents would not be able to discern so easily that they were consistently forming wrong expectations. Thanks to the work of Gumowski and Mira (1980), Mira et al. (1996) and Abraham et al. (1997) we now appreciate that the underlying nonlinear dynamical systems referred to above may display a range of other phenomena such as coexisting attractors with non-connected basins of attraction and various types of local and global bifurcations. This rich array of possible dynamic outcomes implies that a nonlinear model in which agents use adaptive expectations, particularly in an economic environment that is noisy, may not tend to paths along which agents can readily see that they are consistently wrong.

The aim of this paper is to revisit one of the basic models of adaptive expectations in a nonlinear framework and analyze its dynamics in light of the advances in our understanding of nonlinear dynamic maps that has occurred over the last decade. The model that we take is essentially the original one considered by Sargent and Wallace and extended to a nonlinear form by Chiarella (1986, 1990).

On the one hand the model is of historical interest as it is the one within which many of the essential early ideas of the rational expectations revolution developed. On the other hand it is of current interest as it is the basic expectations mechanism at work in the range of descriptive macrodynamic models considered by Chiarella and Flaschel (2000) and Flaschel et al. (2001).

In Section 2 we outline the nonlinear monetary dynamics model and in particular motivate the nonlinear money demand function. In Section 3 we discuss the adaptive expectations mechanism, focusing attention on the point in time when agents form expectations and in particular on the information available to them. We also consider the consequences of allowing the agents a certain degree of rationality in that they might know the price formation rule of the market. In Section 4 we analyze the dynamic map that arises under the various adaptive expectations schemes discussed in Section 3. It turns out that all cases can be reduced to a study of the one two-dimensional map whose local and global stability properties we analyze in detail. This analysis indicates a rich array of dynamic outcomes such as simple cycles, multistability and period doubling bifurcations. Section 5 discusses the economic consequences of this rich array of dynamic behaviour. We trace out the impact of a monetary shock. We also consider the impact of a noisy economic environment in particular noise in the agent's speed of adjustment of expectations. In the standard linear analysis such noise does little more than add a little noise around the underlying monotonic or oscillating price paths. However in the nonlinear model the underlying phenomena of multistability and bifurcations lead to far more complicated paths along which agents would find that expectation errors are unpredictable. Section 6 concludes. The appendices (available on the JEBO website) contain lengthy technical derivations.

2. The nonlinear monetary dynamics model

Economic agents are assumed to allocate their wealth between a physical good and money. The good price adjusts with a lag to excess money demand according to

$$\dot{p} = \alpha[m - p - f(\pi)], \tag{1}$$

where p is the logarithm of the price level, m the logarithm of the money supply (here assumed constant) and π the expected rate of inflation, or

$$\pi(t) = E_t[\dot{p}(t)]. \tag{2}$$

The function f is the logarithm of the demand for real money balances and because of portfolio considerations (see Chiarella, 1990) is assumed to have the nonlinear form shown in Fig. 1. This ensures that agents shift their portfolio allocation between money and the physical good towards the physical good (money) as expected inflation tends to $+\infty$ ($-\infty$).

Agents are assumed to form expectations adaptively according to

$$\tau\dot{\pi} = \dot{p} - \pi, \quad (\tau \geq 0), \tag{3}$$

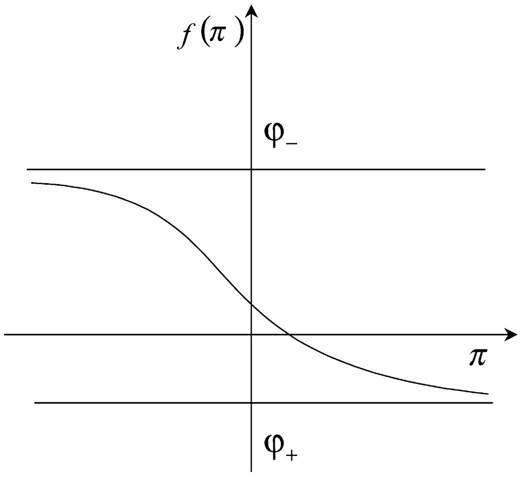

Fig. 1. Nonlinear money demand function.

which when $\tau = 0$ becomes the case of myopic perfect foresight, namely

$$\pi = \dot{p}. \tag{4}$$

Chiarella investigated the continuous time system (1) and (3), relying on the fact that the cases $\tau = 0$ and $\tau = 0^+$ yield qualitatively similar dynamics. He was thus able to obtain insights into the dynamics of the perfect foresight case that could not be obtained by considering (1) and (4) directly. In particular he found that the perfect foresight limit is characterised by relaxation cycles. Flaschel and Sethi further clarified the dynamics of the continuous time price dynamics (1) under the perfect foresight assumption. Chiarella also gave a brief analysis of a particular discrete time version of the perfect foresight case, that is (1) and (4). However to our knowledge a complete analysis of the discrete versions of (1) and (3) has never been undertaken, either in the case of perfect foresight, or in the case of adaptive expectations. In a companion paper (Agliari et al., 2004) we give a detailed analysis of the discrete time perfect foresight model. In this paper we analyze the corresponding adaptive expectations models. We study not only local stability properties but also give an analysis of the global dynamics.

In a certain sense the discrete time analysis of the monetary dynamics model under adaptive expectations is richer than the continuous time analysis because a number of discrete time maps may be obtained depending on what we assume about the information set of agents. In the analysis below we consider a set of maps that can arise in the discrete time setting. Most of these maps allow the agents to have some knowledge of the economic environment that they inhabit (i.e. the model) when they form expectations and hence are at least partially rational in the now traditional sense. We also formulate the map for the case that we call traditional adaptive expectations: agents form expectations based solely on knowledge of past prices (and past expectations). We shall see (what may be a surprising result) that such a map is exactly the same as the one that we obtain assuming that the agents are partially rational, when they are assumed to have some knowledge about the “rule” for the price formation.

We analyze in some detail the qualitative properties of the various maps. The main question we seek to investigate is the extent to which the qualitative dynamics of the monetary dynamics model change (or remain invariant) with the differing assumptions about the information set of agents.

3. Adaptive expectations

The discrete time model for the price evolution is given by

$$p_{t+1} = \alpha m + (1-\alpha)p_t - \alpha f(\pi_{t,t+1}), \tag{5}$$

where, as in the continuous time case, p_t is the logarithm of the price level at time t, $\pi_{t,t+1}$ the expected rate of inflation over the time interval $(t, t+1)$, m the logarithm of the money supply (assumed constant), and α a positive constant, denoting the speed of adjustment of the price to the excess money demand. In fact Eq. (5) can be rewritten as

$$p_{t+1} = p_t + \alpha(m - p_t - f(\pi_{t,t+1})),$$

and $m_t^{\mathrm{d}} = p_t + f(\pi_{t,t+1})$ is the logarithm of the money demand at time t, which depends on the expected rate of inflation for the next period. More precisely, let $I_{t+1} = p_{t+1} - p_t$ be the inflation rate at time $t+1$. At time t the agents do not know the value of I_{t+1}, so they have to consider some expected value of it, which we denote by $\pi_{t,t+1}$ in order to emphasize that it is formed at time t for the next time period $t+1$, that is

$$\pi_{t,t+1} = I_{t+1}^{(\mathrm{e})} = E_t(p_{t+1} - p_t). \tag{6}$$

In the economics literature different mechanisms have been proposed by which agents obtain the expected value in (6). The aim of this work is to analyze and compare the different dynamic models that arise when the expectations mechanism is varied in ways to be made precise below.

3.1. Traditional adaptive expectations—time sequencing

In the case of traditional adaptive expectations we assume that at time t agents form their expectations only making use of past observations of prices and expectations. Following this interpretation, we assume that agents at time t know the price p_t, the most recent inflation rate $I_t = p_t - p_{t-1}$ and the expectation they have made for I_t in the previous period, i.e. $\pi_{t-1,t}$. The agents then calculate the new expectation for I_{t+1}, $\pi_{t,t+1}$, assuming the standard adaptive expectations mechanism that takes a weighted average of the most recent inflation rate and the most recent expectation, namely

$$\pi_{t,t+1} = \omega I_t + (1-\omega)\pi_{t-1,t} \tag{7}$$

$$= \omega(p_t - p_{t-1}) + (1-\omega)\pi_{t-1,t}. \tag{8}$$

In this particular situation the agents modify their previous expectation ($\pi_{t-1,t}$) taking into account the forecast error observed ($I_t - \pi_{t-1,t}$), weighting it with ω, $0<\omega<1$. It is well known (e.g. Gandolfo, 1997) that by a process of continual back-substitution (7) is equivalent to a geometrically decreasing weighted average of past inflation rates. Obviously $\omega = 1$ in (7) gives the particular case known as the *static expectations* scheme:

$$\pi_{t,t+1} = I_t = p_t - p_{t-1}. \tag{9}$$

Such a naive learning scheme implies that agents expect in the next period that the rate of inflation assumes the same value as today, $I^{(e)}_{t+1} = I_t$.

Coupling the *traditional adaptive expectations* scheme (7) with the price adjustment equation in (5) we obtain

$$\begin{cases} \pi_{t,t+1} = \omega(p_t - p_{t-1}) + (1-\omega)\pi_{t-1,t} \\ p_{t+1} = \alpha m + (1-\alpha)p_t - \alpha f(\pi_{t,t+1}), \end{cases} \tag{10}$$

which by substituting the first equation into the second one may be written

$$\begin{cases} \pi_{t,t+1} = \omega(p_t - p_{t-1}) + (1-\omega)\pi_{t-1,t} \\ p_{t+1} = \alpha m + (1-\alpha)p_t - \alpha f(\omega(p_t - p_{t-1}) + (1-\omega)\pi_{t-1,t}). \end{cases} \tag{11}$$

The model (11) is a three-dimensional map in p_{t-1}, p_t and $\pi_{t-1,t}$ (as three values are necessary to start the iterations and to proceed from one time step to the next), that is

$$\begin{cases} \pi_{t,t+1} = F(p_{t-1}, \pi_{t-1,t}, p_t) \\ p_{t+1} = G(p_{t-1}, \pi_{t-1,t}, p_t), \end{cases}$$

where the functions F and G are defined by the right-hand sides of (11).

Note that when considering $\omega = 1$ in (10) we obtain a two-dimensional model for the *static expectations* scheme, namely

$$p_{t+1} = \alpha m + (1-\alpha)p_t - \alpha f(p_t - p_{t-1}), \tag{12}$$

in which the price at time $t+1$ depends on previous prices at time t and $t-1$. The analysis of the model (12) will thus be inserted into a more general setting, as a particular case of the *traditional adaptive expectations* model.

We must take care in interpreting the model in (10), in particular with respect to the information set of the agents when they form the expectation $\pi_{t,t+1}$. From the given initial values $(p_{-1}, \pi_{-1,0}, p_0)$ at time $t = 0$ the system evolves according to

$$(\pi_{0,1}, p_1) \to (\pi_{1,2}, p_2) \to \cdots \to (\pi_{t,t+1}, p_{t+1}) \to (\pi_{t+1,t+2}, p_{t+2}) \to \cdots$$

that is, at time t agents are assumed to know the actual price p_t, that of the previous period p_{t-1}, and the old expectation $\pi_{t-1,t}$, then they first compute the next expectation $\pi_{t,t+1}$ after which the next price p_{t+1} evolves according to the "market rules" that may be totally unknown to the agents.

A different assumption may be made, leading to what may seem a different model. Assuming that at time t agents observe the price p_t and have formed the expectation $\pi_{t,t+1}$, then the next price p_{t+1} evolves according to the "market rules" after which they form the next expectation $\pi_{t+1,t+2}$ taking into account the new price, that is, assuming that the agents "know the market rules" (i.e. the price formation mechanism). The model, in the following denoted as the *forward looking adaptive model*, can be written as

$$\begin{cases} p_{t+1} = \alpha m + (1-\alpha)p_t - \alpha f(\pi_{t,t+1}) \\ \pi_{t+1,t+2} = \omega(p_{t+1} - p_t) + (1-\omega)\pi_{t,t+1}, \end{cases} \tag{13}$$

or also, substituting the first equation into the second one,

$$\begin{cases} p_{t+1} = \alpha m + (1-\alpha)p_t - \alpha f(\pi_{t,t+1}) \\ \pi_{t+1,t+2} = \omega\alpha(m - p_t - \alpha f(\pi_{t,t+1})) + (1-\omega)\pi_{t,t+1}. \end{cases} \tag{14}$$

The model in (14) is a two-dimensional map in p_t and $\pi_{t,t+1}$. From the given initial values $(p_0, \pi_{1,2})$ at time $t=0$ the system evolves according to

$$(p_1, \pi_{1,2}) \to (p_2, \pi_{2,3}) \to \cdots \to (p_t, \pi_{t,t+1}) \to (p_{t+1}, \pi_{t+1,t+2}) \to \cdots$$

When the expectations scheme is viewed in this way agents seem to have "more information" than one assumes in the traditional adaptive expectations scheme. To some extent the difference in viewpoint arises because we imagine in one scheme that the agent stands at the beginning of the time interval, but at the end of the time interval in the other scheme.

The two models (10) and (13) so obtained mathematically also differ in order (the first model being a three-dimensional map and the second one a two-dimensional map). We may expect some relation between the two models when the initial conditions p_{-1} and p_0 at time $t=0$ are suitably chosen. In fact, if in (10) the price p_0 is fixed according to the law of price formation, $p_0 = \alpha m + (1-\alpha)p_{-1} - \alpha f(\pi_{-1,0})$, then the two models behave in the same way. For that reason it is natural to conjecture that their dynamics are correlated in spite of the different dimension of the phase space. However, in Section 4 we shall see a stronger result: that the two models are essentially the same. In fact, we shall see that given arbitrary initial conditions $(p_{-1}, \pi_{-1,0}, p_0)$ (10), in one iteration the agents learn the market rules: the computed value $\pi_{0,1} = \omega(p_0 - p_{-1}) + (1-\omega)\pi_{-1,0}$ with the initial condition p_0 gives rise to the same sequence of values that is obtained with the model (13) with the initial conditions $(p_0, \pi_{0,1})$. Thus, as we shall see in Section 4, we ultimately only have to study a two-dimensional map in order to analyze the traditional adaptive expectation scheme.

3.2. *Delayed adaptive expectations*

The difference in the adaptive expectation learning schemes of the previous subsection is due to the sequential structure of the model. In other words, the two models we have introduced differ in the sequence of computations: in the traditional adaptive model we first compute the expectation and then the price; in the forward looking adaptive model the process is inverted, first we obtain the price and then the expectation. This difference is avoided if we introduce a different assumption on the price formation mechanism. Let us assume that the agents know the actual price p_t, the price of the previous period p_{t-1} and the last expectation $\pi_{t-1,t}$ Then the next price p_{t+1} evolves according to the "market rules", using the known expectation, and agents form the next expectation $\pi_{t,t+1}$ taking into account only known data, so that the model becomes (note the use of $\pi_{t-1,t}$ in the money demand function *f*)

$$\begin{cases} p_{t+1} = \alpha m + (1-\alpha)p_t - \alpha f(\pi_{t-1,t}) \\ \pi_{t,t+1} = \omega(p_t - p_{t-1}) + (1-\omega)\pi_{t-1,t}, \end{cases} \tag{15}$$

where agents use the adaptive expectations mechanism:

$$\pi_{t,t+1} = \omega I_t + (1-\omega)\pi_{t-1,t} \tag{16}$$

$$= \omega(p_t - p_{t-1}) + (1-\omega)\pi_{t-1,t}. \tag{17}$$

It is immediate to see that it is unimportant which one of the two equations in (15) is computed first. The resulting model, which we call *delayed adaptive expectations*, is a true three-dimensional map.

We note that the same model also describes a different assumption on the adaptive expectation scheme, that is, we may assume that the known data at time t are the actual price p_t, the price of the previous period p_{t-1}, and the expectation $\pi_{t,t+1}$. Then the next price p_{t+1} evolves according to the "market rules" using the known expectation, and agents form the next expectation $\pi_{t+1,t+2}$ taking into account only known data:

$$\begin{cases} p_{t+1} = \alpha m + (1-\alpha)p_t - \alpha f(\pi_{t,t+1}) \\ \pi_{t+1,t+2} = \omega(p_t - p_{t-1}) + (1-\omega)\pi_{t,t+1}, \end{cases} \tag{18}$$

where agents now use the delayed adaptive expectations mechanism:

$$\pi_{t+1,t+2} = \omega I_t + (1-\omega)\pi_{t,t+1} = \omega(p_t - p_{t-1}) + (1-\omega)\pi_{t,t+1}. \tag{19}$$

It is clear that mathematically the two models, in (15) and in (18) are the same; they only differ in the "economic interpretation" of one state variable (the expectation). In fact, let us denote by R_t the "known" expectation at time t, and by R_{t+1} the new "computed" expectation, then both models read as

$$p_{t+1} = \alpha m + (1-\alpha)p_t - \alpha f(R_t), \qquad R_{t+1} = \omega(p_t - p_{t-1}) + (1-\omega)R_t \tag{20}$$

where

$$R_{t+1} = \omega I_t + (1-\omega)R_t = \omega(p_t - p_{t-1}) + (1-\omega)R_t.$$

The difference between the two models is that in (15) R_t is taken to be $\pi_{t-1,t}$ and in (20) to be $\pi_{t,t+1}$. A study of the dynamical behaviour of this three-dimensional map is given in (Agliari et al., 2002), who in particular analyze the dynamic consequences of its noninvertibility property.

4. Analysis of adaptive expectations maps

In this section we analyze in detail the characteristics and the dynamic behaviour of the adaptive expectations maps that we set up in Section 3.

4.1. The traditional adaptive expectations map

Let us rewrite the traditional adaptive expectations model in (11) as a three-dimensional map, simplifying the notation by using π_{t+1} to denote $\pi_{t,t+1}$.

Thus we define the map M as

$$M : \begin{cases} q_{t+1} = p_t \\ \pi_{t+1} = \omega(p_t - q_t) + (1-\omega)\pi_t \\ p_{t+1} = \alpha m + (1-\alpha)p_t - \alpha f(\omega(p_t - q_t) + (1-\omega)\pi_t) \end{cases}. \tag{21}$$

From the Jacobian matrix DM of M, given by

$$\mathrm{DM} = \begin{bmatrix} 0 & 0 & 1 \\ -\omega & 1-\omega & \omega \\ \alpha\omega f'(\pi) & -\alpha(1-\omega)f'(\pi) & 1-\alpha-\alpha\omega f'(\pi) \end{bmatrix},$$

it is easy to see that for any vector (q, π, p) the Jacobian determinant vanishes, $\det\mathrm{DM}(q, \pi, p) = 0$, which implies that in any point of the phase space one eigenvalue is equal to zero. From this consideration it follows that there ought to exist a two-dimensional invariant surface on which the dynamics will take place. We can in fact state

Proposition 1. *The set* $A = \{(x, y, \alpha m + (1-\alpha)x - \alpha f(y)) : (x, y) \in \mathbb{R}^2\}$ *is a trapping set for the map M:* $M(A) \subseteq A$.

Proof. The assertion follows by showing that $P' = M(P) \in A$ for any $P \in A$.

Let $P = (x, y, \alpha m + (1-\alpha)x - \alpha f(y))$, then the first and the second components of P' are

$$P'_x = \alpha m + (1-\alpha)x - \alpha f(y),$$
$$P'_y = \omega\alpha(m - x - f(y)) + (1-\omega)y,$$

and the third one is

$$\begin{aligned} P'_z &= \alpha m + (1-\alpha)[\alpha m + (1-\alpha)x - \alpha f(y)] - \alpha f(\omega\alpha(m - x - f(y)) + (1-\omega)y) \\ &= \alpha m + (1-\alpha)P'_x - \alpha f(P'_y). \end{aligned}$$

We conclude that $P' = (P'_x, P'_y, \alpha m + (1-\alpha)P'_x - \alpha f(P'_y)) \in A$.

An example of the two-dimensional surface in $\mathbb{R}^3$ which follows from Proposition 1 is shown in Fig. 2. The function f used to calculate this surface is described in Appendix A. The trajectories starting in A belong to it forever, while any point not belonging to A is mapped into A in one iteration.

The dynamics in the set A are given by the restriction of the map M to A, that is by the map

$$\begin{cases} x_{t+1} = \alpha m + (1-\alpha)x_t - \alpha f(y_t) \\ y_{t+1} = \omega\alpha(m - x_t - f(y_t)) + (1-\omega)y_t \end{cases} \tag{22}$$

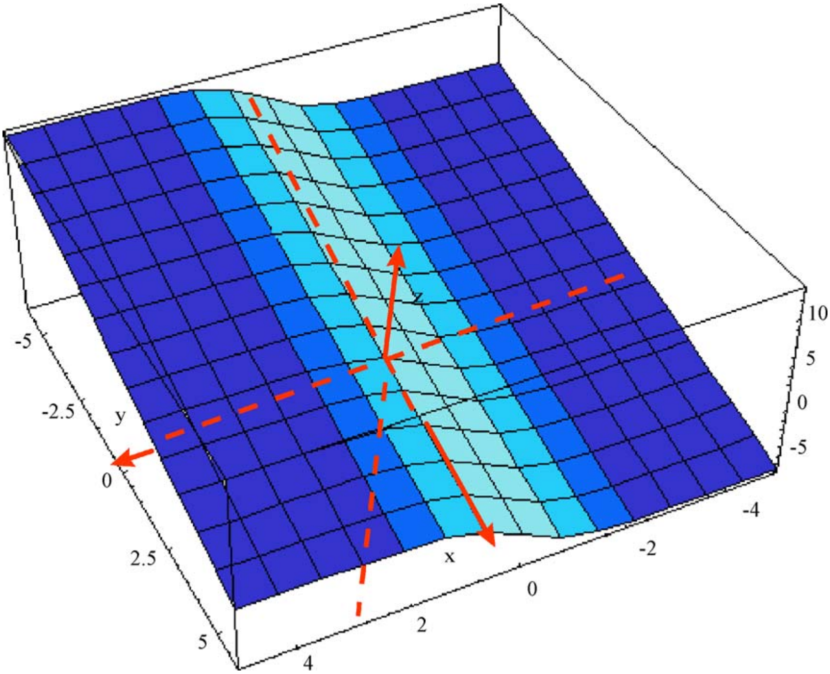

Fig. 2. The trapping set A when $f(\pi)$ is the function described in Appendix A.

which is exactly the map in (14), with $x = p_t$ and $y = \pi_{t,t+1}$ However in the three dimensional map the variables are either $x = q_t = p_{t-1}$ and $y = \pi_t$, or $x = q_{t+1} = p_t$ and $y = \pi_{t+1}$ as we shall explain below.

It follows from the proof of Proposition 1 that on the two-dimensional surface A the map M in (21) can be rewritten as

$$M_1 : \begin{cases} q_{t+1} = \alpha m + (1-\alpha)q_t - \alpha f(\pi_t) \\ \pi_{t+1} = \omega\alpha(m - q_t - f(\pi_t)) + (1-\omega)\pi_t \\ p_{t+1} = \alpha m + (1-\alpha)q_{t+1} - \alpha f(\pi_{t+1}) \end{cases} . \tag{23}$$

The Jacobian matrix of M_1 is

$$J = \begin{bmatrix} 1-\alpha & -\alpha f'(\pi) & 0 \\ -\omega\alpha & 1-\omega-\omega\alpha f'(\pi) & 0 \\ (1-\alpha)\dfrac{\partial q_{t+1}}{\partial q} - \alpha f'(\pi)\dfrac{\partial \pi_{t+1}}{\partial q} & (1-\alpha)\dfrac{\partial q_{t+1}}{\partial \pi} - \alpha f'(\pi)\dfrac{\partial \pi_{t+1}}{\partial \pi} & 0 \end{bmatrix}.$$

Now it is easy to see, on comparing J with the Jacobian matrix of the two-dimensional map in (22) (as we shall see also in the next subsection), that the eigenvalue associated with the transverse direction to the set A is exactly the eigenvalue 0, which is an attracting direction. More precisely, any point of the three-dimensional space belonging to a one-dimensional manifold issuing from a point (x, y) of the surface A, tangent to the eigendirection associated with the eigenvalue zero in (x, y), is mapped onto (x, z) in one iteration.

In other words, let us consider a generic initial condition for the map in (21), say $(p_{-1}, \pi_{-1,0}, p_0)$ at time $t = 0$,[1] then the first computed value $\pi_{0,1} = \omega(p_0 - p_{-1}) + (1 - \omega)\pi_{-1,0}$ with the other initial condition p_0 being such that the three dimensional model can be studied with the two-dimensional map in (22) with the initial conditions $(p, \pi) = (p_0, \pi_{0,1})$. Alternatively, given any initial condition $(p_{-1}, \pi_{-1,0}, p_0)$ in (21) at time $t = 0$ belonging to the surface A (so that it is true that $p_0 = \alpha m + (1 - \alpha)p_{-1} - \alpha f(\pi_{-1,0}) = \alpha m + (1 - \alpha)q_0 - \alpha f(\pi_{-1,0})$) then the same sequence of values of the three-dimensional map can be obtained from the two-dimensional map in (22) with the initial conditions $(p, \pi) = (p_{-1}, \pi_{-1,0})$.

From the above arguments we conclude that the dynamical properties of the map M are the same as those of the map T below, which gives the dynamics on the trapping set A. The analysis of the two-dimensional map will be the object of the next subsection.

4.2. The forward looking adaptive expectations maps

We consider the two-dimensional model in (13), which we rewrite, for the sake of simplicity, using the advancement operator "′" as

$$T : \begin{cases} p' = \alpha m + (1 - \alpha)p - \alpha f(\pi) \\ \pi' = \omega\alpha(m - p - f(\pi)) + (1 - \omega)\pi \end{cases} \tag{24}$$

where $\alpha > 0$, $0 < \omega < 1$ are the parameters of the model, and $f(\pi)$ is a generic function whose qualitative shape is given in Fig. 1. To study the model given in (14) we set $(p, \pi) = (p_t, \pi_{t,t+1})$ and $(p', \pi') = (p_{t+1}, \pi_{t+1,t+2})$, while in the study of the map given in (21) this two-dimensional map (24) governs the dynamics on the invariant two-dimensional surface A so that given any initial condition $(p_{-1}, \pi_{-1,0}, p_0)$ in (21) at time $t = 0$, then either it belongs to A (so that it is true that $p_0 = \alpha m + (1 - \alpha)p_{-1} - \alpha f(\pi_{-1,0})$ and the dynamics are studied by the map in (24) with the initial conditions $(p, \pi) = (p_{-1}, \pi_{-1,0})$), or not (in which case the first computed value is $\pi_{0,1} = \omega(p_0 - p_{-1}) + (1 - \omega)\pi_{-1,0}$ and the dynamics are studied by the map in (24) with the initial conditions $(p, \pi) = (p_0, \pi_{0,1})$).

In this section we consider the dynamic properties of the map T in (24) and in the local and global bifurcations eventually arising in the phase plane.

4.2.1. Fixed point and local stability

As usual, the fixed points of the map T are obtained by looking for the solutions of the system (24), such that $p' = p = p^*$ and $\pi' = \pi = \pi^*$:

$$\begin{cases} p^* = \alpha m + (1 - \alpha)p^* - \alpha f(\pi^*) \\ \pi^* = \omega\alpha(m - p^* - f(\pi^*)) + (1 - \omega)\pi^*. \end{cases}$$

[1] For the sake of clarity, in the following comments we maintain the full notation of indices in the expectations, as used in Section 2.

It is easy to see that the map T has a unique fixed point given by

$$P^* = (m - f(0), 0). \tag{25}$$

Observe that *if the function $f(\pi)$ is symmetric with respect to the point* (0,f(0)), *then the map T in* (24) *is a symmetric map with respect to the fixed point.* In fact the map T is topologically conjugate to the map

$$T_S : \begin{cases} p' = (1-\alpha)p - \alpha(f(\pi) - f(0)) \\ \pi' = -\omega\alpha(p + (f(\pi) - f(0))) + (1-\omega)\pi \end{cases},$$

via the homeomorphism $\varphi\,(p, \pi) = (p + m + f(0), \pi)$, and the map T_S, which has the fixed point (0, 0), is symmetric with respect to it since

$$\begin{aligned} T_S(-p, -\pi) &= [-(1-\alpha)p - \alpha(f(-\pi) - f(0)), -\omega\alpha(-p + (f(-\pi) - f(0))) \\ &\quad -(1-\omega)\pi] = [-(1-\alpha)p + \alpha(f(\pi) - f(0)), \omega\alpha(p + (f(\pi) - f(0))) - (1-\omega)\pi], \\ &= -T_S(p, \pi). \end{aligned}$$

In order to study the local stability of the fixed point, we use the Jacobian matrix of T, given by

$$J(p, \pi) = \begin{bmatrix} 1-\alpha & -\alpha f'(\pi) \\ -\omega\alpha & -\omega\alpha f'(\pi) + 1 - \omega \end{bmatrix}, \tag{26}$$

(note that the Jacobian matrix does not depend on m). Evaluating J at the fixed point we obtain

$$J^* = J(P^*) = \begin{bmatrix} 1-\alpha & -\alpha f'(0) \\ -\omega\alpha & -\omega\alpha f'(0) + 1 - \omega \end{bmatrix},$$

so that $\det J^* = (1-\omega)\,(1-\alpha) - \omega\alpha f'(0)$, $\operatorname{tr} J^* = 2 - \alpha - \omega - \omega\alpha f'(0)$. The local stability conditions are thus given by

$$\begin{cases} \text{(a) } 1 - \operatorname{tr} J^* + \det J^* = \omega\alpha > 0 \\ \text{(b) } 1 + \operatorname{tr} J^* + \det J^* = (2-\alpha)(2-\omega) - 2\omega\alpha f'(0) > 0 \\ \text{(c) } 1 - \det J^* = -\omega\alpha + \alpha + \omega + \omega\alpha f'(0) > 0. \end{cases} \tag{27}$$

Since condition (27a) is always satisfied, the stability conditions of the fixed point P^* reduces to the region of the parameter plane (α,ω) in which (27b) and (27c) hold. Thus the stability region is bounded by the two hyperbolas

$$[(1 - 2f'(0))\alpha - 2]\omega = 2(\alpha - 2), \qquad [(1 - f'(0))\alpha - 1]\omega = \alpha. \tag{28}$$

A qualitative sketch of the stability region in the parameter plane (α, ω) is given in Fig. 3, and we can see that a stable fixed point may become unstable either via a flip bifurcation when the parameters (α, ω) cross the first hyperbola (the dark grey curve whose equation is given in (28a)), or via a Neimark–Hopf bifurcation (with complex eigenvalues) when the

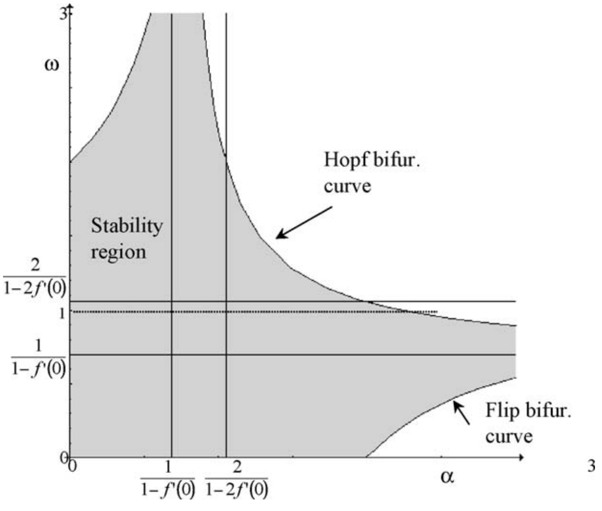

Fig. 3. Stability region of P^* when the function f is the one described in Appendix A, with $b = 1$, $g = 4.5$ and $\mu = 0.4$.

parameters (α, ω) cross the second hyperbola (the light grey curve whose equation is given in (28b)).

Proposition 2. *The flip bifurcation arising when the curve*

$$(2-\alpha)(2-\omega) - 2\omega\alpha f'(0) = 0$$

is transversely crossed is of subcritical type.

A proof of Proposition 2 is given in Appendix B.

As we shall see in the numerical examples, the unstable 2-cycle, which must be of saddle type when the parameters are close to the flip bifurcation curve, plays an important role in the global dynamics as its stable set bounds the basin of attraction of the stable fixed point.

4.2.2. Invertibility of T

In the study of the global properties of the map T, an important role is played by the invertibility of the map T. In fact, if the map is noninvertible, some global bifurcations can be explained by the folding action of the map on the plane. In particular, the global bifurcations, also called "*contact bifurcations*" (e.g. Mira et al., 1996) and arising when the frontier of a basin or the boundary of an attractor has some contact with the *critical lines* of the map, cause important changes in the topological structure of the basin (in the first case) or in the attractor (in the second case). This happens because the critical lines separate zones of the plane whose points have a different number of rank-1 preimages,[2] and then

[2] Given a n-dimensional map $F: \mathbf{R}^n \to \mathbf{R}^n$ and a positive integer r we say that the point y is a rank-r preimage of the point x if $F^r(y) = x$, that is if y is mapped into x in r iterations.

after the occurrence of each contact a set, say H, of points of a basin (or of an attractor) belongs to a different zone. Then the points of the set H have a different number of rank-1 preimages; that is, preimages may appear or disappear. These preimages, which have the same asymptotic behaviour as the points of H, may be located far from H, creating, for example, new disconnected components of a basin or holes in some old basin, thus causing the transition of a basin into a disconnected or a multiply-connected set.

In order to study the invertibility of the map T in (24), we have to find the preimages of a given point (u, v), that is, to look for the solutions (p, π) of the system

$$\begin{cases} u = \alpha m + (1-\alpha)p - \alpha f(\pi) \\ v = \omega\alpha(m - p - f(\pi)) + (1-\omega)\pi \end{cases}$$

which is equivalent to

$$\begin{cases} p = u - \dfrac{v}{\omega} + \dfrac{1-\omega}{\omega}\pi \\ f(\pi) = \dfrac{(1-\omega)(1-\alpha)}{\omega\alpha}\pi + h(u, v) \end{cases} \tag{29}$$

where

$$h(u, v) = m - u - \frac{1-\alpha}{\omega\alpha}v.$$

From the second equation in (29) we see that the number of preimages depends on the number of intersections between the graph of the function $f(\pi)$ and the straight line whose equation is given on the right side, with slope $\frac{(1-\omega)(1-\alpha)}{\omega\alpha}$. Thus we deduce that the map may be invertible (for example when $(1-\omega)(1-\alpha) > 0$, as then only one solution exists for any point (u, v)), or three solutions may exist, at least in some region of the phase plane. This means that for a suitable choice of the parameters (α, ω) the map is non-invertible and of type Z_1–Z_3–Z_1, that is; in the phase plane there is a zone of points with three preimages, Z_3, while a unique preimage exists in Z_1. Such zones are separated by the critical line LC, the locus of points having merging rank-1 preimages.

The preimages of the points of LC belong to the *critical line of rank*-0, LC_{-1}, which in our case, given the differentiability of the map T, is the set

$$\mathrm{LC}_{-1} = \{(p, \pi) : |J(p, \pi)| = 0\}$$

where $|J(p, \pi)|$ is the Jacobian matrix determinant. By straightforward computations, from (26) we obtain that LC_{-1} exists, with equation $\pi = k$ (a constant), when the equation

$$f'(\pi) = \frac{(1-\omega)(1-\alpha)}{\omega\alpha}$$

has a solution k.

From the properties of the function f, we deduce that the map T is noninvertible if

$$-a < \frac{(1-\omega)(1-\alpha)}{\omega\alpha} < 0$$

where $-a$ is the minimum of the derivative of f.

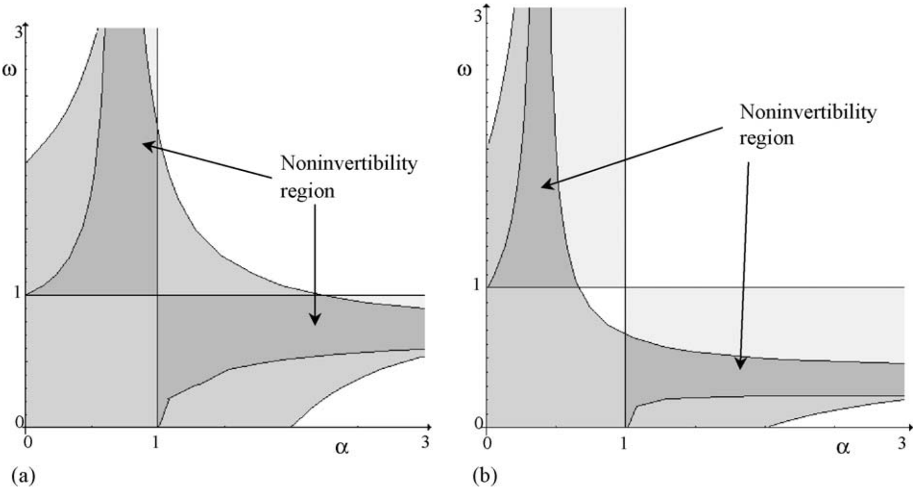

Fig. 4. Noninvertibility and stability region in the case (a): $f'(0) > -\frac{1}{2}$ and (b) $f'(0) < -\frac{1}{2}$. The darker grey region is the subset of the stability region in which the map is noninvertible.

Fig. 4 shows both the noninvertibility region and the stability region in the case $f'(0) > -\frac{1}{2}$ (Fig. 4a) and the case $f'(0) < -\frac{1}{2}$ (Fig. 4b).

Using as a specific example the class of functions $f(\pi)$ introduced in Appendix A, the critical line LC_{-1} is obtained by looking for the solutions of the equation

$$-\frac{g\mu\,\mathrm{e}^{-\mu\pi}}{(b+\mathrm{e}^{-\mu\pi})^2} = \frac{(1-\omega)(1-\alpha)}{\omega\alpha}, \tag{30}$$

which exist if

$$\frac{g\mu}{4b} > \frac{(1-\omega)(\alpha-1)}{\omega\alpha}.$$

In such a case straightforward computations show that there exist two solutions of Eq. (30), given by

$$\pi_1 = -\frac{1}{\mu}\ln\frac{-2b(1-\alpha)(1-\omega) - g\mu\alpha\omega + \sqrt{\Delta}}{(1-\alpha)(1-\omega)}$$

$$\pi_2 = -\frac{1}{\mu}\ln\frac{-2b(1-\alpha)(1-\omega) - g\mu\alpha\omega - \sqrt{\Delta}}{(1-\alpha)(1-\omega)}$$

where $\Delta = g\mu\left(g\mu + 4b\frac{(1-\alpha)(1-\omega)}{\alpha\omega}\right)$, so that LC_{-1} is given by the union of two vertical lines

$$LC_{-1} = LC_{-1}^{(a)} \cup LC_{-1}^{(b)} = \{\pi = \pi_1\} \cup \{\pi = \pi_2\}$$

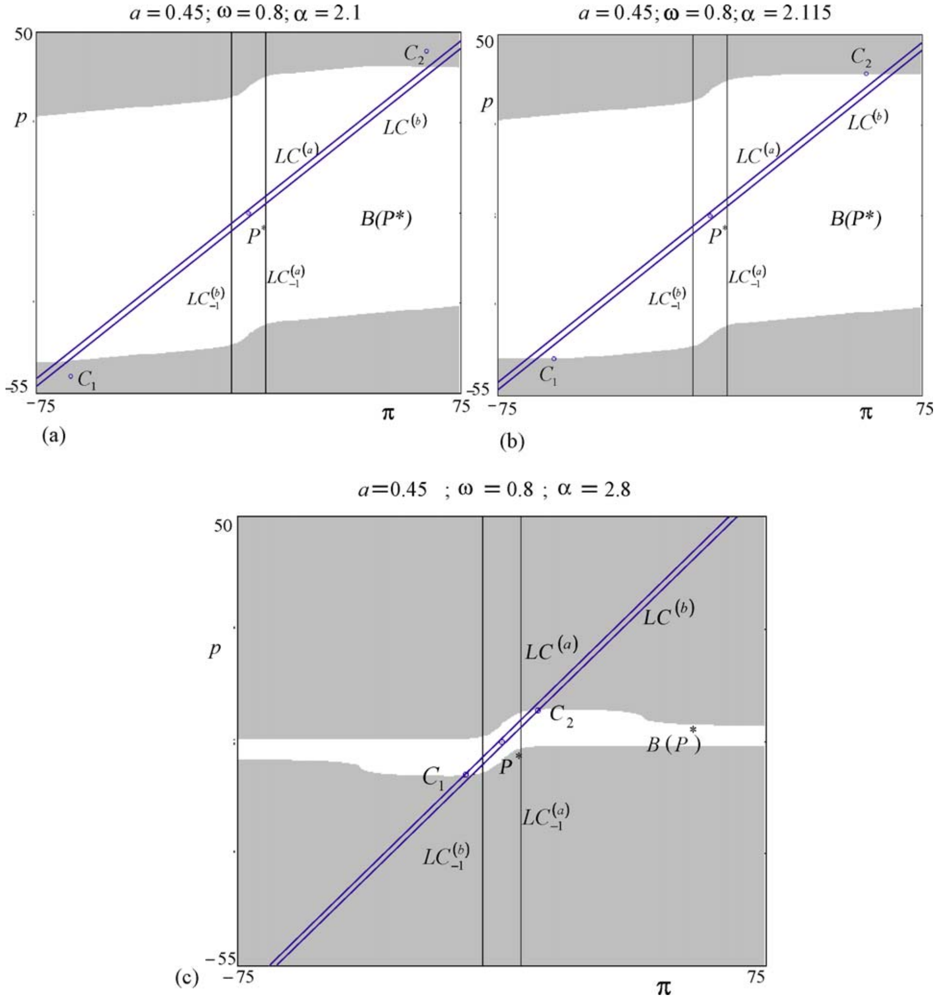

Fig. 5. The subcritical flip bifurcation.

and, as a consequence, the critical line LC is given by two distinct branches, being

$$\mathrm{LC} = T(\mathrm{LC}_{-1}) = T(\mathrm{LC}_{-1}^{(a)}) \cup T(\mathrm{LC}_{-1}^{(b)}) = \mathrm{LC}^{(a)} \cup \mathrm{LC}^{(b)}.$$

These lines are indicated in Fig. 5 (as well as in Figs. 6 and 7).

With the same class of functions we have investigated some aspects of the dynamic behaviour of the map. As we know from the properties of the stability region, the fixed point may become unstable either via a flip bifurcation or via a Neimark–Hopf bifurcation. Examples of both these situations is briefly described below. These examples are obtained with T symmetric (i.e. with $b = 1$), but analogous dynamics can also be found in the generic case.

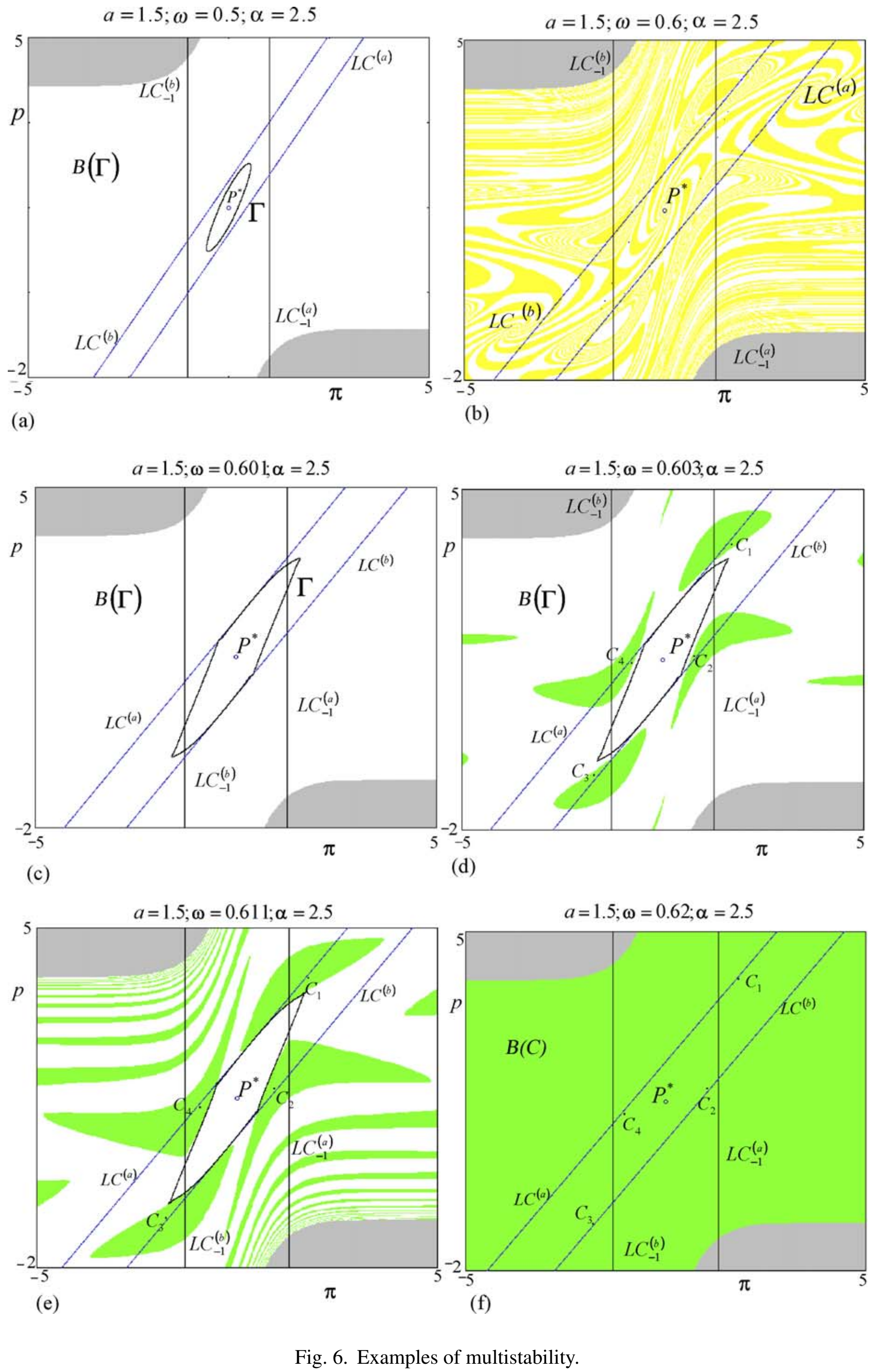

Fig. 6. Examples of multistability.

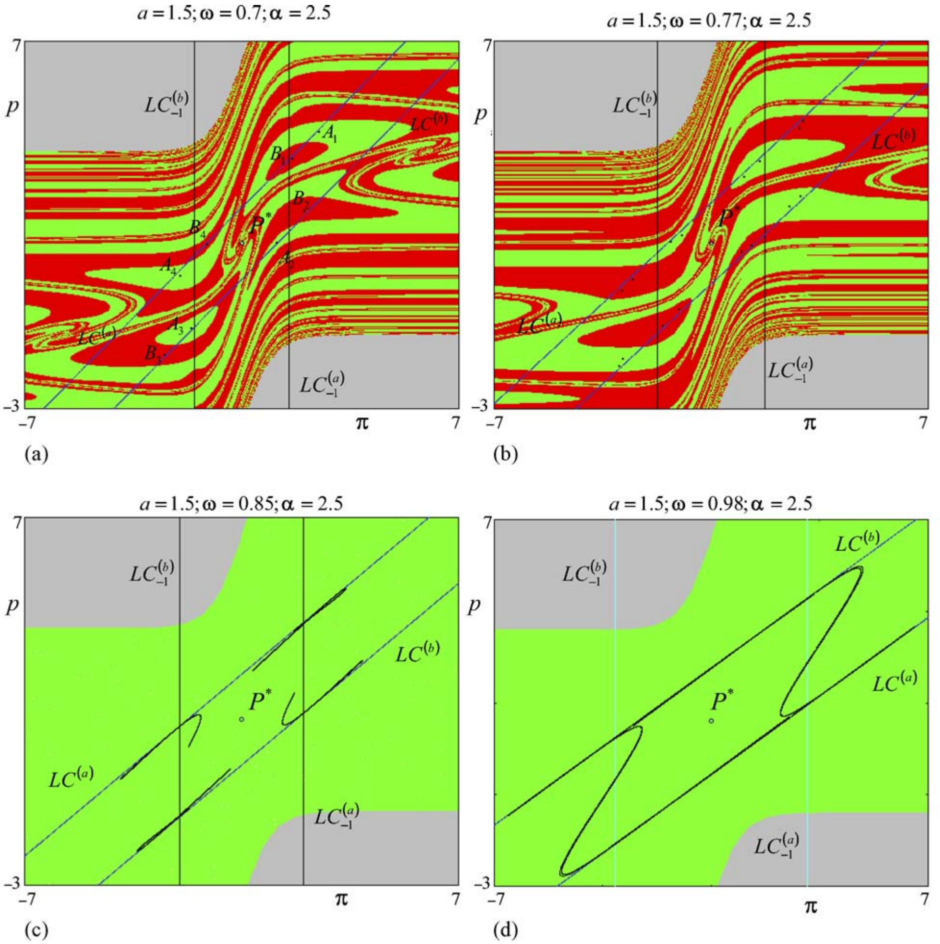

Fig. 7. Sequence of period doubling bifurcations.

4.2.3. *Crossing the flip bifurcation curve*

In the previous subsection we have seen that the flip bifurcation is of sub-critical type: when the fixed point P^* is stable there exists also an unstable 2-cycle so that after the flip bifurcation, locally (in a neighbourhood of the fixed point) we have no attracting set. The asymptotic behaviour of the points starting in a neighbourhood of the unstable fixed point depend on the global properties of the map. The trajectories may be divergent or convergent to some attractor existing far from the fixed point, but this case may occur only when close to the flip bifurcation curve the *stable* fixed point *coexists* with a different attractor. However, in our simulations we have always found the first situation; that is, close to the flip bifurcation value the dynamics are divergent (although it may occur that far from the bifurcation curve some other attracting set appears).

An example of the global dynamics, as the parameters are varied, is shown in Fig. 5. In Fig. 5a, with the parameters $b = 1$, $g = 4.5$, $\mu = 0.4$ (so that $|f'(0)| = a = 0.45$), $m = 1$, $\alpha = 2.1$, $\omega = 0.8$, we are inside the stability region. The basin of attraction of the stable fixed point, $B(P^*)$, is given by the connected white region. The grey points denote the states having divergent trajectories. The 2-cycle C is a repelling node, and the two periodic points belong to the grey region. In Fig. 5b ($\alpha = 2.115$) we can see that the 2-cycle C belongs to the boundary of the basin of attraction of the fixed point, and it will belong to this boundary also for higher values of α, up to the flip bifurcation. The local bifurcation causing the transition of the 2-cycle from repelling node (in Fig. 5a) to saddle (in Fig. 5b) is associated with a 4-cycle saddle existing on the frontier of the basin in Fig. 5a, which undergoes a flip bifurcation, merging (and disappearing) with the 2-cycle C, which then becomes a saddle and whose stable set gives the frontier of the basin $B(P^*)$. In Fig. 5c ($\alpha = 2.8$) we can see that the basin $B(P^*)$ becomes smaller and smaller as the flip bifurcation curve is approached, reducing to a thin strip. At the flip bifurcation value the 2-cycle merges with P^*, leaving a saddle fixed point, and after this bifurcation in our numerical simulations the trajectories have been found to be divergent.

4.2.4. Crossing the Neimark–Hopf bifurcation curve, and multistability

We describe here some of the possible dynamic outcomes that may occur after the Neimark–Hopf bifurcation, when the fixed point is a repelling focus. For parameter values close to the bifurcation curve, the unstable focus is surrounded by an attracting closed invariant curve, which means that the bifurcation is of supercritical type.

As the parameters are varied, we have found different situations of multistability between different cycles or between the closed curve and other cycles. Figs. 6 and 7 show examples of such situations. The examples are associated with the function f, with $\mu = 2$, $g = 3$ and $b = 1$ (so that $|f'(0)| = a = 1.5$), and the parameters $m = 1$ and $\alpha = 2.5$ while the value of ω is varied. When ω is smaller than the bifurcation value ($\hat{\omega} \simeq 0.47619$), the fixed point P^* is the unique attracting set of the system with a wide basin of attraction. After the crossing of the bifurcation curve, the attracting closed invariant curve that appears around P^* is the ω-limit set of a large portion of points of the plane (white points in Fig. 6a). The grey points in Fig. 6 denote points having divergent trajectories. For $\omega = 0.6$ we see a first multistability situation between cycles: two attracting cycles of period nine appear due to a saddle-node connection on the invariant closed curve that gives rise to the appearance of two attracting cycles and two saddle cycles of the same period. The appearance of a pair of cycles is typical of symmetric maps, and this is the case we are considering since the function f is symmetric (because $b = 1$). The basins of attraction of the two cycles (in different colors) are separated by the stable manifold of the saddle cycles (see Fig. 6b).

As the parameter ω is further increased the two attracting cycles merge with the saddles, so disappearing and leaving again an invariant attracting closed curve Γ (see Fig. 6c), but another bifurcation now occurs: in fact at $\omega = 0.603$ we find the coexistence of the curve Γ with a cycle C of period 4, whose points are denoted C_1, C_2, C_3, C_4 in Fig. 6d. This cycle seems to be born via a saddle-node bifurcation, associated with a saddle 4-cycle, whose stable manifold separates the basins of the two attractors at finite distance. As the parameter ω increases, the basin of attraction of the cycle C becomes wider and wider because larger portions of its points enter the Z_3 region (the region between $LC^{(a)}$ and $LC^{(b)}$

and its boundary approaches the closed curve Γ, as in Fig. 6e). When the boundary of B(C) has a contact with the curve Γ, we have a so-called *contact bifurcation* between the stable set on the boundary of B(C) and the attracting closed curve Γ, whose effect is to destroy the attracting set Γ. In Fig. 6f, after the bifurcation, we observe that the points having bounded trajectories are now almost all converging to the 4-cycle C; only looking at the transient part of a trajectory can we find traces of the "old" attracting set. In fact the trajectory of the points previously belonging to the set B(Γ), "recalls" the shape of the curve Γ ("ghost of the attractor") before approaching its ω-limit set.

Also the surviving cycle undergoes several bifurcations, as we can see in Fig. 7. In fact, for $\omega = 0.7$, in Fig. 7a, we note the coexistence of two attracting cycles A = $\{A_1, A_2, A_3, A_4\}$ and B = $\{B_1, B_2, B_3, B_4\}$ of period 4: they are born from the cycle C via a pitchfork bifurcation of the cycle C, which now is a saddle cycle with stable manifold separating the basins of attraction of A and B. If we compare Figs. 6e and 7a, we can observe that now the frontier of the basins is much more complex; this is due to the presence of the chaotic repeller "ghost of the attractor" surviving from the curve Γ as described above. As the parameter ω increases, the two cycles undergo a flip bifurcation after which the ω-limit set of bounded trajectories is given by two attracting 8-cycles (see Fig. 7b), but the structure of the basins of attraction remains similar. This first flip bifurcation opens the classical route toward chaos; a sequence of period doubling bifurcations is now observable for larger values of ω, giving rise to cycles of period 4×2^k, for any k, and concluding in some chaotic attractors. In Fig. 7c we observe a 4-piece chaotic attractor as the unique attracting set of the dynamical system and, finally, in Fig. 7d a chaotic attractor surrounding the fixed point P^*.

4.3. Static expectations

As we have seen in Section 2.3, the naive case in which the expectations are static is given by

$$\pi_{t,t+1} = I_t = p_t - p_{t-1}.$$

This is obtained as a particular case, with $\omega = 1$, in the traditional adaptive expectation scheme, that is,

$$\begin{cases} p_{t+1} = \alpha m + (1-\alpha)p_t - \alpha f(\pi_t) \\ \pi_t = p_t - p_{t-1}, \end{cases}$$

or equivalently,

$$\begin{cases} p_{t+1} = \alpha m + (1-\alpha)p_t - \alpha f(\pi_t) \\ \pi_{t+1} = p_{t+1} - p_t = \alpha(m - p_t - f(\pi_t)), \end{cases}$$

which corresponds to the two-dimensional map (24) with $\omega = 1$. It follows that this particular case can be analyzed making use of the results of Section 4.2. For example, from Fig. 3 we deduce the stability region of the fixed point on the line $\omega = 1$, realizing that the only possible bifurcation occurs crossing the Neimark–Hopf curve, so that after this bifurcation the state variable p_t has cyclical dynamics around the unstable fixed point.

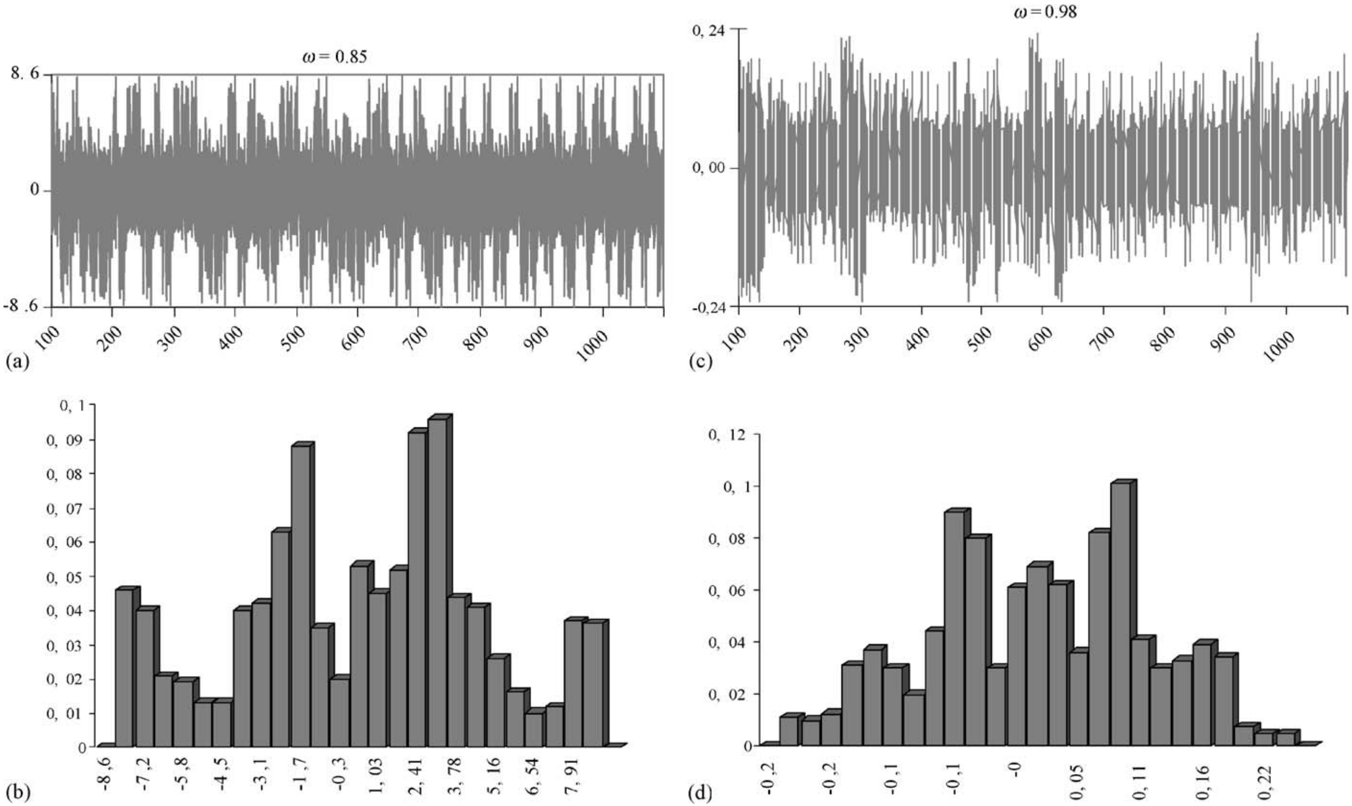

Fig. 8. Expectational errors in two chaotic regimes. (a) and (c): time series; (b) and (d) frequencies.

5. Some economic considerations

The analysis of Section 4 has laid out quite clearly the very rich structure of the nonlinear monetary dynamics model under adaptive expectations, certainly much richer than one would be able to anticipate merely by analyzing the linearized version of the model. In this section we consider some of the economic consequences of the rich dynamic structure of the model.

Consider first the issue that economic agents would recognize that they are making consistent forecast errors. Clearly this depends on the parameter constellation. Suppose the economy were operating with a parameter set corresponding to Fig. 7c or 7d. Here the agents would observe chaotic time series for their expectational error (see Fig. 8) and would be hard pressed to realize that they are using the "wrong" expectational scheme.

Consider next the issue of a noisy economic environment. It is in this context that the phenomenon of multistability becomes important. Suppose for example that the speed of expectations adjustment ω varies stochastically on a small interval around some average value.

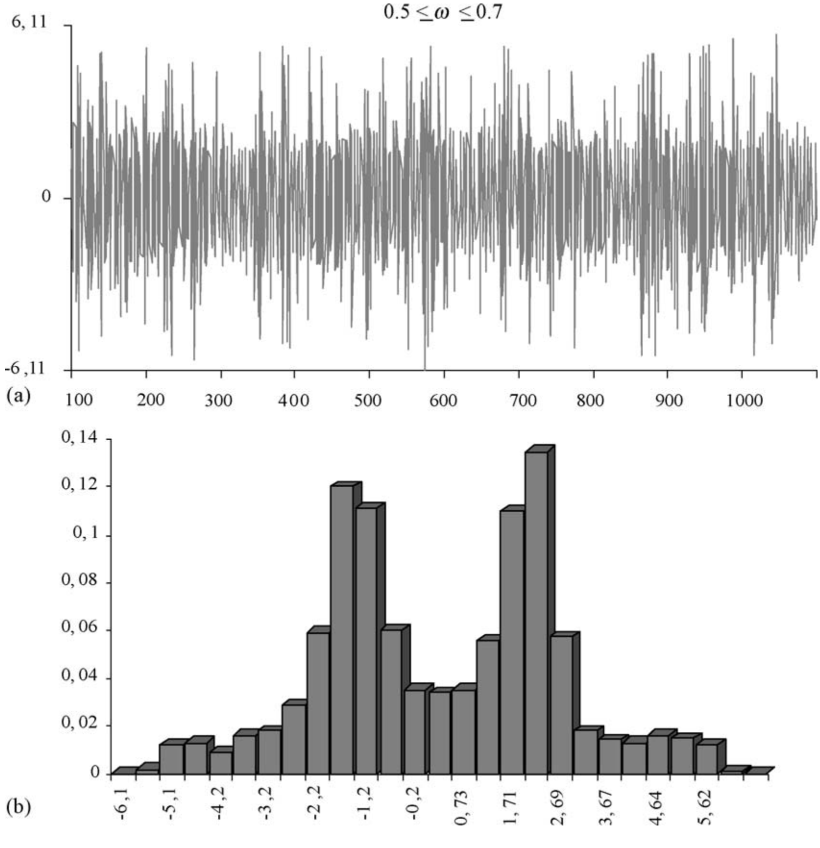

Fig. 9. Expectational error when the parameter ω is allowed to vary in the range [0.58, 0.62] following a uniform distribution; (a) time series, (b) frequencies.

This could be explained by the fact that agents adjust their expectations more or less aggressively around the average value of ω according to the random arrival of good or bad news. To see the impact of this phenomenon we have allowed ω to vary uniformly on the interval [0.5, 0.7] with other parameters being those relevant to Fig. 6. What essentially happens in this situation is that in every period there may be a switching among the different regimes that are influencing the dynamics. Fig. 9 displays the expectational error and its distribution. Here again we see that, at least at first glance, this appears to be a random process, though detailed statistical analysis may reveal some structure. The important point here is that agents do not so easily see that their expectations are wrong in some predictable way. This behaviour should be contrasted with what the agents would see in a linearized model with a noisy ω. Provided that ω moves in a range that does not disturb the stability of the steady state the expectational error would display a strong predictable component with some noise around it.

Finally consider the effect of an unanticipated monetary shock. Here again take $0.5 \leq \omega \leq 0.7$ so that only the phenomenon of multistability plays a role, but rather than being moved amongst different regimes, now the economy may move to a different basin of attraction. Suppose for example the parameters are such that the economy is operating in Fig. 6e and initial points are in the white region so that the curve Γ, which implies quasi-periodic attractors, is the long run outcome of the economy. A change in the money supply m is equivalent to moving the steady state to a new point in Fig. 6e, so it causes

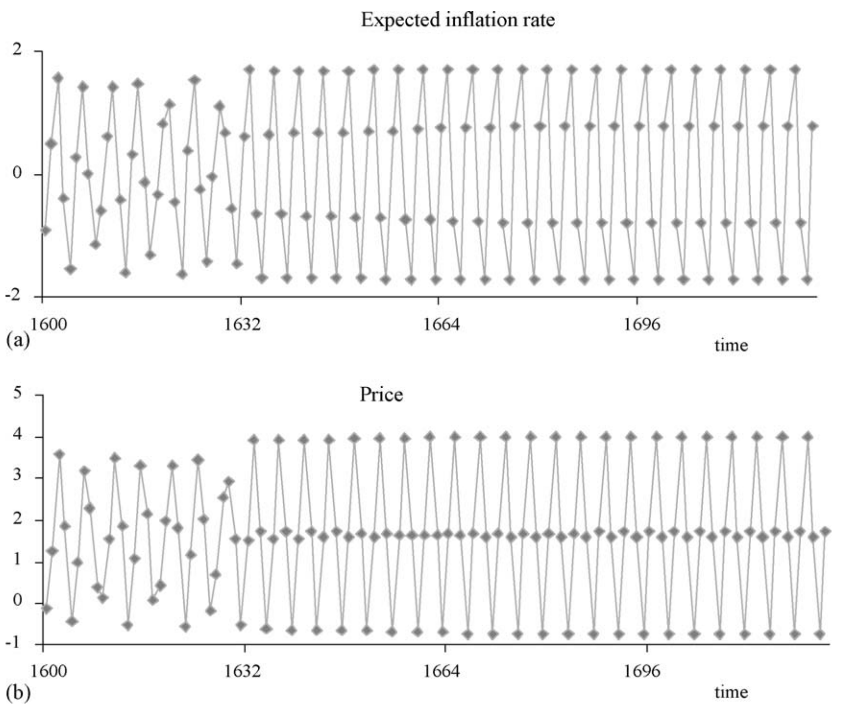

Fig. 10. Time series resulting from a monetary shock occurring at time $t = 1632$. The parameter m, initially 1, becomes 1.14.

a change in initial values. This could result in the initial point being moved to the other basin of attraction, and hence the 4-cycle becomes the long run outcome of the economy. Such an outcome is shown in Fig. 10 when a money supply shock of 15 percent occurs at time $t = 1632$. Of course in this situation the expectation error on the four cycle would be predictable and agents would then likely change behaviour; however we do not model this situation here. The point we want to emphasize is that a monetary shock could lead to a different dynamic attractor. Because of the multistability phenomenon a wide range of attractor shifts is possible, from quasi periodic to periodic (as discussed above) and vice versa, from periodic to periodic with a change in the periodic points.

6. Conclusion

We have analyzed a discrete time nonlinear monetary dynamics model under various adaptive expectations schemes. These schemes differ according to the information set of agents and the knowledge they are assumed to have about the market price setting rule. These schemes ultimately result in the same map driving prices and expectations. We have studied in detail the local and global dynamic behaviour of the dynamics. We have found that the dynamics displays a rich array of possible outcomes from stable fixed point to periodic, quasi periodic and chaotic fluctuations, as well as situations of multistability (i.e. coexistence of multiple attractors) with connected and non-connected basins of attraction, we have discussed the economic significance of these various dynamic phenomena for the expectational error of the economic agents in the model. In particular we have shown that there are many situations (especially if the economic environment is noisy) in which the expectational error would appear to the agents as a stochastic process with no particular structure that could be exploited for prediction purposes. There also exist situations in which the expectational error would be predictable, such as periodic cycles.

Overall our analysis shows that there can be a wide range of situations in which adaptive expectations does not necessarily tend to a situation of consistent prediction errors. Hence the jettisoning of it as a useful expectations scheme may have been premature. Our analysis also sits comfortably with some empirical economics literature that finds that economic agents often display adaptive type expectations.

Acknowledgments

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Appendix A. A possible choice for function $f(\pi)$

In this appendix we describe a possible choice for the function $f(\pi)$ (whose qualitative graph is shown in Fig. 1), which is the function used in our numerical simulations. The function $f(\pi)$ we consider is defined as

$$f(\pi) = 1 - \frac{g}{b + \mathrm{e}^{-\mu\pi}}, \tag{31}$$

where the parameters g, b and μ are positive.

The function f is defined in $\mathbb{R}$, and it is always positive if $g/b < 1$.

Its two horizontal asymptotes are

$$\varphi_- = 1 \quad \text{as } x \to -\infty$$
$$\varphi_+ = 1 - \frac{g}{b}, \quad \text{as } x \to +\infty.$$

The function f is decreasing, since its derivative is given by

$$f'(\pi) = -\frac{g\mu\,\mathrm{e}^{-\mu\pi}}{(b + \mathrm{e}^{-\mu\pi})^2}.$$

Furthermore it has an inflection point at

$$\pi_f = -\frac{\ln b}{\mu}.$$

Then the minimum value of $f'(\pi)$ in absolute value is

$$a = |f'(\pi_f)| = \frac{g\mu}{4b},$$

and the absolute value of the derivative at 0 is

$$|f'(0)| = \frac{g\mu}{(b+1)^2}.$$

Moreover we can observe that *the function $f(\pi)$ is symmetric with respect to the point* $(0, f(0))$ *iff* $b = 1$; in fact the symmetry condition $f(\pi) - f(0) = f(0) - f(\pi)$ for any π reduces to

$$(b - 1)(\mathrm{e}^{\mu\pi} + \mathrm{e}^{-\mu\pi} - 2) = 0,$$

which is true for any π iff $b = 1$.

Appendix B. Proof of Proposition 2

We seek the conditions for the existence of a 2-cycle $C = \{(p, \pi_1), (q, \pi_2)\}$. Recalling the definition of the map T, the points of C must be such that

$$\begin{cases} q = \alpha m + (1-\alpha)p - \alpha f(\pi_1) \\ \pi_2 = \omega\alpha(m - p - f(\pi_1)) + (1-\omega)\pi_1, \\ p = \alpha m + (1-\alpha)q - \alpha f(\pi_2) \\ \pi_1 = \omega\alpha(m - q - f(\pi_2)) + (1-\omega)\pi_2. \end{cases} \tag{32}$$

The second and the fourth of these equations can be rewritten as

$$\pi_2 = \omega(q - p) + (1-\omega)\pi_1,$$
$$\pi_1 = -\omega(q - p) + (1-\omega)\pi_2,$$

and it is immediate to see that $\pi_2 = -\pi_1$ must hold; we denote by π this common value. Then the system in (32) becomes

$$\begin{cases} q = \alpha m + (1-\alpha)p - \alpha f(-\pi) \\ p = \alpha m + (1-\alpha)q - \alpha f(\pi) \\ (2-\omega)\pi = \omega(q-p) \end{cases}$$

which for $\alpha \neq 2$ is equivalent to

$$q = m + \frac{(1-\alpha)f(\pi) + f(-\pi)}{\alpha - 2}, \qquad p = m + \frac{(1-\alpha)f(-\pi) + f(\pi)}{\alpha - 2},$$
$$(2-\omega)\pi = \omega(q-p).$$

Finally, substituting the value of p and q into the last equation, we obtain the condition on π that guarantees the existence of a 2-cycle, namely

$$\frac{(2-\alpha)(2-\omega)}{\omega\alpha}\pi = f(\pi) - f(-\pi). \tag{33}$$

Eq. (33) admits the solution $\pi = 0$ (the fixed point P^* is a solution of the system (32)), but it may also have two extra solutions, which are symmetric, one positive and one negative. These two further solutions exist if the function defined on the left side of (33), which is a straight line issuing from the origin, intersects in two more points (besides the origin) the function defined on the right side (which is a symmetric function with respect to the origin, decreasing with two horizontal asymptotes). Thus the conditions are that the slope $\frac{(2-\alpha)(2-\omega)}{\omega\alpha}$ of the straight line be negative and greater than the slope at 0 of the tangent to the function on the right side of (33) (see Fig. 11).

It follows that the conditions for the existence of a 2-cycle are

$$\frac{(2-\alpha)(2-\omega)}{\omega\alpha} < 0, \qquad \frac{(2-\alpha)(2-\omega)}{\omega\alpha} > 2f'(0). \tag{34}$$

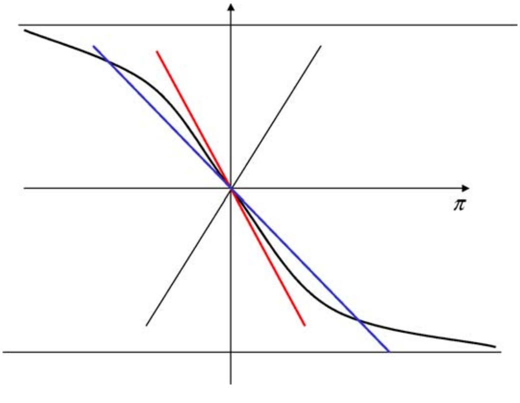

Fig. 11. Existence of the 2-cycle.

We see immediately that the second condition in (34) is exactly the same condition given in (27b) defining the stability region. Summarizing, we have demonstrated that a 2-cycle exists when the fixed point is stable, merging onto it at the bifurcation value, because the three solutions merge into the origin when the slope $\frac{(2-\alpha)(2-\omega)}{\omega\alpha}$ of the straight line becomes equal to the slope, $2f'(0)$, of the tangent in 0.

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