Journal of Economic Theory **87**, 243–253 (1999)
Article ID jeth.1999.2508, available online at http://www.idealibrary.com on IDEAL®

NOTES, COMMENTS, AND LETTERS TO THE EDITOR

Endogenous Fluctuations in a Bounded Rationality Economy: Learning Non-perfect Foresight Equilibria

Emilio Barucci

Dipartimento di Statistica e Matematica Applicata all'Economia, Università di Pisa, 56124 Pisa, Italy
ebarucci@ec.unipi.it

and

Gian Italo Bischi and Laura Gardini

Istituto di Scienze Economiche, Università di Urbino, 61029 Urbino, Italy
bischi@econ.uniurb.it, gardini@econ.uniurb.it

Received February 21, 1997; revised December 22, 1998

We study forward-looking economic models assuming that agents take one step ahead expectations looking back k time periods. We show that the dynamics of the economy with such an expectation function are characterized by the *coexistence* of perfect foresight and nonperfect foresight cycles. The stability of all these periodic solutions under bounded rationality is related to the stability of the perfect foresight cycles. *Journal of Economic Literature* Classification Numbers: C62, D83, D84, E31, E32.

1. INTRODUCTION

In this note we analyze the class of forward looking economic models

$$x_t = F(x^e_{t+1}) \tag{1}$$

under bounded rationality. The variable x_t represents the *state* of the economic system at time t and x^e_{t+1} is the *expected state* for time $t+1$ according to the information set at time t. F is a continuously differentiable map from an open interval of the real line into itself.

This class of models can be characterized by many perfect foresight stationary (PFS) solutions and perfect foresight cycles (PFC). In [4] and then in many other papers (e.g. [5, 6]) the relevance of a perfect foresight (PF) solution has been investigated by analyzing the dynamics of the model under bounded rationality learning. If a PF solution is stable under

0022-0531/99 $30.00

learning then the agents endogenously learn it as the economy evolves. Since we want the PF solution to be a trapping set for the dynamics with learning, then an *ad hoc* learning mechanism for each PF solution is assumed. For example, in order to analyze the stability of a PFC of period k with an expectation function based on past observations of the state, then we require that the expectation function *detects period* k, see [4] for a definition and [4, 5] for the stability conditions of PFC. The expectation function detecting period k considered in our analysis is the simplest one:

$$x^e_{t+1} = x_{t+1-k}. \tag{2}$$

Then the dynamics under learning is governed by a difference equation of order $n = k-1$,

$$x_t = F(x_{t+1-k}), \qquad k \geqslant 2, \tag{3}$$

which is equivalent to a system of first order difference equations

$$(y'_1, ..., y'_{k-1}) = \tilde{W}(y_1, ..., y_{k-1}),$$

where

$$\tilde{W}: \begin{cases} y'_1 = F(y_{k-1}) \\ y'_2 = y_1 \\ \vdots \\ y'_{k-1} = y_{k-1} \end{cases} \tag{4}$$

and $'$ is the unit time advancement operator. A stable cycle for the dynamics (4) is said to be W-stable, see [4]. Such learning dynamics have three interesting properties:

- there are many non-PFC of period different from k associated with each k periodic PFC;
- only a PFC of period k can be a cycle of period k for the dynamics with learning;
- if a PFC is locally stable under the backward PF dynamics (F-stable, see [5]) then it is locally stable for the dynamics in (4), as stated in [4], and also the associated non-PFC are stable.

These results contribute to the literature showing that under bounded rationality the dynamics of a forward looking economic model may converge to a non-PFC attractor, e.g. see [2]. The expectation function is different from the one considered in [2] and moreover in our framework non-PFC coexist with PF equilibria, whereas in [2] non-PF equilibria appear only when the PF ones lose stability.

2. THE LEARNING DYNAMICS

Given the expectation function (2) which detects period $k \geqslant 2$, the map (4) has the property that its iterate of order $n = k-1$ has separated variables, i.e.,

$$\widetilde{W}^n(y_1, ..., y_n) = (F(y_1), ..., F(y_n)). \tag{5}$$

The main results, stated in the next three propositions, are consequences of this property. Proposition 2.1 characterizes the values of the cycles of the dynamics under learning. Propositions 2.2–2.3 analyze the stability and the period of the cycles for the learning dynamics.

PROPOSITION 2.1. *Every cycle of period p of the forward dynamics with learning* (4) *takes values belonging to some PFC*:

- *if p is a multiple of n then every cycle of period p of the forward dynamics with learning* (4) *is made up of points belonging to PFC of period p/n* (*or its divisors*);
- *if p is not a multiple of n then every cycle of period p of the forward dynamics with learning* (4) *is made up of points belonging to PFC of period p* (*or its divisors*).

Proof. Let p be a multiple of n, i.e., $p = nj$, $j \geqslant 1$. From (5) it follows that $\widetilde{W}^{nj}(y_1, ..., y_n) = (F^j(y_1), ..., F^j(y_n))$. Hence $\widetilde{W}^{nj}(y_1, ..., y_n) = (y_1, ..., y_n)$ if and only if $y_1 = F^j(y_1)$, ..., $y_n = F^j(y_n)$, i.e. cycles of $\widetilde{W}$ of period $p = nj$ are associated with PFC of period j or its divisors.

Let p be not a multiple of n, i.e. $p = nj + i$, $1 \leqslant i \leqslant (n-1)$, $j \geqslant 0$. If $(y_1, ..., y_n)$ belongs to a p-cycle of $\widetilde{W}$, then $\widetilde{W}^{n(nj+i)}(y_1, ..., y_n) = (F^{nj+i}(y_1), ..., F^{nj+i}(y_n)) = (y_1, ..., y_n)$ and therefore $y_k = F^{nj+i}(y_k)$ for each $k = 1, ..., n$ i.e. each y_k is a point of a cycle of period $p = nj + i$ (or its divisors) of the map F. ▮

An F-stable (F-unstable) PFC generates cycles for the *forward dynamics with learning* which are W-stable (W-unstable). We prove this result in two propositions. The first one concerns a PFC of period k.

PROPOSITION 2.2. *Let* $\{\alpha_1, ..., \alpha_k\}$ *be a k-PFC with eigenvalue* $\lambda_\alpha = \prod_{i=1}^{k} DF(\alpha_i)$ *and assume that expectations are formed according to* (2). *Then the forward dynamics with learning* (4) *has* $(k^{k-2} + k - 2)/(k-1)$ *cycles*: *one of period k corresponding to the PFC with eigenvalues given by the* $(k-1)$ *complex roots* $\sqrt[k-1]{\lambda_\alpha}$, *and the remaining* $(k^{k-2} - 1)/(k-1)$ *cycles of period* $k(k-1)$ *with eigenvalues* $\lambda_1 = \cdots \lambda_{k-1} = \lambda_\alpha$.

Proof. Given the array $\tilde{\alpha}=\{\alpha_1, \alpha_2, ..., \alpha_{k-1}\}$, formed by the first $k-1$ points of a k *periodic* cycle of F, i.e., $F(\alpha_1)=\alpha_2, ..., F(\alpha_{k-1})=\alpha_k$ and $F(\alpha_k)=\alpha_1$, it is straightforward to verify that $\tilde{\alpha}$ is a k periodic cycle of $\tilde{W}$. In fact, from (4) we have: $\tilde{W}(\tilde{\alpha})=\{\alpha_k, \alpha_1, \alpha_2, ..., \alpha_{k-2}\}$, $\tilde{W}^2(\tilde{\alpha})=\{\alpha_{k-1}, \alpha_k, \alpha_1, ..., \alpha_{k-3}\}, ...,$ $\tilde{W}^k(\tilde{\alpha})=\{\alpha_1, \alpha_2, ..., \alpha_{k-1}\}$. Any array formed by a permutation with repetition of $(k-1)$ points taken from $\{\alpha_1, ..., \alpha_k\}$ is obtained after $k(k-1)$ applications of $\tilde{W}$. Let $\{\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{k-1}}\}$ be one of such permutations, from (5) we have that

$$\begin{aligned}
\tilde{W}^n\{\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{k-1}}\} &= \{F(\alpha_{i_1}), F(\alpha_{i_2}), ..., F(\alpha_{i_{k-1}})\},\\
\tilde{W}^{2n}\{\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{k-1}}\} &= \{F^2(\alpha_{i_1}), F^2(\alpha_{i_2}), ..., F^2(\alpha_{i_{k-1}})\},\\
&\vdots\\
\tilde{W}^{kn}\{\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{k-1}}\} &= \{F^k(\alpha_{i_1}), F^k(\alpha_{i_2}), ..., F^k(\alpha_{i_{k-1}})\}\\
&= \{\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{k-1}}\}.
\end{aligned}$$

The number of the permutations with repetition from $\tilde{\alpha}$ is k^{k-1}; this number, deprived of the k different permutations generating the k periodic cycle obtained above, becomes $k^{k-1}-k$. Since every cycle of period $k(k-1)$ contains exactly $k(k-1)$ of these permutations, then the number of distinct cycles generated in this way is

$$\frac{k^{k-1}-k}{k(k-1)}=\frac{k^{k-2}-1}{k-1}.$$

Since the Jacobian matrix of (4) is

$$D\tilde{W}(y_1, y_2, ..., y_n)=\begin{bmatrix} 0 & 0 & \cdots & DF(y_n)\\ 1 & \ddots & \ddots & \vdots\\ & \ddots & 0 & 0\\ 0 & \cdots & 1 & 0 \end{bmatrix} \tag{6}$$

and

$$D\tilde{W}^n(y_1, y_2, ..., y_n)=\begin{bmatrix} DF(y_1) & 0 & \cdots & 0\\ 0 & DF(y_2) & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & DF(y_n) \end{bmatrix} \tag{7}$$

where $n=k-1$, then for the PFC we have

$$D\tilde{W}^k(\tilde{\alpha})=\begin{bmatrix} 0 & 0 & \cdots & DF(\alpha_1)\,DF(\alpha_k)\\ DF(\alpha_2) & 0 & & 0\\ 0 & DF(\alpha_3) & \ddots & \vdots\\ 0 & 0 & DF(\alpha_{k-1}) & 0 \end{bmatrix}, \tag{8}$$

so that the corresponding characteristic equation is $\lambda^n = \lambda_\alpha$.

For the stability of the cycles of period $k(k-1) = kn$ we have

$$D\tilde{W}^{kn}(\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{(k-1)}}) = \begin{bmatrix} \prod_{j=1}^{k} DF(\alpha_j) & 0 & \cdots & 0 \\ 0 & \prod_{j=1}^{k} DF(\alpha_j) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \prod_{j=1}^{k} DF(\alpha_j) \end{bmatrix},$$

where $(\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_{(k-1)}})$ is a permutation of $(k-1)$ of the k points of the k periodic cycle of F. Therefore for the cycles of period $k(k-1)$ we have $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda_\alpha$. ∎

Let us consider now the case in which a PFC of period $p \neq k$ exists.

PROPOSITION 2.3. *Let $\mathscr{C}_p = \{\alpha_1, ..., \alpha_p\}$ be a PFC of period $p \neq k$ with eigenvalue $\lambda_\alpha = \prod_{i=1}^{p} DF(\alpha_i)$ and suppose that expectations are formed according to the expectation function* (2). *Then*:

(i) *if p is multiple of $k-1$ then the system* (4) *has* $(p^{k-2})/(k-1)$ *cycles of period $p(k-1)$, all made up of the points of $\mathscr{C}_p$*;

(ii) *if p is not multiple of $k-1$ then the system* (4) *has* $(p^{k-2}-1)/(k-1)$ *cycles of period $p(k-1)$ and one cycle of period p all made up of the points of $\mathscr{C}_p$*;

(iii) *A cycle made up of points of $\mathscr{C}_p$ is W-stable if and only if $\mathscr{C}_p$ is F-stable.*

Proof. Let us consider a vector $\tilde{y} \in \mathscr{R}^n$, $\tilde{y} = (\alpha_{i_1}, ..., \alpha_{i_n})$, with $\{i_1, ..., i_n\} \in \{1, ..., p\}$ and $n = k-1$, i.e. the components of $\tilde{y}$ are a permutation with repetition of the p points of the cycle $\mathscr{C}_p$. From (5) we have that $\tilde{W}^{np}(\tilde{y}) = (F^p(\alpha_{i_1}), ..., F^p(\alpha_{i_n}))$, since $(\alpha_{i_1}, ..., \alpha_{i_n})$ are p-periodic points of F we have $\tilde{W}^{np}(\tilde{y}) = \tilde{y}$. This implies that $\tilde{y}$ is a periodic point of $\tilde{W}$ of period np. In fact it cannot be a periodic point of period which divides np since $W^{np/u}(\tilde{y}) = \tilde{y}$ would imply, from (5), that $F^{p/u}(\alpha_i) = \alpha_i$, i.e. α_{i_j}, $j = 1, ..., n$, belongs to a cycle of F of period less than p, a contradiction. The same argument holds for every permutation of n points taken from the periodic points of $\mathscr{C}_p$, so that the total number of np periodic points of $\tilde{W}$ is p^n. Since each cycle of period np of $\tilde{W}$ is made up of np periodic points then the number of distinct cycles of $\tilde{W}$ of period np is $p^n/(np) = p^{n-1}/n$. This is the number of the cycles of period np if p is a multiple of n.

If p is not multiple of n then from Proposition 2.1 we know that a cycle of $\tilde{W}$ with the same period p exists and therefore the p periodic points of such cycle must be subtracted from the total number of np-periodic points.

Summing up, the number of distinct cycles of $\tilde{W}$ of period np is $(p^n - p)/(np) = (p^{n-1} - 1)/n$.

To prove part (iii) we consider the Jacobian matrix of (4), given by (6). A straightforward computation gives

$$D\tilde{W}^i(y_1, ..., y_n) = \begin{bmatrix} 0 & 0 & \cdots & DF(y_{n-i}) & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & DF(y_{n-i+1}) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & DF(y_n) \\ 0 & 1 & 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \end{bmatrix}$$

and when $i = n$ we obtain (7). Hence, for p multiple of n, say $p = nj$, $j \geqslant 1$, the matrix $D\tilde{W}^{nj}(\alpha_{i_1}, ..., \alpha_{i_n})$ is diagonal with eigenvalues $\lambda_1 = \cdots = \lambda_n = \lambda_\alpha$. The same holds for the cycles of period np generated by a cycle $\mathscr{C}_p$ when p is not multiple of n, say $p = nj + i$, $j \geqslant 0$ and $1 \leqslant i \leqslant (n-1)$, whereas for the cycle of period p the matrix $D\tilde{W}^{nj+i}(\alpha_{i_1}, ..., \alpha_{i_n})$ has the same structure as the matrix $D\tilde{W}^i(\alpha_{i_1}, ..., \alpha_{i_n})$, hence its characteristic equation is $\lambda^n = \lambda_\alpha$. In both cases, if $|\alpha_\alpha| < 1$ then the eigenvalues of all the cycles are inside the unit circle of the complex plane. ∎

A interesting case is obtained when $k = 3$ which has been fully investigated in [1, 7]. In this case, given a F-stable PFC of period three we have that it is stable under learning together with a non-PFC of period six. Given a PFC of period $p \neq 3$ we have that if p is even then (4) has $p/2$ cycles of period $2p$, if p is odd then (4) has $(p-1)/2$ cycles of period $2p$ and one cycle of period p.

In addition to the cycles described above (*homogeneous* type), W-stable cycles of *mixed* type (cycles made up of points belonging to different PFC) can be obtained with an expectation function of the type described in (2) if the map F has more than one F-stable cycles. Following [1, 7], the following Proposition can be stated for $k = 3$.

PROPOSITION 2.4. *Let $\mathscr{C}_3^{(\alpha)} = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathscr{C}_p^{(\beta)} = \{\beta_1, ..., \beta_p\}$ be two cycles of the pap F of period three and p, respectively, with eigenvalues $\lambda_\alpha = \prod_{i=1}^{3} DF(\alpha_i)$ and $\lambda_\beta = \prod_{i=1}^{p} DF(\beta_i)$, and assume that expectations are formed according to the expectation function* (2) *with $k = 3$. Then the forward dynamics with learning* (4) *has the two cycles $\tilde{\mathscr{C}}_3$ and $\tilde{\mathscr{C}}_6$, $p/2$ cycles* *of period* $2p$ *if* p *is even, or* $(p-1)/2$ *cycles of period* $2p$ *and one of period* p *if* p *is odd, all of homogeneous type, and* $3p/m$ *mixed cycles of period* $2m$, *where* m *is the least common multiple between three and* p. *The mixed type cycles are generated as follows* $\tilde{C}^{(s)}_{2m} = \{\tilde{W}^i(\beta_1, \alpha_s), i = 1, ..., 2m\}$, $s = 1, ..., 3p/m$, *the eigenvalues associated with the mixed type cycles are*: $\lambda_1 = \lambda_\alpha^{n_1}$ *and* $\lambda_2 = \lambda_\beta^{n_2}$, *where* $n_1 = m/3$ *and* $n_2 = m/p$.

The existence of *mixed* type cycles can be easily verified iterating $2m$ times each point (β_1, α_s), $s = 1, ..., 3p/m$; it can be easily verified that for $s > 3p/m$ one of the cycles described above is obtained again. A similar result for a generic p, q and k can be stated in our setting. As an example, suppose that the map F is characterized by the stable PFSE β and the stable PFC of period three $C_3^{(\alpha)}$. Then a stable cycle of *mixed* type of period six exists for the *forward dynamics with learning*, i.e. $\tilde{C}_6 = \{(\beta, \alpha_1), (\alpha_2, \beta), (\beta, \alpha_2), (\alpha_3, \beta), (\beta, \alpha_3), (\alpha_1, \beta)\}$, the corresponding six period cycle for the state x_t is $C_6 = \{\beta, \alpha_2, \beta, \alpha_3, \beta, \alpha_1\}$.

3. TWO EXAMPLES

Let us consider a model of the type in (1) with the standard logistic map $F(x) = \mu x(1-x)$, $\mu > 0$. Such a map has been considered by many authors as the prototype of hill-shaped function, often appearing in overlapping generations models, see [4, 6]. For $\mu = 3.83$ the map has a stable cycle of period three $\mathscr{C}_3 = \{\alpha_1, \alpha_2, \alpha_3\} = \{0.1561494..., 0.5046667..., 0.9574166\cdots\}$ with eigenvalue $\lambda_\alpha = 0.325$. Let us assume that the agents believe in a three periodic evolution of the economy and that they form their expectations as in (2) with $k = 3$. Then the *dynamics with learning* is described by the second order difference equation

$$x_t = \mu x_{t-2}(1 - x_{t-2}) \tag{9}$$

and the equivalent system of two first order difference equations becomes

$$\tilde{W}: \begin{cases} y_1' = \mu y_2(1 - y_2) \\ y_2' = y_1 \end{cases}. \tag{10}$$

According to the analysis developed above, the map (10) has the PFC $\{(\alpha_1, \alpha_2), (\alpha_3, \alpha_1), (\alpha_2, \alpha_3)\}$ and the non-PFC of period six, $\{(\alpha_1, \alpha_1), (\alpha_2, \alpha_1), (\alpha_2, \alpha_2), (\alpha_3, \alpha_2), (\alpha_3, \alpha_3), (\alpha_1, \alpha_3)\}$. The points of the corresponding cycles of (9) are obtained by taking the first coordinate of each point of the cycles of (10). Since the *forward dynamics with learning* has two coexisting attractors, it is important to delimitate their basins of attraction. In Fig. 1 the two basins of attraction are represented by different colors in

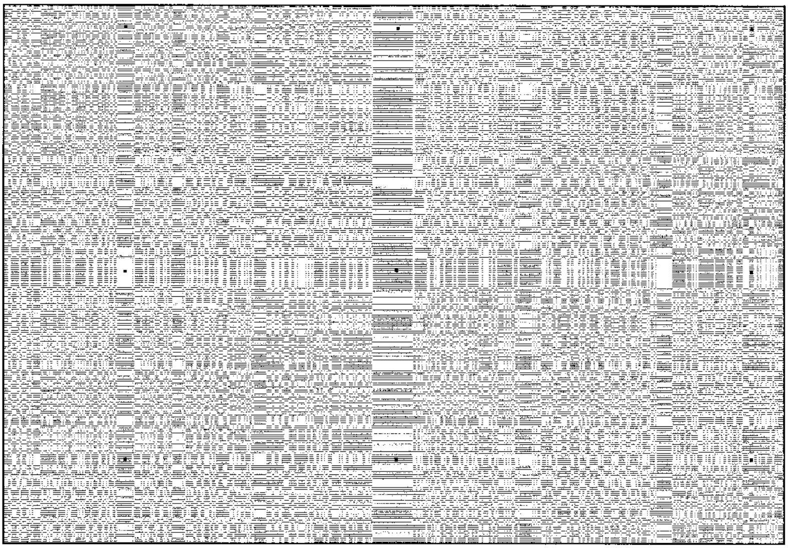

FIG. 1. Numerical representation, in the two-dimensional phase space of the dynamical system (10), of the basins of attraction of the two stable cycles existing for $\mu = 3.83$: the white region represents the basin of the cycle of period 3 (PFC), the grey region represents the basin of the cycle of period 6. The black dots represent the periodic points.

the phase plane of the two-dimensional dynamical system (10): the white region represents the set of points belonging to the basin of the cycle of period three and the grey-shaded region represents the basin of the stable cycle of period six (the black dots inside the white region and the block dots inside the grey region represent the periodic points of the cycle of period three and six, respectively).

From this figure it turns out that the basin of attraction of the three periodic PFC is smaller than the basin of the non-PFC of period six. Therefore, starting from a random initial condition, the *dynamics with learning* has a large probability of converging to the not PFC.

Similar results can be obtained if the logistic map has a stable PFC of period $k > 3$ and the agents use the expectation function (2) which detects cycles of period k. For example, with $\mu = 3.5$ the logistic map has a stable cycle of period four, $\mathscr{C}_4 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{0.5009..., 0.8741..., 0.3838..., 0.8269...\}$ with eigenvalue $\lambda_\alpha = -0.03$. If the expectation function (2) with $k = 4$ is considered then, according to Proposition 2.2, the three-dimensional dynamical system (4) has six distincts cycles: the PFC of period four $\tilde{C}_4 = \{\tilde{W}^i(\alpha_1, \alpha_2, \alpha_3),\ i = 1, ..., 4\}$, with eigenvalues given by the three

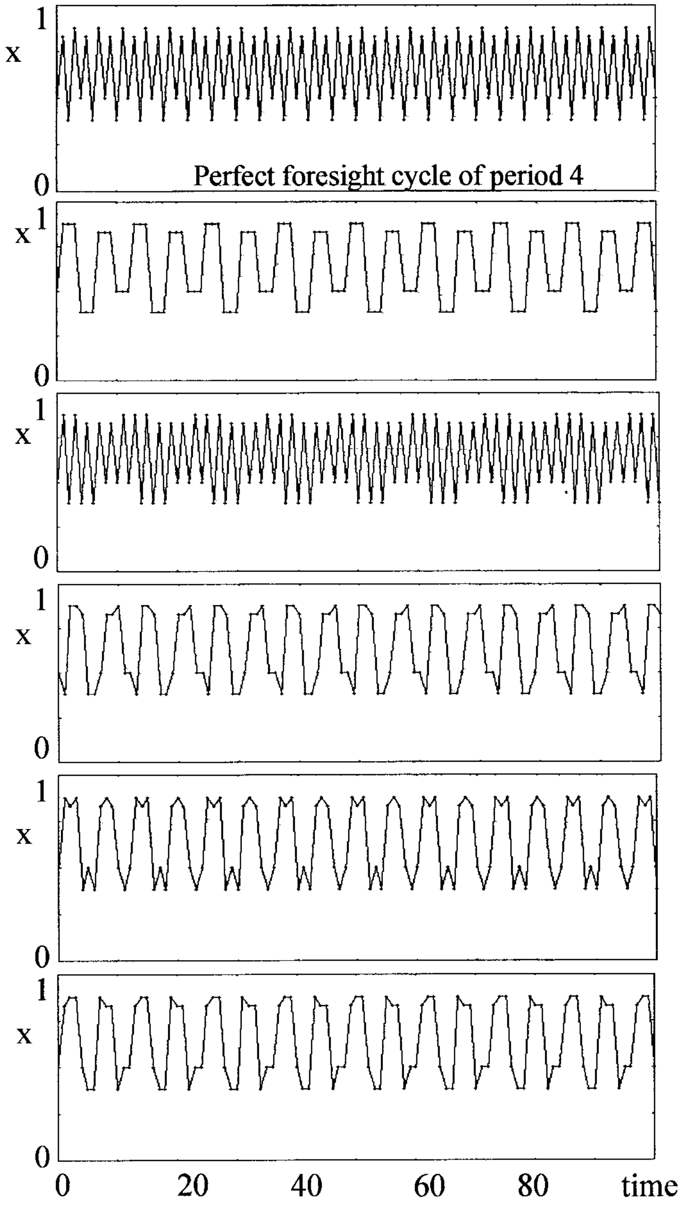

FIG. 2. Versus-time representation of six trajectories of the forward dynamics with learning obtained by the iteration of (4) with logistic map with $\mu = 3.5$ and expectation function (2) with $k = 4$. The six trajectories converge to six different stable cycles: one of period 4 (PFC) and five of period 12.

complex roots $\sqrt[3]{\lambda_\alpha}$, and five cycles of period twelve given by $\tilde{C}_{12}^{(1)} = \{\tilde{W}^i(\alpha_1, \alpha_1, \alpha_1),\ i=1, ..., 12\}$, $\tilde{C}_{12}^{(2)} = \{\tilde{W}^i(\alpha_1, \alpha_2, \alpha_1),\ i=1, ..., 12\}$, $\tilde{C}_{12}^{(3)} = \{\tilde{W}^i(\alpha_1, \alpha_1, \alpha_2),\ i=1, ..., 12\}$, $\tilde{C}_{12}^{(4)} = \{\tilde{W}^i(\alpha_1, \alpha_3, \alpha_1),\ i=1, ..., 12\}$, $\tilde{C}_{12}^{(5)} = \{\tilde{W}^i(\alpha_1, \alpha_1, \alpha_3),\ i=1, ..., 12\}$, with eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_\alpha$. By taking six different initial conditions we can easily obtain convergence to the six different cycles, represented in Fig. 2. The cycle shown in the upper graph is the PFC of period 4, the other graphs show the five cycles of period 12. Each of these stable cycles has its own basin of attraction which belongs to $\mathfrak{R}^3$. The basins of attraction look like six three-dimensional boxes with an intermingled pattern similar to that in Fig. 1.

To highlight the results on *mixed* type non-PFC we can consider a model characterized by multiple PFS, for example the one in [3] where an overlapping generations model with production and increasing returns to labor is analyzed. In this setting the forward-looking map (1) has multiple PFSE. In particular, if strong enough increasing returns are considered then we have four stationary equilibria, say $\{x_0, x_L, x_U, x_H\}$, of which x_L and x_H are F-stable and x_0 and x_U are F-unstable. If the agents compute x_{t+1}^e using an expectation function of the form in (2) with $k > 2$, then the model with learning has stable cycles of period $(k-1)$ made up of sequences of x_L and x_H. For example, with $k=3$, the *dynamics with learning* can converge to the periodic sequence $\{... x_L, x_H, x_L, x_H, ...\}$, whereas with $k=4$, the *dynamics with learning* has two possible three periodic cycles, namely $\{... x_L, x_H, x_L, x_L, x_H, x_L, ...\}$ and $\{... x_H, x_L, x_H, x_H, x_L, x_H, ...\}$. Both these periodic solutions of the model with learning are stable, each with its own basin of attraction.

ACKNOWLEDGMENTS

The first author thanks the DIMADEFAS (University of Florence) for the kind hospitality during the preparation of this work. The second and the third author were supported by a MURST grant for the research project. "Dinamiche non lineari ed applicazioni alle scienze economiche e sociali" and by a CNR grant. We thank an associate editor for useful comments and suggestions. The usual disclaimers apply.

REFERENCES

1. G. I. Bischi, C. Mammana, and L. Gardini, Multistability and cyclic attractors in duopoly games, *Chaos, Solitons Fractals*, forthcoming (1998).
2. J. Bullard, Learning equilibria, *J. Econ. Theory* **64** (1994), 468–485.
3. G. Evans and S. Honkapohja, Increasing social returns, learning, and bifurcation phenomena, *in* "Learning and Rationality in Economics" (A. Kirman and M. Salmon, Eds.), pp. 216–235, Blackwell, Oxford, 1995.

4. J. Grandmont, On endogenous competitive business cycle, *Econometrica* **53** (1985), 995–1045.
5. J. Grandmont and G. Laroque, Stability of cycles and expectations, *J. Econ. Theory* **40** (1986), 138–151.
6. R. Guesnerie and M. Woodford, Endogeneous fluctuations, *in* “Advances in Economic Theory” (J. Laffont, Ed.), pp. 288–412, Cambridge University Press, Cambridge, UK, 1993.
7. L. Lupini, S. Lenci, and L. Gardini, Bifurcations and multistability in a class of two-dimensional endomorphisms, *Nonlinear Anal. Theory Appl.* (1997), 61–85.