Applied Mathematical Finance,
Vol. 12, No. 1, 17–52, March 2005

The Dynamic Interaction of Speculation and Diversification

CARL CHIARELLA*, ROBERTO DIECI** & LAURA GARDINI†

**School of Finance and Economics, University of Technology Sydney, P.O. Box 123, Sydney, NSW 2007, Australia, **Dipartimento di Matematica per le Scienze Economiche e Sociali, University of Bologna Viale Q. Filopanti, 5, I-40126 Bologna, Italy, †Istituto di Scienze Economiche, University of Urbino, I-61029 Urbino, Italy*

(Received 28 November 2003; revised 16 May 2004)

ABSTRACT *A discrete time model of a financial market is developed, in which heterogeneous interacting groups of agents allocate their wealth between two risky assets and a riskless asset. In each period each group formulates its demand for the risky assets and the risk-free asset according to myopic mean-variance maximizazion. The market consists of two types of agents:* fundamentalists, *who hold an estimate of the fundamental values of the risky assets and whose demand for each asset is a function of the deviation of the current price from the fundamental, and* chartists, *a group basing their trading decisions on an analysis of past returns. The time evolution of the prices is modelled by assuming the existence of a* market maker, *who sets excess demand of each asset to zero at the end of each trading period by taking an offsetting long or short position, and who announces the next period prices as functions of the excess demand for each asset and with a view to long-run market stability. The model is reduced to a seven-dimensional nonlinear discrete-time dynamical system, that describes the time evolution of prices and agents' beliefs about expected returns, variances and correlation. The unique steady state of the model is determined and the local asymptotic stability of the equilibrium is analysed, as a function of the key parameters that characterize agents' behaviour. In particular it is shown that when chartists update their expectations sufficiently fast, then the stability of the equilibrium is lost through a supercritical Neimark–Hopf bifurcation, and self-sustained price fluctuations along an attracting limit cycle appear in one or both markets. Global analysis is also performed, by using numerical techniques, in order to understand the role played by the chartists' behaviour in the transition to a regime characterized by irregular oscillatory motion and coexistence of attractors. It is also shown how changes occurring in one market may affect the price dynamics of the alternative risky asset, as a consequence of the dynamic updating of agents' portfolios.*

Introduction

One of the key ideas in modern finance is that of diversification among a choice of risky assets. The standard framework is that of one-period mean-variance optimization. A key assumption in this framework is that of rational, homogeneous

Correspondence Address: Roberto Dieci, Dipartimento di Matematica per le Scienze Economiche e Sociali, University of Bologna, Viale Q. Filopanti, 5, I-40126 Bologna, Italy. Email: rdieci@rimini.unibo.it

1350-486X Print/1466-4313 Online/05/010017–36
DOI: 10.1080/1350486042000260072

agents who have complete knowledge of the distribution of asset returns in the next period. Studies involving surveys of market participants indicate that investors may not be homogeneous and that at least two groups (or two groups of strategies) coexist in the market (Taylor and Allen, 1992; Lui and Mole, 1998). The rationality assumption, particularly with regard to perfect knowledge of the future distribution of returns, has started to look tenuous as some economists have come to accept that extreme versions of rationality give agents more ability to learn that may be possible in real markets (Gallegati and Kirman, 2000). Thus interest has grown in recent years in models in which agent heterogeneity is allowed and the different groups of agents seek to learn about the future return distribution using different information sets. We cite in particular the models of asset price dynamics based on the interaction of *heterogeneous agents* that have been proposed by Caginalp and Ermentrout (1990, 1991), Day and Huang (1990), Brock and Hommes (1998), Lux (1998), Chen and Yeh (1997), Gaunersdorfer (2000), Chiarella and He (2001, 2002, 2003), Chiarella, Dieci and Gardini (2001, 2002), and Fernandez-Rodriguez *et al.*, (2002). These models in general consider a financial market with one risky asset and one risk-free asset and focus on the effect of agents' heterogeneous beliefs about expected return and volatility of the risky asset and different risk attitudes on the dynamics of asset prices and wealth.[1] They show how the interaction of these factors with the heterogeneity of the agents and the market trading mechanism can, even in the absence of external random events, cause sustained deviations in prices away from their equilibrium. Such sustained deviations in the underlying deterministic driving dynamics constitute one possible source of the fat tails and volatility clustering that are a key feature of asset returns in financial markets. The heterogeneous agents paradigm of asset price fluctuations views the financial market as being the result of the interaction between nonlinear deterministic elements and stochastic elements.

In the development of portfolio theory the one risky/one risk-free asset model is merely a first step to understanding how investors will spread their investment dollar among several risky assets. Once several risky assets are available to the investor then correlation (or rather the beliefs about correlation) also becomes a factor in the investors' decision process. A natural question that arises in this context is *whether diversification tends to dampen or to amplify the price fluctuations that arise due to the interaction of heterogeneous agents*, and whether agents' beliefs about correlation of returns may generate comovements in the prices of the risky assets. Another question is to what extent markets may become interdependent (i.e. price dynamics in one market may be affected by changes in agents' behaviour and beliefs in the alternative market), as a result of agents' portfolio diversification.

In this paper we develop a discrete time model of financial market dynamics, which combines the essential elements of the interacting heterogeneous agents paradigm with the classical model of *diversification* between two risky assets and a risk-free asset. In common with the earlier cited literature, we assume that the market consists of two types of traders: *fundamentalists*, who hold an estimate of the fundamental value of the risky assets and whose demand for each asset is a function of the deviation of the current price from the fundamental, and *chartists*, a group basing their trading decisions on an analysis of past returns. Each group forms expectations about asset returns and their variance–covariance structure and allocates its wealth between two risky assets and a riskless asset. The time evolution of the prices of the risky assets is modelled by assuming the existence of a *market maker*, who sets excess demand to zero at the end of each trading period by taking an offsetting long or short position, and who announces the next period prices on the basis of the excess demand. The model is reduced to a seven-dimensional nonlinear discrete-time dynamical system that describes the time evolution of prices and agents' beliefs about expected returns, variances and correlation. The local asymptotic stability conditions of the unique equilibrium are investigated using both analytical and numerical techniques: in particular we clarify how the local stability is affected by the key parameters, namely the *strength* of fundamentalist and chartist demands at the steady state (inversely related to agents' risk aversion and beliefs about volatility), the speed of reaction of market prices, the chartist extrapolation parameter, and the long-run variance–covariance structure. The local stability analysis, together with the global analysis performed through numerical experiments, also help us to understand how chartists' beliefs and behavior may cause price fluctuations to become more and more irregular and to be transmitted from one market to the other.

The structure of the paper is as follows. Section 2 derives the asset demand functions for each asset by each investor type. Section 3 describes the schemes used by each group to revise expectations. Section 4 describes how demands are aggregated by the market maker via a price adjustment rule in the market for each asset. Section 5 describes the resulting dynamical system for the dynamic evolution of prices, expected returns, variances and correlation. Section 6 considers the particular case of 'zero long-run correlation' between returns, outlines the main analytical results about the conditions of local asymptotic stability of the unique steady state of the model and their dependence on the key parameters, and explores the dynamic behaviour of the model when the parameters are such that the steady state is unstable. Section 7 considers the general case of 'nonzero long-run correlation' between returns and gives numerical examples where this parameter can cause the transmission of price fluctuations from one market to the other. Section 8 considers, via numerical simulations, the effect of changes in agents' perceptions about long-run risk and return. It turns out that such changes can have a dramatic effect on the price dynamics. Section 9 contains some conclusions and final remarks.

Asset Demand

Our starting point is the fundamentalist/chartist model studied in Chiarella, Dieci and Gardini (2001, 2002) and Fernandez-Rodriguez *et al.* (2002), whose antecedents are Zeeman (1974), Beja and Goldman (1980), Day and Huang (1990), and Chiarella (1992). It should be stressed that the framework we use in this section to derive the asset demands is in fact that of the standard one-period CAPM; the major difference being that we allow agents to have heterogeneous beliefs about the distribution of future returns and that these beliefs are updated dynamically as a function of observed returns.

We denote by $P_{i,t}$ the logarithm of the price of the ith risky asset at time t ($i=1, 2$), and use the subscript $j \in \{f, c\}$ to denote fundamentalists or chartists. In each time period each group of agents is assumed to invest some of its wealth in the risky assets and some in the risk-free asset. Denote, respectively, by $\Omega_t^{(j)}$ and $Z_{i,t}^{(j)}$ the wealth of agent j at time t and the fraction that agent j decides to invest in the ith risky asset. The evolution of the wealth of agent j can then be written

$$\Omega_{t+1}^{(j)} = \Omega_t^{(j)} + \Omega_t^{(j)}\left(1 - Z_t^{(j)}\right)g + \Omega_t^{(j)}\left[Z_{1,t}^{(j)}(P_{1,t+1} - P_{1,t} + G_{1,t+1}) + Z_{2,t}^{(j)}(P_{2,t+1} - P_{2,t} + G_{2,t+1})\right]$$

where $Z_t^{(j)} = Z_{1,t}^{(j)} + Z_{2,t}^{(j)}$ is the fraction invested in the risky assets, g is the (constant) risk-free rate of return, $G_{i,t+1}$, $(P_{i,t+1} - P_{i,t})$ and $(P_{i,t+1} - P_{i,t} + G_{i,t+1})$, are the *dividend yield*, the *capital gain* and the *return* of the ith asset in period $(t, t+1)$, respectively.

We denote by $E_t^{(j)}$, $Var_t^{(j)}$, $Cov_t^{(j)}$ the 'beliefs' of investor type j, at time t, about conditional expectation, variance, and covariance, respectively. We assume that investor type j has exponential utility of wealth function $u(\Omega) = -\exp(-\alpha^{(j)}\Omega)$, where $\alpha^{(j)}$ is agent j's risk aversion coefficient. Agent j seeks the fractions $Z_{i,t}^{(j)}$, so as to maximize expected utility of wealth at time $t+1$

$$E_t^{(j)}\left[-\exp\left(-\alpha^{(j)}\Omega_{t+1}^{(j)}\right)\right]$$

As is well known, under the assumption of conditional normality of returns, this problem is equivalent to

$$\max_{Z_{1,t}^{(j)}, Z_{2,t}^{(j)}} \left\{ E_t^{(j)}\left[\Omega_{t+1}^{(j)}\right] - \frac{\alpha^{(j)}}{2} Var_t^{(j)}\left[\Omega_{t+1}^{(j)}\right] \right\}$$

The first order conditions of the foregoing optimisation problem lead to the demand functions for the risky assets, given by

$$\zeta_{1,t}^{(j)} = Z_{1,t}^{(j)}\Omega_t^{(j)} = \frac{V_{2,t}^{(j)}\left(m_{1,t}^{(j)} + g_{1,t}^{(j)} - g\right) - \rho_t^{(j)}\sqrt{V_{1,t}^{(j)}V_{2,t}^{(j)}}\left(m_{2,t}^{(j)} + g_{2,t}^{(j)} - g\right)}{\alpha^{(j)}\left(1 - \rho_t^{(j)^2}\right)V_{1,t}^{(j)}V_{2,t}^{(j)}} \tag{1}$$

$$\zeta_{2,t}^{(j)} = Z_{2,t}^{(j)}\Omega_t^{(j)} = \frac{V_{1,t}^{(j)}\left(m_{2,t}^{(j)} + g_{2,t}^{(j)} - g\right) - \rho_t^{(j)}\sqrt{V_{1,t}^{(j)}V_{2,t}^{(j)}}\left(m_{1,t}^{(j)} + g_{1,t}^{(j)} - g\right)}{\alpha^{(j)}\left(1 - \rho_t^{(j)^2}\right)V_{1,t}^{(j)}V_{2,t}^{(j)}} \tag{2}$$

where

$$m_{i,t}^{(j)} \equiv E_t^{(j)}[P_{i,t+1} - P_{i,t}], \quad g_{i,t}^{(j)} \equiv E_t^{(j)}[G_{i,t+1}]$$
$$V_{i,t}^{(j)} \equiv Var_t^{(j)}[P_{i,t+1} - P_{i,t} + G_{i,t+1}]$$

and $\rho_t^{(j)}$ is agent j's 'belief', at time t, about the correlation between the risky returns over the next trading period, i.e.

$$\rho_t^{(j)} = \frac{Cov_t^{(j)}[(P_{1,t+1} - P_{1,t} + G_{1,t+1}), (P_{2,t+1} - P_{2,t} + G_{2,t+1})]}{\sqrt{V_{1,t}^{(j)}V_{2,t}^{(j)}}}$$

The demand functions (1) and (2) can be rewritten as

$$\zeta_{1,t}^{(j)} = \frac{1}{\left(1-\rho_t^{(j)^2}\right)} \frac{\left(m_{1,t}^{(j)} + g_{1,t}^{(j)} - g\right)}{\alpha^{(j)} V_{1,t}^{(j)}} - \frac{\rho_t^{(j)} \sqrt{V_{2,t}^{(j)}}}{\left(1-\rho_t^{(j)^2}\right)\sqrt{V_{1,t}^{(j)}}} \frac{\left(m_{2,t}^{(j)} + g_{2,t}^{(j)} - g\right)}{\alpha^{(j)} V_{2,t}^{(j)}}$$

$$\zeta_{2,t}^{(j)} = \frac{1}{\left(1-\rho_t^{(j)^2}\right)} \frac{\left(m_{2,t}^{(j)} + g_{2,t}^{(j)} - g\right)}{\alpha^{(j)} V_{2,t}^{(j)}} - \frac{\rho_t^{(j)} \sqrt{V_{1,t}^{(j)}}}{\left(1-\rho_t^{(j)^2}\right)\sqrt{V_{2,t}^{(j)}}} \frac{\left(m_{1,t}^{(j)} + g_{1,t}^{(j)} - g\right)}{\alpha^{(j)} V_{1,t}^{(j)}}$$

with a fairly standard interpretation as a *direct* demand (the first term) and a *hedging* demand (the second term). The demand for each risky asset is a linear combination of the expected risk adjusted excess return on each asset, with coefficients being determined by the expected correlation coefficient.

In the next section we describe how agents update these 'beliefs' about future asset returns and so generate different demand functions.

Expectation Formation

The two groups of agents differ in the way they update their expectations of the means, variances and correlation between returns over successive time intervals. For simplicity it is assumed that the dividend yields are i.i.d. and uncorrelated with price changes in agents' beliefs, and that agents share the same beliefs about the dividend yields, with $E_t(G_{i,t+1}) \equiv g_i$, $Var_t(G_{i,t+1}) \equiv \sigma_i^2$, $i=1, 2$, $Cov_t(G_{1,t+1}, G_{2,t+1}) \equiv \delta\sigma_1\sigma_2$. The common beliefs about variances (σ_i^2, $i=1, 2$) and correlation (δ) of the dividend yields determine the 'long-run' or 'equilibrium' variance/covariance structure of returns in this model.

Agents' heterogeneity is introduced by assuming that the different agent-types use different ways to form expectations about the 'price' component of the return ($P_{i,t+1} - P_{i,t}$), i.e. fundamentalists and chartists are assumed to have heterogeneous beliefs about expected prices changes, as well as their volatility and correlation.

3.1. Fundamentalist Expectations

We assume that fundamental values of the risky assets grow at constant rates over time, i.e. for each asset the log fundamental value evolves according to

$$W_{i,t+1} = W_{i,t} + \gamma_i$$

so that there is an underlying growth rate of γ_i in each market ($\gamma_i \geq 0$).[2] Fundamentalists are assumed to know both the fundamental price $W_{i,t}$ and the fundamental growth rate γ_i. They believe that the expected return of asset i contains a 'long-run' or 'equilibrium' component and a 'short-run' component, the latter being proportional to the difference between the log asset price $P_{i,t}$ and the log fundamental value $W_{i,t}$. Hence they calculate expected change in log-price according to

$$m_{i,t}^{(f)} \equiv E_t^{(f)}[P_{i,t+1} - P_{i,t}] = \eta_i(W_{i,t} - P_{i,t}) + \gamma_i$$

where $\eta_i > 0$ represents the estimate of the speed of reversion to the fundamental price. We also assume that fundamentalist beliefs about variances and correlation do not vary over time, being given by the long-run variance/covariance structure that is determined by that of the dividend yield process. Thus we set

$$V_{i,t}^{(f)} = \sigma_i^2, \quad \rho_t^{(f)} = \delta \quad \text{so that} \quad Cov_t^{(f)} = \delta\sigma_1\sigma_2$$

The fundamentalist demand functions thus become

$$\zeta_{1,t}^{(f)} = a_1(W_{1,t} - P_{1,t}) - b_2(W_{2,t} - P_{2,t}) + h_1$$
$$\zeta_{2,t}^{(f)} = a_2(W_{2,t} - P_{2,t}) - b_1(W_{1,t} - P_{1,t}) + h_2$$

where

$$a_1 = \frac{\eta_1}{\alpha^{(f)}(1-\delta^2)\sigma_1^2} \quad a_2 = \frac{\eta_2}{\alpha^{(f)}(1-\delta^2)\sigma_2^2}$$
$$b_1 = \frac{\delta\eta_1}{\alpha^{(f)}(1-\delta^2)\sigma_1\sigma_2} \quad b_2 = \frac{\delta\eta_2}{\alpha^{(f)}(1-\delta^2)\sigma_1\sigma_2}$$
$$h_1 = \frac{\sigma_2^2\pi_1 - \delta\sigma_1\sigma_2\pi_2}{\alpha^{(f)}(1-\delta^2)\sigma_1^2\sigma_2^2} \quad h_2 = \frac{\sigma_1^2\pi_2 - \delta\sigma_1\sigma_2\pi_1}{\alpha^{(f)}(1-\delta^2)\sigma_1^2\sigma_2^2} \tag{3}$$

and the $\pi_i \equiv (\gamma_i + g_i - g)$, represent the long-run expected excess returns (risk premia) for each asset, determined by the growth of fundamentals and dividend yields. Notice also that h_1 and h_2 represent the long-run or equilibrium component of fundamentalist demand for each asset.

These demand functions generalize in a straightforward way the corresponding fundamentalist demand function derived in Chiarella, Dieci and Gardini (2002) for the case of one risky and one riskless asset. In fact if we set $\delta = 0$, so that fundamentalists expect no correlation between the risky assets, then the demand function for each risky asset is merely the one obtained in Chiarella, Dieci and Gardini (2002).

3.2. Chartist expectations

Chartists are assumed to compute expected returns by extrapolating past price changes according to the weighted average (with exponentially decreasing weights) scheme

$$\psi_{i,t} = m_{i,t}^{(c)} \equiv E_t^{(c)}[P_{i,t+1} - P_{i,t}] = \sum_{s=0}^{\infty} c_i(1-c_i)^s(P_{i,t-s} - P_{i,t-s-1})$$

which results in the well-known adaptive expectations scheme:

$$\psi_{i,t} = (1 - c_i)\psi_{i,t-1} + c_i(P_{i,t} - P_{i,t-1}) \tag{4}$$

The chartist *extrapolation parameter* c_i ($0<c_i<1$) represents the weight given to the most recent price change in the computation of the next period expected price change: the higher is c_i, the more sensitive are chartists to recent data in their expectation formation.

The chartists' beliefs about variances and correlation have two components. One component is the long-run variance–covariance structure of the dividend yield process, the second component is one that varies as a function of the evolution of returns. In particular we assume that

$$V_{i,t}^{(c)} = v_{i,t} + \sigma_i^2 \tag{5}$$

and

$$\rho_t^{(c)} = \frac{K_t + \delta\sigma_1\sigma_2}{\sqrt{\left(v_{1,t} + \sigma_1^2\right)\left(v_{2,t} + \sigma_2^2\right)}} \tag{6}$$

where σ_i^2 and $\delta\sigma_1\sigma_2$ are the long-run components of the conditional variances and covariance, respectively. In this way, chartists' estimates of the variances and covariance include time-varying components that may change in each period according to the observed volatility and the observed comovements in prices. We assume that the time varying components, $v_{i,t}$ and K_t in Equations 5 and 6 are computed by chartists by extrapolating past deviations from expected returns according to

$$v_{i,t} = \sum_{s=0}^{\infty} c_i(1 - c_i)^s \left(P_{i,t-s} - P_{i,t-s-1} - \psi_{i,t}\right)^2 \tag{7}$$

$$K_t = \sum_{s=0}^{\infty} c_K(1 - c_K)^s \left(P_{1,t-s} - P_{1,t-s-1} - \psi_{1,t}\right)\left(P_{2,t-s} - P_{2,t-s-1} - \psi_{2,t}\right) \tag{8}$$

Equations 5 and 7 state that chartists increase their estimate of the variance of each risky asset in proportion to the historical volatility; Equations 6 and 8 state that they adjust their estimate of correlation between the two risky assets in proportion to the historical correlation (calculated using geometrically declining weighting functions with parameters c_i and c_K).[3]

Equations 7 and 8 (see the Appendix) result in the updating rules

$$v_{i,t} = (1 - c_i)v_{i,t-1} + c_i(1 - c_i)\left(P_{i,t} - P_{i,t-1} - \psi_{i,t-1}\right)^2$$

$$\begin{aligned}
K_t = {} & (1 - c_K)K_{t-1} + c_K(1 - c_1)(1 - c_2)\left(P_{1,t} - P_{1,t-1} - \psi_{1,t-1}\right)\left(P_{2,t} - P_{2,t-1} - \psi_{2,t-1}\right) \\
& + (1 - c_K)\Big[c_1\left(P_{1,t} - P_{1,t-1} - \psi_{1,t-1}\right)\left(\psi_{2,t-1} - \widetilde{\psi}_{2,t-1}\right) \\
& + c_1c_2\left(P_{1,t} - P_{1,t-1} - \psi_{1,t-1}\right)\left(P_{2,t} - P_{2,t-1} - \psi_{2,t-1}\right) \\
& + c_2\left(P_{2,t} - P_{2,t-1} - \psi_{2,t-1}\right)\left(\psi_{1,t-1} - \widetilde{\psi}_{1,t-1}\right)\Big]
\end{aligned}$$

where the subsidiary quantities $\widetilde{\psi}_{i,t}(i=1,2)$ are defined recursively as

$$\widetilde{\psi}_{i,t} = (1-c_K)\,\widetilde{\psi}_{i,t-1} + c_K(P_{i,t}-P_{i,t-1})$$

In terms of these updating rules the chartist demand functions for the risky assets thus become

$$\zeta_{1,t}^{(c)} = \frac{\left(v_{2,t}+\sigma_2^2\right)\left(\psi_{1,t}+g_1-g\right) - \rho_t^{(c)}\sqrt{\left(v_{1,t}+\sigma_1^2\right)\left(v_{2,t}+\sigma_2^2\right)}\left(\psi_{2,t}+g_2-g\right)}{\alpha^{(c)}\left(1-\left(\rho_t^{(c)}\right)^2\right)\left(v_{1,t}+\sigma_1^2\right)\left(v_{2,t}+\sigma_2^2\right)} \tag{9}$$

$$\zeta_{2,t}^{(c)} = \frac{\left(v_{1,t}+\sigma_1^2\right)\left(\psi_{2,t}+g_2-g\right) - \rho_t^{(c)}\sqrt{\left(v_{1,t}+\sigma_1^2\right)\left(v_{2,t}+\sigma_2^2\right)}\left(\psi_{1,t}+g_1-g\right)}{\alpha^{(c)}\left(1-\left(\rho_t^{(c)}\right)^2\right)\left(v_{1,t}+\sigma_1^2\right)\left(v_{2,t}+\sigma_2^2\right)} \tag{10}$$

where $\rho_t^{(c)}$ is given by Equation 6. As with the fundamentalist demand functions, the chartist demand functions (9) and (10) also generalize to the case of two risky assets the one derived for the one risky asset case by Chiarella, Dieci and Gardini (2002), the difference being the time varying correlation $\rho_t^{(c)}$ that determines the hedging demand components.

Clearing the Market: The Role of the Market Maker

We assume that the market clearing function is performed by a *market maker*, who is sufficiently rational to know the fundamental price, as well as the dividend growth rate in each market. The market maker is also assumed to be able to estimate the 'equilibrium' demand for each asset.[4] The market maker adjusts the prices in each market according to the price setting rules

$$P_{i,t+1} = P_{i,t} + \beta_i\left[\zeta_{i,t}^{(f)} + \zeta_{i,t}^{(c)} - \lambda_{i,t}\right]$$

which imply that he/she increases (decreases) the price of the ith asset, $i=1, 2$, when the demand for the asset is higher (lower) than a certain threshold. The 'thresholds' $\lambda_{i,t}$ are chosen by the market maker in a way such that prices are equal to fundamentals in the long run. In this way the market maker ensures long-run market stability.

The market clearing mechanism that we have chosen here is of course highly schematized. It captures the essence of stated role of the market-maker of ensuring 'orderly market conditions'. In reality of course market makers would also be giving consideration to their inventory positions. However, taking account of these aspects of market maker behaviour would result in a far more complicated model and must be left for future research. We also leave for future research consideration of other market clearing mechanisms such as limit-order markets. One possible framework into which one could incorporate the ideas of this paper is provided by Chiarella and Iori (2002).

It is convenient to define the new variables

$$q_{i,t} = P_{i,t} - W_{i,t}$$

the deviation of (log) price from (log) fundamental value, and

$$\xi_{i,t} = \psi_{i,t} - \gamma_i$$
$$\widetilde{\xi}_{i,t} = \widetilde{\psi}_{i,t} - \gamma_i$$

the expected deviations from the growth trend, calculated using weights c_i and c_K, respectively. These latter two state variables cater for the fact that chartists use different weights for each asset and for the correlation between them. In terms of these new variables, the time evolution of prices and agents' beliefs about expected returns, variances and correlation is described by the nine-dimensional dynamical system

$$\begin{aligned} q_{1,t+1} &= q_{1,t} - \gamma_1 + \beta_1 \left[-a_1 q_{1,t} + b_2 q_{2,t} + h_1 + \zeta_{1,t}^{(c)} - \lambda_{1,t} \right] \\ \xi_{1,t+1} &= (1-c_1)\xi_{1,t} + c_1(q_{1,t+1} - q_{1,t}) \end{aligned} \tag{11}$$

$$\begin{aligned} q_{2,t+1} &= q_{2,t} - \gamma_2 + \beta_2 \left[-a_2 q_{2,t} + b_1 q_{1,t} + h_2 + \zeta_{2,t}^{(c)} - \lambda_{2,t} \right] \\ \xi_{2,t+1} &= (1-c_2)\xi_{2,t} + c_2(q_{2,t+1} - q_{2,t}) \end{aligned} \tag{12}$$

$$\begin{aligned} v_{1,t+1} &= (1-c_1)v_{1,t} + c_1(1-c_1)\left(q_{1,t+1} - q_{1,t} - \xi_{1,t}\right)^2 \\ v_{2,t+1} &= (1-c_2)v_{2,t} + c_2(1-c_2)\left(q_{2,t+1} - q_{2,t} - \xi_{2,t}\right)^2 \\ K_{t+1} &= (1-c_K)K_t + c_K(1-c_1)(1-c_2)\left(q_{1,t+1} - q_{1,t} - \xi_{1,t}\right)\left(q_{2,t+1} - q_{2,t} - \xi_{2,t}\right) \\ &\quad + (1-c_K)[c_1\left(q_{1,t+1} - q_{1,t} - \xi_{1,t}\right)\left(\xi_{2,t} - \widetilde{\xi}_{2,t}\right) \\ &\quad + c_1 c_2\left(q_{1,t+1} - q_{1,t} - \xi_{1,t}\right)\left(q_{2,t+1} - q_{2,t} - \xi_{2,t}\right) \\ &\quad + c_2\left(q_{2,t+1} - q_{2,t} - \xi_{2,t}\right)\left(\xi_{1,t} - \widetilde{\xi}_{1,t}\right)] \\ \widetilde{\xi}_{1,t+1} &= (1-c_K)\widetilde{\xi}_{1,t} + c_K(q_{1,t+1} - q_{1,t}) \\ \widetilde{\xi}_{2,t+1} &= (1-c_K)\widetilde{\xi}_{2,t} + c_K(q_{2,t+1} - q_{2,t}) \end{aligned}$$

As far as the price adjustment rule is concerned, we assume that the market maker chooses $\lambda_{1,t}$, $\lambda_{2,t}$ so that prices, in equilibrium, are equal to the (growing) fundamental values, i.e. the equilibrium values of q_1 and q_2 are overbarq_1=overbarq_2=0. With $\zeta_i^{(c)} (i=1, 2)$ denoting the chartist demand for the ith asset in equilibrium, this latter assumption implies that the price setting rules $\lambda_{1,t}$ and $\lambda_{2,t}$ must satisfy

$$\lambda_{1,t} = \lambda_1 = h_1 + \overline{\zeta}_1^{(c)} - \gamma_1/\beta_1$$

$$\lambda_{2,t} = \lambda_2 = h_2 + \overline{\zeta}_2^{(c)} - \gamma_2/\beta_2$$

Clearly we are assuming that the market maker has sufficient knowledge of the market fundamentals and behaviour of the two groups to undertake all the requisite calculations. Some economists might even use the term ‘rational’ to describe the market maker in this framework. Under the foregoing assumptions the dynamic Equations 11 and 12 may be rewritten

$$q_{1,t+1} = q_{1,t} + \beta_1\left[-a_1 q_{1,t} + b_2 q_{2,t} + \left(\zeta_{1,t}^{(c)} - \overline{\zeta}_1^{(c)}\right)\right] \tag{13}$$

$$q_{2,t+1} = q_{2,t} + \beta_2\left[-a_2 q_{2,t} + b_1 q_{1,t} + \left(\zeta_{2,t}^{(c)} - \overline{\zeta}_2^{(c)}\right)\right] \tag{14}$$

These equations show the price deviations from fundamental of each asset in any period as being determined by the price deviations of both assets in the previous period and the deviation of asset demand from its own equilibrium level. The same equations could also be derived by assuming that the market maker announces the new price in the ith market according to

$$P_{i,t+1} = P_{i,t} + \gamma_i + \beta_i\left(\zeta_{i,t}^{(f)} + \zeta_{i,t}^{(c)} - N_{i,t}\right)$$

where $N_{i,t}$ represents the (nominal) supply of outside shares at time t and therefore $\zeta_{i,t}^{(f)} + \zeta_{i,t}^{(c)} - N_{i,t}$ represents the excess demand. This means that the market maker increases the price according to the fundamental trend in the case of zero excess demand, while the price trend is higher (lower) than the fundamental trend in the case of positive (negative) excess demand. By assuming that the supply $N_{i,t}$ is constant and equal to agents’ equilibrium demand, $N_{i,t} = N_i = \overline{\zeta}_i^{(f)} + \overline{\zeta}_i^{(c)}$, where $\overline{\zeta}_i^{(f)} = h_i$, and by using the changes of variables described before, one again obtains the price setting rules (13) and (14).

The Dynamical System

In order to reduce the dimension of the system and to obtain analytical results about the dynamic behaviour of the model and its dependence on the key parameters, we will focus on the particular case where chartists update their calculations of expected returns, variances and covariance with the same weighting parameters, i.e. $c_1 = c_2 = c_K = c$, so that the dynamic equations for the state variables $\widetilde{\xi}_{i,t}$ and $\xi_{i,t}$ turn out to be the same. Numerical computations performed in the general case, where chartist weighting parameters are different across markets, show that the dynamics and the bifurcation structure of the system are very similar to the ones of the simplified case $c_1 = c_2 = c_K = c$.

Under this assumption, the updating rule for the 'time-varying portion' of the covariance becomes

$$K_{t+1}=(1-c)K_t+c(1-c)\left(q_{1,t+1}-q_{1,t}-\xi_{1,t}\right)\left(q_{2,t+1}-q_{2,t}-\xi_{2,t}\right)$$

Denoting by the symbol $'$ the unit time advancement operator,[5] the time evolution of the model is thus given by the iteration of the seven-dimensional nonlinear map $T:(q_1,\xi_1,q_2,\xi_2,v_1,v_2,K)\mapsto\left(q_1',\xi_1',q_2',\xi_2',v_1',v_2',K'\right)$ that may be written as

$$T:\begin{cases} q_1'=q_1+\beta_1\left[-a_1q_1+b_2q_2+\left(\zeta_1^{(c)}-\overline{\zeta}_1^{(c)}\right)\right]\\ \xi_1'=(1-c)\xi_1+c\left(q_1'-q_1\right)\\ q_2'=q_2+\beta_2\left[-a_2q_2+b_1q_1+\left(\zeta_2^{(c)}-\overline{\zeta}_2^{(c)}\right)\right]\\ \xi_2'=(1-c)\xi_2+c\left(q_2'-q_2\right)\\ v_1'=(1-c)v_1+c(1-c)\left(q_1'-q_1-\xi_1\right)^2\\ v_2'=(1-c)v_2+c(1-c)\left(q_2'-q_2-\xi_2\right)^2\\ K'=(1-c)K+c(1-c)\left(q_1'-q_1-\xi_1\right)\left(q_2'-q_2-\xi_2\right)\end{cases}\tag{15}$$

where the chartist demand functions $\zeta_1^{(c)}$, $\zeta_2^{(c)}$ are given by

$$\zeta_1^{(c)}=\frac{\left(v_2+\sigma_2^2\right)(\xi_1+\pi_1)-(K+\delta\sigma_1\sigma_2)(\xi_2+\pi_2)}{\alpha^{(c)}\left[\left(v_1+\sigma_1^2\right)\left(v_2+\sigma_2^2\right)-\delta^2\sigma_1^2\sigma_2^2-K^2-2K\delta\sigma_1\sigma_2\right]}\tag{16}$$

$$\zeta_2^{(c)}=\frac{\left(v_1+\sigma_1^2\right)(\xi_2+\pi_2)-(K+\delta\sigma_1\sigma_2)(\xi_1+\pi_1)}{\alpha^{(c)}\left[\left(v_1+\sigma_1^2\right)\left(v_2+\sigma_2^2\right)-\delta^2\sigma_1^2\sigma_2^2-K^2-2K\delta\sigma_1\sigma_2\right]}\tag{17}$$

and we recall that $\pi_i=(\gamma_i+g_i-g)$ can be interpreted as the long-run expected excess return on asset i. Notice that the chartist optimal demand for each asset at each time t is a function of the π_i as well as the state variables ξ_1, ξ_2, v_1, v_2, and K. The map (15) is nonlinear due to the variance–covariance updating rules and the functional form of chartists demand functions.

Our dynamic analysis starts from the determination of the unique steady state of the system that we denote by O. It is easy to see that the steady state is characterized by the equilibrium levels of the state variables given by

$$\overline{q}_i=\overline{\xi}_i=\overline{v}_i=0\quad(i=1,2)$$
$$\overline{K}=0$$

The equilibrium values for q_i and ξ_i imply that, in equilibrium, prices are equal to fundamentals, $P_{i,\infty}=W_{i,\infty}$, chartists' expected price trends are equal to the rates of growth of fundamentals, $\overline{\psi}_i=\gamma_i$, and chartists' beliefs about variances and covariance are determined by the long-run variance/covariance structure.

The chartist demand for each asset in equilibrium is thus given by

$$\overline{\zeta}_1^{(c)}=\frac{\sigma_2^2\pi_1-\delta\sigma_1\sigma_2\pi_2}{\alpha^{(c)}\left(1-\delta^2\right)\sigma_1^2\sigma_2^2}\quad\overline{\zeta}_2^{(c)}=\frac{\sigma_1^2\pi_2-\delta\sigma_1\sigma_2\pi_1}{\alpha^{(c)}\left(1-\delta^2\right)\sigma_1^2\sigma_2^2}\tag{18}$$

Notice from Equations 3 and 18 that the demands of agent-type j in equilibrium may be written as

$$\bar{\zeta}_1^{(j)} = \frac{1}{\left(1-\delta^2\right)} \frac{\pi_1}{\alpha^{(j)}\sigma_1^2} - \frac{\delta\sigma_2/\sigma_1}{\left(1-\delta^2\right)} \frac{\pi_2}{\alpha^{(j)}\sigma_2^2} \tag{19}$$

$$\bar{\zeta}_2^{(j)} = \frac{1}{\left(1-\delta^2\right)} \frac{\pi_2}{\alpha^{(j)}\sigma_2^2} - \frac{\delta\sigma_1/\sigma_2}{\left(1-\delta^2\right)} \frac{\pi_1}{\alpha^{(j)}\sigma_1^2} \tag{20}$$

Equations 19 and 20 indicate that the agent j's demand in equilibrium is a weighted average of risk-adjusted equilibrium expected excess returns on each asset with the weight determined by the long-run correlation. These demands may also be interpreted as consisting of direct and hedging components.

The Case of Zero Long-Run Correlation Between Returns

In this section we investigate an important particular case where analytical conditions can be derived for the local asymptotic stability of the equilibrium, namely the case of zero 'long-run' correlation between returns ($\delta=0$). In this case the dividend yields of the two risky assets are not correlated in agents' beliefs, but agents (chartists) may expect a nonzero correlation between returns when the system is out-of-equilibrium, depending on the observed past comovements in prices.

Since in this case $b_1=b_2=0$, the map T in eq. (15) becomes:

$$T: \begin{cases} q_1' = q_1 + \beta_1\left[-a_1 q_1 + \left(\zeta_1^{(c)} - \bar{\zeta}_1^{(c)}\right)\right] \\ \xi_1' = (1-c)\xi_1 + c\left(q_1' - q_1\right) \\ q_2' = q_2 + \beta_2\left[-a_2 q_2 + \left(\zeta_2^{(c)} - \bar{\zeta}_2^{(c)}\right)\right] \\ \xi_2' = (1-c)\xi_2 + c\left(q_2' - q_2\right) \\ v_1' = (1-c)v_1 + c(1-c)\left(q_1' - q_1 - \xi_1\right)^2 \\ v_2' = (1-c)v_2 + c(1-c)\left(q_2' - q_2 - \xi_2\right)^2 \\ K' = (1-c)K + c(1-c)\left(q_1' - q_1 - \xi_1\right)\left(q_2' - q_2 - \xi_2\right) \end{cases}$$

where

$$a_1 = \frac{\eta_1}{\alpha^{(f)}\sigma_1^2} \qquad a_2 = \frac{\eta_2}{\alpha^{(f)}\sigma_2^2}$$

denote the *strength of fundamentalist demand* for asset 1 and 2, respectively.

Chartist optimal demands in Equations 16 and 17 reduce to

$$\zeta_1^{(c)} = \frac{\left(v_2+\sigma_2^2\right)(\xi_1+\pi_1) - K(\xi_2+\pi_2)}{\alpha^{(c)}\left[\left(v_1+\sigma_1^2\right)\left(v_2+\sigma_2^2\right) - K^2\right]}$$

$$\zeta_2^{(c)} = \frac{\left(v_1+\sigma_1^2\right)(\xi_2+\pi_2) - K(\xi_1+\pi_1)}{\alpha^{(c)}\left[\left(v_1+\sigma_1^2\right)\left(v_2+\sigma_2^2\right) - K^2\right]}$$

with equilibrium levels:

$$\overline{\zeta}_1^{(c)} = \frac{\pi_1}{\alpha^{(c)}\sigma_1^2} \quad \overline{\zeta}_2^{(c)} = \frac{\pi_2}{\alpha^{(c)}\sigma_2^2}$$

From an analytical point of view, the main feature of this particular case is the existence of lower dimensional *invariant* subsets of the phase space, associated with asset 1 and asset 2, respectively. Assume, for example, that the state variables associated with asset 1 are at their equilibrium levels, i.e. $q_1 = \xi_1 = v_1 = 0$, and assume, in addition, $K=0$. It is immediate to see that the iteration of the map does not move the system away from the three-dimensional subset of the phase-space defined by $q_1 = \xi_1 = v_1 = K = 0$, since

$$T(0, 0, q_2, \xi_2, 0, v_2, 0) = \left(0, 0, q_2', \xi_2', 0, v_2', 0\right)$$

This means that such a subset of the phase space (let us denote it by I_2) is *invariant*, and that along such an invariant manifold the dynamics of the system are obtained by iteration of a three-dimensional map, say $T_{(2)} : (q_2, \xi_2, v_2) \mapsto \left(q_2', \xi_2', v_2'\right)$. Similarly, along the invariant set I_1 defined by $q_2 = \xi_2 = v_2 = K = 0$ (i.e. with market 2 'in equilibrium') the dynamics are governed by the three-dimensional map $T_{(1)} : (q_1, \xi_1, v_1) \mapsto \left(q_1', \xi_1', v_1'\right)$. More precisely, the three-dimensional map $T_{(i)}$ ($i=1, 2$) governing the dynamics along the invariant manifold associated with the ith risky asset is given by

$$T_{(i)} : \begin{cases} q_i' = q_i + \beta_i \left[-a_i q_i + \frac{\xi_i + \pi_i}{\alpha^{(c)}\left(v_i + \sigma_i^2\right)} - \frac{\pi_i}{\alpha^{(c)}\sigma_i^2} \right] \\ \xi_i' = (1-c)\xi_i + c\left(q_i' - q_i\right), \\ v_i' = (1-c)v_i + c(1-c)\left(q_i' - q_i - \xi_i\right)^2 \end{cases} \tag{21}$$

As one can verify, this *invariancy* property no longer holds in the general case of nonzero long-run correlation, $\delta \neq 0$.

The economic story behind this case is that when agents expect no correlation between the dividend yields, and one of the markets starts from an equilibrium situation, then it will remain in equilibrium, no matter what is occuring in the other market. Of course, as our numerical simulations will suggest, this does not mean that this 'partial' equilibrium outcome will always be attained by the system when both markets start out-of-equilibrium.

6.1. Local Stability Conditions

As usual, the local stability analysis of a fixed point is performed through the location, in the complex plane, of the eigenvalues of the Jacobian matrix evaluated at the fixed point. The case of zero long-run correlation between the risky assets is characterized by a very simple structure for the Jacobian matrix computed at the steady state, which allows us to derive analytical conditions for the local asymptotic stability of the steady state, and to study analytically its dependence on the key parameters of the model. Denoting by $\zeta_{i,\xi_1}^{(c)}, \zeta_{i,\xi_2}^{(c)}, \zeta_{i,v_1}^{(c)}, \zeta_{i,v_2}^{(c)}, \zeta_{i,K}^{(c)}$, $i=1,2$, the partial derivatives (at the steady state) of the chartist optimal demands with respect to the state variables, we can write the Jacobian matrix computed at the steady state O as

$$DT(O)=\begin{bmatrix} 1-a_1\beta_1 & \beta_1\zeta^{(c)}_{1,\xi_1} & 0 & 0 & \beta_1\zeta^{(c)}_{1,v_1} & 0 & \beta_1\zeta^{(c)}_{1,K} \\ -c\beta_1 a_1 & 1-c+c\beta_1\zeta^{(c)}_{1,\xi_1} & 0 & 0 & c\beta_1\zeta^{(c)}_{1,v_1} & 0 & c\beta_1\zeta^{(c)}_{1,K} \\ 0 & 0 & 1-a_2\beta_2 & \beta_2\zeta^{(c)}_{2,\xi_2} & 0 & \beta_2\zeta^{(c)}_{2,v_2} & \beta_2\zeta^{(c)}_{2,K} \\ 0 & 0 & -c\beta_2 a_2 & 1-c+c\beta_2\zeta^{(c)}_{2,\xi_2} & 0 & c\beta_2\zeta^{(c)}_{2,v_2} & c\beta_2\zeta^{(c)}_{2,K} \\ 0 & 0 & 0 & 0 & 1-c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-c \end{bmatrix}.$$

Calculation of the partial derivatives (at the steady state) yields

$$\zeta^{(c)}_{1,\xi_1}=\frac{1}{\alpha^{(c)}\sigma_1^2},\quad \zeta^{(c)}_{2,\xi_2}=\frac{1}{\alpha^{(c)}\sigma_2^2}$$

$$\zeta^{(c)}_{1,v_1}=\frac{-\pi_1}{\alpha^{(c)}\left(\sigma_1^2\right)^2},\quad \zeta^{(c)}_{2,v_2}=\frac{-\pi_2}{\alpha^{(c)}\left(\sigma_2^2\right)^2}$$

$$\zeta^{(c)}_{1,K}=\frac{-\pi_2}{\alpha^{(c)}\sigma_1^2\sigma_2^2},\quad \zeta^{(c)}_{2,K}=\frac{-\pi_1}{\alpha^{(c)}\sigma_1^2\sigma_2^2}$$

and $\zeta^{(c)}_{1,\xi_2}, \zeta^{(c)}_{1,v_2}, \zeta^{(c)}_{2,\xi_1}, \zeta^{(c)}_{2,v_1}=0$.

Since the Jacobian matrix is upper block triangular, its eigenvalues may be computed as roots of the characteristic polynomials associated with each block on the diagonal (Gantmacher, 1990). It follows that the first two eigenvalues, say λ_1 and λ_2, are the roots of the two-dimensional block associated with the variables q_1 and ξ_1, while the third and fourth eigenvalues, say λ_3 and λ_4, are the roots of the block associated with the variables q_2 and ξ_2. The remaining eigenvalues are $\lambda_5=\lambda_6=\lambda_7=(1-c)$, and thus they are smaller than one in modulus.

In order to alleviate the notation, we denote by $\theta_i\equiv\zeta^{(c)}_{i,\xi_i}=\frac{1}{\alpha^{(c)}\sigma_i^2}$ the partial derivative of the chartist demand for the ith risky asset, with respect to the expected return of that asset, evaluated at the steady state. We shall see later that the parameters θ_i, which we will refer to as *strengths of chartist demand* (at the steady state) play a determining role in the dynamical outcomes of the financial market. We also denote by A_i the two-dimensional submatrices associated with the state variables q_i and ξ_i (i=1, 2), i.e.

$$A_i=\begin{bmatrix} 1-a_i\beta_i & \beta_i\theta_i \\ -c\beta_i a_i & 1-c+c\beta_i\theta_i \end{bmatrix}$$

Let Tr_i and Det_i be the trace and the determinant of A_i, and $\wp_i(\lambda)=\lambda^2-Tr_i\lambda+Det_i$ the associated characteristic polynomial. A well known necessary and sufficient condition for all of the roots of $\wp_i(\lambda)$ to be smaller than 1 in absolute value consists of the inequalities[6]

$$\begin{cases} \wp_i(1) = 1 - Tr_i + Det_i > 0 \\ \wp_i(-1) = 1 + Tr_i + Det_i > 0 \\ \wp_i(0) = Det_i < 1 \end{cases} \tag{22}$$

Since the eigenvalues λ_5, λ_6, λ_7 are all equal to $(1-c)$, $0<(1-c)<1$, it follows that when the set of inequalities (22) holds for both the blocks A_i ($i=1, 2$) then the equilibrium O is locally attracting. The conditions (22) may be rewritten, respectively, as:

$$\begin{cases} a_i\beta_i c > 0 \\ a_i\beta_i(2-c) < 2(2-c) + 2c\beta_i\theta_i \\ a_i\beta_i(1-c) > c[\beta_i\theta_i - 1] \end{cases} \tag{23}$$

These conditions are very similar to the ones obtained for the simpler two-dimensional, single asset model analyzed in Chiarella, Dieci and Gardini (2002). Figure 1 represents, in the space of the parameters (c, a_i), $0<c<1$, $a_i>0$, the region S_i where the system of inequalities (22) is satisfied for asset i (and thus the eigenvalues of A_i are less than one in modulus). Depending on the parameters β_i and θ_i, the region S_i may have qualitatively different shapes, as shown by Fig. 1*a* (for the case $\beta_i\theta_i \leq 1$) and Fig. 1*b* ($\beta_i\theta_i > 1$).

In the particular case of a single risky asset, where the dynamical system would be described by the three-dimensional map (21), the system of inequalities (23) would precisely define, in the space of the parameters, the region of local asymptotic stability of the steady state. In the case of two risky assets, the local stability and bifurcations of the steady state can be analysed by 'combining' the two regions S_i ($i=1, 2$) associated with the two markets.

Assume that for $i=1, 2$, i.e. for both assets, the parameters c, a_i are inside the region S_i, and therefore the steady state O is locally asymptotically stable. Since the first inequality of (23) is always satisfied for economically meaningful values of the parameters, it follows that when one of the parameters c, a_i, $i=1, 2$, is varied a loss of stability may occur in one of two ways. The first is due to the crossing of the 'flip' curve whose equation is given by

$$\wp_i(-1) = 1 + Tr_i + Det_i = 0 \quad \text{i.e.} \quad a_i = \frac{2}{\beta_i} + \frac{2c\theta_i}{2-c} \tag{24}$$

along which one of the eigenvalues of A_i is equal to -1. The second is due to the crossing of the 'Neimark–Hopf' curve whose equation is given by

$$\wp_i(0) = Det_i = 1 \quad \text{i.e.} \quad a_i = \frac{c[\beta_i\theta_i - 1]}{\beta_i(1-c)} \tag{25}$$

along which a pair of complex eigenvalues of A_i are equal to 1 in modulus. In this second case a Neimark–Hopf bifurcation occurs, changing the equilibrium from an attracting focus into a repelling focus, and we will find numerical evidence of the *supercritical* nature of this bifurcation by showing the existence of an invariant

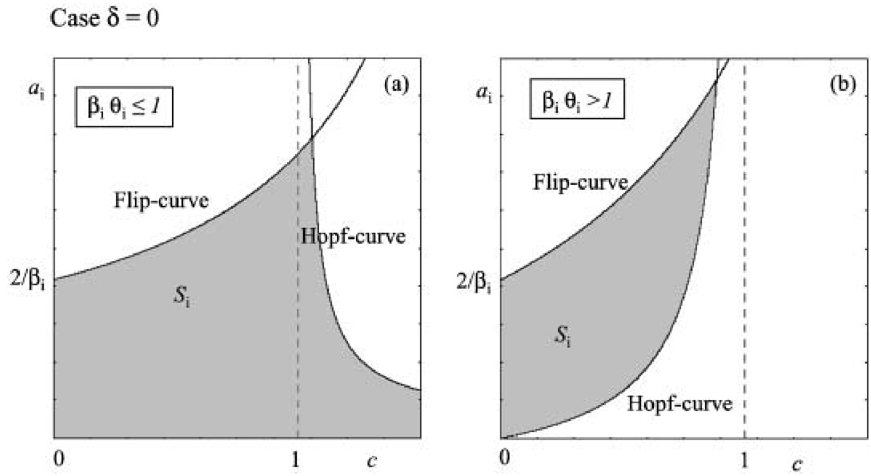

Figure 1. Conditions of local stability and bifurcation curves in the space of the parameters, in the case of zero 'long-run' correlation between returns ($\delta=0$). The parameters $\alpha^{(f)}$ and $\alpha^{(c)}$ denote fundamentalist and chartist risk aversion, respectively; c is the chartist extrapolation parameter. For $i=1,2, \sigma_i^2$ denotes agents' belief about 'long-run' variance of asset i, η_i is the fundamentalist parameter and a_i is the 'strength' of fundamentalist demand $(a_i \equiv \eta_i/(\alpha^{(f)}\sigma_i^2))$, θ_i is the 'strength' of chartist demand $(\theta_i \equiv 1/(\alpha^{(c)}\sigma_i^2))$, β_i is the price reaction coefficient of asset i. For fixed values of the parameters β_i and θ_i, the grey area S_i represents the combinations of the parameters (c, a_i) for which both the eigenvalues 'associated' with the ith risky asset are of modulus smaller than one (i=1, 2). The 'shape' of the region S_i depends on the aggregate parameter $\beta_i\theta_i=\beta_i/(\alpha^{(c)}\sigma_i^2)$: (*a*) shows the case $\beta_i\theta_i \le 1$, while the case $\beta_i\theta_i > 1$ is shown in (*b*).

attracting closed curve around the repelling focus soon after the crossing of the bifurcation curve (25).[7]

As far as the economic interpretation of the local stability conditions is concerned, we notice that by assuming the market maker's speed of adjustment of the ith price $\beta_i>0$ as fixed, the shape of region S_i in the parameter plane (c, a_i) (indicated in grey in Fig. 1) is greatly affected by the strength of chartists' demand at the steady state (θ_i).[8] In particular, when $\beta_i\theta_i \le 1$, i.e. when the risk aversion $\alpha^{(c)}$ or the variance σ_i^2 are sufficiently high (Fig. 1*a*) the region appears wider than in the opposite case (Fig. 1*b*). Figure 1*a* shows that, when the strength of chartist demand is relatively weak ($\theta_i <1/\beta_i$), at a given level of chartists reaction speed (c) the eigenvalues associated with the ith market are less than one in modulus for sufficiently low values of the fundamentalists reaction parameter a_i, but fundamentalists can cause instability by reacting too strongly to the deviation from the fundamental value. Figure 1*b* shows that when the strength of chartist demand is relatively strong ($\theta_i>1/\beta_i$), the ability of fundamentalists' demand to stabilise the ith market is restricted to a fairly narrow range of the parameter a_i.

In Chiarella, Dieci and Gardini (2002) we have observed a similar phenomenon in the case of a single risky asset. The foregoing analysis confirms the economic intuition that we would expect a similar picture in the case of multiple risky assets amongst which there is no ‘long-run’ correlation. Of course it is still possible that out of equilibrium dynamic behaviour is different, and that is indeed the case, as we shall see below.

6.2. Out-of-Equilibrium Dynamics

In this section we investigate the dynamics of the map when the equilibrium is not locally asymptotically stable, mainly in order to analyse the dynamic effect of increasing values of the chartist extrapolation parameter (c). Precisely, we will show how chartists can destabilize the system by reacting too quickly to recent price changes, and how high values of the parameter c may cause persistent irregular price oscillations and complex dynamic scenarios with coexisting attracting sets. Since we are considering a high-dimensional system, the numerically obtained trajectories will be visualized by representing their projections in the planes of the state variables (q_1, ξ_1), in black, and (q_2, ξ_2), in grey. This will allow us to evaluate and compare the effects of the dynamics on each market. Moreover, throughout the numerical simulations of Sections 6.2.1, 6.2.2 and 7 we will assume that asset 1 has higher expected risk premium and volatility than asset 2 in equilibrium ($\pi_1>\pi_2$, $\sigma_1>\sigma_2$) and therefore the strength of chartist demand in market 2 ($\theta_2=\frac{1}{\alpha^{(c)}\sigma_2^2}$) is greater than the strength of chartist demand in market 1 ($\theta_1=\frac{1}{\alpha^{(c)}\sigma_1^2}$). We will also assume that chartists are less risk averse than fundamentalists. We will fix the values of the following parameters: $\pi_1=0.05$, $\pi_2=0.025$, $\sigma_1^2=0.005$, $\sigma_2^2=0.0025$, $\alpha^{(f)}=100$, and we will vary the values for the remaining parameters.

6.2.1. Price fluctuations in a single market. Let us begin with the case where $\alpha^{(c)}=75$, $\beta_1=\beta_2=0.3$, $\eta_1=\eta_2=0.3$. In this case the strengths of fundamentalist demand for the two assets become $a_1=0.6$ and $a_2=1.2$, respectively, while the strengths of chartist demand at the steady state are characterized by $\theta_1\simeq 2.667$, $\theta_2\simeq 5.333$. The regions S_1 and S_2 where, respectively, the eigenvalues associated with asset 1 and asset 2 are smaller than one in modulus are represented by the grey areas in Fig. 2. From the shape of the region S_1 we can see that for sufficiently low values of the fundamentalists reaction parameter in market 1 (a_1), the eigenvalues associated with asset 1 will be less than one in modulus for any value of the chartist parameter c, $0<c<1$ (Fig. 2*a*). On the contrary, by increasing the parameter c starting from inside the region S_2, the crossing of the Neimark-Hopf curve (occurring at $c=c_2^*=0.375$) will cause the couple of (complex) eigenvalues associated with asset 2 to become greater than one in modulus (Fig. 2*b*). Figure 3*a*, *b* (where $c=0.5$) represents the effect of this bifurcation on the dynamics of the markets for the two assets: while in market 2 an invariant attracting closed curve appears (in grey), ‘trajectories’ in market 1 converge with dampened oscillations to the ‘equilibrium’ (in black). Of course the system as a whole is not in equilibrium, but the existing attracting limit cycle is located in the three-dimensional invariant manifold I_2 characterized by $q_1=0$, $\xi_1=0$, $v_1=0$, $K=0$ (and therefore the projection of the trajectory in the plane (q_1, ξ_1) shows convergence to the fundamental). By increasing further the speed of adjustment c, we notice that the shape of the invariant closed curve in market 2 becomes more and more irregular, until it changes into a chaotic attractor (see

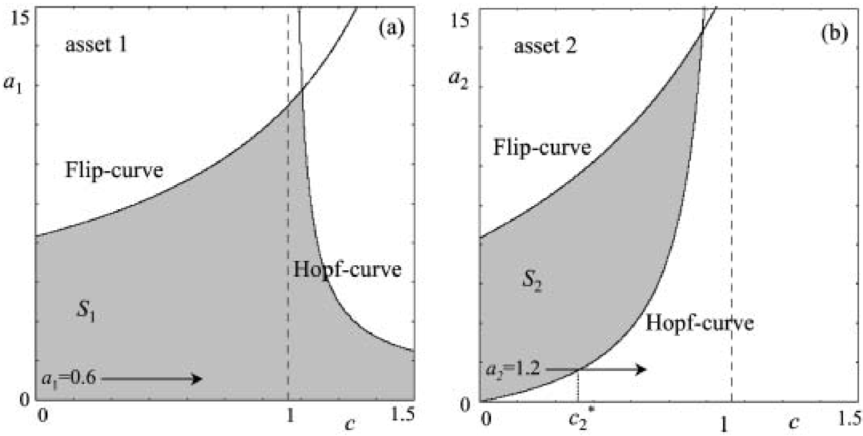

Figure 2. Case of zero 'long-run' correlation (δ=0). A situation where the structure of the domain of stability of the steady state in the space of parameters allows a Neimark–Hopf bifurcation to occur (only) in one of the two markets. The values of the parameters are: $\alpha^{(f)}=100$, $\alpha^{(c)}=75$, $\sigma_1^2=0.005$, $\sigma_2^2=0.0025$, $\theta_1\equiv 1/(\alpha^{(c)}\sigma_1^2)\simeq 2.667$, $\theta_2\equiv 1/(\alpha^{(c)}\sigma_2^2)\simeq 5.333$, $\beta_1=\beta_2=0.3$. The regions S_1 (*a*) and S_2 (*b*) are the projections of the stability domain in the (c, a_1) and (c, a_2) parameter planes, respectively. A typical bifurcation path is the one obtained by choosing $\eta_1=\eta_2=0.3$, i.e. $a_1\equiv\eta_1/(\alpha^{(f)}\sigma_1^2)=0.6$, $a_2\equiv\eta_2/(\alpha^{(f)}\sigma_2^2)=1.2$, and by increasing the chartist extrapolation parameter c. When c becomes greater than the bifurcation value c_2^*, the pair of (complex) eigenvalues associated with market 2 become of modulus greater than one, determining a Neimark–Hopf bifurcation (*b*); both the eigenvalues associated with market 1 remain smaller than one in modulus for any economically meaningful value of c, $0<c<1$, as shown in (*a*).

Fig. 3*c*, *e*, where c=0.75 and c=0.9, respectively). However, these global bifurcations only determine qualitative changes of the asymptotic dynamics in market 2, but they do not affect the asymptotic dynamics in market 1, that still converges to the fundamental price (Fig. 3*d*, *f*). Of course the increasing complexity of the asymptotic dynamics in market 2 may affect the *transient* part of the price adjustment process in market 1.

6.2.2. Price fluctuations in both markets. A different situation is met in the case where we increase the market price reaction coefficients to $\beta_1=\beta_2=0.6$, and decrease the chartist risk aversion, by setting $\alpha^{(c)}=50$. The strengths of chartist demand in the two markets is now given by $\theta_1=4$, $\theta_2=8$. Also in this case it is useful to look at the shapes of the regions S_1 and S_2 where the eigenvalues associated with asset 1 and asset 2, respectively, are less than one in modulus (Fig. 4*a*, *b*). Starting from parameters a_1, a_2, c such that (c, a_1) is inside the region S_1 and (c, a_2) is inside S_2, and by increasing the chartist parameter c, first we observe the crossing of the Neimark-Hopf curve of the region S_2 (occurring at $c=c_2^*\simeq 0.159$, Fig. 4*b*). Similarly to the previous numerical example, the effect of this bifurcation is the creation of an attracting invariant closed curve in market 2, while trajectories in market 1 converge to the 'equilibrium' (see Fig. 5*a*, *b*, where c=0.175). We will denote this limit cycle by Γ_a. At the crossing of the Neimark-Hopf curve of the region S_1 (occurring at

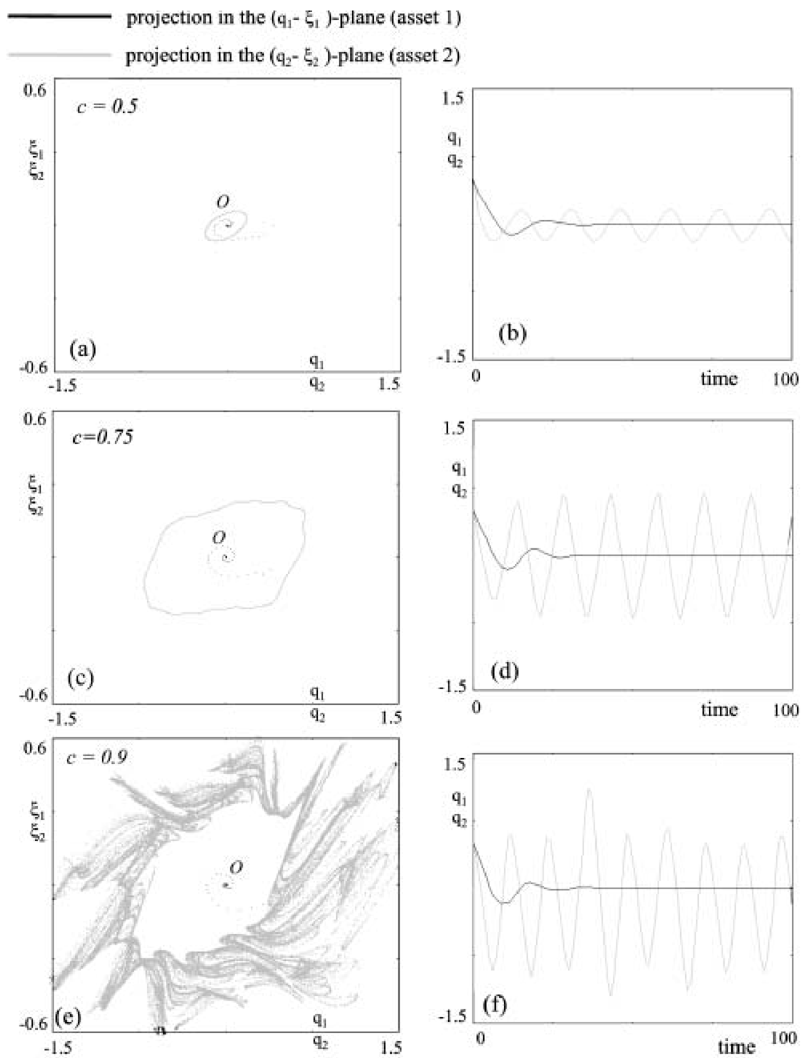

Figure 3. Dynamic effect observed by following the path represented in Fig. 2, after the crossing of the Neimark-Hopf bifurcation curve in Fig. 2*b*. The values of the parameters are: $\delta=0$, $\alpha^{(f)}=100$, $\alpha^{(c)}=75$, $\sigma_1^2=0{,}005$, $\sigma_2^2=0.0025$, $\eta_1=\eta_2=0.3$ (i.e. $\theta_1\simeq 2.667$, $\theta_2\simeq 5.333$, $a_1=0.6$, $a_2=1.2$), $\beta_1=\beta_2=0.3$, $\pi_1=0.05$, $\pi_2=0.025$. In (*a*), (*c*), (*e*) the numerically obtained trajectories are visualized by means of their projections in the planes of the state variables q_1, ξ_1 (i.e. market 1), in black, and q_2, ξ_2 (market 2), in grey, while (*b*), (*d*), (*f*) represent the time series of the log price/fundamental ratios in the two markets. For $c=0.5$, market 2 is characterized by long-run fluctuations of price (q_2) and chartists' expected return (ξ_2), while in market 1 motion is to steady state (see (*a*), (*b*)). For increasing values of c, fluctuations become more and more irregular in market 2, with no effect on the long-run behavior of market 1(see (*c*), (*d*), where $c=0.75$, (*e*), (*f*), where $c=0.9$).

$c = c_1^* \simeq 0.205$, Fig. 4*a*) also the (complex) eigenvalues associated with asset 1 become greater than one in modulus: this second crossing is not associated with a *local* bifurcation of the equilibrium, which is already unstable (a repelling focus), and therefore at the exact 'bifurcation' value $c = c_1^*$ no structural change affects the attracting limit cycle Γ_a. However, strictly related to this second crossing, a sequence of *global* bifurcations can be numerically observed, in the parameter range where two couples of eigenvalues are greater than one in modulus. Although such phenomena are illustrated through a particular numerical example, they are quite general and can be obtained with several different parameter regimes.

The first bifurcation occurs at a parameter value $c^* \simeq 0.281 > c_1^*$, determining the structural change of the limit cycle Γ_a into a *torus* (Fig. 5*c*, *d*). After this bifurcation, the asymptotic oscillatory dynamics are no longer restricted to the invariant manifold I_2 and the effect is that long-run fluctuations (on the torus) also appear in market 1. A second bifurcation occurs at a parameter value $c^{**} > c^* > c_1^*$, whose effect is the abrupt appearance of a new attracting limit cycle (denoted by Γ_b), characterized by fluctuations in both markets. The global bifurcation that creates the

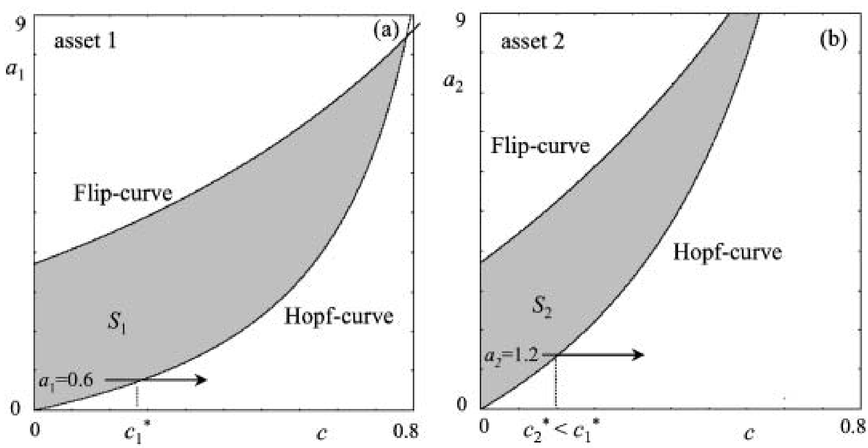

Figure 4. Case of zero 'long-run' correlation (δ=0). A situation where the structure of the domain of stability of the steady state in the space of parameters allows a sequence of two Neimark–Hopf bifurcations to occur. The values of the parameters are: $\alpha^{(f)}=100$, $\alpha^{(c)}=50$, $\sigma_1^2=0.005$, $\sigma_2^2=0.0025$, $\theta_1 \equiv 1/\left(\alpha^{(c)}\sigma_1^2\right)=4$, $\theta_2 \equiv 1/\left(\alpha^{(c)}\sigma_2^2\right)=8$, $\beta_1=\beta_2=0.6$. The regions S_1 (*a*) and S_2 (*b*) are the projections of the stability domain in the (c, a_1) and (c, a_2) parameter planes, respectively. A typical bifurcation path is the one obtained by choosing $\eta_1=\eta_2=0.3$, i.e. $a_1 \equiv \eta_1/\left(\alpha^{(f)}\sigma_1^2\right)=0.6$, $a_2 \equiv \eta_2/\left(\alpha^{(f)}\sigma_2^2\right)=1.2$, and by increasing the chartist extrapolation parameter c. When c becomes greater than the first bifurcation value c_2^*, the pair of (complex) eigenvalues associated with market 2 become of modulus greater than one (*b*), determining a first Neimark–Hopf bifurcation (and creating an attracting closed orbit); at the second bifurcation value $c_1^* > c_2^*$, also the pair of (complex) eigenvalues associated with market 1 become of modulus greater than one (*a*); the second 'crossing' anticipates a 'secondary' Hopf bifurcation (taking place from the existing closed orbit), that will occur at a higher value of the chartist extrapolation rate c.

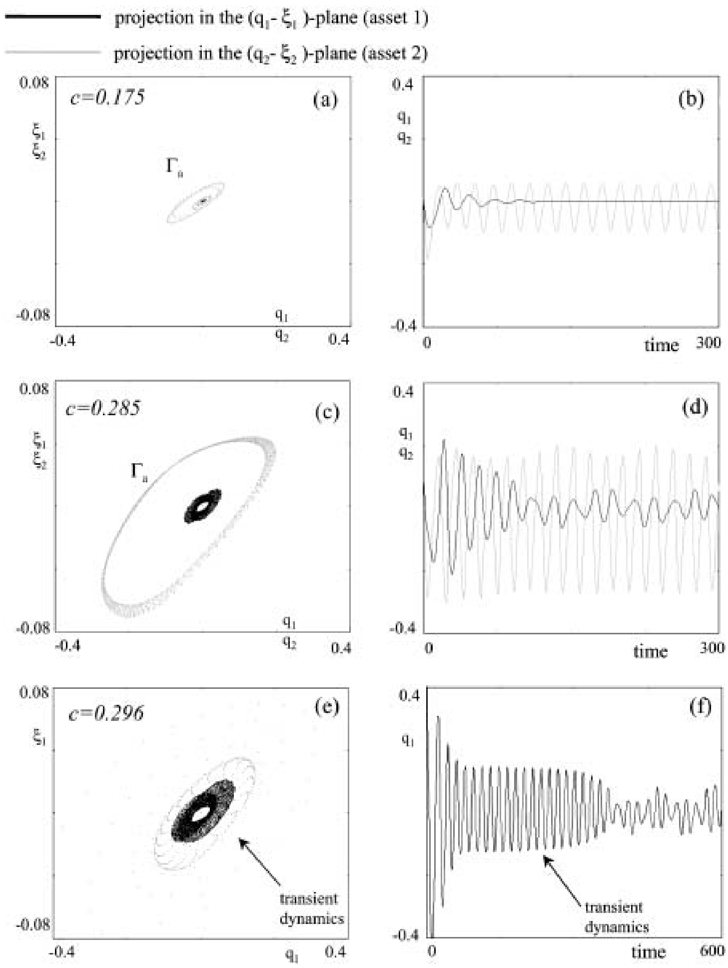

Figure 5. Dynamic effect observed by following the path represented in Fig. 4, after the crossing of the Neimark–Hopf bifurcation curve in Fig. 4*b*. The values of the parameters are: $\alpha^{(f)}=100$, $\alpha^{(c)}=50$, $\sigma_1^2=0.005$, $\sigma_2^2=0.0025$, $\eta_1=\eta_2=0.3$ (i.e. $\theta_1=4$, $\theta_2=8$, $a_1=0.6$, $a_2=1.2$), $\beta_1=\beta_2=0.6$, $\pi_1=0.05$, $\pi_2=0.025$. In (*a*), (*c*), (*e*) the numerically obtained trajectories are visualized by means of their projections in the planes of the state variables q_1, ξ_1 (market 1), in black, and q_2, ξ_2 (market 2), in grey. The Neimark–Hopf bifurcation occurring at $c=c_2^*\simeq0.159$ generates long-run fluctuations only in market 2, while market 1 'converges' to steady state (see (*a*), (*b*), where $c=0.175$). On the right of the Hopf-curve in Fig. 4*a*, for $c=c^*\simeq0.281 > c_1^*\simeq0.205$, a secondary Hopf bifurcation changes the limit cycle into a torus, and long-run fluctuations appear also in market 1 (see (*c*), (*d*), where $c=0.285$). A new, competing limit cycle appears, via *global* bifurcation, at a higher value of the chartist extrapolation rate: the creation of the new attractor (at $c=c^{**}\simeq0.2965$) is 'anticipated' by the transient dynamics of the system before the convergence to the torus Γ_a (see (*e*), (*f*) where $c=0.296$).

new attracting closed curve Γ_b may be a so-called *saddle-node* bifurcation for closed curves; in this example the bifurcation value of the parameter is $c^{**} \simeq 0.2965$ At $c=0.296$, immediately before the bifurcation, the appearance of the new attractor is anticipated by the transient part of some trajectories, that fluctuate for a high number of iterations where the new limit cycle will appear, before converging to the existing attractor (torus) (Fig. 5*e*, *f*).

Soon after this bifurcation ($c=0.3$), two different coexisting attractors, the torus Γ_a and the new limit cycle Γ_b, share the phase-space. The two coexisting attractors

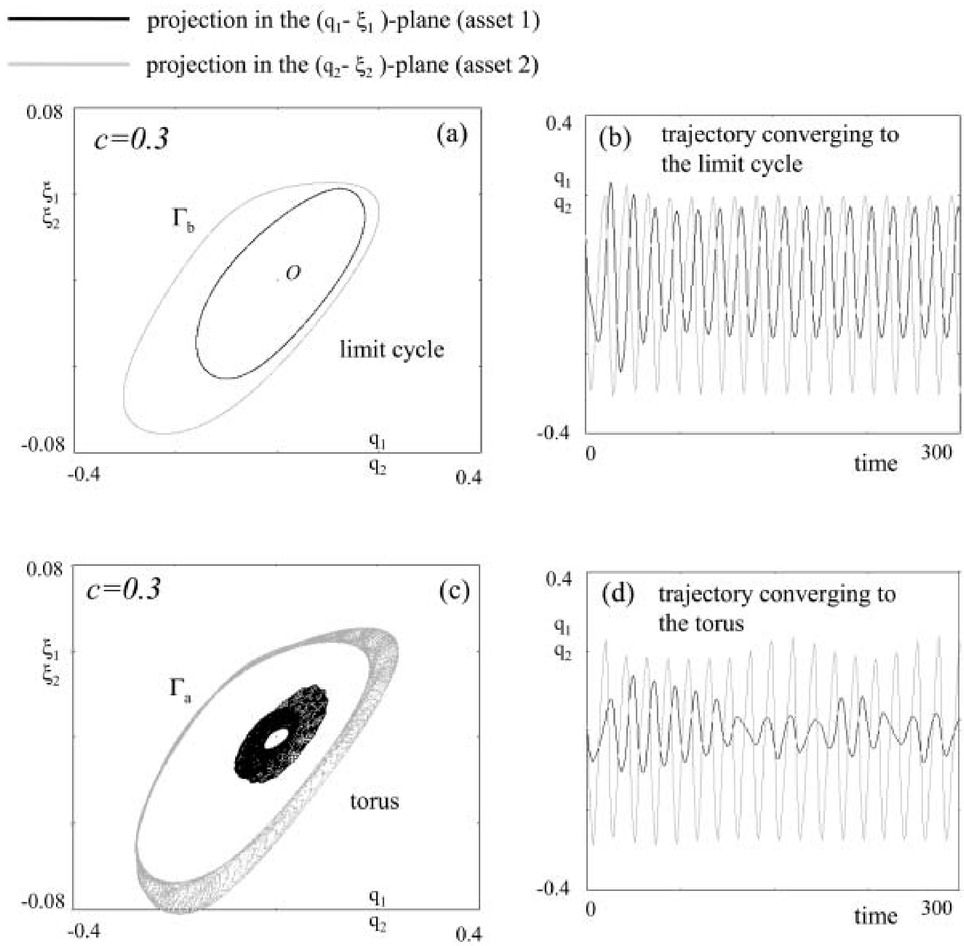

Figure 6. Coexistence of attractors and role of the *basins* of attraction. In the same parameter situation as in Fig. 5, with $c=0.3$, the coexisting attractors are a newly appeared limit cycle Γ_b (*a*) and a torus Γ_a (*c*). A trajectory of the system may converge to the limit cycle Γ_b (see (*b*)), or to the torus Γ_a (see (*d*)), according as it starts with market 1 sufficiently 'far from equilibrium' or 'close to equilibrium', respectively: the trajectory (*b*) is obtained with the initial condition $q_{1,0}=q_{2,0}=0.1$, $\xi_{1,0}=\xi_{2,0}=0.01$, $v_{1,0}=v_{2,0}=K_0=0.005$, while the trajectory (*d*) is obtained by choosing $q_{1,0}=q_{2,0}=0.01$, $\xi_{1,0}=\xi_{2,0}=v_{1,0}=v_{2,0}=K_0=0.001$.

are both characterized by long-run fluctuations in the two markets. Looking at the projections of the coexisting attractors in the planes (q_1, ξ_1) and (q_2, ξ_2) (Fig. 6*a*, *e*) and at the time series of q_1 and q_2 (Figs. 6*b*, *c*), we can observe that both attractors generate oscillations of greater amplitude in market 2 (the one with higher strength of chartist demand) than in market 1. We also notice that the range of fluctuations in market 1 is wider when the system fluctuates on the newly appeared limit cycle Γ_b, than on the torus Γ_a. Of course in the case where different asymptotic states coexist in the phase-space, the study of the basins of attraction becomes important, i.e. the identification of the sets of initial conditions generating trajectories converging to each of the different coexisting attractors. For dynamical systems of dimension greater than one, this kind of analysis can only be performed through numerical simulations: in the case of Fig. 6, we have numerically checked that only the trajectories starting with initial values of ξ_1, q_1, v_1 sufficiently close to 0 (i.e. with market 1 sufficiently close to its 'equilibrium') converge to the torus Γ_a, while in the opposite case trajectories converge to the limit cycle Γ_b. When the parameter c is further increased, both attractors increase in size, and evolve towards a chaotic structure (Fig. 7*a*, *b* represent the projections in market 1 of the coexisting attractors for $c=0.415$), while the structure of their basins of attraction becomes more and more complex, as can be verified numerically.

The economic intuition behind the above sequence of global bifurcations is that when price reaction coefficients (β_i, $i=1, 2$) are sufficiently high and chartist risk aversion ($\alpha^{(c)}$) sufficiently low, the chartist sensitivity to recent price movements (c) can be the reason for the onset of sustained price fluctuations in both markets, even though the two risky assets are not 'intrinsically' correlated in agents' beliefs ($\delta=0$). Phase space transitions similar to the one presented in this example could also have been obtained by taking the chartist parameter c as fixed and decreasing the chartist risk aversion ($\alpha^{(c)}$).

From an analytical point of view we remark that, whilst the steady state and the local stability conditions of the model with two risky assets, in the case $\delta=0$, are merely a multiple copy of the corresponding single risky asset model analyzed in Chiarella, Dieci and Gardini (2002), the out-of-equilibrium dynamics are in general quite different. As we have shown in this section, due to the agents' diversification policy and the updating of expected covariance by chartists, the dynamics of prices and expected returns in the two markets may interact in a very complicated way.

The Case of Nonzero Long-Run Correlation

The case of zero long-run correlation ($\delta=0$) studied in the previous section is a good starting point from which to understand the local and global dynamics in the general case of nonzero long-run correlation between returns. However, different from the previous case, in the case with $\delta\neq0$ we cannot have situations with one market 'in equilibrium' and the second market 'out of equilibrium'. Rather, as our previous analysis suggests and our numerical experiments will confirm, the Neimark–Hopf bifurcation of the first couple of eigenvalues will cause the simultaneous appearance of persistent oscillations in both markets.

7.1. Local Stability Conditions

The Jacobian matrix computed at the steady state has a more complicated structure than in the previous case. Using, as before, $\zeta^{(c)}_{i,\xi_1}$, $\zeta^{(c)}_{i,\xi_2}$, $\zeta^{(c)}_{i,v_1}$, $\zeta^{(c)}_{i,v_2}$, $\zeta^{(c)}_{i,K}(i=1, 2)$ to denote the partial derivatives of the chartist demand for each asset with respect to the state variables, we can write the Jacobian matrix computed at the steady state O as

$$DT(O)=\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ 0 & (1-c)\mathbf{I} \end{bmatrix}$$

where $\mathbf{0}$ is the null (3×4) matrix, $\mathbf{I}$ is the three-dimensional identity matrix. The submatrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\boldsymbol{A}=\begin{bmatrix} 1-a_1\beta_1 & \beta_1\zeta^{(c)}_{1,\xi_1} & \beta_1 b_2 & \beta_1\zeta^{(c)}_{1,\xi_2} \\ -c\beta_1 a_1 & 1-c+c\beta_1\zeta^{(c)}_{1,\xi_1} & c\beta_1 b_2 & c\beta_1\zeta^{(c)}_{1,\xi_2} \\ \beta_2 b_1 & \beta_2\zeta^{(c)}_{2,\xi_1} & 1-a_2\beta_2 & \beta_2\zeta^{(c)}_{2,\xi_2} \\ c\beta_2 b_1 & c\beta_2\zeta^{(c)}_{2,\xi_1} & -c\beta_2 a_2 & 1-c+c\beta_2\zeta^{(c)}_{2,\xi_2} \end{bmatrix}$$

and

$$\boldsymbol{B}=\begin{bmatrix} \beta_1\zeta^{(c)}_{1,v_1} & \beta_1\zeta^{(c)}_{1,v_2} & \beta_1\zeta^{(c)}_{1,K} \\ c\beta_1\zeta^{(c)}_{1,v_1} & c\beta_1\zeta^{(c)}_{1,v_2} & c\beta_1\zeta^{(c)}_{1,K} \\ \beta_2\zeta^{(c)}_{2,v_1} & \beta_2\zeta^{(c)}_{2,v_2} & \beta_2\zeta^{(c)}_{2,K} \\ c\beta_2\zeta^{(c)}_{2,v_1} & c\beta_2\zeta^{(c)}_{2,v_2} & c\beta_2\zeta^{(c)}_{2,K} \end{bmatrix}$$

where

$$\zeta^{(c)}_{1,\xi_1}=\frac{1}{\alpha^{(c)}\left(1-\delta^2\right)\sigma_1^2} \qquad \zeta^{(c)}_{2,\xi_2}=\frac{1}{\alpha^{(c)}\left(1-\delta^2\right)\sigma_2^2}$$

$$\zeta^{(c)}_{1,\xi_2}=\zeta^{(c)}_{2,\xi_1}\,\frac{-\delta}{\alpha^{(c)}\left(1-\delta^2\right)\sigma_1\sigma_2}$$

$$\zeta^{(c)}_{1,v_1}=\frac{-\pi_1\sigma_2^2+\delta\pi_2\sigma_1\sigma_2}{\alpha^{(c)}\left(1-\delta^2\right)^2\left(\sigma_1^2\right)^2\sigma_2^2} \qquad \zeta^{(c)}_{2,v_2}=\frac{-\pi_2\sigma_1^2+\delta\pi_1\sigma_1\sigma_2}{\alpha^{(c)}\left(1-\delta^2\right)^2\sigma_1^2\left(\sigma_2^2\right)^2}$$

$$\zeta^{(c)}_{1,v_2}=\frac{-\pi_1\delta^2\sigma_2^2+\pi_2\delta\sigma_1\sigma_2}{\alpha^{(c)}\left(1-\delta^2\right)^2\sigma_1^2\left(\sigma_2^2\right)^2} \qquad \zeta^{(c)}_{2,v_1}=\frac{-\pi_2\delta^2\sigma_1^2+\pi_1\delta\sigma_1\sigma_2}{\alpha^{(c)}\left(1-\delta^2\right)^2\left(\sigma_1^2\right)^2\sigma_2^2}$$

$$\zeta^{(c)}_{1,K}=\frac{2\pi_1\delta\sigma_1\sigma_2-\pi_2\sigma_1^2\left(1+\delta^2\right)}{\alpha^{(c)}\left(1-\delta^2\right)^2\left(\sigma_1^2\right)\sigma_2^2} \qquad \zeta^{(c)}_{2,K}=\frac{2\pi_2\delta\sigma_1\sigma_2-\pi_1\sigma_2^2\left(1+\delta^2\right)}{\alpha^{(c)}\left(1-\delta^2\right)^2\sigma_1^2\left(\sigma_2^2\right)^2}$$

Again the Jacobian matrix is upper block triangular, but in this case the eigenvalues λ_1, λ_2, λ_3 and λ_4 associated with the variables q_1, ξ_1, q_2, and ξ_2, respectively, cannot be computed separately as roots of second-order characteristic polynomials, but the

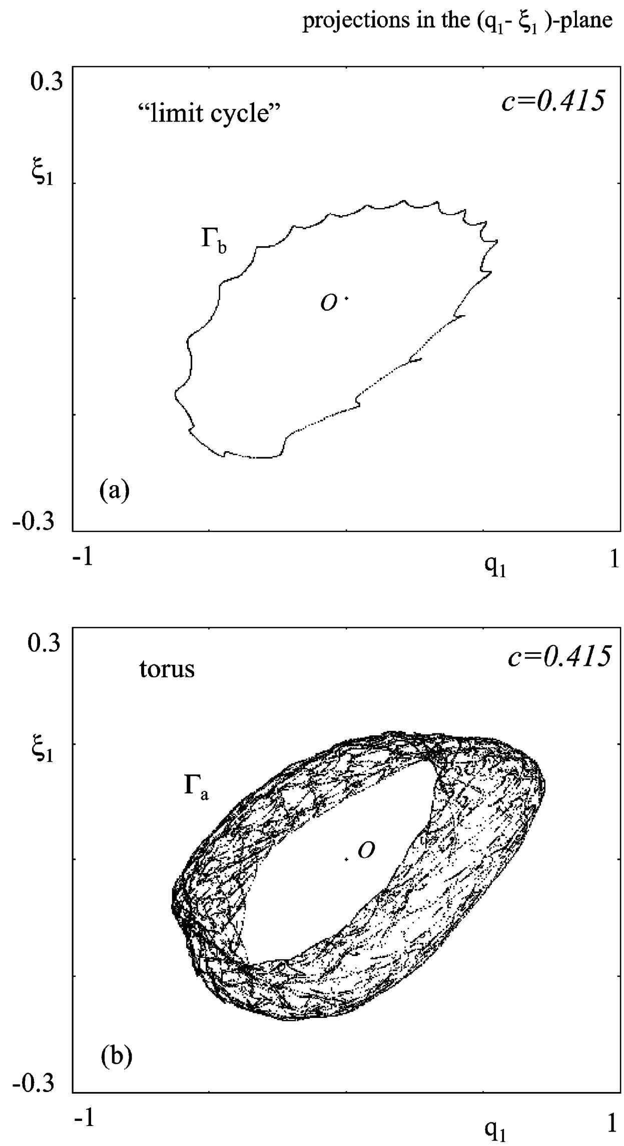

Figure 7. The coexisting attractors considered in Fig. 6 (as well as their basins of attraction) evolve towards a more complex structure for higher values of c, as shown by the projections in the (q_1, ξ_1)-plane of the 'limit cycle' Γ_b (*a*), and of the torus Γ_a (*b*), obtained for c=0.415. In this case, starting from the same initial conditions used in Fig. 6, the system behaves exactly in the opposite way: the initial condition $q_{1,0}=q_{2,0}=0.01$, $\xi_{1,0}=\xi_{2,0}=v_{1,0}=v_{2,0}=K_0=0.001$ generates a trajectory converging to the 'limit cycle' Γ_b, now characterized by a more irregular shape, while the initial condition $q_{1,0}=q_{2,0}=0.01$, $\xi_{1,0}=\xi_{2,0}=0.01$, $v_{1,0}=v_{2,0}=K_0=0.005$ generates a trajectory converging to the torus Γ_a.

four-dimensional submatrix **A** must be considered as a whole. Of course the eigenvalues of this matrix can be computed numerically. As in the case $\delta=0$, the remaining eigenvalues are $\lambda_5=\lambda_6=\lambda_7=(1-c)$, and thus are less than one in modulus. This implies that a sufficient condition to have a locally attracting equilibrium is that all the eigenvalues of the submatrix **A** be less than one in modulus. Although it is difficult to derive analytical conditions for this, and to study their dependence on the key parameters of the model, the case of zero long-run correlation studied in the previous section allows us to understand the local and global dynamics of the system.

7.2. Out-of-Equilibrium Dynamics

As outlined before, also in the general case with $\delta\neq0$ a supercritical Neimark–Hopf bifurcation occurs for sufficiently high values of the chartist parameter c, creating an attracting limit cycle. But in this case the bifurcation generates fluctuations simultaneously in both markets. As an example, Fig. 8*a*, *b*, *c* and Fig. 8*d*, *e*, *f* are obtained with the same parameters as Fig. 3*a* and Fig. 3*e*, respectively except that now the long-run correlation assumes the values $\delta=0.15$ (Fig. 8*b*, *e*) and $\delta=0.3$ (Fig. 8*c*, *f*). The corresponding basic cases with $\delta=0$ are reported in Fig. 8*a* and Fig. 8*d*, respectively. Figure 8*a*, *b*, *c* show the projections in market 1 (in black) and market 2 (in grey) of the attracting limit cycle existing soon after the Neimark-Hopf bifurcation: while in the case $\delta=0$ (Fig. 8*a*) market 1 is 'in equilibrium' when the system fluctuates on the limit cycle, this is no longer true in the cases with $\delta\neq0$ (Fig. 8*b*, *c*) so that in these latter cases oscillations appear simultaneously in both markets. However, in some sense the asymptotic behavior of the general case 'keeps track' of the basic case $\delta=0$, in that the size of fluctuations is wider in market 2 (the market characterized by stronger chartist demand). Similar pictures characterize the case of negative correlation. Notice also that the absolute value of δ seems to have an effect on the relative amplitude of the fluctuations in the two markets, in the sense that the higher is $|\delta|$, the more similar is the range of the fluctuations in the two markets. Similarly to the previous example, Fig. 8*e*, *f* represent the effect of long-run correlation on the basic case of Fig. 3*e* (reported in Fig. 8*d*), where market 2 fluctuates on a chaotic attractor.

We have also performed numerical simulations in order to see the effect of a nonzero long-run correlation on the bifurcation sequence analysed in Section 6.2.2. Again, the main difference is that in this case oscillations on a limit cycle appear in both markets simultaneously at the first Neimark–Hopf bifurcation (differently from the case of Fig. 5*a*). The important result is that all of the rich dynamics shown in Figs. 5, 6, 7 still exist, in particular the appearance of a coexisting attractor and the transition to complexity associated with increasing values of the chartists extrapolation parameter.

Of course it is not possible to state any general result concerning the influence of δ on the dynamics. The important message of the simulations of this section is that the existence of long-run correlation seems to have a significant effect both on the local stability properties of the steady state, and on the structure of the attracting sets that determine the out-of-equilibrium behaviour.

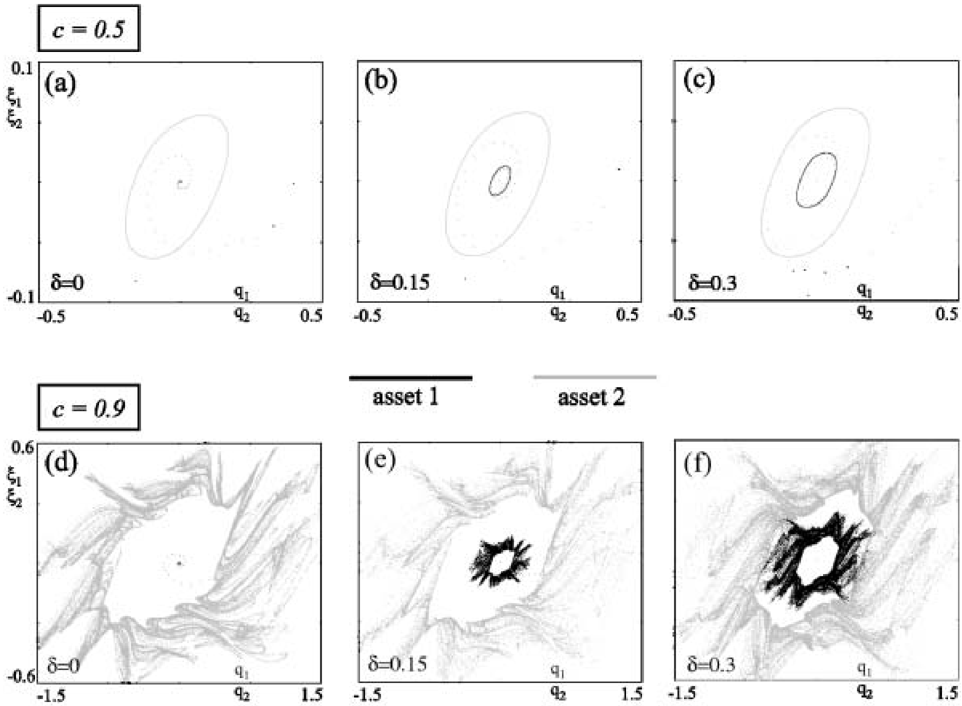

Figure 8. Dynamic effect of non-zero 'long-run' correlation ($\delta \neq 0$), compared with the basic case of zero 'long-run' correlation of Fig. 3*a* (enlarged in (*a*)), and Fig. 3*e* (reported in (*d*)). In the case $\delta \neq 0$ the Neimark–Hopf bifurcation causes the simultaneous appearance of oscillations in both markets, and it cannot happen that one market is in equilibrium while in the other market price fluctuations are occurring: the two markets are always characterized by the same qualitative behaviour (see (*b*), the case of a limit cycle and (*e*), the case of a chaotic attractor, where $\delta = 0.15$). Moreover, the higher is $|\delta|$, the more similar is the range of the fluctuations in the two markets (see (*c*) and (*f*), respectively, where $\delta = 0.3$).

Effect of Agents' Beliefs about Risk and Return

The aim of this section is to provide some initial insights concerning the role that agents' beliefs about the long-run risk/return structure of the two assets play in the asymptotic behaviour of the system, and especially in the transition from regular to irregular fluctuations of prices and returns. In particular, we are interested in how the irregular price behaviour of one market may propagate to the market for the alternative risky asset, as the result of changes in beliefs about the ratio (σ_1^2/σ_2^2) of the long-run variances of the returns, or about the ratio (π_1/π_2) of the expected long-run risk premia. To do this, we consider a case with zero long-run correlation, and start from the parameter situation of Fig. 9*a*, *b*, where prices exhibit fluctuations in market 2 (see the projection of the attractor in Fig. 9*a*), while in market 1 motion is to 'steady state', with dampened fluctuations (Fig. 9*b*). Such a parameter regime is qualitatively similar to the one associated with the bifurcation curves of Fig. 2, or Fig. 4, in the parameter range where only the pair of (complex) eigenvalues associated with market 2 are of modulus greater than one. We notice that the effect,

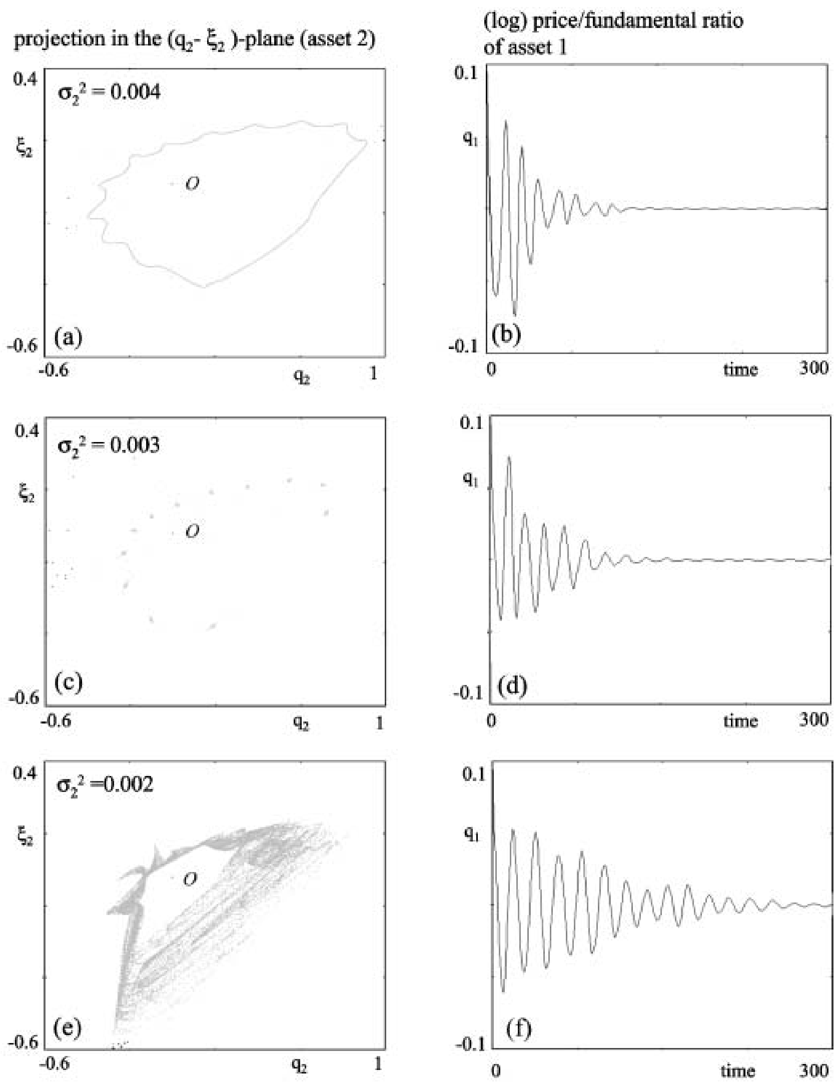

Figure 9. Effect of changes in agents' beliefs about the ratio σ_1^2/σ_2^2 of the 'long-run' variances of the returns, in the case of zero 'long-run' correlation (δ=0). The values of the parameters are: $\alpha^{(f)}=100$, $\alpha^{(c)}=50$, $\eta_1=\eta_2=0.4$, $\sigma_1^2=0.01$ (i.e. $\theta_1=2$, $a_1=0.4$), $c=0.65$, $\beta_1=\beta_2=0.6$, $\pi_1=0.05$, $\pi_2=0.025$. The parameter regime is characterized by wide price fluctuations in market 2 (*a*) and motion to steady state in market 1 (*b*). The parameter σ_2^2 is gradually diminished (and therefore $\theta_2\equiv 1/\left(\alpha^{(c)}\sigma_2^2\right)$ and $a_2\equiv\eta_2/\left(\alpha^{(f)}\sigma_2^2\right)$ are increased). Under decreasing values of σ_2^2 from $\sigma_2^2=0.004$ to $\sigma_2^2=0.002$, the dynamics in market 2 become increasingly irregular and chaotic: see the projections in the (q_2, ξ_2)-plane in (*c*), where $\sigma_2^2=0.003$, $\theta_2\simeq 6.667$, $a_2\simeq 1.333$, and (*e*), where $\sigma_2^2=0.002$, $\theta_2=10$, $a_2=2$); on the other hand, both the transient and the asymptotic behaviour in market 1 remain qualitatively unchanged (see the corresponding time series of the log fundamental/price ratio in market 1 in (*d*) and (*f*)).

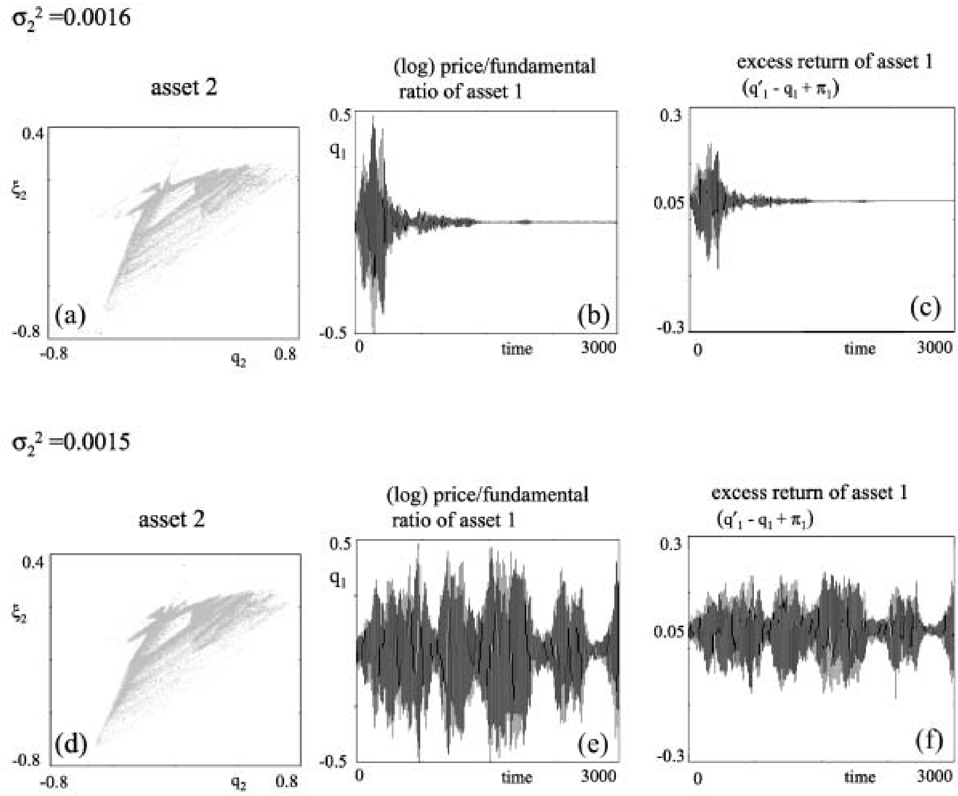

Figure 10. Effect of changes in agents' beliefs about the ratio σ_1^2/σ_2^2 of the 'long-run' variances of the returns, starting from the situation represented in Fig. 9*e*, *f*. The parameter regime is characterized by highly irregular price fluctuations in market 2 and motion to steady state in market 1. Under decreasing values of σ_2^2 from $\sigma_2^2=0.002$ to $\sigma_2^2=0.0015$, the dynamics in market 2 remain qualitatively much the same (see (*a*), (*d*)), while complex behaviour 'propagates' from market 2 to market 1: first, the transient part of the trajectories is affected, before 'convergence' to steady state (see (*b*), (*c*), where $\sigma_2^2=0.0016$, $\theta_2=12.5$, $a_2=2.5$); then a dramatic change in the asymptotic dynamics occurs, which results in oscillatory behaviour characterized by *intermittency* (see (*e*), (*f*), where σ_2^2 is slightly decreased to $\sigma_2^2=0.0015$, and thus $\theta_2 \simeq 13.333$, $a_2 \simeq 2.667$).

on such bifurcation curves, of changing the ratio σ_1^2/σ_2^2, may be quite complicated because both the *strengths of the chartist demand*$\theta_i = 1/\left(\alpha^{(c)}\sigma_i^2\right)$ and the *strengths of fundamentalist demand*$a_i = \eta_i/\left(\alpha^{(f)}\sigma_i^2\right)$, $i=1$, 2, are in general simultaneously affected by such a change. The numerical experiments of Figs. 9 and 10 consist in simulating the long-run behaviour of the system under decreasing values of the (common) belief about risk in market 2 (σ_2^2), that is the market of the asset that exhibits the more irregular price path. In Fig. 9 the parameter σ_2^2 is decreased from $\sigma_2^2=0.004$ to $\sigma_2^2=0.003$ and then to $\sigma_2^2=0.002$. It can be seen that such changes only affect the dynamics in market 2, which become increasingly irregular and chaotic (Fig. 9*a*, *c*, *e*) while both the *transient* and the *asymptotic* dynamics in market

1 remain qualitatively much the same, with the price converging to the fundamental with dampened fluctuations (Fig. 9*b*, *d*, *f*). Further reduction of the value of σ_2^2 down to $\sigma_2^2 = 0.0016$ has no remarkable effect in market 2 (Fig. 10*a*), but it starts to affect the *transient* part of the trajectories in market 1, where the time paths of prices and returns become more and more irregular, before convergence occurs (Figs. 10*b* and 10*c*, respectively). In Figs. 10*d*, *e*, *f* the value of σ_2^2 is decreased to $\sigma_2^2 = 0.0015$. However this slight change leads to a dramatic change in the *asymptotic* dynamics in market 1, where prices and returns are successively attracted to the 'steady state' and then pushed away (Figs. 10*e* and 10*f*, respectively). The price behaviour in market 2 remains qualitatively much the same (Fig. 10*d*). We stress that the foregoing numerical example shows an abrupt change occurring (only) in market 1, even if the unique assumed parameter change concerns agents' beliefs about asset 2. Such a phenomenon could be interpreted in the sense that a reduced perception of risk in market 2 (the one with more irregular price behaviour in our numerical example) causes agents to increase their average wealth invested in asset 2. But this in turns causes more and more irregular changes in agents' portfolios and thus also in agents' demands for asset 1. As a consequence, due to the price reaction to an increasingly unpredictable excess demand in market 1, irregular price and return behavior propagates also to asset 1.

Dynamic phenomena very similar to the ones illustrated in the foregoing numerical example are observed if, *ceteris paribus*, the ratio π_1/π_2 between the expected long-run risk premia is modified (via changes in agents beliefs about expected dividend yields). For instance, the same dynamic effect represented in Fig. 10 can also be obtained by starting from parameter values of Figs. 10*a*, *b*, *c* and by increasing the long-run risk premium for asset 2 (π_2) from the value π_2=0.025 to π_2=0.026.

Conclusions

We have set up a model of heterogenous agents (fundamentalists and chartists) investing in a portfolio of a risk-free asset and two risky assets. The investors differ with respect to attitudes to risk as well as to how they form expectations about the conditional means, variances and covariance of the returns on the risky assets. Market clearing is effected by a market maker whose price adjustment rules ensure that long-run equilibrium prices in each market grow at exogenously determined fundamental rates.

We have set up the dynamical system arising from the interaction and dynamic updating of beliefs of the various agents across the markets for the two risky assets. Without loss of generality in terms of the dynamic analysis, we have focused on the special case where chartists weight equally returns in both markets when updating their beliefs. The dynamic interaction between the two markets is driven by a seven-dimensional dynamical system. Despite the high dimension of the system, we have been able to characterize the steady state of the model, and to analyse with both analytical tools and numerical simulations the dependence of the local stability property of the steady state from the key parameters of the model.

In order to study the dynamics we have considered first the case when there is zero long-run correlation between the returns on the two risky assets. We have found that the characteristic feature of this case is that the local dynamics near the steady state are driven by two lower-dimensional dynamical systems associated with each market. As might have been expected the steady state and eigenvalue structure are merely a double copy of the corresponding situation for the single risky asset case analysed in Chiarella, Dieci and Gardini (2002). However the out-of-equilibrium dynamics can be quite different. In particular it is possible to observe that price movements in one market are tranquil while in the other market quite complex price patterns are occurring. Particularly important is the appearance of complex dynamics associated with increasing values of the chartists extrapolation parameter, as well as with decreasing values of the chartists' risk aversion coefficient.

We then considered the case in which there is non-zero long-run correlation between the two markets. Here analytical results seem impossible, though we were able to give some broad characterization of the eigenvalue structure. Numerical simulations of the out-of-equilibrium behaviour reveal a parameter dependence very similar to the basic case of zero long-run correlation. The main peculiarity is the existence of similar price patterns in both markets simultaneously, presumably brought about by the non-zero long-run correlation.

Finally we have considered the impact on the price dynamics in both markets of changes in agents' beliefs about the long-run risk/return structure of the two assets. Our simulations here suggest that a sufficiently large change in this belief in one market can cause volatility to spill-over into the other market.

This has been a very preliminary study of the effect of agents heterogeneity on portfolio diversification. It still remains to undertake a more thorough numerical study of the effect of changes of key parameters such as strength of fundamentalist and chartist demand, and the market-maker's price adjustment parameters. Furthermore we need to consider the impact of exogenous stochastic factors. The analysis here has focused on the underlying deterministic trend, which interacts with the exogenous stochastic factors to produce the volatility patterns observed in real markets. This kind of analysis will require an interplay among theoretical and numerical methods, which is typical for the study of the global dynamic properties of nonlinear dynamical systems of dimension greater than one, as stressed in Mira *et al.* (1996) and Brock and Hommes (1997). Another topic for future research is to study how the capital asset pricing model relationships are modified in this dynamic framework. For instance the time varying variances and correlation could provide a basis for a theory of time varying beta, which is widely reported as an empirical fact but poorly explained in the standard CAPM framework.

Notes

[1] See however Böhm and Chiarella (2004) for a heterogeneous agent framework that allows for multiple risky assets. Caginalp and Balenovich (1996) also give some discussion of this issue. Westerhoff (2004) considers a fundamentalist-chartist model with multiple assets. The framework of these papers and questions addressed are somewhat different from those studied in this paper.

[2] This assumption can be justified by considering our model as a *deterministic skeleton* of a stochastic model with a dividend process characterized by a constant expected dividend growth rate.

[3] Under the assumption that chartists update variances and covariance consistently with the way they update expected returns, the extrapolation parameters c_1 and c_2 used in Equation 7 are the same used to update expected returns in Equation 4, while the parameter c_K of Equation 8 should take some average

value between c_1 and c_2. In particular, it can be shown that by taking $c_K = 1 - \sqrt{(1-c_1)(1-c_2)}$, the return deviations of asset *i, i*=1, 2, in Equation 8 are discounted at the same rate as in the expected return and variance calculations (Equations 4 and 7, respectively).

[4] We may assume that he/she is able to compute the equilibrium demands for each asset as long-run averages, based for example on the past order flow.

[5] If y is the value of a state variable at time t, then y' denotes the value of the same variable at time (t+1).

[6] See, for instance, Gumowski and Mira (1980), p. 159.

[7] Of course, in the case represented in Fig. 1. ($\beta_i\theta_i \leq 1$), the crossing of the Neimark–Hopf bifurcation curve does not occur for economically meaningful values of the parameter c.

[8] We recall that the strength of chartist demand for the i-th asset at the steady state $\theta_i \equiv \zeta^{(c)}_{i,\xi_i} = \frac{1}{\alpha^{(c)}\sigma_i^2}$ decreases as long as the chartists risk aversion coefficient $\alpha^{(c)}$ or the long-run variance σ_i^2 increase; similarly, the strength of fundamentalist demand for the ith asset $a_i = \frac{\eta_i}{\alpha^{(f)}\sigma_i^2}$ is a decreasing function of the fundamentalist risk aversion coefficient $\alpha^{(f)}$ and of the long-run variance σ_i^2.

References

Beja, A. and Goldman, M. B. (1980) On the dynamic behavior of prices in disequilibrium, *Journal of Finance*, 35, pp. 235–48.

Böhm, V. and Chiarella, C. (2004) Mean variance preferences, expectations formation and the dynamics of random asset prices, *Mathematical Finance*, forthcoming.

Brock, W. and Hommes, C. (1997) A rational route to randomness, *Econometrica*, 65, pp. 1059–95.

Brock, W. and Hommes, C. (1998) Heterogeneous beliefs and routes to chaos in a simple asset pricing model, *Journal of Economic Dynamics and Control*, 22, pp. 1235–74.

Caginalp, G. and Balenovich, D. (1996) Trend-based asset flow in technical analysis and securities marketing, *Psychology and Marketing*, 13, pp. 407–44.

Caginalp, G. and Ermentrout, G. B. (1990) A kinetic thermodynamic approach to the psychology of fluctuations in financial markets, *Applied Mathematical Letters*, 4, pp. 17–19.

Caginalp, G. and Ermentrout, G. B. (1991) Numerical studies of differential equations related to theoretical financial markets, *Applied Mathematical Letters*, 4, pp. 35–38.

Chen, S. H. and Yeh, C. H. (1997) Modelling speculators with genetic programming, in: P. Angeline *et al.* (Eds) *Evolutionary Programming VI, Lecture Notes in Computer Science*, Vol. 1213, pp. 137–147 (Springer-Verlag).

Chiarella, C. (1992) The dynamics of speculative behaviour, *Annals of Operations Research*, 37, pp. 101–23.

Chiarella, C., Dieci, R. and Gardini, L. (2001) Asset price dynamics in a financial market with fundamentalists and chartists, *Discrete Dynamics in Nature and Society*, 6, pp. 69–99.

Chiarella, C., Dieci, R. and Gardini, L. (2002) Speculative behaviour and complex asset price dynamics: a global analysis, *Journal of Economic Behavior and Organization*, 49, pp. 173–97.

Chiarella, C. and He, X. (2001) Heterogeneous beliefs, risk and learning in a simple asset pricing model, *Computational Economics*, 19(1), pp. 95–132.

Chiarella, C. and He, X. (2002) Asset pricing and wealth dynamics under heterogeneous expectations, *Quantitative Finance*, 1, pp. 509–26.

Chiarella, C. and He, X. (2003) Heterogeneous beliefs, risk and learning in a simple asset pricing model with a market maker, *Macroeconomic Dynamics*, 7, pp. 503–36.

Chiarella, C. and Iori, G. (2002) A simulation analysis of the microstructure of double auction markets, *Quantitative Finance*, 2, pp. 346–53.

Day, R. H. and Huang, W. (1990) Bulls, bears and market sheep, *Journal of Economic Behavior and Organization*, 14, pp. 299–329.

Fernandez-Rodriguez, F., Garcia-Artiles, M. D. and Martin Gonzalez, J. M. (2002) A model of speculative behaviour with a strange attractor, *Applied Mathematical Finance*, 9, pp. 143–61.

Gallegati, M. and Kirman, A. (Eds) (2000) *Beyond the Representative Agent* (Cheltenham: Edward Elgar).

Gantmacher, F. R. (1990) *The Theory of Matrices* (New York: Chelsea).

Gaunersdorfer, A. (2000) Endogenous fluctuations in a simple asset pricing model with heterogeneous agents, *Journal of Economic Dynamics and Control*, 24, pp. 799–831.

Gumowski, I. and Mira, C. (1980) *Dynamique Chaotique* (Toulouse: Cepadues Ed.).
Lui, Y.-H. and Mole, D. (1998) The use of fundamental and technical analysis by foreign exchange dealers: Hong Kong evidence, *Journal of International Money and Finance*, 17, pp. 535–45.
Lux, T. (1998) The socio-economic dynamics of speculative markets: interacting agents, chaos and the fat tails of return distributions, *Journal of Economic Behaviour and Organization*, 33, pp. 143–65.
Mira, C., Gardini, L., Barugola, A. and Cathala, J. C. (1996) *Chaotic Dynamics in Two-Dimensional Noninvertible Maps* (Singapore: World Scientific).
Taylor, M. and Allen, H. (1992) The use of technical analysis in the foreign exchange market, *Journal of International Money and Finance*, 11, pp. 304–14.
Westerhoff, F. (2004) Multi-asset-market dynamics, *Macroeconomic Dynamics*, in press.
Zeeman, E. C. (1974) On the unstable behaviour of stock exchanges, *Journal of Mathematical Economics*, 1, pp. 39–49.

Appendix

Derivation of the chartist updating rule for the variance

The time varying components, $v_{i,t}$, $i=1$, 2, of the variances of the returns are computed by extrapolating past deviations from expected returns according to Equation 7, i.e.

$$v_{i,t}=\sum_{s=0}^{\infty} c_i(1-c_i)^s\left(P_{i,t-s}-P_{i,t-s-1}-\psi_{i,t}\right)^2$$

where the expected return $\psi_{i,t}$ is defined as

$$\psi_{i,t}\equiv E_t^{(c)}[P_{i,t+1}-P_{i,t}]=\sum_{s=0}^{\infty} c_i(1-c_i)^s(P_{i,t-s}-P_{i,t-s-1})$$

or, recursively as

$$\psi_{i,t}=(1-c_i)\psi_{i,t-1}+c_i(P_{i,t}-P_{i,t-1}) \tag{26}$$

In this Appendix we show that $v_{i,t}$ satisfies the recurrence relation

$$v_{i,t}=(1-c_i)v_{i,t-1}+c_i\left(P_{i,t}-P_{i,t-1}-\psi_{i,t}\right)^2+(1-c_i)\left(\psi_{i,t}-\psi_{i,t-1}\right)^2 \tag{27}$$

or, equivalently

$$v_{i,t}=(1-c_i)v_{i,t-1}+c_i(1-c_i)\left(P_{i,t}-P_{i,t-1}-\psi_{i,t-1}\right)^2 \tag{28}$$

In fact, starting from Equation 7 we can write:

$$\begin{aligned}v_{i,t}=&\,c_i\left(P_{i,t}-P_{i,t-1}-\psi_{i,t}\right)^2+c_i(1-c_i)\left(P_{i,t-1}-P_{i,t-2}-\psi_{i,t}\right)^2\\&+c_i(1-c_i)^2\left(P_{i,t-2}-P_{i,t-3}-\psi_{i,t}\right)^2+\dots\end{aligned}$$

and

$$\begin{aligned}(1-c_i)v_{i,t-1}=&\,c_i(1-c_i)\left(P_{i,t-1}-P_{i,t-2}-\psi_{i,t-1}\right)^2\\&+c_i(1-c_i)^2\left(P_{i,t-2}-P_{i,t-3}-\psi_{i,t-1}\right)^2+\dots\end{aligned}$$

By summing up we obtain

$$v_{i,t}-(1-c_i)v_{i,t-1}=c_i\left(P_{i,t}-P_{i,t-1}-\psi_{i,t}\right)^2 + c_i(1-c_i)\left[\left(P_{i,t-1}-P_{i,t-2}-\psi_{i,t}\right)^2-\left(P_{i,t-1}-P_{i,t-2}-\psi_{i,t-1}\right)^2\right] + c_i(1-c_i)^2\left[\left(P_{i,t-2}-P_{i,t-3}-\psi_{i,t}\right)^2-\left(P_{i,t-2}-P_{i,t-3}-\psi_{i,t-1}\right)^2\right]+\ldots$$

i.e.

$$v_{i,t}-(1-c_i)v_{i,t-1}=c_i\left(P_{i,t}-P_{i,t-1}-\psi_{i,t}\right)^2 + c_i(1-c_i)\left[\psi_{i,t}^2-\psi_{i,t-1}^2-2(P_{i,t-1}-P_{i,t-2})\left(\psi_{i,t}-\psi_{i,t-1}\right)\right] + c_i(1-c_i)^2\left[\psi_{i,t}^2-\psi_{i,t-1}^2-2(P_{i,t-2}-P_{i,t-3})\left(\psi_{i,t}-\psi_{i,t-1}\right)\right]+\ldots$$

and finally

$$v_{i,t}=(1-c_i)v_{i,t-1}+c_i\left(P_{i,t}-P_{i,t-1}-\psi_{i,t}\right)^2 + (1-c_i)\left(\psi_{i,t}^2-\psi_{i,t-1}^2\right)\sum_{s=0}^{\infty}c_i(1-c_i)^s - 2(1-c_i)\left(\psi_{i,t}-\psi_{i,t-1}\right)\sum_{s=0}^{\infty}c_i(1-c_i)^s(P_{i,t-s-1}-P_{i,t-s-2})$$

Notice that

$$\sum_{s=0}^{\infty}c_i(1-c_i)^s=1$$

$$\sum_{s=0}^{\infty}c_i(1-c_i)^s(P_{i,t-s-1}-P_{i,t-s-2})=\psi_{i,t-1}$$

from which we obtain

$$v_{i,t}=(1-c_i)v_{i,t-1}+c_i\left(P_{i,t}-P_{i,t-1}-\psi_{i,t}\right)^2 + (1-c_i)\left[\psi_{i,t}^2-\psi_{i,t-1}^2-2\left(\psi_{i,t}-\psi_{i,t-1}\right)\psi_{i,t-1}\right]$$

and finally Equation 27, i.e.

$$v_{i,t}=(1-c_i)v_{i,t-1}+c_i\left(P_{i,t}-P_{i,t-1}-\psi_{i,t}\right)^2+(1-c_i)\left(\psi_{i,t}-\psi_{i,t-1}\right)^2$$

To obtain the alternative expression (Equation 28) notice that, from the recurrence relation (26) of the expected return $\psi_{i,t}$ we have:

$$\psi_{i,t}-\psi_{i,t-1}=c_i\left(P_{i,t}-P_{i,t-1}-\psi_{i,t-1}\right)$$

and

$$P_{i,t}-P_{i,t-1}-\psi_{i,t}=(1-c_i)\left(P_{i,t}-P_{i,t-1}-\psi_{i,t-1}\right)$$

and thus Equation 27 may be rewritten in the alternative form, Equation 28:

$$v_{i,t}=(1-c_i)v_{i,t-1}+c_i(1-c_i)\left(P_{i,t}-P_{i,t-1}-\psi_{i,t-1}\right)^2.$$

Derivation of the chartist updating rule for the covariance

The time varying component, K_t, of the covariance between returns is computed according to Equation 8, i.e.

$$K_t=\sum_{s=0}^{\infty}c_K(1-c_K)^s\left(P_{1,t-s}-P_{1,t-s-1}-\psi_{1,t}\right)\left(P_{2,t-s}-P_{2,t-s-1}-\psi_{2,t}\right)$$

Here we prove that Equation (8) results in the updating rule

$$\begin{aligned}K_t=&(1-c_K)K_{t-1}+c_K\left(P_{1,t}-P_{1,t-1}-\psi_{1,t}\right)\left(P_{2,t}-P_{2,t-1}-\psi_{2,t}\right)\\&+(1-c_K)\left[\left(\psi_{1,t}-\psi_{1,t-1}\right)\left(\psi_{2,t}-\widetilde{\psi}_{2,t-1}\right)+\left(\psi_{2,t}-\psi_{2,t-1}\right)\left(\psi_{1,t-1}-\widetilde{\psi}_{1,t-1}\right)\right]\end{aligned}\tag{29}$$

or, equivalently

$$\begin{aligned}K_t=&(1-c_K)K_{t-1}\\&+c_K(1-c_1)(1-c_2)\left(P_{1,t}-P_{1,t-1}-\psi_{1,t-1}\right)\left(P_{2,t}-P_{2,t-1}-\psi_{2,t-1}\right)\\&+(1-c_K)\Big[c_1\left(P_{1,t}-P_{1,t-1}-\psi_{1,t-1}\right)\left(\psi_{2,t-1}-\widetilde{\psi}_{2,t-1}\right)\\&+c_1c_2\left(P_{1,t}-P_{1,t-1}-\psi_{1,t-1}\right)\left(P_{2,t}-P_{2,t-1}-\psi_{2,t-1}\right)\\&+c_2\left(P_{2,t}-P_{2,t-1}-\psi_{2,t-1}\right)\left(\psi_{1,t-1}-\widetilde{\psi}_{1,t-1}\right)\Big]\end{aligned}\tag{30}$$

where the quantities $\widetilde{\psi}_{i,t}$, $i=1,2,$, are defined recursively as

$$\widetilde{\psi}_{i,t}=(1-c_K)\widetilde{\psi}_{i,t-1}+c_K(P_{i,t}-P_{i,t-1})\quad(i=1,\,2)$$

In fact, by following the same reasoning as for the variance, we obtain

$$\begin{aligned}K_t=&(1-c_K)K_{t-1}+c_K\left(P_{1,t}-P_{1,t-1}-\psi_{1,t}\right)\left(P_{2,t}-P_{2,t-1}-\psi_{2,t}\right)\\&+(1-c_K)\left(\psi_{1,t}\psi_{2,t}-\psi_{1,t-1}\psi_{2,t-1}\right)\sum_{s=0}^{\infty}c_K(1-c_K)^s\\&-(1-c_K)\left(\psi_{1,t}-\psi_{1,t-1}\right)\sum_{s=0}^{\infty}c_K(1-c_K)^s(P_{2,t-s-1}-P_{2,t-s-2})\\&-(1-c_K)\left(\psi_{2,t}-\psi_{2,t-1}\right)\sum_{s=0}^{\infty}c_K(1-c_K)^s(P_{1,t-s-1}-P_{1,t-s-2})\end{aligned}\tag{31}$$

where $\sum_{s=0}^{\infty} c_K(1-c_K)^s = 1$. By defining

$$\widetilde{\psi}_{i,t} = \sum_{s=0}^{\infty} c_K(1-c_K)^s (P_{i,t-s} - P_{i,t-s-1}) \quad (i=1,\, 2)$$

or, recursively

$$\widetilde{\psi}_{i,t} = (1-c_K)\,\widetilde{\psi}_{i,t-1} + c_K(P_{i,t} - P_{i,t-1}) \tag{32}$$

Equation 31 becomes

$$\begin{aligned} K_t = {} & (1-c_K)K_{t-1} + c_K\big(P_{1,t} - P_{1,t-1} - \psi_{1,t}\big)\big(P_{2,t} - P_{2,t-1} - \psi_{2,t}\big) \\ & + (1-c_K)\Big[\big(\psi_{1,t}\psi_{2,t} - \psi_{1,t-1}\psi_{2,t-1}\big) - \big(\psi_{1,t} - \psi_{1,t-1}\big)\,\widetilde{\psi}_{2,t-1} \\ & - \big(\psi_{2,t} - \psi_{2,t-1}\big)\,\widetilde{\psi}_{1,t-1}\Big] \end{aligned}$$

and may finally be rewritten in the form of Equation 29. Since the following relations hold for $i=1,\, 2$ (from Equation 26 and 32)

$$\begin{aligned} \psi_{i,t} - \psi_{i,t-1} &= c_i\big(P_{i,t} - P_{i,t-1} - \psi_{i,t-1}\big) \\ P_{i,t} - P_{i,t-1} - \psi_{i,t} &= (1-c_i)\big(P_{i,t} - P_{i,t-1} - \psi_{i,t-1}\big) \\ \psi_{i,t} - \widetilde{\psi}_{i,t-1} &= \psi_{i,t-1} - \widetilde{\psi}_{i,t-1} + c_i\big(P_{i,t} - P_{i,t-1} - \psi_{i,t-1}\big) \end{aligned}$$

Equation 29 is easily reduced to the alternative form of Equation 30, that only contains the expectations at time $(t-1)$.

If in particular we assume $c_1=c_2=c_K=c$, we obtain $\psi_{i,t}=\widetilde{\psi}_{i,t}$ $i=1,2$, and Equation 30 can be reduced to the simpler form

$$K_t = (1-c)K_{t-1} + c(1-c)\big(P_{1,t} - P_{1,t-1} - \psi_{1,t-1}\big)\big(P_{2,t} - P_{2,t-1} - \psi_{2,t-1}\big)$$

which is the one used in our model.