ITERATION THEORY (ECIT '08)
A.N. Sharkovsky, I.M. Sushko (Eds.)
Grazer Math. Ber., ISSN 1016–7692
Bericht Nr. 354 (2009), 53-72

Connection between bifurcations on the Poincaré Equator and the dangerous bifurcations

Laura Gardini, Viktor Avrutin
and Michael Schanz

Abstract

The object of the present work is to describe some bifurcations occurring in the 2D piecewise linear continuous map in canonical form involving the Poincaré Equator (i.e. periodic points at infinity) and its relation to the regions of so called *dangerous bifurcations* (following Hassouneh et al. [15] and Ganguli and Banerjee [12]). It will be shown that such regions are related not only to stable fixed points and repelling saddle cycles, and a more general definition is proposed. The boundaries of such regions are curves related to *border collision bifurcations* of cycles on the Poincaré Equator.

1 Introduction

Recently many works have been published showing applied models (in engineering, economics, etc.) described by continuous piecewise smooth or piecewise linear maps. Examples may be found in [16], [23], [24], [19]-[21], [33], [2]-[4], [10], [32], [11], [13], [14], [27]-[29], [35], [36]. The bifurcations occurring in these maps, different from those studied in smooth models, are denoted as *border collision bifurcations* (BCB for short henceforth) after Nusse and Yorke [23]. This is a quite recent research subject, although some works by Feigen date back to the 70th, and were rediscovered only a few years ago, see [5]. The first works associated with discontinuous linear maps

Mathematics Subject Classification 2000: Primary 37G15, Secondary 37G35, 70K05.
Keywords and phrases: border collision bifurcations, 2D normal forms, dangerous bifurcations, piecewise-linear 2D maps.

are due to Leonov almost fifty year ago (see [17], [18]), and were rediscovered by Gumowski and Mira, see [22] and Maistrenko et al. [19]-[21].

In the two-dimensional case the works are more recent, and several of them deal with the two-dimensional canonical form proposed in Nusse and Yorke [23]. This piecewise linear map is defined by two linear functions and its analysis is quite important because its dynamic behavior is at the basis also of the BCB occurring in piecewise smooth systems. The two-dimensional canonical form has been mainly considered in dissipative cases associated with *real* eigenvalues of the point which undergoes the BCB. Among the effects studied up to now we recall the uncertainty about the occurrence after the BCB (see e.g., [25], [9]), multistability and unpredictability of the number of coexisting attractors (see e.g. in [35]). The *center bifurcation* associated with complex eigenvalues has been considered in [31], [30], [26].

A special phenomenon, the so-called *dangerous* BCB (introduced in [15] and in [12]), is related to the case in which a fixed point is attracting before and after the BCB, while close to the bifurcation value the basin of attraction shrinks up to a unique point and at the bifurcation value the dynamics are divergent, except at most a set of zero Lebesgue measure (as noticed also in [8], [26]). This specific subject will be reconsidered in the present paper. In the recent paper by Avrutin et al. [1] it is shown the importance of the bifurcations involving infinity, which we refer as occurring on the Poincaré Equator (PE for short henceforth), because they involve cycles existing on the PE, and may be considered as well as border collision, that is: *PE-collision bifurcations*. It was partly observed in [12] that the *dangerous BCB* may occur in regions bounded by the curves of PE-collision bifurcation. In the present paper we discuss how these two types of bifurcations are related to each other. Additionally, it is possible that cycles different from fixed points are involved in the dangerous BCB (examples can be found in [7]). We also present several examples which lead us to the conclusion that the definition of dangerous BCB suggested in the cited works should be extended.

The plan of the work is as follows. In Section 2 we shall consider the two-dimensional piecewise linear map which is a normal form to study BCB in piecewise smooth two-dimensional maps, recalling some known bifurcations involving cycles of the map and introducing the bifurcations involving the PE. In Section 3 we shall illustrate the known results and its dynamic behaviors in the case of dangerous bifurcations, giving an extended definition, and showing the regions in the parameter space where they may occur. Section 4 gives some conclusion.

2 Piecewise linear 2D normal form

As proposed in [23], the normal form for the border-collision bifurcation in a 2D phase space, the real plane, is given by a family of two-dimensional piecewise linear maps $F : \mathbb{R}^2 \to \mathbb{R}^2$ made up of two linear maps F_R and F_L which are defined in two half planes L and R:

$$F : X' = \begin{cases} F_l(X) = A_l X + B & \text{if} \quad x \leq 0 \\ F_r(X) = A_r X + B & \text{if} \quad x \geq 0 \end{cases}$$

where

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \quad , \quad A_l = \begin{bmatrix} \tau_l & 1 \\ -\delta_l & 0 \end{bmatrix} \quad , \quad A_r = \begin{bmatrix} \tau_r & 1 \\ -\delta_r & 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} \mu \\ 0 \end{bmatrix}$$

τ_l, τ_r are the traces and δ_l, δ_r are the determinants of the Jacobian matrix of the map F in the left and right halfplanes, i.e., in L and R, respectively, $\mathbb{R}^2 = L \cup R$.

Following [2] we denote by O_L^* and O_R^* the fixed points of F_l and F_r given, respectively, by

$$O_{L/R}^* = \left(\frac{\mu}{1 - \tau_{l/r} + \delta_{l/r}}, \frac{-\delta_{l/r}\mu}{1 - \tau_{l/r} + \delta_{l/r}} \right) \tag{1}$$

(where the notation l/r denotes l or r). Obviously, O_L^* and O_R^* are fixed points of the map F only when they belong to the related partitions L and R. Namely, O_L^* is the fixed point of the map F if $\mu/(1 - \tau_l + \delta_l) \leq 0$, otherwise it is a so-called *virtual* fixed point. Similarly, O_R^* is the fixed point of F if $\mu/(1 - \tau_r + \delta_r) \geq 0$, otherwise it is a virtual fixed point. Clearly, if the parameter μ varies through 0, the fixed points (including virtual fixed points) cross the border $x = 0$, so that the collision with it occurs at $\mu = 0$, value at which O_L^* and O_R^* merge with the origin $(0, 0)$. As shown in [1], there is also another bifurcation which leads a fixed point to become virtual, and it occurs when the denominator in (1) becomes zero, that is at:

$$\chi_{L/R} = \left\{ \left(\tau_{l/r}, \delta_{l/r} \right) \mid \tau_{l/r} = 1 + \delta_{l/r} \right\} \tag{2}$$

At this bifurcation one real eigenvalue of the matrix $A_{l/r}$ becomes equal to $+1$, and the real fixed point $O_{L/R}^*$ is merging with a fixed point on the PE (the limit point on the eigenvector associated with the eigenvalue $\lambda = 1$), becoming virtual after, for $\tau_{l/r} > 1 + \delta_{l/r}$.

Let μ vary from a positive to a negative value. As it was noted in [2], if some bifurcation occurs for μ decreasing through 0, then the same bifurcation occurs also for μ increasing through 0 if we exchange the parameters of the maps F_l and F_r, i.e., there is a symmetry of the bifurcation structure with respect to $\tau_r = \tau_l$, $\delta_r = \delta_l$

in the $(\tau_l, \tau_r, \delta_l, \delta_r)$-parameter space. Thus, it is enough to consider μ varying from positive to negative (or *vice versa*).

The stability of the fixed points $O^*_{L/R}$ comes from the eigenvalues $\lambda^{L,R}_{1,2}$ of the Jacobian matrix of the map $F_{l/r}$, which are

$$\lambda^{L,R}_{1,2} = \frac{1}{2}\left(\tau_{l/r} \pm \sqrt{\tau^2_{l/r} - 4\delta_{l/r}}\right) \tag{3}$$

The stability of $O^*_{L/R}$ is thus given by:

$$S_{O^*_{L/R}} = \left\{\left(\tau_{l/r}, \delta_{l/r}\right) \mid -(1+\delta_{l/r}) < \tau_{l/r} < (1+\delta_{l/r}) \;\;,\;\; \delta_{l/r} < 1\right\} \tag{4}$$

The bifurcation curves

$$\xi_{l/r} = \left\{\left(\tau_{l/r}, \delta_{l/r}\right) \mid \tau_{l/r} = -(1+\delta_{l/r})\right\}$$

are associated to the eigenvales $\lambda_1 = -\delta_{l/r}$ and $\lambda_2 = -1$.

At $\mu = 0$ we have $O^*_L = O^*_R = (0,0)$, i.e., the fixed points collide with the border line $x = 0$, and for $\mu > 0$ (i.e., before the border-collision) the fixed point O^*_R is real and O^*_L is virtual, while the viceversa occurs for $\mu < 0$.

Turning to the cycles of the map different from the fixed points, their existence is related to the applications of the linear maps $F_{l/r}$ in the proper order, and their stability is governed by the eigenvalues of the matrix product of the matrices $A_{l/r}$ involved. For example, for a $2-$cycle denoted O_{RL} we obtain that its points (x^{RL}_i, y^{RL}_i) $i = 1, 2$ are given by

$$\begin{aligned}(x^{RL}_1, y^{RL}_1) &= \left(\frac{\mu(1+\delta_l+\tau_l)}{(1+\delta_l)(1+\delta_r) - \tau_l\tau_r} \;,\; -\delta_l x^{RL}_2\right) \\ (x^{RL}_2, y^{RL}_2) &= \left(\frac{\mu(1+\delta_r+\tau_r)}{(1+\delta_l)(1+\delta_r) - \tau_l\tau_r} \;,\; -\delta_r x^{RL}_1\right)\end{aligned} \tag{5}$$

The $2-$cycle O_{RL} exists iff the points given in (5) are located in the proper half-planes, which means that one of them must be in L and the other one in R. The $2-$cycle is stable when the eigenvalues of the matrix $A_r A_l$, given by

$$\lambda_{1,2} = \frac{1}{2}\left(\tau_l\tau_r - \delta_r - \delta_l \pm \sqrt{(\tau_l\tau_r - \delta_r - \delta_l)^2 - 4\delta_l\delta_r}\right)$$

are less than 1 in modulus. Notice that when the denominator in (5) becomes zero, i.e.at the *PE*-collision bifurcation curve

$$\chi_{LR} = \{(\tau_l, \tau_r, \delta_l, \delta_r) \mid \tau_l\tau_r = (1+\delta_l)(1+\delta_r)\} \tag{6}$$

one eigenvalue becomes equal to 1 and we have the merging of the real 2−cycle with a 2−cycle on the PE (and then it becomes virtual). Notice that only one kind of 2−cycle can exist, whereas for all higher periods there are several orbits with the same periods but corresponding to different symbolic sequences. Especially important for us are pairs of so-called *complementary* cycles ([12], [1]). In the most simple case such a pair is given by $O_{RL^{n-1}}$ and $O_{R^2L^{n-2}}$. These orbits appear via a so-called *fold BCB*: At this bifurcation a cycle with one point on the boundary $x = 0$ emerge, which splits after into two points on opposite sides of the boundary. Therefore, the two complementary cycles differ only by one application of the linear maps, or with other words, by one letter in the symbolic sequence.

The explicit formulation of the complementary 3−cycles (with periodic points $(x_i^{RL^2}, y_i^{RL^2})$ and $(x_i^{R^2L}, y_i^{R^2L})$, $i = 1, 2, 3$) can be found in [1] (see also in [12]). Here we recall that from the expression of the periodic points

$$x_i^{RL^2} = \mu \frac{N_i^{RL^2}}{D^{RL^2}} \text{ and } x_i^{R^2L} = \mu \frac{N_i^{R^2L}}{D^{R^2L}} \tag{7}$$

we can obtain directly the parameter subspaces where these orbits undergo some bifurcations. The condition $N_3^{RL^2} = 0$ and $N_1^{R^2L} = 0$ determine the border collision bifurcation curve associated with the existence of the cycles, given by

$$\begin{aligned} \xi &= \left\{(\tau_l, \tau_r, \delta_l, \delta_r)|\ N_3^{RL^2} = 0\right\} \equiv \left\{(\tau_l, \tau_r, \delta_l, \delta_r)|\ N_1^{R^2L} = 0\right\} \\ &= \{(\tau_l, \tau_r, \delta_l, \delta_r)|\ \tau_l\tau_r + \tau_l + \tau_r\delta_l + \delta_r\delta_l - \delta_r + 1 = 0\} \end{aligned} \tag{8}$$

and shown in Fig.1.

Further, from the condition that the denominators in (7) vanish, we obtain the surfaces of the PE-collision bifurcations of the 3−cycles, which are given explicitly by

$$\begin{aligned} \chi_{RL^2} &= \left\{(\tau_l, \tau_r, \delta_l, \delta_r)|\ D^{RL^2} = 0\right\} \\ &= \left\{(\tau_l, \tau_r, \delta_l, \delta_r)|\ 1 + \delta_l(\tau_l + \tau_r) + \tau_l\delta_r + \delta_l^2\delta_r - \tau_l^2\tau_r = 0\right\} \end{aligned} \tag{9}$$

$$\begin{aligned} \chi_{R^2L} &= \left\{(\tau_l, \tau_r, \delta_l, \delta_r)|\ D^{R^2L} = 0\right\} \\ &= \left\{(\tau_l, \tau_r, \delta_l, \delta_r)|\ 1 + \tau_r(\delta_l + \delta_r) + \tau_l\delta_r + \delta_l\delta_r^2 - \tau_l\tau_r^2 = 0\right\} \end{aligned} \tag{10}$$

We recall that at such bifurcation curves (χ) a finite 3−cycle expands up to merging with a 3−cycle located on the PE (whereby one of its eigenvalues becomes equal to +1). Hence, when a stable 3−cycle expands up to merging with a 3−cycle located

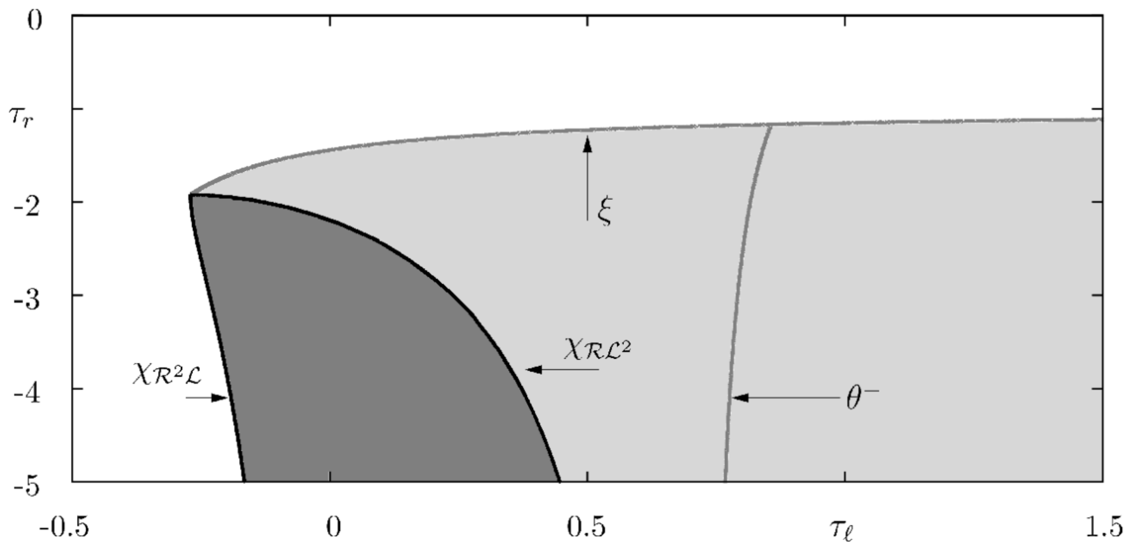

Figure 1: *Two-dimensional bifurcation diagram in the* (τ_l, τ_r) *parameter plane at* $\delta_r = \delta_l = 0.52$ *fixed, associated with the complementary* $3-$*cycles* O_{RL^2} *and* O_{R^2L}.

on the PE (approaching the curve χ_{RL^2}) becoming virtual after the bifurcation and leaving only one unstable $3-$cycle in the plane, we have a dynamic behavior that is typical for transcritical bifurcations. When a single unstable $3-$cycle expands up to merging with a $3-$cycle located on the PE (approaching the curve χ_{R^2L}) disappearing after the bifurcation, we have a dynamic behavior that is typical of a saddle-node bifurcation.

To accomplish the description of the complementary orbits O_{RL^2} and O_{R^2L} we have to consider their stability. Straight forward calculations show that at the curve θ^- (see Fig.1) the matrix $A_rA_lA_l$ (responsible for the stability of the $3-$cycle O_{RL^2}) has an eigenvalue equal to -1. On the left side of θ^- the orbit O_{RL^2} is a stable node and O_{R^2L} is a saddle, whereas on the right hand side of θ^- both the orbits O_{RL^2} and O_{R^2L} are unstable (in the case shown in Fig.1 both orbits are saddles).

The relative location of the curves ξ, χ_{RL^2}, χ_{R^2L} in the parameter space with respect to each other, as illustrated in Fig.1, show that there is a whole region (the dark gray one) in which, of the complementary pair of $3-$cycles, only one saddle exists. As one can clearly see, there is a region between the curves θ^- and χ_{RL^2} where the orbit O_{RL^2} is stable. It is known that the saddle cycle O_{R^2L} is located on the boundary of the basin of the stable orbit O_{RL^2}. By contrast, between the curves χ_{R^2L} and χ_{RL^2} the stable orbit O_{RL^2} no longer exists (destroyed by the PE-collision bifurcation) whereas the saddle O_{R^2L} still persists in the phase-plane. As shown in

[12] and [1], this is the reason why in this region dangerous bifurcations may occur.

As it follows from eq.(7), the parameter μ is a scale parameter, so that without loss of generality we can fix $\mu = 1$ for the generic case $\mu > 0$ and $\mu = -1$ for the generic case $\mu < 0$. This means that if the parameter μ is changed from positive to negative values (or viceversa), all the cycles existing in the phase plane shrink up to the origin. Consequently, the bifurcation occurring at $\mu = 0$ is a peculiar one: the unique cycle in the phase plane is the fixed point O in the origin, which may be attracting, or we may have divergent trajectories (which means that there exists an attractor on the PE). It follows that a peculiar behavior may occur when we have divergent trajectories at $\mu = 0$, separating the transition to some attractor (or attractors), existing for $\mu > 0$ to one attractor (or attractors), existing for $\mu < 0$. When this occurs we say that a *dangerous BCB* takes place.

It is worth noticing that this definition includes (and thus extend) the one previously given in [15] and [12]. In the cited works, the definition of the dangerous bifurcation was restricted to the situation in which a stable fixed point exists both for $\mu > 0$ and $\mu < 0$. However it will be shown that also in such a case, a stable fixed point may be not the unique attracting set. Moreover, we shall see that for $\mu < 0$ (resp. $\mu > 0$) we may have the fixed point which is stable while for $\mu > 0$ (resp. $\mu < 0$) the divergent phenomena coexists with some attractors different from a fixed point, and the dangerous behavior exists at $\mu = 0$.

3 Dangerous bifurcations

Let us first consider the examples proposed in [15] and [12], which assume the parameters $0 < \delta_i < 1$ and

$$-(1 + \delta_i) < \tau_i < (1 + \delta_i) \quad , \quad i = l/r$$

so that before and after the BCB we have a stable fixed point: O_R^* is locally stable for $\mu > 0$ and O_L^* is locally stable for $\mu < 0$. Then in the two-dimensional parameter plane (τ_L, τ_R) we may have regions corresponding to divergent trajectories at $\mu = 0$ which are called regions of dangerous BCB, shown in dark gray in Fig.2.

The example shown in [15] and [12] refers to the case $\delta_l = \delta_r = 0.9$ which is the same used in Fig.2, and in the cited references the authors were interested only in the rectangle $[-1.9, 1.9] \times [-1.9, 1.9]$, where the dark gray regions are such peculiar regions of dangerous BCB. However, in Fig.2 we show a wider region, and there the dark gray regions are bounded by the curves of the PE-collision bifurcations for specific pairs of complementary orbits, as explained in the pervious section. Hence, for $\mu \neq 0$ in these parameter regions correspond to there exist an unstable (saddle)

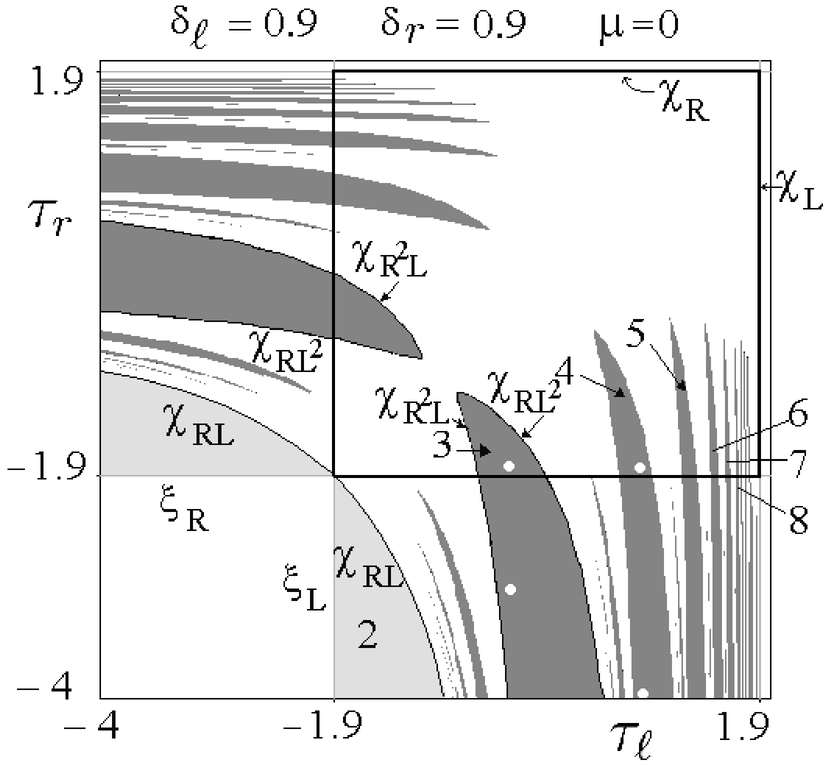

Figure 2: *Two-dimensional bifurcation diagram in the* (τ_l, τ_r) *parameter plane at* $\delta_r = \delta_l = 0.9$ *fixed, and* $\mu = 0$. *The boundaries of the dark gray regions are the PE-collision bifurcation curves. There is divergence in the white region* $[-4, 1.9] \times [-4, 1.9]$ *and convergence to the origin in the light gray region bounded by the bifurcation curves* χ_R, χ_L *and* χ_{LR}.

cycle whose stable set separates the basin of attraction of the stable fixed point from the trajectories which are diverging (i.e. attracted from an attractor on the PE).

As illustrative example consider the case shown in Fig.3. For $\mu > 0$ (Fig.3b) the fixed point O_R^* is locally stable (and its basin of attraction is formed by the stable set of the unique 3-cycle, a saddle), while for $\mu < 0$ (Fig.3a) the fixed point O_L^* is locally stable (and here also its basin of attraction is formed by the stable set of the unique 3-cycle saddle). Due to the scaling of the state space linear in μ, at $\mu = 0$ the 3-cycles, and thus the basin of attraction of the fixed points, reduce to the unique fixed point in the origin, and all over around the generic trajectory is divergent. Thus, the existence of only one saddle cycle (not in pair with a complementary stable cycle), seems related with the dangerous behavior. This was evidenced in

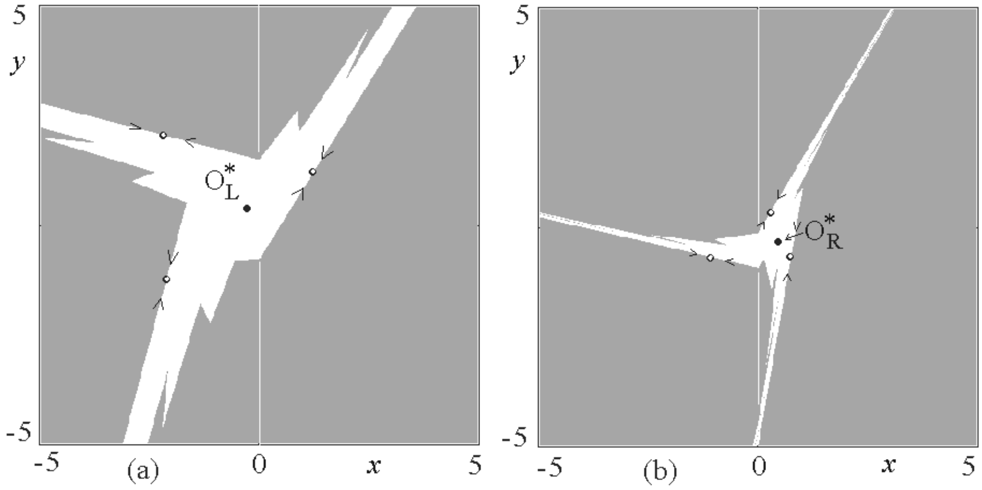

Figure 3: *Two dimensional phase plane at* $\delta_r = \delta_l = 0.9$, $\tau_l = -0.3$, $\tau_r = -1.85$ *fixed. In (a)* $\mu = -1$, *the stable fixed point* O_L^* *is the unique attractor. In (b)* $\mu = +1$, *the stable fixed point* O_R^* *is the unique attractor. In both figures dark gray points denote the basin of divergent trajectories (whose boundary is the stable set of the* 3−*cycle saddle), while white points denote the basin of attraction of the fixed point.*

Ganguli and Banerjee [2005], where the equations bounding the region associated with a saddle 3-cycle were given, suggesting that a similar behavior is going to occur in the other dark regions, associated with so-called principal $k-$cycles saddles, for $k = 4, 5, ...$ having one point in one region and the remaining $(k-1)$ points in the other (also called basic or maximal orbits). Indeed we shall see that this behavior occurring in the region associated with the saddle 3−cycle can be generalized only in part.

As shown in [1] all the cycles which undergo a fold BCB appearing in pair, give rise to two bifurcations with the PE and the parameters between these two bifurcation curves (for example the dark gray region in Fig.1) are those which may be associated with the dangerous bifurcation. To be more precise, consider the case shown in Fig.2 at $\tau_r = -1.85$ fixed and decreasing τ_l from 0.5. Before entering the dark-gray region in the dynamics of the map, for $\mu > 0$ there are two coexisting attractors: the stable fixed point O_R^* and a stable 3-cycle, the two basins of attraction are separated by the stable set of the 3-cycle saddle. When the parameter τ_l has a contact with the right side boundary of the region, the stable 3-cycle has a PE-bifurcation (curve χ_{RL^2}), and disappears (becoming virtual), but leaving in the

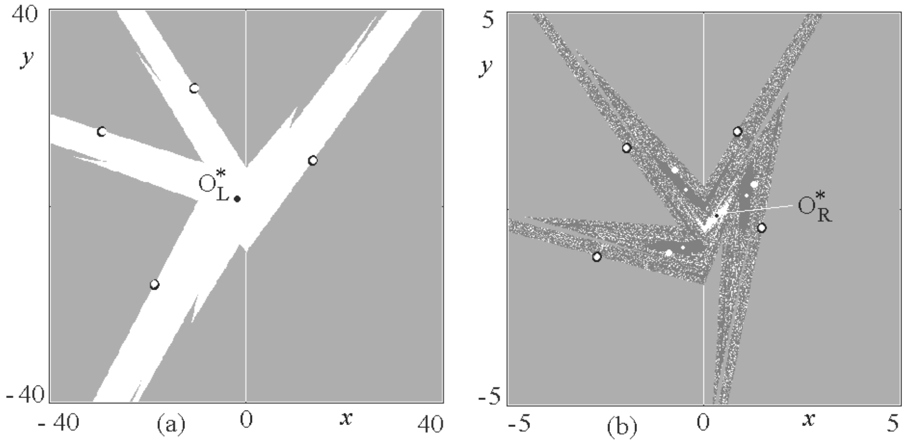

Figure 4: *Two dimensional phase plane at* $\delta_r = \delta_l = 0.9$, $\tau_l = 0.9$, $\tau_r = -1.85$ *fixed. In (a)* $\mu = -1$, *the stable fixed point* O_L^* *is the unique attractor. In (b)* $\mu = +1$, *the stable fixed point* O_R^* *coexists with a stable cycle of period 3. In both figures light gray points denote the basin of divergent trajectories, white points denote the basin of attraction of the fixed point, and dark gray points in (b) denote the basin of the* $3-$*cycle.*

phase-plane the saddle 3-cycle, which exists, on decreasing the parameter τ_l (and the 3-cycle becoming wider and wider), up to the bifurcation on its turn with the PE occurring at the contact with the curve χ_{R^2L}. So that (as evidenced also in [30]) for values of τ_l between these two curves (χ_{RL^2} and χ_{R^2L}) besides the stable fixed points there exist a unique saddle 3-cycle, and it is on the boundary of the basin of attraction of the origin. It follows that for the parameters (τ_L, τ_R) inside this region we are lead to a dangerous bifurcation for $\mu = 0$. The reasoning in the other dark gray tongues is similar but not completely the same. As it is shown in Fig.2 the other dark gray regions on the right of the one associated with the 3-cycle saddle are denoted as $k-$regions for $k = 4, 5, ...$ because indeed the boundaries of these regions are related to the existence of a $k-$cycle saddle without the existence of the complementary k orbit. That it to say: the equations of the tongues are given by the PE-bifurcation curves $\chi_{RL^{k-1}}$ (contact with the PE of the stable k-cycle) and $\chi_{R^2L^{k-2}}$ (contact with the PE of the saddle k-cycle), and for parameter values inside such regions only a saddle $k-$cycle exists, without the complementary orbit, and this $k-$cycle saddle is responsible of the dangerous behavior. However it is not the unique cycle existing for parameter in such $k-$regions at $\mu > 0$. Considering for example a point inside the region of the 4-cycle, in Fig.4 we show the phase portrait of the map for positive and negative values of μ. It is shown in Fig.4b that for $\mu > 0$, besides the locally stable fixed point O_R^* a pair of 3−cycles exist, one locally stable and one saddle (i.e. we are inside the region of existence of the two complementary 3-cycles, before the collision bifurcation of the 3-cycle node with the PE but after the collision bifurcation with the PE of the 4-cycle). Moreover, both the 3-cycle saddle and the 4-cycle saddle have transverse homoclinic points, and all the three basins (of the origin, of the 3-cycle and of the PE) have a fractal structure. Anyhow, decreasing μ, at $\mu = 0$ all the infinitely many cycles merge in the origin, and only the most external basin (that of the PE) is left, thus causing the dangerous behavior. It can be seen that the behavior is similar also in the other regions: between the curves $\chi_{RL^{k-1}}$ and $\chi_{R^2L^{k-2}}$ for $k = 4, 5, ...$ at least one pair of $(k-1)-$cycles exist in some subregions.

We have commented up to now the dark-gray regions below the main diagonal of Fig.2, and it is clear that we can reason in the same way also in the symmetric tongues, as exchanging the values of the parameters the comments in the regions $\mu > 0$ and $\mu < 0$ are just exchanged. For example keeping the $\delta_r = \delta_l = 0.9$ as in Fig.4, but symmetric values with respect to the diagonal of Fig.2 for the two other parameters: $\tau_l = -1.85$, $\tau_r = 0.9$, then for $\mu = 1$ the stable fixed point O_R^* is the unique attractor while for $\mu = -1$ the stable fixed point O_L^* coexists with a stable cycle of period 3. The stable set of the 4-cycles always bounds the basin of divergent trajectories.

Thus our extended definition of dangerous bifurcations can be applied to all the cases discussed above. But more, there is no reason to limit the analysis to the regions in which the fixed points are locally stable. In fact, *whichever is the finite attracting set, the occurrence of a dangerous bifurcation may be related to some finite attractors for* $\mu > 0$ *and* $\mu < 0$ *coupled with divergence at* $\mu = 0$ (as all the cycles and all the basins of attraction shrink up to a single point for μ tending to zero, so that there is mainly divergence at $\mu = 0$). Fig.2 shows in fact a wider region: $[-4, 1.9] \times [-4, 1.9]$. The upper and right limits are related with the bifurcations of the fixed points with the PE (χ_R and χ_L curves, given in (2)), and for points in the parameter space on the other side of such borders, for $\tau_l > 1 + \delta_l$ or $\tau_r > 1 + \delta_r$, we mainly have divergent dynamics at least on one side of $\mu = 0$. So we are interested in the extension of the region on the other side (where $\tau_{l/r} < 1 + \delta_{l/r}$). But below the χ_{LR} curve, given in (6), we have an unstable 2-cycle with divergent behavior on one side of $\mu = 0$, and divergence in the white region below the χ_{LR} curve (where the fixed points are unstable both for positive and negative values of μ). Thus we are left to the other region (see Fig.2): above the χ_{LR} curve and before crossing the $\chi_{L/R}$ curse, in which we have either $-(1 + \delta_l) < \tau_l < (1 + \delta_l)$ or $-(1 + \delta_r) < \tau_r < (1 + \delta_r)$, *so that we can conclude that either for* $\mu > 0$ *or for* $\mu < 0$ *we have a locally*

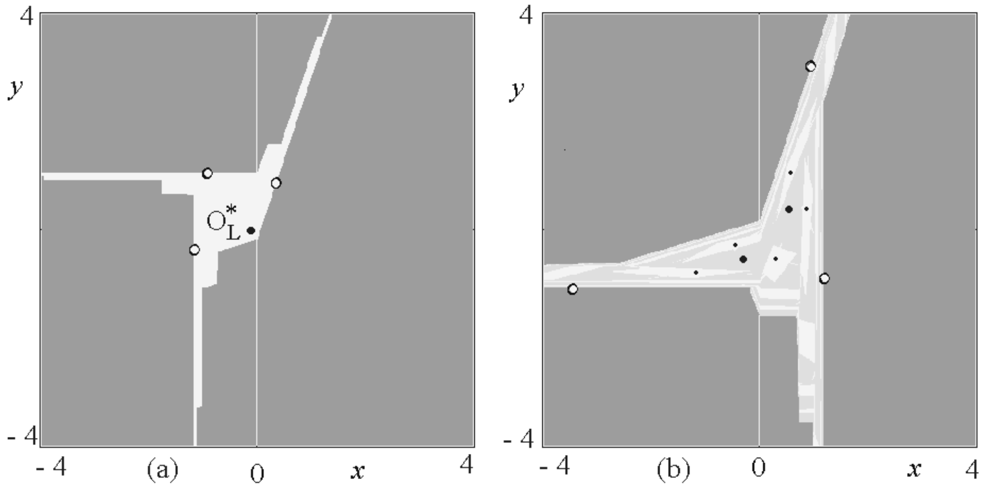

Figure 5: *Two dimensional phase plane at* $\delta_r = \delta_l = 0.9$, $\tau_l = -0.3$, $\tau_r = -3$ *fixed (white point in Fig.2 inside the gray tongue of the* $3-$*cycle saddle). In (a)* $\mu = -1$, *the stable fixed point* O_L^* *is the unique attractor, and gray points denote its basin of attraction. In (b)* $\mu = +1$, *the fixed point* O_R^* *is unstable, and two attractors coexist: a stable* $2-$*cycle and a stable* $5-$*cycle, whose basins are in two different gray tonalities, while dark gray points denote the basin of divergent trajectories, whose boundary is the stable set of the* $3-$*cycle saddle.*

attracting fixed point. In these strips (see Fig.2) we have dark gray tongues bounded by pairs of $\chi_{RL^{k-1}}$ and $\chi_{R^2L^{k-2}}$ bifurcation curves of collision with the PE. Around such regions we see light gray points denoting that we have a stable fixed point at least on one side of $\mu = 0$. Inside such dark gray regions we may have bounded attractors for $\mu > 0$ and $\mu < 0$ coupled with generic divergence at $\mu = 0$, that is: dangerous BCB. In fact, Fig.5 shows an example at $\tau_L = -3$ inside the tongue associated with the 3-cycle and Fig.6 shows an example at $\tau_L = -4$ inside the region associated with the 4-cycle. As we are outside the stability region of the fixed point O_R^* we have a different attracting set existing for $\mu > 0$. In Fig.5b (inside the 3-cycle tongue) we have a stable 2-cycle coexisting with a stable 5-cycle and their basins are separated by the stable set of the saddle 5-cycle, whose limit set is the stable set of the saddle 3-cycle bounding the basin of divergent trajectories. While in Fig.6b (inside the 4-cycle tongue) we have a stable 3-cycle. Anyhow, the crucial fact is the existence of a k-saddle cycle whose stable set bounds the basin of attraction of the PE and this causes a dangerous bifurcation, as on the other side, for $\mu < 0$, the fixed point is locally stable.

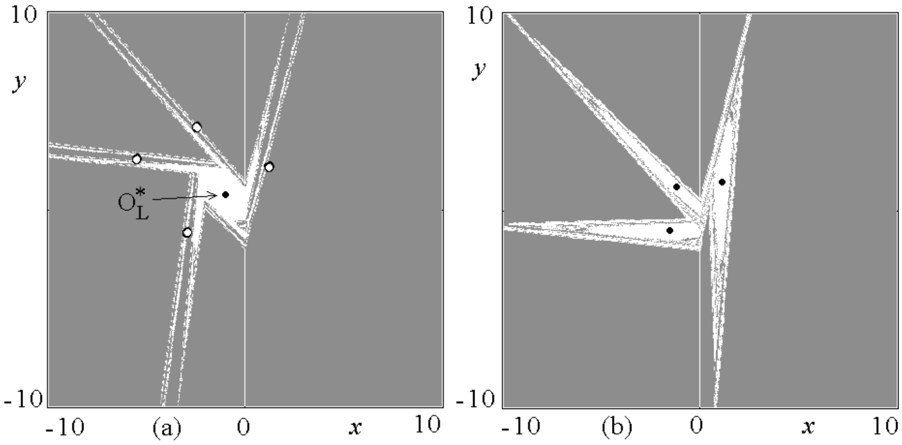

Figure 6: *Two dimensional phase plane at* $\delta_r = \delta_l = 0.9$, $\tau_l = 0.9$, $\tau_r = -4$ *fixed (white point in Fig.2 inside the gray tongue of the* $4-$*cycle saddle). In (a)* $\mu = -1$, *the stable fixed point* O_L^* *is the unique attractor, and white points denote its basin of attraction. In (b)* $\mu = +1$, *the fixed point* O_R^* *is unstable and a stable* $3-$*cycle exists whose basin has a fractal boundary. In both figures dark gray points denote the basin of divergent trajectories, and the boundary is the stable set of the* $4-$*cycle saddle.*

The extension of the definition of the *dangerous BCBs* can be appreciated at a different set of parameters for δ_L and δ_R. Fig.7 shows the two-dimensional bifurcation diagram in the case $\delta_L = \delta_R = 0.55$ at $\mu = 0$. We remark that the bifurcation diagrams in Fig.2 and Fig.7 are numerically detected setting $\mu = 0$ in the map, and considering an initial condition close to the origin. As we know, a dark gray region (bounded by bifurcation curves of collision with the PE) denotes the region of existence of a $k-$cycle saddle, whose related $k-$cycle node belongs to the PE and at $\mu = 0$ the points of the $k-$cycle saddle are reduced to the origin, however the $k-$invariant lines belonging to the related unstable set of the $k-$cycle saddle at $\mu = 0$ give $k-$invariant lines belonging to the unstable set of the origin, but also the $k-$invariant lines belonging to the stable set of the $k-$cycle saddle at $\mu = 0$ give $k-$invariant lines belonging to the stable set of the origin. Thus at least a set of points of zero Lebesgue measure whose trajectories are convergent to the origin always exist, also at $\mu = 0$.

In Fig.7 it can be seen that now inside the rectangle $[-1.55, 1.55] \times [-1.55, 1.55]$ (where O_R^* is locally stable for $\mu > 0$ and O_L^* is locally stable for $\mu < 0$) we have no *dangerous regions* associated with fixed points locally stable before and after the bifurcation. However, the tongues associated with the $k-$cycle saddle for $k =$

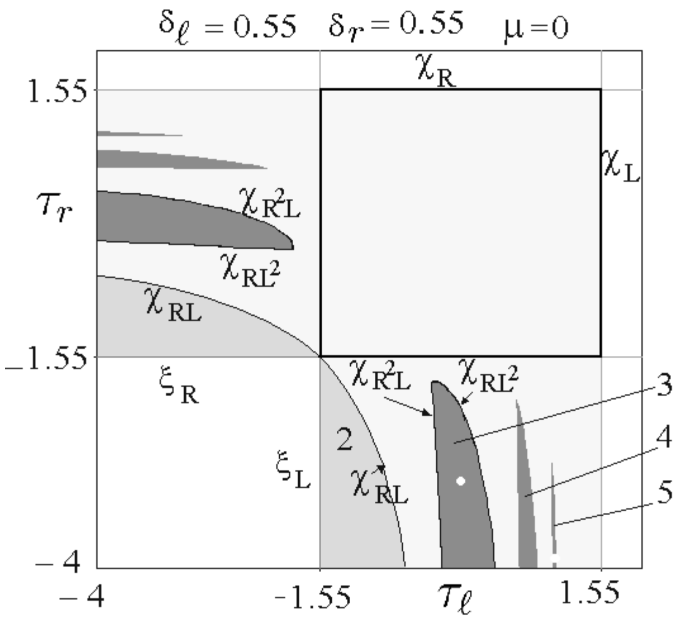

Figure 7: *Two-dimensional bifurcation diagram in the* (τ_l, τ_r) *parameter plane at* $\delta_r = \delta_l = 0.55$ *fixed, and* $\mu = 0$.

$3, 4, 5$ can be clearly seen, and the dark gray tongues may be regions of dangerous bifurcations occurring between a locally stable fixed point on one side and some other attracting set on the other side. Fig.8 shows an example of phase plane for parameter values inside the 3−cycle tongue.

However, it is worth noticing that inside such dark gray regions bounded by pairs of $\chi_{RL^{k-1}}$ and $\chi_{R^2L^{k-2}}$ bifurcation curves, *it is not true that all the points denote a dangerous bifurcation.* In fact, it is easy to find points for which we have no stable cycle on one side of $\mu = 0$ (so that the bifurcation is a transition from a stable fixed point on one side and divergence on the other side). In Fig.9 we show an example in which for $\mu = -1$ the fixed point O_L^* is attracting while for $\mu = 1$ there exists a chaotic attractor, which is very close to its basin boundary, i.e. close to the homoclinic bifurcation causing its transition to a chaotic repellor, as it occurs for example at lower values of τ_R.

It is clear that in order to detect the proper points leading to the transition to another attractor(s) we have to "overlap" the stability regions of the $k-$cycles, as well as the regions having some bounded attracting set, which may also be chaotic. And the examples given above show that many regions may coexist in those tongues.

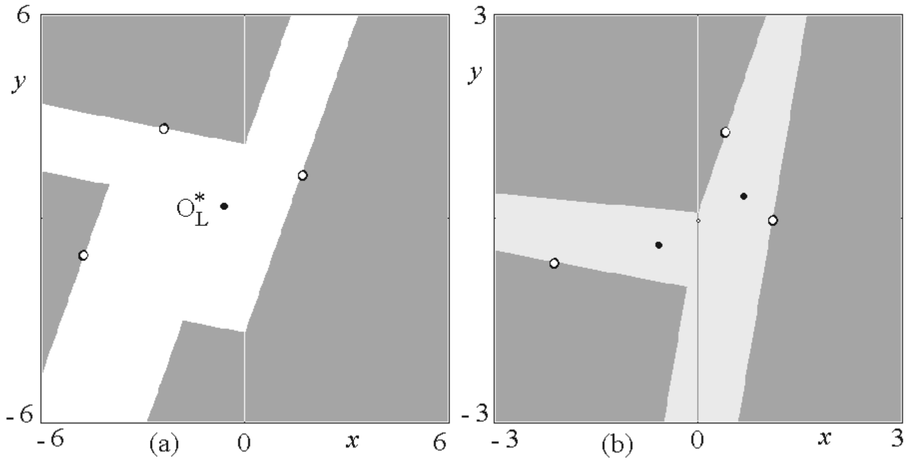

Figure 8: *Two dimensional phase plane at* $\delta_r = \delta_l = 0.55$, $\tau_l = 0$, $\tau_r = -3$ *fixed (white point in Fig.7 inside the gray tongue of the* $3-$*cycle saddle). In (a)* $\mu = -1$, *the stable fixed point* O_L^* *is the unique attractor, and white points denote its basin of attraction. In (b)* $\mu = +1$, *the fixed point* O_R^* *is unstable, and a stable* $2-$*cycle is the unique attractor, with basin in light gray tonality. In both figures dark gray points denote the basin of divergent trajectories, whose boundary is the stable set of the* $3-$*cycle saddle.*

4 Conclusions

In this work we have considered the problem related with the so-called *dangerous bifurcation* in the two-dimensional map in canonical form commonly used to study the border collision bifurcations in two-dimensional piecewise smooth maps. We have shown that the previous definition may be too strict to consider all the possible cases associated with two stable fixed points on both sides of the border collision occurring as the parameter μ crosses the value $\mu = 0$. In fact, inside the regions of *dangerous bifurcation* known up to now we have seen that coexistence of stable attractors is a common event, as thus it is common to have two saddle-cycles (or many more), and a problem arise in order to detect which one is related with the *dangerous bifurcation.* We have shown that such points belong to regions in the parameter space bounded by two special bifurcation curves, called $\chi_{RL^{k-1}}$ and $\chi_{R^2L^{k-2}}$ (denoting a contact of a cycle with the PE), and we have shown that such cycles, which exist as saddle without the complementary stable cycles, are those related with the *dangerous bifurcations.* Moreover, our definition of *dangerous bifurcation*,

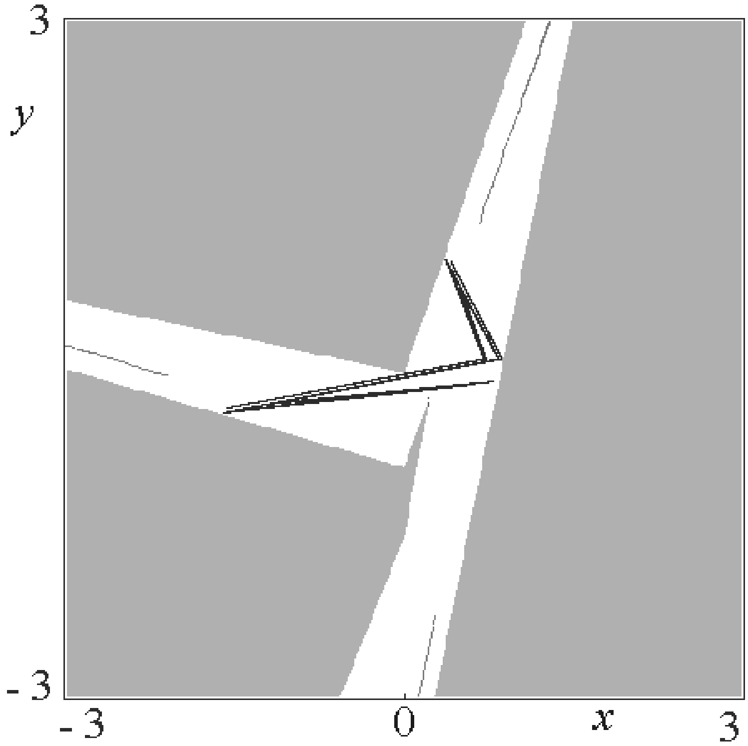

Figure 9: *Two dimensional phase plane at* $\delta_r = \delta_l = 0.55$, $\tau_l = 0.1$, $\tau_r = -3$, *and* $\mu = +1$ *(point inside the gray tongue of the* $3-$*cycle saddle of Fig.7). The attractor is a one-piece chaotic set. Dark gray points denote the basin of divergent trajectories, whose boundary is the stable set of the* $3-$*cycle saddle.*

which consists in giving this label whenever we have attractors at finite distance both for $\mu > 0$ and $\mu < 0$ coupled with divergence at $\mu = 0$ is an extension of the previous one, and leaves open the problem of its full determination and description. By using several examples we have shown how many different cases may occur.

References

[1] Avrutin, V., M. Schanz and L. Gardini, *On a special type of border-collision bifurcations occurring at infinity*, (in submission) (2009).

[2] Banerjee, S. and C. Grebogi, *Border-collision bifurcations in two-dimensional piecewise smooth maps*, Physical Review E, 59(4) (1999), 4052–4061.

[3] Banerjee, S., M.S. Karthik, G. Yuan and J.A. Yorke, *Bifurcations in One-Dimensional Piecewise Smooth Maps - Theory and Applications in Switching Circuits,* IEEE Trans. Circuits Syst.-I: Fund. Theory Appl. 47(3) (2000), 389–394.

[4] Banerjee, S., P. Ranjan and C. Grebogi, *Bifurcations in 2D Piecewise Smooth Maps - Theory and Applications in Switching Circuits,* IEEE Trans. Circuits Syst.-I: Fund. Theory Appl. 47(5) (2000), 633–64.

[5] di Bernardo M., M.I. Feigen, S.J. Hogan and M.E. Homer, *Local analysis of C-bifurcations in n-dimensional piecewise smooth dynamical systems*, Chaos, Solitons & Fractals 10(11) (1999), 1881–1908.

[6] di Bernardo M., C.J. Budd, A.R. Champneys, P. Kowalczyk, *Piecewise-smooth Dynamical Systems Theory and Applications*, Springer-Verlag, London 2008.

[7] Do, Y. and H.K. Baek, *Dangerous border-collision bifurcations of a piecewise-smooth map*, Communications on Pure and Applied Analysis, 5(3), (2006) 493–503.

[8] Do, Y., *A mechanism for dangerous border collision bifurcations*, Chaos, Solitons & Fractals, 32 (2007), 352–362.

[9] Dutta, M., H. Nusse, E. Ott, J.A. Yorke and G.H. Yuan, *Multiple attractor bifurcations: A source of unpredictability in piecewise smooth systems*, Phys. Rev. Lett. 83 (1999), 4281–4284.

[10] Feely, O., D. Fournier-Prunaret, I. Taralova-Roux and D. Fitzgerald, *Nonlinear dynamics of bandpass sigma-delta modulation. An investigation by means of the critical lines tool*, International Journal of Bifurcation and Chaos, 10(2) (2000), 303–327.

[11] Gallegati, M., L. Gardini, T. Puu, and I. Sushko, *Hicks's trade cycle revisited: cycles and bifurcations*, Mathematics and Computers in Simulations, 63 (2003), 505–527.

[12] Ganguli, A. and S. Banerjee, *Dangerous bifurcation at border collision: when does it occur?*, Phys. Rev.E 71 057202 (1-4), (2005).

[13] Gardini, L., T. Puu and I. Sushko, *The Hicksian Model with Investment Floor and Income Ceiling,* In: *Business Cycles Dynamics. Models and Tools* (T. Puu and I. Sushko Ed.s), Springer-Verlag, Berlin 2006.

[14] Gardini, L., T. Puu and I. Sushko, *A Goodwin-type Model with a Piecewise Linear Investment Function,* In: *Business Cycles Dynamics. Models and Tools* (T. Puu and I. Sushko Ed.s), Springer-Verlag, Berlin 2006.

[15] Hassouned, M.A., E.H. Abed and H.E. Nusse, *Robust dangerous Border-Collision Bifurcations in piecewise smooth systems*, Phys. Rev. Letters, 92(7) 070201 (1-4), (2004).

[16] Hommes, C.H. and H. Nusse, *Period three to period two bifurcations for piecewise linear models*, Journal of Economics 54(2) (1991) 157–169.

[17] Leonov, N.N., *Map of the line onto itself*, Radiofisica 3(3) (1959), 942–956.

[18] Leonov, N.N., *Discontinuous map of the straight line*, Dohk. Ahad. Nauk. SSSR. 143(5) (1962), 1038–1041.

[19] Maistrenko Yu.L., V.L. Maistrenko and L.O. Chua, *Cycles of chaotic intervals in a time-delayed Chua's circuit*, International Journal of Bifurcation and Chaos 3(6) (1993), 1557–1572.

[20] Maistrenko Yu.L., V.L. Maistrenko, S.I. Vikul and L. Chua, *Bifurcations of attracting cycles from time-delayed Chua's circuit*, International Journal of Bifurcation and Chaos 5(3) (1995), 653–671.

[21] Maistrenko Yu.L., V.L. Maistrenko and S.I. Vikul, *On period-adding sequences of attracting cycles in piecewise linear maps*, Chaos, Solitons & Fractals 9(1) (1998) 67–75.

[22] Mira C., *Chaotic Dynamics*, World Scientific, Singapore 1987.

[23] Nusse, H.E. and J.A. Yorke, *Border-collision bifurcations including period two to period three for piecewise smooth systems*, Physica D 57 (1992), 39–57.

[24] Nusse, H.E. and J.A. Yorke, *Border-collision bifurcations for piecewise smooth one-dimensional maps*, International Journal of Bifurcation and Chaos 5(1) (1995), 189–207.

[25] Kapitaniak, T. and Yu.L. Maistrenko, *Multiple choice bifurcations as a source of unpredictability in dynamical systems*, Phys. Rev. E 58(4) (1998), 5161–5163.

[26] Simpson, D.J.W. and J.D. Meiss, *Neimark–Sacker Bifurcations in Planar, Piecewise-Smooth, Continuous Maps*, SIAM J. Applied Dynamical Systems, Vol. 7 (2008), No. 3, 795–824.

[27] Sushko, I., A. Agliari and L. Gardini, *Bistability and border-collision bifurcations for a family of unimodal piecewise smooth maps*, Discrete and Continuous Dynamical Systems, Serie B, 5(3) (2005), 881–897.

[28] Sushko, I., A. Agliari and L. Gardini, *Bifurcation Structure of Parameter Plane for a Family of Unimodal Piecewise Smooth Maps: Border-Collision Bifurcation Curves*, Chaos, Solitons & Fractals, 29(3), (2006), 756–770.

[29] Sushko, I. and L. Gardini, *Center Bifurcation for a 2D Piecewise Linear Map,* In: *Business Cycles Dynamics. Models and Tools* (T. Puu and I. Sushko Ed.s), Springer-Verlag, Berlin 2006.

[30] Sushko, I. and L. Gardini, *Center Bifurcation for Two-Dimensional Border-Collision Normal Form*, Int. J. Bifurcation and Chaos 18(4) (2008), 1029–1050.

[31] Sushko I., T. Puu and L. Gardini, *The Hicksian floor-roof model for two regions linked by interregional trade*, Chaos Solitons & Fractals 18 (2003), 593–612.

[32] Taralova-Roux, I., D. Fournier-Prunaret, *Dynamical study of a second order DPCM transmission system modeled by a piece-wise linear function*, IEEE Transactions of Circuits and Systems I, 49(11) (2002), 1592–1609.

[33] Yuan, G.H., *Shipboard crane control, simulated data generation and border collision bifurcation*, Ph. D. dissertation, Univ. of Maryland, College Park, USA 1997.

[34] Zhusubaliyev, Z.T. and E. Mosekilde, *Bifurcations and Chaos in Piecewise - Smooth Dynamical Systems*, World Scientific, Singapore 2003.

[35] Zhusubaliyev, Z.T., E. Mosekilde, S. Maity, S. Mohanan and S. Banerjee, *Border collision route to quasiperiodicity: Numerical investigation and experimental confirmation*, Chaos 16, 023122 (2006), 1–11.

[36] Zhusubaliyev, Z.T., E. Soukhoterin and E. Mosekilde, *Quasiperiodicity and torus breakdown in a power electronic dc/dc converter,* Mathematics and Computers in Simulation 73 (2007), 364–377.

Laura Gardini
Dept. of Economics and Quantitative Methods
University of Urbino
Via A. Saffi n.42
61029 Urbino, Italy
e-mail: *laura.gardini@uniurb.it*

Viktor Avrutin and Michael Schanz
Institute of Parallel and Distributed Systems
University of Stuttgart
Universitätstrasse 38
70569 Stuttgart, Germany
e-mail: *viktor.avrutin@ipvs.uni-stuttgart.de*
michael.schanz@ipvs.uni-stuttgart.de