

Memory Effects on Binary Choices with Impulsive Agents: Bistability and a new BCB structure

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Abstract

After the seminal works by Schelling, several authors have considered models representing binary choices by different kinds of agents or groups of people. The role of the memory in these models is still an open research argument, on which scholars are investigating. The dynamics of binary choices with impulsive agents has been represented, in the recent literature, by a one-dimensional piecewise smooth map. Following a similar way of modelling, we assume a memory effect which leads the next output to depend on the present and the last state. This results in a two-dimensional piecewise smooth map with a limiting case given by a piecewise linear discontinuous map, whose dynamics and bifurcations are investigated. The map has a particular structure, leading to trajectories belonging only to a pair of straight lines. The system can have, in general, only attracting cycles, but the related periods and periodicity regions are organized in a complex structure of the parameter space. We show that the period adding structure, characteristic for the one-dimensional case, also persists in the two-dimensional one. The considered cycles have a symbolic sequence which is obtained by concatenation of the symbolic sequences of cycles which play the role of basic cycles in the bifurcation structure. Moreover, differently from the one-dimensional case, coexistence of two attracting cycles is now possible. The bistability regions in the parameter space are investigated, evidencing the role of different kinds of codimension-two bifurcation points, as well as in the phase space, the related basins of attraction are described.

Lead Paragraph

Since the seminal papers by Schelling on social interactions that influences the behavior of individuals, several contributions have expanded the literature on the topic, providing mathematical formalizations. The analysis of the dynamics of models, under different behavioral assumptions, has been done, also considering different real world situations. In particular, the introduction of memory in the decision process of minority games has been proposed by several authors. However, a comparison of the titles "Irrelevance of memory in the minority game" of [11], and "Relevance of memory in minority games" of [12] shows that the effects of memory are debated. As described in the present work, we can have models leading to a behavior in between: depending on some parameters, the memory may or not have an effect on the choices. Following the ideas and the model proposed in [10], represented by a one-dimensional discontinuous piecewise smooth map, additional information has been introduced. We consider a binary choices model with impulsive agents, when also information about the previous state is considered. Specifically, we analyze how the relative weights between the present and the past states affect the dynamics. The limiting case here studied is represented by a two-dimensional discontinuous piecewise linear map, which has the advantage to keep the system simpler to analyze. The discontinuity leads to a class of maps still not well studied and we observe new bifurcation phenomena. In the proposed system, we show that the particular bifurcation structure typical of the one-dimensional case can exist. This structure is related to cycles which play the role of basic cycles (also known as maximal or principal cycles), in the bifurcation structure. However, the existence of different families of basic cycles leads to the phenomena of bistability which cannot occur in the one-dimensional case. Moreover, we illustrate that the overlapped regions in the parameter space lead to particular codimension-two bifurcation points, which are worth to be studied in more detail.

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1 Introduction

In binary choices the outcomes are endogenously determined rather than exogenously [33]. Models with binary choices offer a good example of how the microscopic behavior of agents affects the macroscopic behavior of a system. A particular case of binary choices is given by the minority game – a highly simplified model of El Farol’s Bar Problem – in which the agents interact and contribute to determine the outcome of the system [1].

Different mathematical formalizations and analysis of systems representing minority games have been considered after the pioneering work by Schelling [47] and Granovetter [26], [27]. A population of impulsive decision makers in the adaptive discrete-time dynamic setting considered in [9] has provided a theoretical justification of some observable phenomena. That is, the system has shown the overreactions of actors who are repeatedly involved in the same decisions, and the emergence of the equilibria selection problem.

According to [6, p.3], among the psychological processes which may lead to impulsive behaviors we can find the inability to store multiple choices in memory in order to evaluate them (so-called working memory). Furthermore, the relationship between impulsivity and working memory has been studied in rats [16] and primates [30], and in the human behavior as well [38]. These considerations lead to the idea that finite memory can affect the dynamics of a population of impulsive agents too, and not only in an evolutionary framework.

More recently, the population of decision makers, embedded in a discrete dynamical system based on exponential replicator dynamics, allowed the study of the effect of memory on the long-run outcomes in minority games repeatedly played [7]. The results confirmed the stabilizing effect of memory. However, they also showed that this is globally true with infinite memory, while with a two-period memory the stability is only local.

Considering binary choices, it has been assumed that the players’ decisions are not only based on the current observed states, but also take account of the past ([17], [29]). The consideration of previous states may also be referred to as introducing memory in the decision process [48]. For example, several papers consider bounded memory (see for instance [13]). The effect of memory in minority games has been explored in evolutionary and even in adaptive settings [51]. Also the Tit For Tat strategy can be considered as a low-memory strategy [41]. However, there is still a missing link between memory and impulsive behavior.

In this paper we are interested in extending the binary choices model in order to take into account some information. Although agents are impulsive and follow their utility almost directly, also a short memory may be taken into account. For instance, we consider the knowledge of the present value and of the previous state. This leads to impulsive behavior with a memory effect, where the present and the previous payoffs contribute in a weighted average. The use of intermediate values of the weight may be considered as representing situations in which all available information is taken into account and memory interacts with decision-making [46].

We start from the analysis performed in [10], where a repeated two-choice minority game has been studied. The system there considered is described by a smooth function which leads, in the limiting case, to a one-dimensional discontinuous map, representing impulsive choices. As we shall recall below, the smooth system possesses a Nash equilibrium, which is mainly unstable, leading to complex behaviors, while in the limiting case the Nash equilibrium may be considered virtual, since in general it cannot be reached, and the dynamic result is that the generic states are periodic (cyclical).

Our goal is to understand whether more information, which we can assume represented by memory of past situations, influences the outcome of the system. As we shall see, depending on the parameter weighting the memory, we may have results similar to those in [11] (when the weight is low) as well as those in [12] (when the weight is high). In particular, some observed differences, limited to particular regions of the parameter space, are related to regions leading to bistability, which confirm results already observed in the literature, such as the multistability commented in models with memory described in [7], [8].

Recall that in smooth systems the dynamics may evolve from a regular behavior to a complex one via a sequence of bifurcations (Feigenbaum cascades of period doubling bifurcations, Neimark-Sacker bifurcations followed by frequency locking, homoclinic bifurcations, etc.), while in piecewise smooth (PWS) systems Border Collision Bifurcations (BCB) may occur, leading to a sharp transition of the dynamics, as first evidenced by Nusse and Yorke [42], [43]. In particular, in piecewise linear (PWL) systems mainly BCBs and contact bifurcations occur. A border collision occurs whenever there is a contact between an invariant set

of a map with the border of its region of definition. Such a contact may, or may not, produce a bifurcation. For a k -cycle colliding with a border of a discontinuous map, the effect of the collision is independent of the eigenvalues of the Jacobian matrix. PWS systems have been studied extensively in the last two decades, due to their wide spectrum of applications in engineering and other fields, see e.g. [18], [54].

Many works are devoted to PWS and PWL one-dimensional maps (see [2], [21], [5] and references therein), where the so-called period adding and period incrementing bifurcation structures, occurring in the parameter space, are described. These kind of bifurcations are known since the works by Leonov ([34], [35]) and well studied also after, since they are related to some families of Lorenz maps (see [28], [31], [23], [3], [5]). Also two-dimensional continuous systems have been investigated (see [49], [4], [25] among others). However, the description of the bifurcations occurring in two-dimensional discontinuous systems is still not well developed (see [37], [45], [32], [20], [36], [50]), and the system considered in our work belongs to this class of maps.

It is worth noting that impulsive choices are also important in other economic contexts, such as the way in which groups of agents follow cyclically some states which are fashionable (see [24] and [40]). In particular, the system considered in [24] is described via a two-dimensional discontinuous piecewise linear map, similar to the model considered in the present work. However, the dynamics of the system here considered are peculiar, keeping the properties of the concatenation rule applied to symbolic sequences of different families of cycles, and leading to new bifurcation phenomena and codimension-two bifurcation points in the parameter space.

After this Introduction, the paper is structured as follows. In Sec.2 we recall the one-dimensional system in which we introduce the memory effect. This is described in Sec.3, devoted to the formalization of the dynamic model with memory and the discussion of some general properties of the map. In particular, we prove that all the available cycles of any period have a well defined symbolic sequence, that multistability is possible, but at most with two attracting cycles. Sec.4 includes the main part of our work, related to the new results, and it is structured in several subsections. We give the explicit BCB conditions, for several cycles and families of existing cycles. These bifurcations are related to the collision of periodic points of the cycles with the two discontinuity points of the map. The relevant effect is the existence of the period adding bifurcation structures also in the considered two-dimensional map, in which the symbolic sequences are obtained as usual, by concatenating the symbolic sequences of other cycles. But the peculiarity is that we have two different families of basic cycles (originating the related period adding structure) which are overlapping pairwise, leading to a new structure, with codimension-two bifurcation points having particular properties. Thus, we show how the coexistence of attractors can emerge when memory is introduced in the model, at a low value of the weight given to the past states the effect of the memory is negligible, while increasing the weight we observe differences, such as the bistability. We describe the structure of the related basins of attraction in the phase space, distinguishing between points converging to one attracting cycle or to the other one. Since unstable cycles cannot exist, the basin boundaries are determined only by the discontinuity points and related preimages. Sec.5 concludes, evidencing that our results just open the way to further researches, especially in the investigation of the bifurcation structures.

2 The model without memory

In this section we introduce the case with no memory in which the decisions are based only on the current payoffs which has been studied in [10].

We consider a population with a unitary continuum of players in $[0, 1]$. Agents choose strategies from a set of two actions $\mathcal{A} = \{A, B\}$. As a result of the binary choices process, a minority game, the agents update their choice each time t . The value $x_t \in [0, 1]$ denotes the proportion of agents playing strategy A at time t , while $(1 - x_t)$ the proportion of those playing strategy B at the same time. We assume that, at time $t + 1$, the fraction x_t is of common knowledge as well as the payoffs. Individual payoffs depend only on the proportion of agents making a given choice, and are assumed to be linear functions, that is:

$$U_A(x) = A(x) = p_A x + q_A, \quad U_B(x) = B(x) = p_B x + q_B \quad (1)$$

Since in a minority game decision makers aim to choose a different action than the majority, it is commonly assumed that congestion is costly, therefore in (1) it will be $p_B > p_A$ and $q_A > q_B$.

We consider linear payoffs (following [47]) as they often serve as proxies for more general curves and because, as we shall see below, with impulsive agents their influence is in the switching rule.

We assume that agents are homogeneous and myopic, that is, each of them is interested to increase its own next period payoff. If at time t the fraction x_t of players are playing strategy A and $U_A(x_t) > U_B(x_t)$ then a fraction of the $(1 - x_t)$ agents that are playing strategy B will switch to A in the following turn. Similarly, if $U_A(x_t) < U_B(x_t)$. In other words, at any time period t agents decide their action for the period $t + 1$ according to

$$x_{t+1} = T(x_t) = \begin{cases} x_t - \delta_B g[\lambda(B(x_t) - A(x_t))]x_t & \text{if } U_B(x_t) > U_A(x_t) \\ x_t + \delta_A g[\lambda(A(x_t) - B(x_t))](1 - x_t) & \text{if } U_B(x_t) < U_A(x_t) \end{cases} \quad (2)$$

where δ_A and δ_B are the propensities to switch to the other strategy, determining how many agents may switch to A and B , and thus $\delta_A, \delta_B \in [0, 1]$; $g : R \rightarrow [0, 1]$ is a continuous and increasing function such that $g(0) = 0$ and $\lim_{z \rightarrow \infty} g(z) = 1$; λ is a positive real number. The function g modulates how the fraction of switching agents depends on the difference between the payoffs, in [10] we have considered as prototype the function $g(z) = \frac{2}{\pi} \arctan(z)$. The parameter λ represents the switching intensity or speed of reaction of agents as a consequence of the difference between the payoffs. Large values of λ can be interpreted in terms of impulsivity. According to the Clinical Psychology literature [44] impulsivity leads agents to act with lack of planning.

An interior Nash equilibrium (NE for short) of the game is a state $x^* \in (0, 1)$ such that no unilateral deviation is profitable, that is, $A(x^*) = B(x^*)$ which leads, assuming $p_A \neq p_B$ in (1), to the point

$$x^* = -\frac{q_B - q_A}{p_B - p_A} = -\frac{\Delta_q}{\Delta_p} \quad (3)$$

where we have defined $\Delta_p = (p_B - p_A)$ and $\Delta_q = (q_B - q_A)$. In the cases with $\Delta_p > 0$ and $\Delta_q < 0$ the payoff functions have a qualitative shape as the one in 1(a), so that the function $T(x)$ is decreasing at the NE and when it is unstable other attracting sets occur (see 1(b)). The effect of the increasing parameter λ

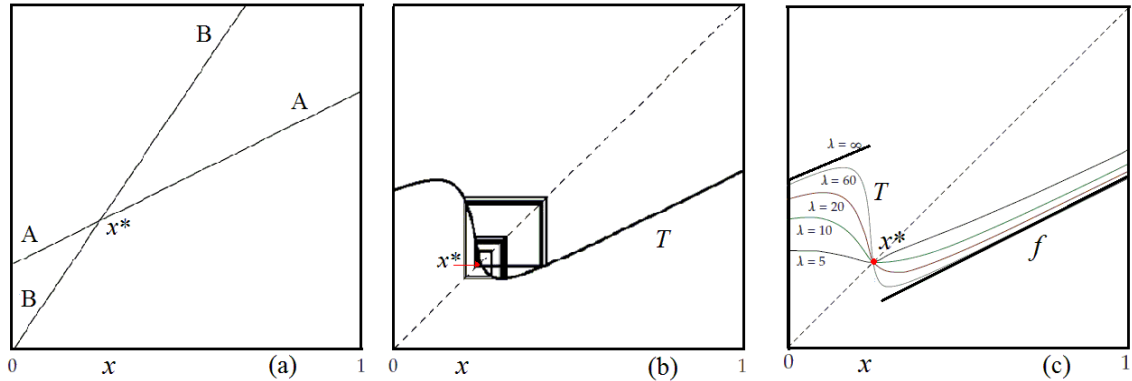


Figure 1: In (a) payoff functions $B(x) = 1.5x$ and $A(x) = 0.25 + 0.5x$, at $\delta_A = \delta_B = 0.5$. In (b) graph of the function $T(x)$ with $\lambda = 35$. In (c) graphs as the parameter λ increases, and limiting discontinuous function f .

is to steepen the function T in a neighborhood of x^* , as shown in 1(c), and the dynamics for large values of λ can be well approximated by the piecewise linear map with a discontinuity in x^* , representing the case of impulsive choices:

$$x_{t+1} = f(x_t) = \begin{cases} x_t - \delta_B x_t & \text{if } U_B(x_t) > U_A(x_t) \\ x_t + \delta_A(1 - x_t) & \text{if } U_B(x_t) < U_A(x_t) \end{cases}$$

which can be rewritten as

$$x_{t+1} = f(x_t) = \begin{cases} (1 - \delta_B) x_t & \text{if } U_B(x_t) > U_A(x_t) \\ (1 - \delta_A) x_t + \delta_A & \text{if } U_B(x_t) < U_A(x_t) \end{cases} \quad (4)$$

In fact, considering for example $\lambda = 900$ we have a very steep graph for map T in a very narrow neighborhood of the NE x^* , as shown in 2(a). The related bifurcation diagram in the two-dimensional parameter

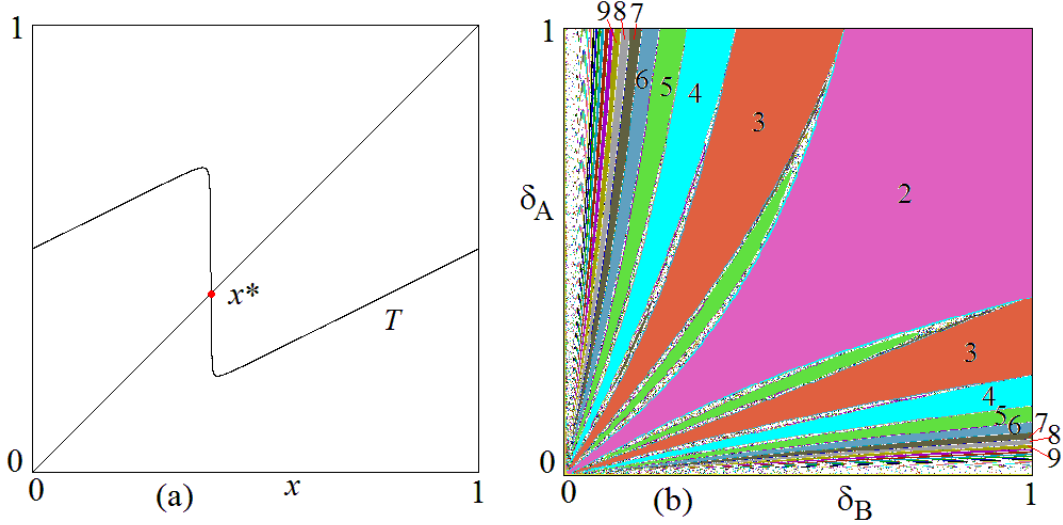


Figure 2: In (a) graph of the function $T(x)$ at $\lambda = 900$. In (b) bifurcation diagram of map T in the parameter plane (δ_B, δ_A) at $\lambda = 900$. In color are represented periodicity regions associated with attracting cycles. Some periods are also indicated in the regions.

plane (δ_B, δ_A) is shown in 2(b), and it can be seen that we have mainly regions related to attracting cycles of different periods. The bifurcation structure is quite similar to the one obtained with the limiting discontinuous function f , as shown in 3. Clearly, for the continuous map $T(x)$ the bifurcation curves in the parameter plane shown in 2(b) are related to fold bifurcations of the cycles and sequences of flip bifurcations. Differently, in the parameter plane of the discontinuous map $f(x)$ the bifurcations shown in 3(b) are related to border collision bifurcations, that is, collision of the periodic points of the cycles with the discontinuity point, the NE x^* in (3). Notice that indeed this point can still be interpreted as a NE, since it is related to the condition $A(x) = B(x)$, however, it is no longer an equilibrium of the dynamic game. Even if we can define, in an artificial way, $f(x^*) = x^*$, its stable set (i.e. the initial conditions which are ultimately mapped in x^*) is a set of zero measure, of a finite number of states.

The bifurcation structure related to this kind of piecewise linear discontinuous map is well known since many years, and most frequently called as period adding bifurcation structure. Since the two slopes are smaller than 1, the system cannot have repelling cycles, only attracting cycles or (in limiting cases) quasi-periodic trajectories dense in the absorbing interval (in red in 3(a)). The periods follow the well known Farey summation rule and the related BCBs can be written explicitly for any period of the cycles. These results were first published by Leonov (more than 50 years ago) ([34], [35]), and then considered by other authors (mainly because they are related to some families of Lorenz maps, see for example, [28], [31]). We also started from the works by Leonov, improving the description of the bifurcation structure both from a theoretical point of view (relating the bifurcations to the first return map in a neighborhood of the discontinuity point) and from a numerical point of view, via the map replacement technique (based on this first return, a kind of renormalization process, see [23], [3], [5]).

It is worth noting that in the figures of this work we keep the parameters $\Delta_p > 0$ and $\Delta_q < 0$ fixed, since a change in these values leads only to slight deformations of the bifurcation curves in the two-dimensional parameter plane (δ_B, δ_A) still maintaining qualitatively the same bifurcation structure. In fact, the bifurcation conditions are detected analytically as a function of all the parameters (including Δ_p and Δ_q).

3 The model with memory

We are interested in extending the binary choices model in order to take into account some information. That is, although agents are impulsive and follow their utility almost directly, the knowledge not only of the

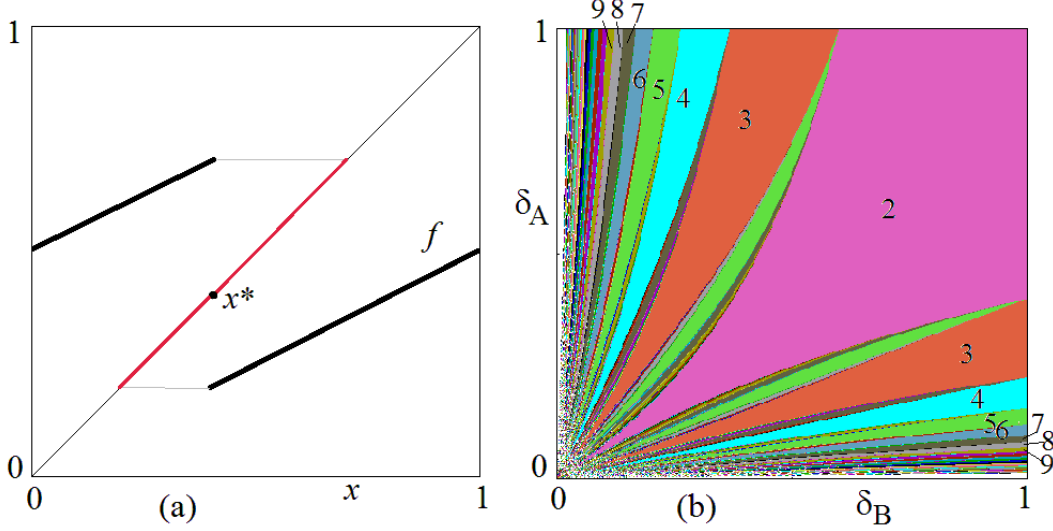


Figure 3: In (a) graph of the function $f(x)$, payoff functions $B(x)$ and $A(x)$ as in Fig.1(a). In (b) bifurcation diagram of map f in the parameter plane (δ_B, δ_A) . In color are represented periodicity regions associated with attracting cycles.

present payoff but also of the previous one may be taken into account, leading to impulsive behaviors with memory effect and to some differences for particular parameter values, which are essentially due to the new phenomena of bistability.

Our system is described, as before, by a population with a unitary continuum of players in $[0, 1]$, and agents choose strategies from a set of two actions $\mathcal{A} = \{A, B\}$, with linear functions as individual payoffs:

$$A(x) = p_A x + q_A, \quad B(x) = p_B x + q_B \quad (5)$$

however, the utility function governing the agents' behavior is modeled as the weighted average of the current payoff and the one previously observed. In this way, they decide their action at time $t+1$ taking into account the payoffs of two states (time t and time $t-1$, which is the memory effect) by using the utility function defined by:

$$U_A(x_t, x_{t-1}) = (1 - \omega)A(x_t) + \omega A(x_{t-1}) \quad (6)$$

$$U_B(x_t, x_{t-1}) = (1 - \omega)B(x_t) + \omega B(x_{t-1}) \quad (7)$$

where the parameter $\omega \in [0, 1]$ is the weight assigned to the past payoff. Thus, agents modify their choices according to the rule

$$x_{t+1} = T(x_t) = \begin{cases} x_t - \delta_B g[\lambda(B(x_t) - A(x_t))]x_t & \text{if } U_B(x_t, x_{t-1}) > U_A(x_t, x_{t-1}) \\ x_t + \delta_A g[\lambda(A(x_t) - B(x_t))](1 - x_t) & \text{if } U_B(x_t, x_{t-1}) < U_A(x_t, x_{t-1}) \end{cases} \quad (8)$$

where, as before, the parameters $\delta_A, \delta_B \in [0, 1]$ represent the agents' propensity to switch to the other strategy. In the case of impulsive agents, which leads to the analysis of a simpler system, we have the map which is of interest in the present work:

$$x_{t+1} = f(x_t) = \begin{cases} (1 - \delta_B) x_t & \text{if } U_B(x_t, x_{t-1}) > U_A(x_t, x_{t-1}) \\ (1 - \delta_A) x_t + \delta_A & \text{if } U_B(x_t, x_{t-1}) < U_A(x_t, x_{t-1}) \end{cases} \quad (9)$$

Obviously, the memory effect can be shaped by assigning different weights to the payoffs for A and B as well as lengthened and assigning different weights to any different time in the past. However, as remarked above, this is not the purpose of the paper. What we are interested in is to see how the memory of only the most recent past might change and add some new features to the impulsive binary choice dynamics. Clearly, the

two extreme cases lead either to consider only the present state (no memory) with $\omega = 0$, briefly recalled in the previous section, or to the case with decisions made only on the past payoffs, modeled with $\omega = 1$ (which we consider less realistic, but it may be interesting from the dynamic point of view).

Let us rewrite the dynamical system as follows:

$$x_{t+1} = f(x_t) = \begin{cases} f_B(x_t) = (1 - \delta_B)x_t & \text{if } U_B(x_t, x_{t-1}) > U_A(x_t, x_{t-1}) \\ f_A(x_t) = (1 - \delta_A)x_t + \delta_A & \text{if } U_B(x_t, x_{t-1}) < U_A(x_t, x_{t-1}) \end{cases} \quad (10)$$

introducing the functions $f_B(x_t)$ and $f_A(x_t)$ depending only on the state x_t while the condition to get one or the other depends on the present state x_t and the previous one x_{t-1} .

From $U_B(x_t, x_{t-1}) - U_A(x_t, x_{t-1}) = (1 - \omega)(B(x_t) - A(x_t)) + \omega(B(x_{t-1}) - A(x_{t-1}))$, defining as in the previous section $\Delta_p = (p_B - p_A)$ and $\Delta_q = (q_B - q_A)$ we have

$$U_B(x_t, x_{t-1}) - U_A(x_t, x_{t-1}) = (1 - \omega)\Delta_p x_t + \omega\Delta_p x_{t-1} + \Delta_q \quad (11)$$

and from the analysis of the case $\omega = 0$ we require

$$\Delta_p = (p_B - p_A) > 0, \quad \Delta_q = (q_B - q_A) < 0 \quad (12)$$

so that for $\omega = 0$ the two branches of map f are increasing, with positive slopes smaller than 1, without real fixed points, and the discontinuity point is $x^* = -\frac{\Delta_q}{\Delta_p}$, which we assume in the interval $(0, 1)$.

For $\omega > 0$, let us introduce the variable $y_t = x_{t-1}$, then we can write our system as a two dimensional map $(x_{t+1}, y_{t+1}) = F(x_t, y_t)$ defined as follows

$$F : \begin{cases} x_{t+1} = \begin{cases} f_B(x_t) = (1 - \delta_B)x_t & \text{if } (1 - \omega)\Delta_p x_t + \omega\Delta_p y_t + \Delta_q > 0 \\ f_A(x_t) = (1 - \delta_A)x_t + \delta_A & \text{if } (1 - \omega)\Delta_p x_t + \omega\Delta_p y_t + \Delta_q < 0 \end{cases} \\ y_{t+1} = x_t \end{cases} \quad (13)$$

and since $(1 - \omega)\Delta_p x_t + \omega\Delta_p y_t + \Delta_q > 0$ iff $y > -\frac{1 - \omega}{\omega}x - \frac{\Delta_q}{\omega\Delta_p}$ we have a discontinuous piecewise smooth map with an upper/lower (u/l for short) definition in the two-dimensional plane (x, y) separated by a straight line with negative slope:

$$(x', y') = F(x, y) = \begin{cases} F_u(x, y) & \text{if } y > -\frac{1 - \omega}{\omega}x - \frac{\Delta_q}{\omega\Delta_p} \\ F_l(x, y) & \text{if } y < -\frac{1 - \omega}{\omega}x - \frac{\Delta_q}{\omega\Delta_p} \end{cases} \quad (14)$$

where

$$F_u(x, y) : \begin{cases} x' = (1 - \delta_B)x \\ y' = x \end{cases}, \quad F_l(x, y) : \begin{cases} x' = (1 - \delta_A)x + \delta_A \\ y' = x \end{cases} \quad (15)$$

where the symbol " ' " denotes the one-period advancement operator.

The discontinuity point x^* of the model without memory is now replaced by the straight line of equation $U_B(x_t, x_{t-1}) - U_A(x_t, x_{t-1}) = 0$ crossing which the map changes its definition. However, due to the fact that the map in the different regions (above and below the straight line) is defined via the functions F_u and F_l which only depend on x_t , we have that only two points of this straight line of discontinuity are involved in the forward dynamics, as described below.

From a dynamical point of view the discontinuity line, which is the separator between the two different definitions of the map, is called (see [37]) set LC_{-1} :

$$LC_{-1} : \quad y = -\frac{1 - \omega}{\omega}x - \frac{\Delta_q}{\omega\Delta_p} = sx + \mu \quad (16)$$

and for our convenience we define, for $\omega > 0$, the slope and the offset of LC_{-1} as

$$s = -\frac{1 - \omega}{\omega} < 0, \quad \mu = -\frac{\Delta_q}{\omega\Delta_p} > 0 \quad (17)$$

Notice that for $\omega = 1$ the discontinuity line LC_{-1} has slope $s = 0$ and offset $\mu = -\frac{\Delta_q}{\Delta_p} = x^*$ (the discontinuity point in the one-dimensional case), while for $0 < \omega < 1$ it is $\mu > x^*$.

As already remarked, the peculiarity of our system is that the different definitions F_u and F_l only depend on x , so that the related Jacobian matrix has always one eigenvalue equal to 0 and one equal to $(1 - \delta_B)$ or $(1 - \delta_A)$. Since the functions are linear this leads to a particular behavior: the upper map $F_u(x, y)$ maps in one iteration the half-plane $y > -\frac{1-\omega}{\omega}x - \frac{\Delta_q}{\omega\Delta_p}$ into the critical line

$$LC_u = F_u(LC_{-1}) : \quad y = \frac{x}{1 - \delta_B} \quad (18)$$

while the lower map $F_l(x, y)$ maps in one iteration the half-plane $y < -\frac{1-\omega}{\omega}x - \frac{\Delta_q}{\omega\Delta_p}$ into the critical line

$$LC_l = F_l(LC_{-1}) : \quad y = \frac{x - \delta_A}{1 - \delta_A} \quad (19)$$

and both have a positive slope larger than 1.

Thus, all the iterates of the two-dimensional map belong to the two lines detected above, LC_u and LC_l , on both of which we apply $F_u(x, y)$ in the partitions above LC_{-1} or $F_l(x, y)$ in the partitions below LC_{-1} . In any case, all the asymptotic trajectories of the map belong to these two straight lines. Moreover, we immediately can say that the application of $F_u(x, y)$ is a contraction towards the virtual fixed point $(0, 0)$ and similarly the application of $F_l(x, y)$ is a contraction towards the virtual fixed point $(1, 1)$. Here *virtual* refers to the fact that the fixed point of the linear map F_u does not belong to the region of definition of the map itself. Similarly for F_l .

For simplicity, all the figures reported from now on are obtained with fixed values $\Delta_p = 1.3$ and $\Delta_q = -0.5$, since these parameters have only a qualitative (or scaling) effect.

As an example of the trajectories in the phase plane, assuming $\delta_B = 0.75$, $\delta_A = 0.5$ and $\omega = 0.4$, the phase space is shown in 4(a), the asymptotic dynamics lead to an attracting 4-cycle, whose points are shown by black circles in 4(a). The discontinuity line (in blue) intersects LC_u and LC_l in two points, which are discontinuity points, related to

$$\begin{aligned} X_u &= \frac{\mu(1 - \delta_B)}{1 - s(1 - \delta_B)} = \frac{-\frac{\Delta_q}{\Delta_p}(1 - \delta_B)}{\omega + (1 - \omega)(1 - \delta_B)} \\ X_l &= \frac{\mu(1 - \delta_A) + \delta_A}{1 - s(1 - \delta_A)} = \frac{-\frac{\Delta_q}{\Delta_p}(1 - \delta_A) + \omega\delta_A}{\omega + (1 - \omega)(1 - \delta_A)} \end{aligned} \quad (20)$$

As remarked above, in our system the cycles can have periodic points only on the two critical lines LC_u and LC_l , and can appear/disappear only via border collision bifurcation with one of the discontinuity points of map F , given by

$$\left(X_u, \frac{X_u}{1 - \delta_B}\right) \in LC_u \quad , \quad \left(X_l, \frac{X_l - \delta_A}{1 - \delta_A}\right) \in LC_l \quad (21)$$

In the example shown in 4(b), for $\delta_B = 0.25$, $\delta_A = 0.75$ and $\omega = 0.4$ only one discontinuity point belongs to the phase plane of interest, the one on LC_u , and the dynamics are converging to a 5-cycle, a periodic point of which is very close to the discontinuity point $\left(X_u, \frac{X_u}{1 - \delta_B}\right)$, and thus close to a border collision bifurcation.

The action of the map is as follows: to points belonging to LC_l above the discontinuity point (i.e. for $x > X_l$) the map F_u applies so that in one iteration the point is mapped on LC_u and F_u is applied as long as a point below the discontinuity is obtained, i.e. up to a point with $x < X_u$, to which F_l applies and in one iteration the point is mapped on LC_l and F_l is applied as long as a point above the discontinuity is obtained, i.e. up to a point with $x > X_l$, and then the process is repeated.

Moreover, since the two linear maps are contractions, our system cannot have any repelling cycle. The existing cycles, of different periods, can only be attracting and, as we shall see, also coexistence with two attracting cycles is possible, which cannot occur for $\omega = 0$, in the one-dimensional case. In fact, the dynamics of our system are more similar to those of a one-dimensional map with two discontinuity points and contractions in the different branches, as considered in [53] (see also [52]).

We can investigate the dynamics of map F introducing a symbolic sequence related to the four pieces in which the two critical lines LC_u and LC_l are separated by the discontinuity points. Let us denote by the

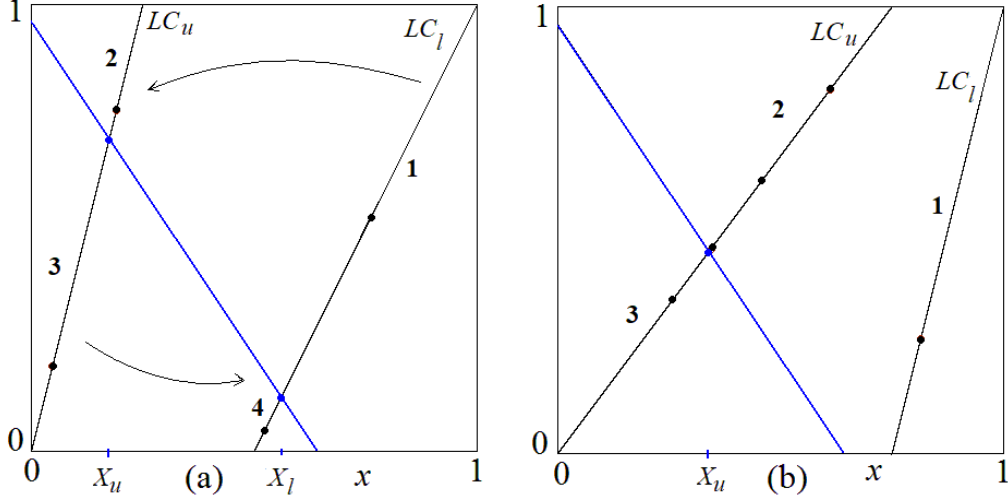


Figure 4: Phase plane of map F , critical lines and attracting sets for $\omega = 0.4$. In (a) $\delta_B = 0.75$, $\delta_A = 0.5$. In (b) $\delta_B = 0.25$, $\delta_A = 0.75$.

symbol **1** (resp. **2**) a point belonging to LC_l (resp. LC_u) in the upper part of the discontinuity line (where the function F_u is applied), and by the symbol **3** (resp. **4**), a point belonging to LC_u (resp. LC_l) in the lower part of the discontinuity line (where the function F_l is applied). This also clarifies the possible border collision bifurcations: a periodic point with symbol **1** can merge with X_l from above, a periodic point with symbol **2** can merge with X_u from above, a symbol **3** can merge with X_u from below, a periodic point with symbol **4** can merge with X_l from below.

The first property of our system is the following:

Property 1. *All the cycles of map F belong to the lines LC_u and LC_l and have the symbolic sequences made up of blocks of type $\mathbf{12}^n\mathbf{34}^m$, with $n \geq 0$ and $m \geq 0$.*

Proof. The proof follows immediately from the properties of the map described above: non existence of real fixed points in the domain of interest $[0, 1] \times [0, 1]$ and the fact that any point belonging to the region denoted by **1** (resp. **3**) is mapped in one iteration by the function F_u (resp. the function F_l) into a point belonging to the critical line LC_u (resp. LC_l). \square

The 4-cycle shown in 4(a) has symbolic sequences $\mathbf{1234}$ while the 5-cycle shown in 4(b) has symbolic sequences $\mathbf{12^33}$. Clearly, a cycle may have a symbolic sequence which consists of the concatenation of a finite number of similar blocks, with different integers n and m . For example, a cycle with symbolic sequence $\mathbf{12^2312^33}$ exists and it consists of the concatenation of $\mathbf{12^23}$ and $\mathbf{12^33}$.

In the following section we shall determine some conditions leading to cycles with specific periods and symbolic sequences. We are interested in the changes occurring for $\omega > 0$ to the stability regions shown in 3b) for $\omega = 0$, in the bifurcation curves in the two-dimensional parameter plane (δ_B, δ_A) as well as in the parameter plane (δ_A, ω) or (δ_B, ω) evidencing the role of the memory.

As already remarked, coexistence of attracting cycles is possible. However, due to the existence of only two different discontinuity points with which a periodic point of a cycle can merge (and undergo a border collision), we have that at most a pair of attracting cycles can coexist. We can so state the following

Property 2. *Map F can have at most two coexisting attracting cycles, and no repelling cycle.*

4 Cycles of map F and related BCBs

Since map F cannot have repelling cycles and the fixed points are only virtual, the dynamics are expected to be cyclical, as it occurs in the one-dimensional case, even if it is known that particular values of the parameters may lead to cases in which quasiperiodic trajectories occur. Such values belong to limit sets of periodicity

regions in the parameter space related to attracting cycles, bounded by border collision bifurcations with the discontinuity points. Thus, we are mainly interested in finding the existence regions in the parameter space of cycles having specific symbolic sequences. Notice that in order to get the periodic points and the related BCBs with the discontinuity points, only the value of x is used (since the functions F_u and F_l do not depend on the second component).

From Property 1 shown above we have immediately that the cycle of map F of minimal period is a 2-cycle, occurring for $n = 0$ and $m = 0$ in the symbolic sequence. That is, with a point $(x_l, y_l) \in LC_l$ having $x_l > X_l$ (whose symbolic sequence is **1**), which is mapped by F_u into a point $(x_u, y_u) \in LC_u$ with $x_u < X_u$ (whose symbolic sequence is **3**), and *vice versa* $((x_u, y_u)$ is mapped by F_l in (x_l, y_l)).

Regarding the other possible cycles, we can see that 3-cycles can have the symbolic sequence **123** and **134** (associated with $n = 1, m = 0$ or $n = 0$ and $m = 1$). Differently, we can have three kinds of 4-cycles, with symbolic sequence **1234**, **12²3**, **134²** (associated with $n = 1, m = 1$, or $n = 2, m = 0$ or $n = 0$ and $m = 2$, respectively).

In general, to determine a cycle with symbolic sequence **12ⁿ34^m** we need the explicit repeated applications of the functions $F_u(x, y)$ and $F_l(x, y)$, which can be easily determined due to the linearity of the functions. It is easy to show the following:

Property 3. *Repeated applications of the functions $F_u(x, y)$ or $F_l(x, y)$ are given by*

$$F_u^n(x, y) = ((1 - \delta_B)^n x, (1 - \delta_B)^{n-1} x), \quad n \geq 1 \quad (22)$$

$$F_l^m(x, y) = ((1 - \delta_A)^m x + \Psi_m, (1 - \delta_A)^{m-1} x + \Psi_{m-1}), \quad m \geq 1 \quad (23)$$

where

$$\Psi_m = 1 - (1 - \delta_A)^m \text{ for } m \geq 1, \quad \Psi_0 = 0 \quad (24)$$

4.1 2-cycle

We have the following result (see the Appendix for the proof):

Proposition 1. *A 2-cycle of map F has symbolic sequence **13**, it exists only for $0 < \omega < 0.5$ and the existence region is bounded by the sets C_{31} and C_{13} of equations:*

$$\begin{aligned} C_{31} : \quad (1 - \delta_B) &= \frac{\mu - \delta_A}{\mu(1 - \delta_A) - s\delta_A} \\ C_{13} : \quad (1 - \delta_B) &= \frac{\mu(1 - \delta_A) + \delta_A - \delta_A(1 - s(1 - \delta_A))}{(1 - \delta_A)[\mu(1 - \delta_A) + \delta_A]} \end{aligned} \quad (25)$$

In the two-dimensional parameter plane (δ_B, δ_A) at $\omega = 0.5$ fixed, shown in 5(a), we can see that there is no existence region related to a cycle of period 2, while two existence regions can be seen, related to the pair of 3-cycles described above. Clearly, an existence region related to a 2-cycle exists for $0 < \omega < 0.5$, and it disappears for larger values of ω . It is worth noting that as for the 2-cycle, also the existence regions of the 3-cycles, as well as for the 4-cycles described below, have to satisfy particular conditions, so that these regions may disappear for larger values of ω , as it is illustrated in 5(b) at $\omega = 0.9$. The periodicity regions related to the 3-cycles no longer exist, and of the three regions related to the three kinds of 4-cycles (corresponding to all the symbolic sequences that the system can have, and existing at $\omega = 0.5$) only the central region is still existing.

In a two-dimensional bifurcation diagram in the parameter plane (δ_B, ω) or (δ_A, ω) it is possible to see the periodicity region related to the 2-cycle, existing for $\omega < 0.5$ as observable in 6. In the same figure we can also see that no 3-cycle exists for $\omega > 0.77$.

4.2 4-cycle 1234

As already remarked, we can have three kinds of 4-cycles, with symbolic sequence **1234**, **12²3**, **134²**. While the 4-cycle **1234** is particular (the related periodicity region is the one in the center of 5(a,b)), the other 4-cycles belong to two families of cycles having the symbolic sequence **12ⁿ3** and **134ⁿ** for $n \geq 1$. So let

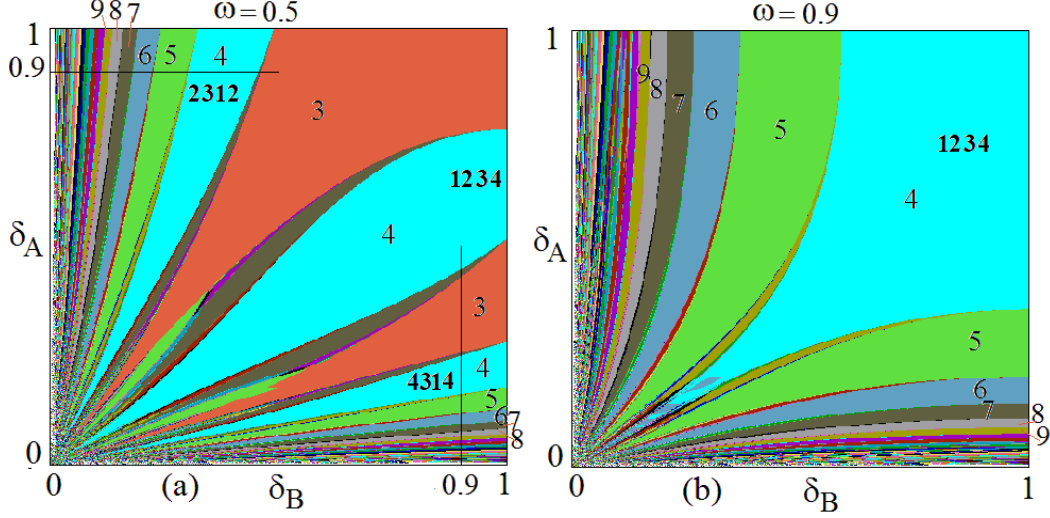


Figure 5: Periodicity regions in the two-dimensional parameter plane (δ_B, δ_A) . In (a) at $\omega = 0.5$ fixed. In (b) at $\omega = 0.9$ fixed.

us determine the bifurcations related to the existence region of the 4-cycle **1234** and then those of the two families (which include the 3-cycles and 4-cycles as particular cases), the proof is given in the Appendix.

Proposition 2. *The existence region of the 4-cycle of map F with symbolic sequence **1234** is bounded by the sets C_{1234} , C_{4123} , C_{3412} and C_{2341} of equations:*

$$C_{1234} : (1 - \delta_B)^2 = \frac{\mu(1 - \delta_A) + \delta_A - [1 - s(1 - \delta_A)][2\delta_A - \delta_A^2]}{(1 - \delta_A)^2[\mu(1 - \delta_A) + \delta_A]} \quad (26)$$

$$C_{4123} : (1 - \delta_B)^2 = \frac{\mu(1 - \delta_A) + \delta_A - \delta_A[1 - s(1 - \delta_A)]}{\delta_A(1 - \delta_A)[1 - s(1 - \delta_A)] + (1 - \delta_A)^2[\mu(1 - \delta_A) + \delta_A]} \quad (27)$$

$$C_{3412} : [1 - s(1 - \delta_B)](1 - \delta_B)[2\delta_A - \delta_A^2] + \mu(1 - \delta_B)^2(1 - \delta_A)^2 - \mu = 0$$

$$C_{2341} : (1 - \delta_A)^2 = \frac{1 - s(1 - \delta_B) - \mu}{1 - s(1 - \delta_B) - \mu(1 - \delta_B)^2} \quad (28)$$

The bifurcation curves of this 4-cycle are evidenced in 7 in the two-dimensional parameter plane (δ_B, δ_A) at $\omega = 0.5$. The periodicity region of the 4-cycle with symbolic sequence **1234** is also shown in 6, and we can see more boundaries, besides those associated with the colliding points **4** and **2**, also the border related to the collision of the periodic points **1** and **3** are visible. In 7 we also show BCB curves of the two families **312ⁿ** and **134ⁿ** for $n = 1, \dots, 10$ and commented below.

4.3 Two families **312ⁿ** and **134ⁿ** for $n \geq 1$

As remarked above, the two kinds of 3-cycles and the other kinds of 4-cycles to be determined belong to two families of cycles having the symbolic sequence **312ⁿ** and **134ⁿ** for any $n \geq 1$, as can be seen in the two-dimensional parameter plane (δ_B, δ_A) at ω fixed.

Consider the first family **312ⁿ**, the existence regions are bounded by two curves related to BCB due to the collision of a periodic point with the same discontinuity point X_u , from above and from below (considering the sequences **312ⁿ** and **2312ⁿ⁻¹**), and only one more boundary exists, due to the collision with X_l from above, via the sequence **12ⁿ3** (a collision with X_l from below cannot occur, since the sequence does not include the symbol **4**). Similarly for the second family, two boundaries are determined via the collision with the same discontinuity point X_l (from above and from below, via **134ⁿ** and **4134ⁿ⁻¹**) and only a third boundary exists, due to the collision with X_u from below, via **34ⁿ1**, but no collision can occur from above (since the sequence does not include the symbol **2**). We have the following (see the Appendix for the proof):

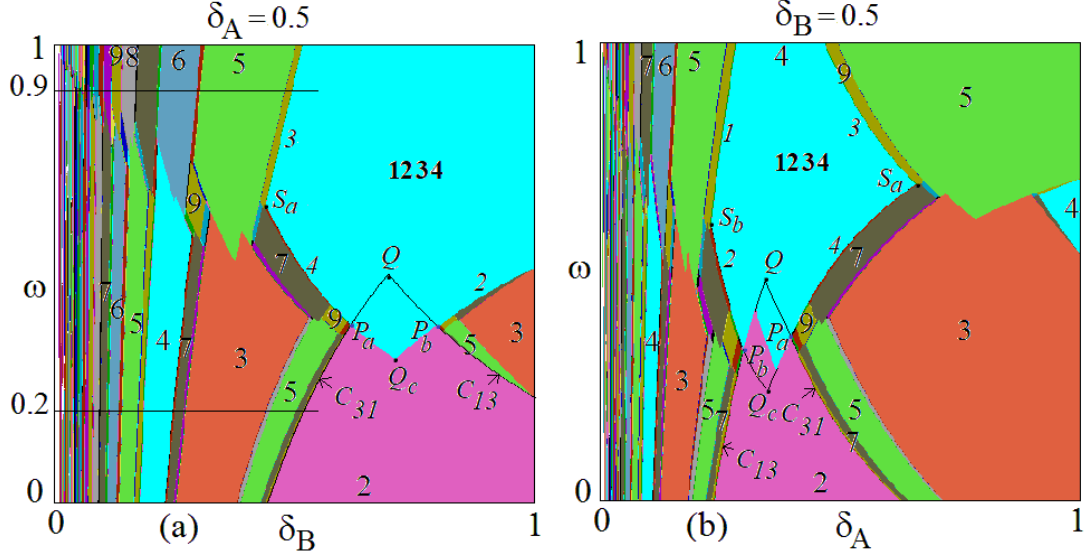


Figure 6: Periodicity regions in the two-dimensional parameter plane. In (a) (δ_B, ω) at $\delta_A = 0.5$ fixed. In (b) (δ_A, ω) at $\delta_B = 0.5$ fixed.

Proposition 3. *The existence regions of the cycle of map F with symbolic sequence $\mathbf{312}^n$ for $n \geq 1$ are bounded by the sets C_{312^n} , $C_{2312^{n-1}}$ and C_{12^n3} of equations:*

$$C_{312^n} : \delta_A = \frac{\mu[1 - (1 - \delta_B)^{n+1}]}{(1 - \delta_B)^n[1 - s(1 - \delta_B) - \mu(1 - \delta_B)]} \quad (29)$$

$$C_{2312^{n-1}} : \delta_A = \frac{\mu[1 - (1 - \delta_B)^{n+1}]}{(1 - \delta_B)^{n-1}[1 - s(1 - \delta_B) - \mu(1 - \delta_B)]} \quad (30)$$

$$C_{12^n3} : (1 - \delta_B)^{n+1} = \frac{\mu + s\delta_A}{\mu(1 - \delta_A) + \delta_A} \quad (31)$$

The existence regions of the cycle of map F with symbolic sequence $\mathbf{134}^n$ for $n \geq 1$ are bounded by the sets C_{134^n} , $C_{4134^{n-1}}$ and C_{34^n1} of equations:

$$C_{134^n} : (1 - \delta_B) = \frac{\mu(1 - \delta_A) + \delta_A - [1 - s(1 - \delta_A)][1 - (1 - \delta_A)^{n+1}]}{(1 - \delta_A)^{n+1}[\mu(1 - \delta_A) + \delta_A]} \quad (32)$$

$$C_{4134^{n-1}} : (1 - \delta_B) = \frac{\mu(1 - \delta_A) + \delta_A - [1 - s(1 - \delta_A)][1 - (1 - \delta_A)^n]}{(1 - \delta_A)^n[(\mu(1 - \delta_A) + \delta_A)(1 - \delta_A) + (1 - s(1 - \delta_A))\delta_A]} \quad (33)$$

$$C_{34^n1} : (1 - \delta_B) = \frac{\mu - 1 + (1 - \delta_A)^{n+1}}{(1 - \delta_A)^{n+1}(\mu - s) + s} \quad (34)$$

Two boundaries of these two families are evidenced in 7 in the (δ_B, δ_A) parameter plane at $\omega = 0.5$, for $n = 1, \dots, 10$.

The cycles are also illustrated in the one-dimensional bifurcation diagrams, as a function of δ_B at $\delta_A = 0.9$ fixed in 8(a) and as a function of δ_A at $\delta_B = 0.9$ fixed in 8(b), keeping $\omega = 0.5$ fixed (along the horizontal and vertical lines in 5(a)). These figures illustrate that the observed cycles follow a period adding bifurcation structure, very similar to the one observable in one-dimensional maps with only one discontinuity point (as for the case $\omega = 0$).

From 7 we can also see a peculiarity associated with the limiting cases $\delta_B = 1$ and $\delta_A = 1$: the intervals of the principal cycles having the symbolic sequence $\mathbf{12}^n\mathbf{3}$ and $\mathbf{134}^n$ for $n \geq 1$ are connected, thus leading to

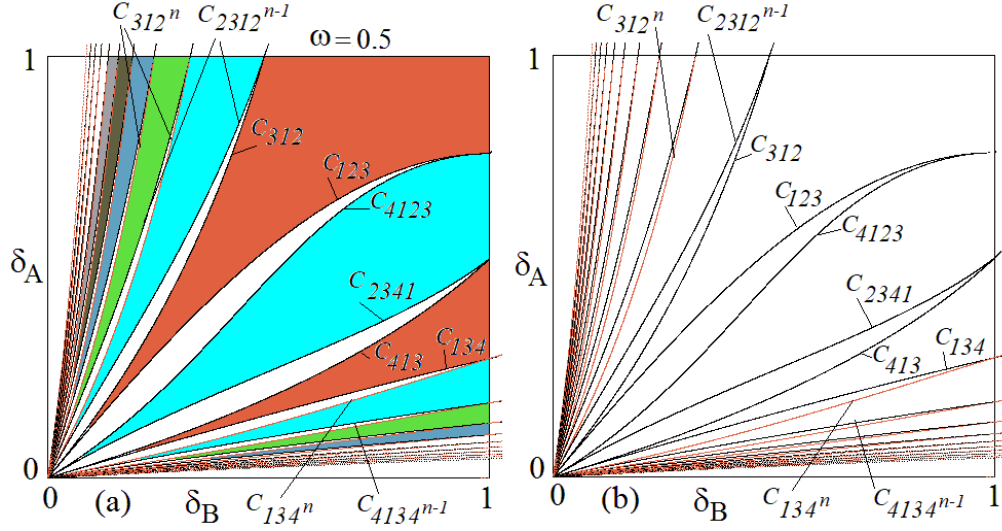


Figure 7: In (a) principal periodicity regions in the two-dimensional parameter plane (δ_B, δ_A) at $\omega = 0.5$. In (b) only the bifurcation curves are shown.

period incrementing structure (no other cycle exists between each consecutive pair). While for $0 < \delta_B < 1$ and $0 < \delta_A < 1$ the white regions between each consecutive pair is filled with infinitely many other periodicity regions, in the period adding bifurcation structure.

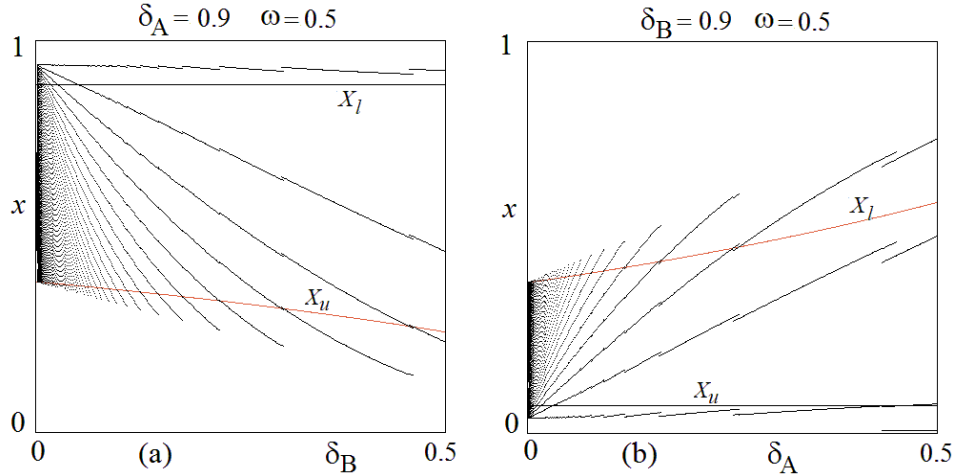


Figure 8: One-dimensional bifurcation diagrams. In (a) as a function of δ_B at $\delta_A = 0.9$ and $\omega = 0.5$ fixed. In (b) as a function of δ_A at $\delta_B = 0.9$ and $\omega = 0.5$ fixed.

4.4 Coexistence in the phase space

From the two-dimensional bifurcation diagrams shown in 6 we can see that all the periodicity regions have a part overlapped with another periodicity region (and overlapping regions can also be seen in 5). Coexistence means that depending on the initial condition the dynamics may converge to a cycle or to the other one. In our simple two-dimensional map the basins can be determined in a simple way. In fact, since the second variable has no influence, the regions belonging to a basin are vertical strips up to the discontinuity line.

The boundaries of the strips are given by x -values obtained by using the discontinuity points, whose x -component is $X_u = \frac{\mu(1-\delta_B)}{1-s(1-\delta_B)}$ and $X_l = \frac{\mu(1-\delta_A)+\delta_A}{1-s(1-\delta_A)}$, and the related preimages on the two critical lines, noticing that we need only the preimages of the first component, given by:

$$F_u^{-1}(z) = \frac{z}{1-\delta_B} \quad , \quad F_l^{-1}(z) = \frac{z-\delta_A}{1-\delta_A} \quad (35)$$

An example in the phase plane is represented in 9(a) at $\omega = 0.3$, $\delta_A = 0.35$ and $\delta_B = 0.5$, where the coexistence of a 2-cycle and a 4-cycle is shown, and the related basins of attraction. The red regions denote the basin of attraction of the 2-cycle, while the white region is the basin of the 4-cycle.

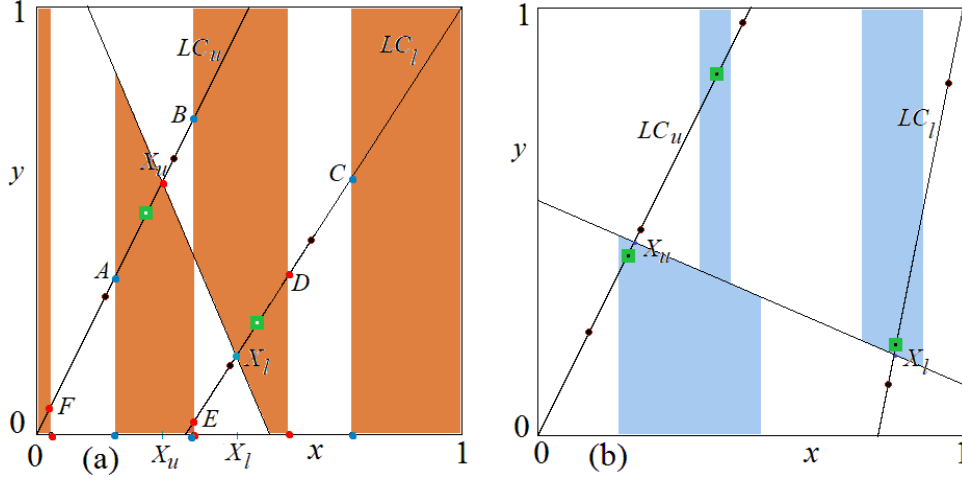


Figure 9: Phase plane. In (a) at $\omega = 0.3$, $\delta_A = 0.35$ and $\delta_B = 0.5$, coexistence of a 2-cycle (basin in orange) and a 4-cycle (basin in white). In (b) at $\omega = 0.7$, $\delta_A = 0.8$ and $\delta_B = 0.5$, coexistence of a 3-cycle (basin in azure) and a 5-cycle (with white basin).

The points on the boundaries which are evidenced by the letters A , B and C with green circles have the x -component given by the preimages of X_l as follows (blue circles also on the x -axis in 9(a)):

$$x(A) \text{ is the preimage of } X_l \text{ on } LC_u \text{ with } F_l^{-1}, x(A) = \frac{X_l - \delta_A}{1 - \delta_A},$$

$$x(B) \text{ is the preimage of } x(A) \text{ on } LC_u \text{ with } F_u^{-1}, x(B) = \frac{x(A)}{1 - \delta_B},$$

$$x(C) \text{ is the preimage of } x(B) \text{ on } LC_l \text{ with } F_u^{-1}, x(C) = \frac{x(B)}{1 - \delta_B}.$$

The points D , E and F with blue circles have the x -component given by the preimages of X_u as follows (red circles also on the x -axis in 9(b)):

$$x(D) \text{ is the preimage of } X_u \text{ on } LC_l \text{ with } F_u^{-1}, x(D) = \frac{X_u}{1 - \delta_B},$$

$$x(E) \text{ is the preimage of } x(D) \text{ on } LC_l \text{ with } F_l^{-1}, x(E) = \frac{x(D) - \delta_A}{1 - \delta_A},$$

$$x(F) \text{ is the preimage of } x(E) \text{ on } LC_u \text{ with } F_l^{-1}, x(F) = \frac{x(E) - \delta_A}{1 - \delta_A}.$$

In this example the two values $x(B)$ and $x(E)$ are very close and cannot be distinguished in 9(a), but $x(B)$ is a little smaller than $x(E)$.

One more example is shown in 9(b) at $\omega = 0.7$, $\delta_A = 0.8$ and $\delta_B = 0.5$, where the coexistence of a 3-cycle (with basin in azure) and a 5-cycle (with white basin) is shown. The boundaries of the basin can be determined as described, considering the preimages of the discontinuity points.

4.5 Coexistence in the parameter space

The possibility of coexisting attractors is perhaps the main difference with respect to the dynamics of the one-dimensional case (for $\omega = 0$). In fact, here, as also in the case $\omega = 0$, no repelling cycles can exist, only attracting cycles. However, for $\omega = 0$ the related periodicity regions cannot overlap, and are organized in

a period adding bifurcation structure. As recalled, each bifurcation curve in 3(b) is the limit set of other periodicity regions (in the period adding bifurcation structure).

Differently, in the two-dimensional map F under study, the dynamics are similar to those occurring in a one-dimensional contracting map with two discontinuity points (see [53]). Due to the possibility of two different discontinuity points in which a cycle can undergo a border collision, as stated in Property 2 a pair of attracting cycles can coexist (and not more). The regions related to the coexistence of a pair of cycles are more evident in the two-dimensional parameter plane (δ_B, ω) and (δ_A, ω) shown in 6. We can observe some peculiarities in the structure. Roughly speaking, we have two sequences of period adding bifurcation structures, a lower one and an upper one, as evidenced by the two one-dimensional bifurcation diagrams in 10, along the two horizontal lines in 6(a), and the related periodicity regions are merging in the middle.

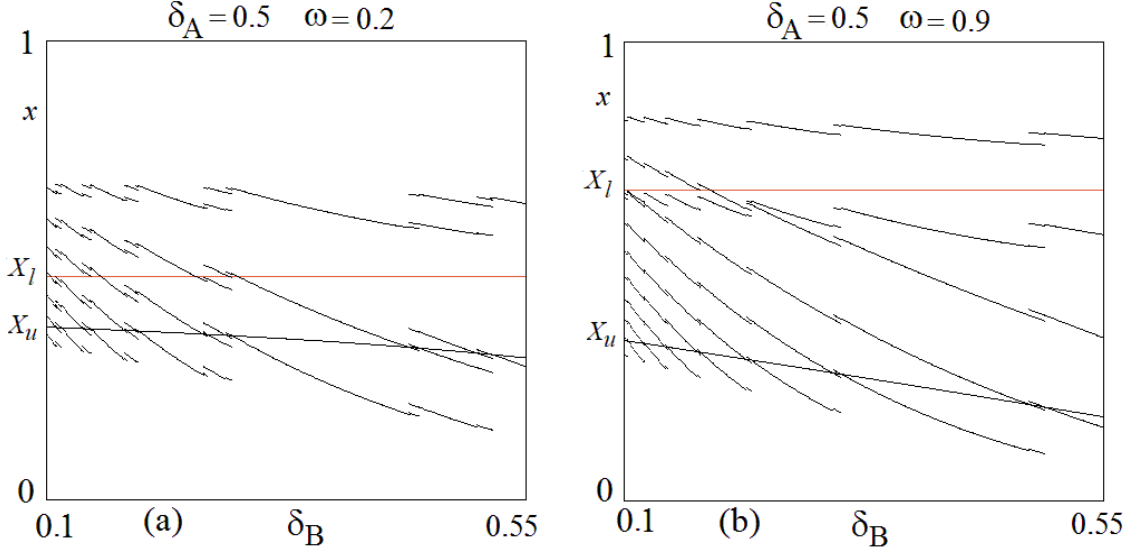


Figure 10: One-dimensional bifurcation diagrams. In (a) along the horizontal line in 6(a) at $\omega = 0.2$. In (b) along the horizontal line in 6(a) at $\omega = 0.9$.

The regions of the principal cycles of the lower family in 6(a) belong to the family with symbolic sequence $\mathbf{312}^n$ for $n \geq 1$ (really we can say for $n \geq 0$ since for $n = 0$ we start from the 2-cycle $\mathbf{31}$), whose bifurcation curves have been determined in Proposition 3. Although to our knowledge the concatenation of the symbolic sequences has been proved only for one-dimensional maps, and we are dealing with a two-dimensional map, we have numerical evidence that it works also in this system. For example, between the 3-cycle $\mathbf{312}$ and the 2-cycle $\mathbf{31}$ we can see the region of a 5-cycle $\mathbf{31231}$, between the 4-cycle $\mathbf{312}^2$ and the 3-cycle $\mathbf{312}$ we can see the region of a 7-cycle $\mathbf{312}^2\mathbf{312}$, and so on.

Similarly for the upper family, whose regions of principal cycles belong to the family with symbolic sequence $\mathbf{3412}^n$ for $n \geq 1$. Farey summation rule related to the concatenation of the symbolic sequences has been numerically observed also here. For example, between the 5-cycle $\mathbf{3412}^2$ and the 4-cycle $\mathbf{3412}$ we can see the region of a 9-cycle $\mathbf{3412}^2\mathbf{3412}$, between the 6-cycle $\mathbf{3412}^3$ and the 5-cycle $\mathbf{3412}^2$ we can see the region of a 11-cycle $\mathbf{3412}^3\mathbf{3412}^2$, and so on.

Not only all the principal regions of the lower and upper families are overlapping (as for the regions of the 2-cycle of the lower family and the 4-cycle of the upper family in 6, and in general of the k -cycle of the lower family and the $(k + 2)$ -cycle of the upper family), the other regions obtained by concatenation of the symbolic sequences are also overlapping with other periodicity regions, in a particular way, still to be investigated.

What we have observed is that the overlapping of the principal regions lead to bistability regions and codimension-two points with particular properties. For the periodicity regions we have that:

- the existence region of a cycle may be bounded by three or four BCB curves, leading to overlapped

parts. The existence of overlapped parts is also related to the existence of cascades of periodicity regions having a quadrilateral shape (recall that a cycle with $n \geq 4$ periodic points may have up to 4 border collision bifurcation boundaries, related to the discontinuity points X_u and X_l from below and from above, when all the 4 symbols are present in the symbolic sequence of the cycle),

- the borders of the periodicity regions related to an overlapped part, are not limit set of other periodicity regions,
- the borders of the periodicity regions not related to an overlapped part, are limit set of other periodicity regions,
- each periodicity region of quadrilateral shape has two opposite corners overlapped with two other regions (called point of type- Q below) and the other two corners (called point of type- S below) which are limit sets of periodicity regions from two sides, both kinds of codimension-2 points are due to the collision of two periodic points of the same cycle with the two different discontinuity points.

In a two-dimensional parameter plane the codimension-2 points are due to the intersection of two different BCB curves (and, as already remarked, are better visualized in the parameter planes (δ_B, ω) or (δ_A, ω)). This leads to codimension-two points of three different kinds: the two curves which are intersecting may be associated with two different cycles, or the two intersecting curves are associated with the same cycle, and this leads to two different types. For the codimension-two points we have the following classification:

- A codimension-two point of type- P is due to the intersection of two BCB curves related to two different cycles, then in the region not related to the existence of the two cycles the point is always a limit sets of other periodicity regions (related to two different period adding bifurcation structures), as the points P_a and P_b in 6.
- A codimension-two point of type- Q is due to the intersection of two BCB curves of the same cycle with the discontinuity points, and belongs to the boundary of an overlapped region: in any neighbourhood of this point we can find only parameter points associated with the to two periodic orbits, as the points Q and Q_c in 6.
- A codimension-two point of type- S is due to the intersection of two BCB curves of the same cycle with the discontinuity points, not belonging to an overlapped region: in any neighbourhood of this point we can find parameter points associated with infinitely many other periodic orbits, that is, in the region not related to the existence of the cycle the point is a limit set of other periodicity regions of different kinds (related to two different period adding bifurcation structures), as the points S in 11.

Examples are shown in the enlargements of 11 where the initial condition has been taken on the critical curve LC_u at points with $X_u \pm 0.0001$. The figures are the same also taking the initial condition on the critical curve LC_l at points with $X_l \mp 0.0001$. The period adding bifurcation structure is evidenced in the two one-dimensional diagrams as a function of δ_B shown in 10 (at $\omega = 0.2$ and $\omega = 0.9$ fixed). Each quadrilateral region related to some n -cycle, $n \geq 4$, is bounded by BCB curves related to the four different kinds of collision which may occur. Two opposite corners are codimension-2 points of type Q belonging to overlapped regions, the other two corner points are of type S . Moreover, the two overlapping regions always lead to 4 points of type P on the four borders. An example is shown in 11(b) related to the quadrilateral region of a 9-cycle. We can state that

Property 4. *In the two-dimensional parameter plane (δ_B, ω) or (δ_A, ω) the coexistence regions are of quadrilateral shape with two BCB curves related to each cycle, and four corners which include two codimension-two points of type Q and two codimension-two points of type P .*

4.6 The memory effect

Recall that in the minority games several individuals are looking for a collective problem when adapting each one's expectations about the future [39]. The impulsive agents (as we have considered) are particularly simple and endowing them with memory is a possible first step in order to provide them with social cognition. Social

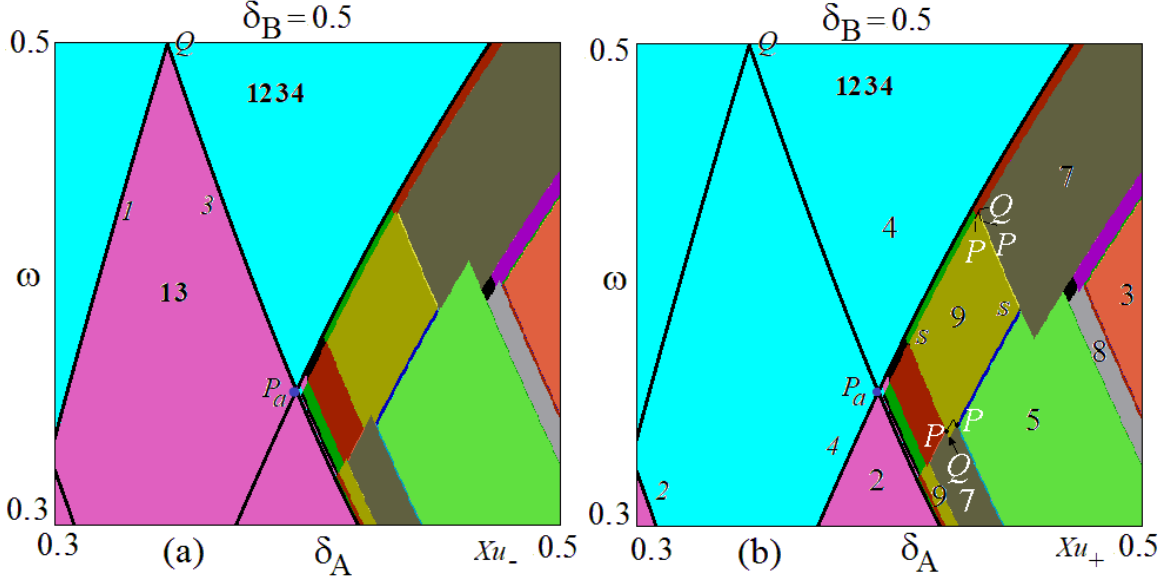


Figure 11: Two-dimensional bifurcation diagrams, enlargements from 6. In (a) the initial condition has been taken on the critical curve LC_u at $x = X_u + 0.0001$. In (b) the initial condition has been taken on the critical curve LC_u at $x = X_u - 0.0001$.

cognition is indeed taken as broad synonym of the Theory of Mind phenomenon [19]. Some contributions study the relationship between memory and Theory of Mind, see e.g. [14], and suggest the relevance of memory when approaching a collective interaction.

In our model, we may expect that the memory effect is negligible when ω is small, that is, when previous period payoff has a small weight, while high values of ω suggest a higher influence, so that memory can change the dynamics of the population. And indeed this is the result from the analysis of our system.

The analysis of the bifurcation structure in the two-dimensional parameter plane (δ_B, ω) or (δ_A, ω) , which we have considered, allows us to understand the role of the memory when all the other parameter are fixed. We can look at the bifurcation diagrams in 6, fixing a vertical straight line at a given value of δ_B or of δ_A , and increasing the value of ω . We can see that if such a straight line crosses only one bistability region, then the memory effect is negligible at low values of ω (we can say that almost nothing changes), while for high values of ω we can have bistability, but with cycles having close periods (k and $k + 2$) or only the cycle with periods $k + 2$ for higher values of ω .

If a vertical straight line is not crossing a bistability region of two principal cycles, we have again that the memory effect is negligible at low values of ω and almost nothing changes, while increasing ω the transition from the periodicity region of a cycle of period k to the periodicity region of a cycle of period $k + 2$ crosses infinitely many intervals (which can also be very small) associated with other cycles and bistability regions, as already observed in other models with memory ([7], [8]).

5 Conclusions

In this paper we have investigated how the bifurcations change with the introduction of memory in a population binary game with impulsive agents. We have considered a minority game already known in the literature, represented by a one-dimensional discontinuous piecewise linear map, characterized by attracting cycles organized in the parameter space by a period adding structure describing the border collision bifurcations of the cycles. In that simple system we have introduced the memory of the past payoffs, in order to see how it affects the decision process, by means of a weighted average of present and past payoffs with weight parameter ω , where $\omega = 0$ means no memory effect. This modifies the utility function, and leads to a two-dimensional discontinuous piecewise linear map with a quite simple structure, since the dynamics

occur only on two straight lines, the two critical lines LC_u and LC_l , on which there are the two discontinuity points $(X_l, \frac{X_l - \delta_A}{1 - \delta_A}) \in LC_l$ and $(X_u, \frac{X_u}{1 - \delta_B}) \in LC_u$. The system cannot have repelling cycles, and attracting cycles can appear/disappear only via border collision bifurcation with one of the discontinuity points. We have shown that the cycles are characterized by the symbolic sequences which consist in blocks of type $\mathbf{12^n34^m}$ with $n \geq 0$ and $m \geq 0$. The main difference with respect to the model without memory consists in the possibility of bistability, that is, two coexisting attracting cycles can now exist, related to border collisions involving two different discontinuity points. We have also shown the related structure of the basins of attraction in the phase space. In the parameter space the bifurcation structure is quite interesting. We have shown the border collision bifurcation conditions for the 2-cycle, 4-cycles, and families of cycles with symbolic sequences $\mathbf{12^n3}$ for $n \geq 1$ and $\mathbf{134^m}$ with $m \geq 1$. Moreover, the existence of two basic families associated with a period adding bifurcation structure, one for low values of ω and one for large values of ω , leads to infinitely many overlapped regions, associated with bistability. The regions in the parameter space related to bistability have peculiar properties, evidenced by codimension-two points which are characterized depending on the kind of the two periodic points undergoing border collision, and may belong to the same cycle or to two different cycles.

In particular, the bifurcation structure in the parameter planes (δ_B, ω) and (δ_A, ω) evidences that when the memory effect is low (for low values of ω), there is almost no difference with respect to the case without memory. The differences appear increasing the value of ω , leading to the possibility of bistability, and also to the possibility of quick changes in the periods of the existing cycles.

The detailed description of the bifurcation structure is still to be investigated, and the existence also in two-dimensional discontinuous maps of a concatenation property in the symbolic sequences of the cycles, is something new and interesting, which is worth to investigate further. Moreover, considering non-linear payoffs in place of the linear functions in (1), even if with impulsive agents the definition of the function does not change, the switching rule is modified, and this may lead to different dynamic behaviors, which also is worth investigating further. An example with a system without memory can be found in [9].

6 Appendix

Proof of Proposition 1. Considering the 2-cycle starting from a point $(x_u, y_u) \in LC_u$, we have the following steps:

$$x_u \xrightarrow{F_l} (1 - \delta_A)x_u + \delta_A \xrightarrow{F_u} (1 - \delta_B)(1 - \delta_A)x_u + (1 - \delta_B)\delta_A$$

so that if a periodic point exists it must be

$$x_u = (1 - \delta_B)(1 - \delta_A)x_u + (1 - \delta_B)\delta_A$$

leading to

$$x_u = \frac{(1 - \delta_B)\delta_A}{1 - (1 - \delta_B)(1 - \delta_A)}$$

which is required to satisfy $x_u < X_u = \frac{\mu(1 - \delta_B)}{1 - s(1 - \delta_B)}$ and a BCB occurs when $x_u = X_u$, that is

$$\frac{\delta_A}{1 - (1 - \delta_B)(1 - \delta_A)} \leq \frac{\mu}{1 - s(1 - \delta_B)} \quad (36)$$

leading to

$$(1 - \delta_B) \leq \frac{\mu - \delta_A}{\mu(1 - \delta_A) - s\delta_A}$$

and the equality gives the bifurcation condition C_{31} . The other periodic point is obtained applying F_l to x_u or we can repeat similar reasoning starting from $x_l \in LC_l$ obtaining

$$x_l \xrightarrow{F_u} (1 - \delta_B)x_l \xrightarrow{F_l} (1 - \delta_A)(1 - \delta_B)x_l + \delta_A$$

so that if a periodic point exists it must be

$$x_l = (1 - \delta_A)(1 - \delta_B)x_l + \delta_A$$

leading to

$$x_l = \frac{\delta_A}{1 - (1 - \delta_B)(1 - \delta_A)} = \frac{x_u}{(1 - \delta_B)}$$

which is required to satisfy $x_l > X_l = \frac{\mu(1 - \delta_A) + \delta_A}{1 - s(1 - \delta_A)}$ and a bifurcation occurs when $x_l = X_l$, that is

$$\frac{\delta_A}{1 - (1 - \delta_B)(1 - \delta_A)} \geq \frac{\mu(1 - \delta_A) + \delta_A}{1 - s(1 - \delta_A)} \quad (37)$$

corresponding to

$$(1 - \delta_B) \geq \frac{\mu(1 - \delta_A) + \delta_A - \delta_A(1 - s(1 - \delta_A))}{(1 - \delta_A)[\mu(1 - \delta_A) + \delta_A]}$$

and the equality gives the bifurcation condition C_{13} .

For the existence of the 2-cycle there are constraints which are to be satisfied, that is $x_u < X_u$ and $x_l > X_l$, which means that the conditions in eq.36 and eq.37 must both be satisfied with the strict inequality. After some algebraic steps this gives the following conditions

$$\frac{\delta_A[1 - s(1 - \delta_B)]}{1 - (1 - \delta_B)(1 - \delta_A)} < \mu < \frac{-s\delta_A + \delta_A(1 - \delta_B)}{1 - (1 - \delta_B)(1 - \delta_A)}$$

and as necessary condition we have

$$\frac{[1 - s(1 - \delta_B)]}{1 - (1 - \delta_B)(1 - \delta_A)} < \frac{-s + 1 - \delta_B}{1 - (1 - \delta_B)(1 - \delta_A)}$$

leading to

$$s < -1$$

and from the definition of the slope s it is

$$s < -1 \text{ iff } \omega < 0.5 \quad (38)$$

Thus, for $\omega \geq 0.5$ the 2-cycle cannot exist. \square

Proof of Proposition 2. Let us consider the 4-cycle with symbolic sequence **1234** (its related periodicity region is shown in the center of 5(a)). The colliding point from below the discontinuity line occurs with the point **4** of the cycle **4123** while from above with the point **2** of the same cycle (i.e. considering **2341**). So to detect the colliding point of the cycle **4123** let us consider the periodic point determined via $F_l \circ F_u^2 \circ F_l(x, y) = (x, y)$ leading to

$$x = (1 - \delta_A)^2(1 - \delta_B)^2x + (1 - \delta_B)^2\delta_A(1 - \delta_A) + \delta_A$$

that is

$$x = \frac{(1 - \delta_B)^2\delta_A(1 - \delta_A) + \delta_A}{1 - (1 - \delta_A)^2(1 - \delta_B)^2}.$$

The related border collision occurs (from below) when $x = X_l$, that is:

$$\frac{(1 - \delta_B)[1 - (1 - \delta_A)^2]}{1 - (1 - \delta_A)^2(1 - \delta_B)} = \frac{\mu(1 - \delta_A) + \delta_A}{1 - s(1 - \delta_A)}$$

leading to the bifurcation condition

$$C_{4123} : (1 - \delta_B)^2 = \frac{\mu(1 - \delta_A) + \delta_A - \delta_A[1 - s(1 - \delta_A)]}{\delta_A(1 - \delta_A)[1 - s(1 - \delta_A)] + (1 - \delta_A)^2[\mu(1 - \delta_A) + \delta_A]}$$

For the periodic point of the cycle **2341** we consider $F_u \circ F_l^2 \circ F_u(x, y) = (x, y)$ leading to

$$x = (1 - \delta_A)^2(1 - \delta_B)^2x + (1 - \delta_B)[1 - (1 - \delta_A)^2]$$

that is

$$x = \frac{(1 - \delta_B)[1 - (1 - \delta_A)^2]}{1 - (1 - \delta_A)^2(1 - \delta_B)^2}.$$

The related border collision occurs (from above) when $x = X_u$, that is:

$$\frac{[1 - (1 - \delta_A)^2]}{1 - (1 - \delta_A)^2(1 - \delta_B)} = \frac{\mu}{1 - s(1 - \delta_B)}$$

leading to the bifurcation condition

$$C_{2341} : (1 - \delta_A)^2 = \frac{1 - s(1 - \delta_B) - \mu}{1 - s(1 - \delta_B) - \mu(1 - \delta_B)^2}.$$

The other boundaries can be detected in a similar way. From the collision of **1234** we consider $F_l^2 \circ F_u^2(x, y) = (x, y)$ leading to

$$x = \frac{2\delta_A - \delta_A^2}{1 - (1 - \delta_A)^2(1 - \delta_B)^2}$$

The related border collision occurs (from above) when $x = X_l$, that is:

$$\frac{2\delta_A - \delta_A^2}{1 - (1 - \delta_A)^2(1 - \delta_B)^2} = \frac{\mu(1 - \delta_A) + \delta_A}{1 - s(1 - \delta_A)}$$

leading to the bifurcation condition

$$C_{1234} : (1 - \delta_B)^2 = \frac{\mu(1 - \delta_A) + \delta_A - [1 - s(1 - \delta_A)][2\delta_A - \delta_A^2]}{(1 - \delta_A)^2[\mu(1 - \delta_A) + \delta_A]}.$$

The last boundary can be detected from the collision of **3412** considering $F_u^2 \circ F_l^2(x, y) = (x, y)$ which leads to

$$x = \frac{(1 - \delta_B)^2[2\delta_A - \delta_A^2]}{1 - (1 - \delta_A)^2(1 - \delta_B)^2}$$

The related border collision occurs (from below) when $x = X_u$, that is:

$$\frac{(1 - \delta_B)[2\delta_A - \delta_A^2]}{1 - (1 - \delta_A)^2(1 - \delta_B)^2} = \frac{\mu}{1 - s(1 - \delta_B)}$$

leading to the bifurcation condition C_{3412} given in the proposition. \square

Proof of Proposition 3. Let us consider the first family with **312ⁿ** for $n \geq 1$ and the function $F_u^{n+1} \circ F_l(x, y)$ whose fixed point is obtained from $F_u^{n+1} \circ F_l(x, y) = (x, y)$ leading to $x = (1 - \delta_B)^{n+1}(1 - \delta_A)x + (1 - \delta_B)^{n+1}\delta_A$ that is:

$$x = \frac{(1 - \delta_B)^{n+1}\delta_A}{1 - (1 - \delta_A)(1 - \delta_B)^{n+1}}$$

and the related border collision occurs when $x = X_u$, that is:

$$\frac{(1 - \delta_B)^n\delta_A}{1 - (1 - \delta_A)(1 - \delta_B)^{n+1}} = \frac{\mu}{1 - s(1 - \delta_B)}$$

leading to the bifurcation condition

$$C_{312^n} : \delta_A = \frac{\mu[1 - (1 - \delta_B)^{n+1}]}{(1 - \delta_B)^n[1 - s(1 - \delta_B) - \mu(1 - \delta_B)]} \quad (39)$$

Now we consider the periodic point determined via $F_u^n \circ F_l \circ F_u(x, y) = (x, y)$ leading to $x = (1 - \delta_B)^{n+1}(1 - \delta_A)x + (1 - \delta_B)^n\delta_A$ that is, the periodic point with

$$x = \frac{(1 - \delta_B)^n\delta_A}{1 - (1 - \delta_A)(1 - \delta_B)^{n+1}}$$

and the related border collision occurs when $x = X_u$:

$$\frac{(1 - \delta_B)^{n-1} \delta_A}{1 - (1 - \delta_A)(1 - \delta_B)^{n+1}} = \frac{\mu}{1 - s(1 - \delta_B)}$$

leading to the bifurcation condition

$$C_{2312^{n-1}} : \delta_A = \frac{\mu[1 - (1 - \delta_B)^{n+1}]}{(1 - \delta_B)^{n-1}[1 - s(1 - \delta_B) - \mu(1 - \delta_B)]} \quad (40)$$

Two boundaries of this family are evidenced in 7 in the (δ_B, δ_A) parameter plane at $\omega = 0.5$, for $n = 1, \dots, 10$.

For the third boundary we consider the periodic point determined via $F_l \circ F_u^{n+1}(x, y) = (x, y)$ leading to the periodic point with

$$x = \frac{\delta_A}{1 - (1 - \delta_A)(1 - \delta_B)^{n+1}}$$

and the related border collision occurs when $x = X_l$:

$$\frac{\delta_A}{1 - (1 - \delta_A)(1 - \delta_B)^{n+1}} = \frac{\mu(1 - \delta_A) + \delta_A}{1 - s(1 - \delta_A)}$$

leading to the bifurcation condition

$$C_{12^{n3}} : (1 - \delta_B)^{n+1} = \frac{\mu + s\delta_A}{\mu(1 - \delta_A) + \delta_A} \quad (41)$$

Now let us consider the second family **134ⁿ** for $n \geq 1$ and the condition $F_l^{n+1} \circ F_u(x, y) = (x, y)$ leading to $x = (1 - \delta_A)^{n+1}(1 - \delta_B)x + [1 - (1 - \delta_A)^{n+1}]$, that is

$$x = \frac{1 - (1 - \delta_A)^{n+1}}{1 - (1 - \delta_A)^{n+1}(1 - \delta_B)}$$

and the related border collision occurs when $x = X_l$:

$$\frac{1 - (1 - \delta_A)^{n+1}}{1 - (1 - \delta_A)^{n+1}(1 - \delta_B)} = \frac{\mu(1 - \delta_A) + \delta_A}{1 - s(1 - \delta_A)}$$

leading to the bifurcation condition

$$C_{134^n} : (1 - \delta_B) = \frac{\mu(1 - \delta_A) + \delta_A - [1 - s(1 - \delta_A)][1 - (1 - \delta_A)^{n+1}]}{(1 - \delta_A)^{n+1}[\mu(1 - \delta_A) + \delta_A]} \quad (42)$$

For the second boundary we consider the periodic point determined via $F_l^n \circ F_u \circ F_l(x, y) = (x, y)$ leading to

$$x = (1 - \delta_A)^{n+1}(1 - \delta_B)x + (1 - \delta_B)\delta_A(1 - \delta_A)^n + [1 - (1 - \delta_A)^n]$$

that is

$$x = \frac{(1 - \delta_B)\delta_A(1 - \delta_A)^n + [1 - (1 - \delta_A)^n]}{1 - (1 - \delta_A)^{n+1}(1 - \delta_B)}$$

and the related border collision occurs when $x = X_l$:

$$\frac{(1 - \delta_B)\delta_A(1 - \delta_A)^n + [1 - (1 - \delta_A)^n]}{1 - (1 - \delta_A)^{n+1}(1 - \delta_B)} = \frac{\mu(1 - \delta_A) + \delta_A}{1 - s(1 - \delta_A)}$$

leading to the bifurcation condition

$$C_{4134^{n-1}} : (1 - \delta_B) = \frac{\mu(1 - \delta_A) + \delta_A - [1 - s(1 - \delta_A)][1 - (1 - \delta_A)^n]}{(1 - \delta_A)^n[(\mu(1 - \delta_A) + \delta_A)(1 - \delta_A) + (1 - s(1 - \delta_A))\delta_A]} \quad (43)$$

These bifurcation curves are evidenced in 7 in the (δ_B, δ_A) parameter plane at $\omega = 0.5$, for $n = 1, \dots, 10$.

The third boundary is obtained similarly via $F_u \circ F_l^{n+1}(x, y) = (x, y)$ leading to

$$x = \frac{(1 - \delta_B)(1 - (1 - \delta_A)^{n+1})}{1 - (1 - \delta_A)^{n+1}(1 - \delta_B)}$$

and the related border collision occurs when $x = X_u$:

$$\frac{1 - (1 - \delta_A)^{n+1}}{1 - (1 - \delta_A)^{n+1}(1 - \delta_B)} = \frac{\mu}{1 - s(1 - \delta_B)}$$

leading to the bifurcation condition

$$C_{34^{n+1}} : (1 - \delta_B) = \frac{\mu - 1 + (1 - \delta_A)^{n+1}}{(1 - \delta_A)^{n+1}(\mu - s) + s} \quad (44)$$

□

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