

Introduction to the focus issue “nonlinear economic dynamics”

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Introduction to the focus issue “nonlinear economic dynamics”

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I. INTRODUCTION

The focus issue on “Nonlinear Economic Dynamics” contains a selection of papers presented at the 10th International Conference on Nonlinear Economic Dynamics-Centro Interuniversitario Crescita & Sviluppo Economico (NED-CICSE 2017) that took place on September 7–9, 2017, at University of Pisa, Italy, as well as contributions by invited scholars. NED is a biennial (odd-year) international meeting of scholars interested in the study of economic dynamics, with the aim to bring together different streams of the growing literature in this field and to stimulate a fruitful exchange between theoretical research and applications in economics. The NED one is a very active research group dealing with a diversity of subjects, methods, and applications of nonlinear dynamics in economics and social sciences. NED has a long tradition and the 2017 conference was jointly organised with the Centro Interuniversitario Crescita & Sviluppo Economico (CICSE), with the aim to promote and coordinate theoretical and applied research programmes focused on economic growth and development.

The contributions appearing in this focus issue address a variety of recent theoretical and applied topics in distinct areas such as Oligopoly Theory and Microeconomic Dynamics, Financial and Speculative Markets, Macroeconomics, Economic Growth and Development, and Social Dynamics.

Throughout the various strands of research, it is a well-established fact that the time series describing the behaviour of economic and financial variables in actual economies show irregular patterns often characterised by fluctuations, implying sharp declines and increases of the variables (with regard to an analysis of financial markets, see, for instance, Ref. 11). Since economists started working on these issues, the debate focused essentially on trying to understanding the reasons why economic variables fluctuate and the nature and causes of this irregular behaviour. A particular emphasis was then placed on knowing whether instability is a phenomenon emerging endogenously within an economic system (see, for instance, the book³⁷ for an analysis of endogenous business cycle dynamics) or it is dictated by exogenous stochastic shocks. This focus issue is strongly oriented on the former motivation to explain market dynamics and aims at showing

that the emergence of chaos is consistent with the behaviour of optimising agents and/or some heuristics widespread in the decision-making process of both firms and consumers.⁵ These developments have led in a natural way to the building on nonlinear dynamic models. Thus, the increasing understanding of the dynamics in nonlinear systems (both in continuous and discrete time) in the mathematical and physical literature has also allowed an increasing understanding of the dynamics related to models in applied economics. The fruitful interactions between scholars in the field of dynamic systems, and those expert in the modelling of economics and financial problems, have increased the knowledge in both fields, leading to some new results in the field of dynamic systems theory as well as in applied economics.

Although the introduction of nonlinear dynamics in the understanding of economic and financial markets dates back to Refs. 12, 23, 24, and 15, it is in the last three decades that these studies have been widely used by Scholars in their research works. First, the use of systems of ordinary differential equations has been introduced, followed by nonlinear systems in discrete time. The last decades have been characterised by several results regarding the dynamics of smooth maps, both in the regular and chaotic regimes. In some contributions, we can see properties which are peculiar to noninvertible n -dimensional maps for $n=1$ and $n=2$. Recall the pioneering works on this subject by Gumowski and Mira.^{17,29,30} More recently, attention has been placed to the study of non-smooth systems (the so called piecewise smooth systems, PWS for short). At the same time, systems in continuous time have also been extended to include one or more delays, and also mixed (hybrid) systems have been adopted. Moreover, economic theory has led to consider stochastic components in nonlinear models and to develop some tools for their analysis.

In the works collected in this issue, we can see different kinds of dynamic economic models, described by smooth or non-smooth systems, in continuous or discrete time, deterministic or with uncertainty in some way. In addition, we can see works including the study of local and global bifurcations occurring both in attracting sets and in the related basins of attraction (both in smooth and PWS systems). Recall that a bifurcation means a change in some invariant set (an attracting or repelling set, a frontier, and the like), and a bifurcation occurs whenever the system before/after a particular parameter configuration is no longer in 1–1

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correspondence (because of changes either in an attracting set or in its basin's structure). A local bifurcation is, roughly speaking, one which can be investigated by the local properties of the map (first derivative, Jacobian matrix, eigenvalues, and eigenvectors), while a global bifurcation is related to some global character of the map, as a homoclinic bifurcation, or a contact bifurcation.^{3,35,43} A contact bifurcation is a contact between two invariant sets of different nature, often involving the stable or unstable set of some cycle, and associated with homoclinic bifurcations. In noninvertible maps, these may be related to critical points or critical curves. In other words, the local bifurcation of attracting cycles is related to stability/instability, while the global bifurcations are related to a change in the structure of the invariant attracting sets or in the structure of a basin of attraction (often related to homoclinic bifurcations of cycles).^{14,20} The homoclinic bifurcations may be of different kinds. For example, related to a saddle cycle, and thus with a horseshoe structure, or related to an expanding cycle (i.e., with all the eigenvalues larger than 1 in modulus). While homoclinic bifurcations related to an horseshoe structure are the only possible one occurring in invertible maps, in noninvertible ones both kinds of homoclinic bifurcations may occur. The homoclinic bifurcations are, amongst the global ones, those more widely used to rigorously prove the existence of chaos, which may belong to an attracting set or to a chaotic repeller (some examples are the works of Radi and Gardini; and Naimzada and Pireddu).

In deterministic systems, the equilibria and related stability/instability are analysed not only through the study of local bifurcations but also with global bifurcations. The dynamic behaviours and the result of bifurcations in the systems are often investigated both in the phase space and in the parameter space. Recall that in the phase space, the attracting sets have a related basin of attraction, whose boundary includes unstable cycles, their stable sets, and limit points. So, considering the phase space, it must be identified which kind of attracting sets (if any) the system can have, and the related basins of attraction. Clearly, this analysis depends on the values of the parameters. Thus, it is to be performed by considering the parameter space of the system (see, for instance, the works published in this focus issue by Gardini, Sushko, and Matsuyama; Radi and Gardini; Campisi, Naimzada, and Tramontana; Panchuk, Sushko, and Westerhoff; and Lamantia and Radi).

For example, considering a one-dimensional map depending on two parameters. Then, a complete bifurcation analysis leads to regions in the parameter plane associated with qualitatively similar dynamic behaviours. In the phase space, attention must be paid to the coexistence of attracting sets, and the related basins of attraction. In the parameter space, the bifurcation curves are associated with local or global bifurcations (Radi and Gardini). Of particular relevance are codimension-two points studied by Panchuk, Sushko, and Westerhoff.

Similar arguments clearly hold for two-dimensional maps (see, for instance, Campisi, Naimzada, and Tramontana). Recall that if the map is invertible then invariant sets cannot have self-intersections, the basins of attraction are necessarily

connected, and only homoclinic bifurcation of saddle cycles can occur. Differently, in the noninvertible case, a smooth map can have in addition invariant sets with self-intersections, basins with disconnected components; homoclinic bifurcations of expanding cycles.

If the map is nonsmooth, then also kink points or switching manifolds and other sets may be responsible of the basin boundaries and related bifurcations and, as already remarked, besides smooth systems, some model may lead to PWS systems (Gardini, Sushko, and Matsuyama). The existence of a border in a PWS map, at which the map changes its definition, may lead to a collision of an invariant set of the map with the border under variation of some parameter that may cause a drastic change in the dynamic behaviors. This phenomenon leads to a border collision bifurcation (BCB for short), and many recent research efforts have focused on the classification of possible BCBs in various classes of non-smooth maps (see, e.g., the books^{4,44} and references therein). For continuous PWS maps, a generic BCB can be seen as a local bifurcation. In this case, many essential results on the classification have been obtained by means of the related normal forms. For a one-dimensional continuous map with one border point, a complete classification of BCBs has been proposed using a one-dimensional Border Collision normal form represented by the skew tent map (Refs. 25, 28, 33, 45, and 41 and references therein). For two-dimensional continuous non-smooth systems, the known results are still very few, mainly related to the two-dimensional Border Collision normal form defined by two linear maps (first used in Ref. 32). Differently, for discontinuous maps, a BCB is not a local phenomenon, because crossing the switching manifold a trajectory is subject to a jump which may be small or not, and the result depends on the global properties of the map, see the work of Gardini, Sushko, and Matsuyama for the study of a system of this kind.

Some of the works published in the NED focus issue dealt with topics that are now classical in the nonlinear economic dynamics literature. In particular, with regard to the Oligopoly Theory and Microeconomic Dynamics, since the works of Refs. 40 and 36, a literature was developed that has focused on several aspects of strategic interactions in articles that essentially accounted for the analysis of (continuous time or discrete time) dynamics of games. In this regard, we summarise here the works of Refs. 9 and 26.

- Cerboni-Baiardi and Naimzada (Imitative and best response behaviors in a nonlinear Cournotian setting). The authors of this work consider a deterministic nonlinear dynamic oligopoly model with quantity setting firms, where the marginal willingness to pay of consumers is captured by an isoelastic demand curve. Firms choose output in different way: some of them are rational-like players and adjust their production towards the direction dictated their own best reply; the remaining part is comprised imitators. The authors find that both the total number of players and the relative fraction of imitators affect stability outcomes of the Cournot-Nash equilibrium by playing an ambiguous role and instability thresholds can be observed.

From the side of the global analysis, the articles show a variety complex scenarios, ranging from multistability to chaos.

- Lamantia and Radi (Evolutionary technology adoption in an oligopoly market with forward-looking firms). In this work, the authors study an evolutionary game described by a two-dimensional piecewise differentiable map. In this game, firms have to choose the kind of (linear) technology with fixed and variable costs to be adopted in production. One of these technologies implies higher marginal costs. The authors show that if firms are forward-looking, the coexistence of mechanism to select technologies, adaptive expectations, and best reply dynamics can generate dynamics non-convergent towards more efficient technologies. Results show that different attractors (with distinct economic characteristics) can coexist. The last part of the work concentrates on the definition of similarities and differences with the deterministic model when the size of the market is affected by a stochastic component.

The study of nonlinear dynamics in macroeconomic models has a long and important tradition originated in the leading (already cited) works of Refs. 12, 23, 24, and 15 aimed at studying business cycle problems. The contributions to this focus issue belonging to the field of Macroeconomics are Refs. 8, 13, 22, and 31. A brief summary of them now follows:

- Cavalli, Naimzada, Pecora, and Pireddu (Agents' beliefs and economic regimes polarization in interacting markets). By taking the 2008–2009 crisis as a starting point, the authors build on a macroeconomic model to integrate real market and financial market behaviours. It is assumed that agents are not able to observe the true underlying fundamental in the stock market and their beliefs are biased by either optimism or pessimism. These beliefs allow agents for selecting the best performing strategy in an evolutionary perspective. The functioning of the real market is based on a nonlinear version of the multiplier-accelerator framework, where there is a nonlinear bounded investment function. The main aim of this contribution is to study and characterise the effects of real and financial market integration. Results go into the direction of showing that strongly polarized beliefs can be responsible for multiple equilibria with high or low income levels. These equilibria mimic the optimistic or pessimistic nature of the agents' beliefs. In addition, by making use of a mixture between analytical and numerical tools, the authors show the emergence of complex attractors characterised by endogenously fluctuating pessimistic and optimistic prices and different levels of national income. They also add a stochastic perturbation into the analysis to capture in a better way the functioning of financial markets (e.g., bubbles and crashes for stock prices).
- Gardini, Sushko, and Matsuyama (2D discontinuous piecewise linear map: Emergence of fashion cycles). In this work, the authors analyse a model with two types of agents: conformists, who want to act or look the same with the others, and nonconformists, who want to act or look different from the others. The dynamics are described by a two-dimensional discontinuous piecewise linear map, for which the authors classify new kinds of bifurcation structures in the parameter space. In particular, they show that the map is characterised by possibly several coexisting attracting cycles and show both analytically and numerically how the period of these cycles vary depending on parameter values. In the last part of the work, the authors also show the link between the discrete time and continuous time versions of the model. The study identifies some properties that seem to be extended to other fields.
- Jungeilges, Ryazanova, Mitrofanova, and Popova (Sensitivity analysis of consumption cycles). In this contribution, the authors propose a model to highlight how a stochastic component in aggregate consumption works in a non-linear deterministic framework. In particular, the authors show the similarities and differences in consumption patterns in these two different frameworks.
- Naimzada and Pireddu (Fashion cycle dynamics in a model with endogenous discrete evolution of heterogeneous preferences). The authors present a dynamic general equilibrium model to study the joint effect of a mechanism of price formation and an endogenous mechanism that describes the interactions and the evolution in the population of groups of heterogeneous agents affected by bandwagon and snob behaviours. In particular, the authors identify cases of multistability and the onset of cyclical phenomena.

With regard to problems of Economic Growth and Development, the work of Ref. 16, analysed the interactions between population dynamics and economic growth in a continuous-time Solow-like⁴² growth model, whereas the works of Refs. 18 and 27 were framed in the Ramsey-like growth literature.³⁹ The former work made a comparison between continuous-time and discrete-time versions of the Ramsey model. The latter one instead considered a growth model with random pollution externalities. The summary of these contributions can be found below:

- Gori, Guerrini, and Sodini (Time delays, population, and economic development). In this work, the authors stress the importance of the link between economic and demographic variables as a relevant explanation for economic development of nations by studying a Solow-like growth model that may explain cycles in income and total population. This is done by augmenting the standard Solow model with time delays in both technology and population dynamics. This assumption has allowed the authors to account for a lag from the time the initial investment was carried out to the time it actually becomes productive and a lag from the time people were born to the time they will be economically active (belonging to the working population) and sexually active. The Solow model, therefore, becomes able to explain the convergence towards a high equilibrium or a Malthusian trap as well as long-term fluctuations in income and population.
- Guzowska and Michetti (Local and global dynamics of Ramsey model: from continuous to discrete time). The authors of this work highlight the importance of the choice

of the modelling approach used (i.e., discrete time or continuous time) in determining the stability properties of the system. In the continuous-time version of the model of Ramsey, the steady-state equilibrium is locally saddle-path stable and there is monotonic convergence. Differently, in the corresponding discrete-time counterpart of the model, the steady-state equilibrium may be unstable and oscillatory dynamics may occur. This work studies a hybrid discretisation of the Ramsey model that preserves the main properties of the continuous-time counterpart. Results show that by using a general backward and forward discretisation, the main properties of the continuous-time version of Ramsey hold are preserved. The authors also identify a small region where the two systems generate different qualitative long-term dynamics.

- La Torre, Marsiglio, and Privileggi (Fractal attractors in economic growth models with random pollution externalities). The authors analyse a discrete time two-sector stochastic growth model where the production of goods and human capital are affected by random shocks generated by both “classical” economic mechanisms (i.e., productivity and factor shares) and a pollution externality. The article characterises the dynamics in the decentralised economy and identifies a suitable parameter configuration able to generate the Barnsley’s fern as the attractor of the log-linearised dynamic system. It also shows how structures and properties usually found in physical and/or biological contexts also arise in social economic frameworks.

The study of Financial and Speculative Markets in a dynamic framework has seen a tremendous increase in the last decades. The works published in this focus issue belonging to this field contribute to the advancement of our knowledge of the functioning of asset price formation as well as the behaviour of heterogeneous agents in the market. The works are Refs. 7, 19, 21, and 34. They are summarised below:

- Campisi, Naimzada, and Tramontana (Local and global analysis of a speculative housing market with production lag). This work takes into account a speculative housing market as in Ref. 10 with heterogeneous interacting agents, whose dynamics is characterised by a two-dimensional nonlinear discrete time map. The basic model already shows complicated dynamics for some parameter values. The authors then provide some extensions by introducing time lags in the supply side of the market and then by considering different expectations formation mechanisms of economic agents in two distinct scenarios. The former scenario accounts for naive expectations instead of perfect foresight. The latter instead takes into account a mixture between Ref. 10 and the one proposed by the author in their own basic model. The main aim is to explain through analytical and numerical techniques the emergence of instability in the housing market. Finally, the authors also provide a global study by analysing the structure of the basins of attraction. The article may help in providing reasons for the observed instability in the housing market.

- He, Li, and Wang (Time-varying economic dominance in financial markets: A bistable dynamics approach). This work builds on a stochastic continuous-time financial market model with heterogeneous agents. The main aim is the study of the behaviour of multi-assets traded by fundamental and momentum investors by providing a mechanism able to generate time-varying dominance between fundamental and non-fundamental in a financial market. The existence of constraints in investments may imply the existence of multiple long-term scenarios, characterised by a locally stable fundamental steady-state equilibrium and a locally stable limit cycle around the fundamental. The model is able to generate some financial market stylized facts, such as market booms and crashes and the theoretical results and in line with the evidence based on the U.S. financial market behaviour.
- Huang and Huang (Connectionist agent-based learning in bank-run decision making). This work aims at studying the probability of bank-run (PBR for short). In this regard, it provides on simulation results to explain the nonlinear dynamic probabilities of bank runs based on the global games approach. This is done by assuming that heterogeneous agents hold highly correlated but non-identical beliefs about the true payoffs. Results show that agents can be affected by the cognitive-affective network to react to bad news, which may actually explain the bank-run.
- Panchuk, Sushko, and Westerhoff (A financial market model with two discontinuities: Bifurcation structures in the chaotic domain). In this paper, the authors explore some of the dynamic properties of a financial market model characterised by the presence of agents with different beliefs about the functioning of the market (chartists and fundamentalists). In particular, the authors investigate some properties of a piecewise linear one-dimensional map with two discontinuities and clarify the mechanism for the emergence and formation of some phenomena of multiband chaotic attractors and coexistence of chaotic attractors.

In the last decades, there has been a burgeoning interest in the study of social phenomena and the interactions between social studies and economics. The works of this focus issue belonging to Social Dynamics aim at clarifying the evolution of the behaviours of both human and non-human populations by identifying the hidden structures behind the data. The works are Refs. 1, 2, 6, and 38. A summary of these works now follows:

- Antoci, Bellanca, Galdi, and Sodini (Narrative dynamics in social groups: A discrete choice model). In this article, the authors are concerned with the study of the interactions between two populations with different characteristics in terms of norms, worldviews and beliefs (i.e., narratives). In particular, they stress the conditions under which there exist coexistence of these (different) populations with distinct characteristics or situations in which one of the two population prevails over the other. Result also show that under what conditions the encounter between the two populations can favour the emergence of complex evolutions in the adoption of the narratives.

- Antoci, Galeotti, Russu, and Sacco (The cultural implications of growth: modeling nonlinear interaction of trait selection and population dynamics). This work takes into account a nonlinear model to study the interaction between trait selection and population dynamics. After the analysis of some properties of the model, the authors provide some numerical simulations of the fine-grained structure of alternative dynamic regimes. The role of the parameters that govern the reinforcement/corruption of maladaptive vs. adaptive traits is of special importance in determining the model's dynamic evolution. A possible implication is the need to pay special attention to the structural forces that may favor the emergence and consolidation of maladaptive traits in contemporary socio-economies.
 - Buscema, Sacco, Della Torre, Massini, Breda, and Ferilli (Theory of impossible worlds. Toward a physics of information). In this paper, the authors introduce an approach for the fusion of datasets in terms of attributes and observations, even when they are not related between them. Starting from datasets representing independent worlds, the authors show that by letting them communicate with each other allows getting predictive results and knowledge of the different worlds. More specifically, the authors suggest a new heuristic to reveal the hidden structure in the data and to test it on datasets of different nature.
 - Radi and Gardini (A piecewise smooth model of evolutionary game for residential mobility and segregation). In this paper, the authors propose a Schelling-type model to study problems of residential segregation. Different from the standard setting in which the model dynamics are linked to the different levels of tolerance of the agents, in the present work it is assumed that agents benefit from integration. The authors show, similar to the original model of Schelling, that an equilibrium with segregation is always stable, whereas there exist equilibria with agents' integration and they are stable only for specific configurations of parameters. In particular, by using the global analysis techniques for the study of piecewise smooth maps, results show that some public policy interventions may be in order to lead the system towards situations with low levels of segregation.
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