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Hicks' trade cycle revisited: cycles and bifurcations

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Abstract

In the Trade Cycle, Hicks introduced the idea that endogenous fluctuations could be coupled with a growth process via nonlinear processes. To argue for this hypothesis, Hicks used a piecewise-linear model. This paper shows the need for a reinterpretation of Hicks' contribution in the light of a more careful mathematical investigation. In particular, it will be shown that only one bound is needed to have non explosive outcome if the equilibrium point is an unstable focus. It will also be shown that when the fixed point is unstable the attracting set has a particular structure: It is a one-dimensional closed invariant curve, made up of a finite number of linear pieces, on which the dynamics are either periodic or quasi-periodic. The conditions under which the model produces periodic or quasi-periodic trajectories and the related bifurcations as a function of the main economic parameters are determined.

Keywords: Business cycle models; Piecewise-linear maps; Tongues of periodicity; Bifurcation diagram

1. Introduction

The accelerator-multiplier class of models in economics dates back to [1] (the origin of these models is described by [2]). Samuelson's article, which is very short and formal, provides a linear second-order difference equation that produces either stable or diverging trajectories, but no self-sustained business cycles except at very particular values of the parameters, and in such cases it produces cycles of fixed period and amplitude. Something which is at odds with the empirical evidence. In [38], Hicks makes some important changes to the model: by adding a floor and a ceiling to a linear model, he formulates

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a piecewise-linear framework that can produce bounded oscillations. Moreover, the model shows how a growth process can be coupled with the business cycles. Hicks' piecewise-linear model attracted a plethora of comments (see. e.g. [3–13]). Recent literature (e.g. [11,12,14]) has shown that "quasi-periodic attractors" can occur in the basic Hicks model, but its mathematical properties have not been fully investigated.

The use of piecewise-linear models is not new in the applied context, specially in economics (see e.g. [15,16]), but also in engineering (see e.g. [17–23]). The phenomena and bifurcations occurring in piecewise-linear maps are close related to those occurring in piecewise-smooth maps. The main feature, as it will appear also in the model studied in this paper, is that the bifurcations are of so called "border-collision" type (following the terminology introduced in [24] (see also in [25]), or "C-bifurcation" following an earlier definition (see for example in [26]). Applied models are often described also by piecewise-smooth maps, and their study is widely increased in the last years (see e.g. [27–29]).

In the following, having introduced the basic ideas of Hicks' trade cycle model, we will deal in more detail with its mathematical properties. In Section 3 we shall review the properties of the basic linear model. In Section 4 we shall consider the piecewise-linear model with only a "floor", showing for which parameter values bounded and economically feasible oscillations (periodic and quasi-periodic) occur and demostrating the stucture of the attracting set and the involved bifurcations as a function of the parameters of the business cycle. The equivalent dynamical investigation is performed in Section 5 for the case of a piecewise-linear model with only a "roof". In Section 6 we shall consider the piecewise-linear model with two constraints, a "floor" and a "roof", putting to evidence the stucture of the attracting set and its bifurcations, as descibed by the bifurcation diagram.

2. The model

In the Samuelson–Hicks business cycle model, investment is determined by the growth in income, through the principle of acceleration. More precisely, investment is taken to be proportional to the rate of change in income, or $I_t = k(Y_{t-1} - Y_{t-2})$, where I_t denotes investment in time period t, Y_{t-1} and Y_{t-2} income one and two periods back, respectively, and the parameter k (>0) is the accelerator (this parameter is the technical coefficient for capital, it is the volume of capital needed to produce one unit of goods during one time period). In past empirical measurement a realistic estimate was considered to be about 2–4. Obviously, this numerical value depends on the choice of the length of the basic time period for the model. It is important in this context to realize that economics distinguishes between concepts of stocks and flows. Stocks are variables which are meaningful by reference to a point of time, such as a stock of capital or the labour force. Flows, on the other hand, are concepts referring to time periods, defined by a beginning and a finishing point on the time scale. To the flows belong investments, consumption, and income/production. A produced quantity, given inputs in form of capital and labour stocks, obviously depends on the length of the time period. During half the period chosen, output is half, during double the period it double that for a period. In some economic models the period length may be a bit ambiguous. This is, however, not the case in macro economic models, which arose in connection with Keynesian theory and national accounting in the 1930's. The reference period is always one year, and whenever accounts or forecasts are made for shorter periods, they are always evaluated to a yearly basis. Hence, the time period in macro economic models such as the present is always the calendar year.

As for the consumption function, a Robertsonian formulation is assumed, $C_t = bY_{t-1}$ where $0 < b < 1$ is the marginal propensity to consume (realistic values belongs to the range (0.6, 1)). Any additional constant term is absorbed in the autonomous expenditure term to be introduced below. This is composed of all the expenditures not dependent on the business cycle, i.e. government expenditures, non-induced investments, due to for instance innovations, and any non-income dependent consumption expenditures. It is common knowledge that consumers need to consume something to survive even if they earn nothing. When population/labour force grows, then the autonomous expenditures for this minimum subsistence consumption must grow along with the labour force.

The Keynesian income formation identity $Y_t = C_t + I_t$, allows us to obtain a simple recurrence relation in the income variable: $Y_t = (b+k)Y_{t-1} - kY_{t-2}$. This has a closed form solution and models the business cycles mechanism. The solution is the product of an exponential growth, or decline factor, and, in case of complex roots to the characteristic equation, a harmonically varying cyclic function. The cyclical element provides a possibility for explaining recurrent cycles, but the exponential factor poses a problem. As it is an unlikely coincidence for the exponential factor to become a constant, the cycles are either exponentially increasing or decreasing in amplitude. If they are decreasing, there is no dynamic theory at all, because the theory only explains how the system goes to eternal equilibrium. [30] suggests that external shocks have to be introduced to keep an oscillating system going, even when it is damped. The dynamical system itself would then provide for a periodicity. Hicks chooses another possibility: to introduce bounds for the linear accelerator. At the same time [31] tries to model a growth process within the Keynesian tradition. As it is well known, the Harrodian equilibrium growth path is unstable, and so is the business cycles mechanism in the empirically relevant parameters space of the multiplier-accelerator mechanism. Hicks therefore develops a Harrodian multiplier-accelerator framework with ceilings and floors, thereby constraining the instability problems and producing cyclical behaviour (technically he moves from a linear to a piecewise-linear system). According to Hicks, when income decreases with fixed proportions technology, more capital could be dispensed with than normallywhat disappears during one period through natural wear. As the capitalists are assumed not actively to destroy capital, just abstain from reinvesting, there is a lower bound to disinvestments, the floor (i.e. net investment cannot be negative). Likewise, if income is rising, other factors of production, labour force or raw materials, become limiting, then there would be no point in pushing investment any further, so there is also a ceiling to investments (full employment, e.g.). Accordingly, a lower and an upper bound to investment limit the action of the linear acceleration principle. This in fact makes the investment function piecewise-linear, i.e. nonlinear. The result could be self-sustained oscillations of limited amplitude. In his book, Hicks introduces autonomous expenditures which are not constant, but may be growing exponentially, i.e. $A_t = A_0(1+g)^t$, where g is a given growth rate and A_0 a positive constant. Summarizing we have:

$$Y_t = C_t + I_t$$

$$C_t = bY_{t-1}$$

$$I_t = I'_t + I''_t = k(Y_{t-1} - Y_{t-2}) + A_0(1+g)^t$$

so that

$$Y_t = bY_{t-1} + k(Y_{t-1} - Y_{t-2}) + A_0(1+g)^t \tag{1}$$

In order to obtain a sensible model, also the floor and ceiling must be regarded as shifting in time, and likewise labour force and other resources are supposed to increase. Hicks does not present a complete formal model for the case with growth in autonomous expenditures, as well as in the bounds in terms of floor and ceiling. The point of departure for the present study is the by now largely accepted interpretation suggested by [9] (cf. also [7,11,12]). In this version, the lower limit (the floor) applies to the induced investment while the upper limit is applied to total expenditures (and related to full employment). Hence, we consider the ceiling: if $bY_{t-1}+k(Y_{t-1}-Y_{t-2})+A_0(1+g)^t > B_0(1+g)^t$ then $Y_t = B_0(1+g)^t$, and the floor: if $k(Y_{t-1}-Y_{t-2}) < -a_t = -a(1+g)^t$ then $I'_t = -a(1+g)^t$, that is, $Y_t = bY_{t-1}+(A_0-a)(1+g)^t$, where B_0 and a are positive constants.

Let us introduce the following definitions for the three parts of the model, namely, for the "ceiling", main model and "floor":

$$U \stackrel{\text{def}}{=} B_0(1+g)^t---\text{the upper bound, or "ceiling"};\; G(Y_{t-1}, Y_{t-2}) \stackrel{\text{def}}{=} bY_{t-1} + k(Y_{t-1}-Y_{t-2}) + A_0(1+g)^t---\text{the main model};\; L(Y_{t-1}) \stackrel{\text{def}}{=} bY_{t-1} + (A_0-a)(1+g)^t---,\textit{the lowerbound}, \text{ or "floor"}. \tag{2}$$

Thus, in the last section we shall consider a family of *non-autonomous* second-order piecewise-linear continuous difference equations given by:

$$Y_t = \begin{cases} U, & \text{if } G(Y_{t-1}, Y_{t-2}) > U; \\ G(Y_{t-1}, Y_{t-2}), & \text{if } L(Y_{t-1}) \le G(Y_{t-1}, Y_{t-2}) \le U; \\ L(Y_{t-1}), & \text{if } G(Y_{t-1}, Y_{t-2}) < L(Y_{t-1}); \end{cases} \tag{3}$$

which depends on six real parameters A_0, B_0, a, b, k, g, such that $B_0 > A_0 > a > 0, k > 0, 0 < b < 1$, and $0 < g < 1$. We are only interested in positive values of the independent variable, i.e. $Y_t > 0$. To prepare for this analysis, we shall investigate the dynamic behavior of the main model when only the floor or only the ceiling is present.

3. Linear model

Let us first consider the main linear model:

$$Y_t = G(Y_{t-1}, Y_{t-2}), \tag{4}$$

where the function $G(Y_{t-1}, Y_{t-2})$ is given in (2). The model (4) is a *non-autonomous* second-order linear difference equation. By using the change of variable:

$$Z_t = \frac{Y_t}{(1+g)^t}, \tag{5}$$

we get an *autonomous* second-order linear difference equation, i.e.

$$Z_t = \frac{b+k}{1+g} Z_{t-1} - \frac{k}{(1+g)^2} Z_{t-2} + A_0, \tag{6}$$

whose *equilibrium* point is given by:

$$Z^* = \frac{A_0(1+g)^2}{(1+g)^2 - b(1+g) - kg}. \tag{7}$$

This means that the original Eq. (4) has the growing solution:

$$Y_t = Z^*(1+g)^t.$$

Clearly, the linear second-order difference equation can also be written as a two-dimensional linear system of equations, by defining:

$$x_t = Z_{t-2}; \quad y_t = Z_{t-1}. \tag{8}$$

Using the symbol "′" to denote the unit-time advancement operator, we get a two-dimensional map F_2 of the form:

$$F_2 : \begin{cases} x' = y; \\ y' = \dfrac{-kx}{(1+g)^2 + ((b+k)y/(1+g)) + A_0}; \end{cases} \tag{9}$$

whose fixed point is $E_2 = (Z^*, Z^*)$ (7). The stability of the equilibrium solution can be studied considering the solution of the characteristic equation associated with (6), or with the eigenvalues of the Jacobian matrix J_2 of (9). It is easy to check that the eigenvalues $\lambda_{1,2}$ of the characteristic equation:

$$\lambda^2 - \left(\frac{b+k}{1+g}\right)\lambda + \frac{k}{(1+g)^2} = 0 \tag{10}$$

are both less then 1 in modulus if

$$\det J_2 = \lambda_1\lambda_2 = \frac{k}{(1+g)^2} < 1.$$

Thus, we can state the following:

Proposition 1. *Let*

$$0 < k < k^* \stackrel{\text{def}}{=} (1+g)^2, \tag{10}$$

then the fixed point $E_2 = (Z^*, Z^*)$ of the map F_2 is stable.

It is also easy to see that the eigenvalues $\lambda_{1,2}$ are complex conjugate as long as:

$$b < 2\sqrt{k} - k \stackrel{\text{def}}{=} \varphi(k). \tag{11}$$

Thus, we can draw the two curves $k = k^*$ (10) and $b = \varphi(k)$ (11) in the (b, k)-parameter plane with fixed values of the other parameters of the model, noticing that if (11) holds then the fixed point E_2 is a focus, either stable (for $k < k^*$) or unstable (for $k > k^*$), otherwise it is a node. Clearly, for $k = k^*$ and $b < \varphi(k)$ the fixed point E_2 is a center (see Fig. 1).

As it is well known (see e.g. [9,32]), for the linear model (6) we can write the analytic solution Z_t making use of the eigenvalues $\lambda_{1,2}$ of the Jacobian matrix J_2, and in terms of the income variable we have $Y_t = Z_t(1+g)^t$.

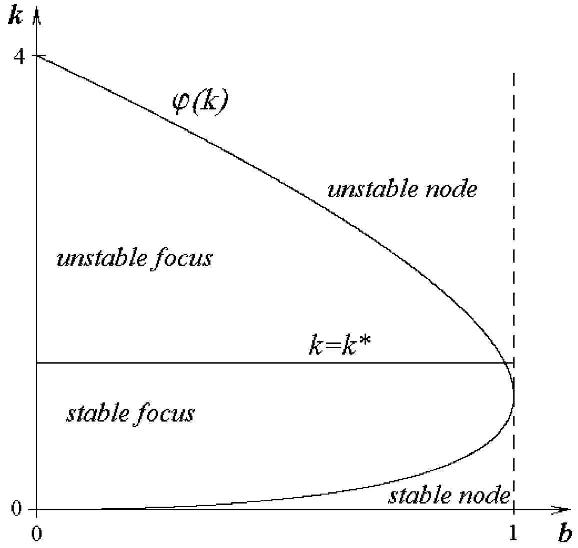


Fig. 1. Stable and unstable regions for the fixed point E_2 in the (b, k)-parameter plane.

4. Piecewise-linear model with a lowed bound

The dynamic behavior of a linear system is well known, so that Z_t becomes explosive (divergent to infinity) when Z^* is unstable. For this reason we introduce a constraint to the dynamics, as already explained in Section 2. Let us start with a lower bound, i.e. a "floor". As we shall see, this constraint is enough to give bounded dynamics when the fixed point becomes unstable. So, let us assume that:

$$Y_t = G(Y_{t-1}, Y_{t-2}), \quad \text{if } G(Y_{t-1}, Y_{t-2}) > L(Y_{t-1});$$

otherwise we have:

$$Y_t = L(Y_{t-1}),$$

where the functions $G(Y_{t-1}, Y_{t-2})$ and $L(Y_{t-1})$ are given in (2).

By using, as before, the variable transformations (5) and (8), we get a two-dimensional piecewise-linear map T_L given by:

$$T_L : \begin{cases} (x', y') = F_2(x, y), & \text{if } (x, y) \in R_2; \\ (x', y') = F_3(x, y), & \text{if } (x, y) \in R_3; \end{cases}$$

where F_2 is given in (9), and

$$F_3 : \begin{cases} x' = y; \\ y' = \dfrac{by}{(1+g) + A_0 - a}; \end{cases} \tag{12}$$

$$R_2 = \{(x, y) : x > 0, y > 0, y \geq r_2(x)\}; \qquad R_3 = \{(x, y) : x > 0, 0 < y < r_2(x)\};$$

$$r_2(x) = \frac{x}{1+g} - \frac{a(1+g)}{k}. \tag{13}$$

We note that for the economic interpretation of the model we require the state vector (x, y) to belong to the positive quadrant of the phase plane, i.e. $x > 0$, $y > 0$. The following conditions on the parameters are also to be fulfilled: $B_0 > A_0 > a$.

The straight line $y = r_2(x)$ (13) separates the two regions R_2 and R_3 where the map T_L is defined by two different linear maps, F_2 and F_3, respectively. The straight line $y = r_2(x)$ intersects the vertical axis in a point $(0, -a(1+g)/k)$ and has slope m_L with $0 < m_L < 1$, so that the fixed point E_3 of the linear map F_3 $(x = y = (A_0 - a)/(1 - b/(1+g))$ which belongs to the diagonal inside the region R_2) is not a fixed point of T_L. We conclude that the only fixed point of T_L is the fixed point E_2 of the map F_2, which is stable for $0 < k < k^*$ (10).

Clearly, at the bifurcation value $k = k^*$ (10) the fixed point becomes a center. For $k > k^*$ and $b < \varphi(k)$ (11), the fixed point is an unstable focus, but now, due to the "floor", the dynamics are bounded. Initial conditions in a neighborhood of E_2 spiral away (iterated by F_2), and enter the region R_3. To the points in the region R_3 the linear map F_3 applies. The eigenvalues of the Jacobian matrix of F_3 are given by:

$$\lambda_1^3 = 0; \qquad \lambda_2^3 = \frac{b}{1+g}, \quad 0 < \lambda_2^3 < 1.$$

The zero eigenvalue entails that F_3 maps certain straight lines into points. The image by F_3 of the whole region R_3 is a half line which is the image under T_L of the constraint $y = r_2(x)$, $x > 0$. Let us redefine such particular lines as follows:

$$\begin{aligned} &\mathrm{LC}_{-1}^L = \{(x, y) \in \mathbb{R}^2 : y = r_2(x)\}; \\ &\mathrm{LC}_0^L = T_L(\mathrm{LC}_{-1}^L) = \left\{(x, y) \in \mathbb{R}^2 : y = \frac{bx}{(1+g) - a + A_0}\right\}; \end{aligned} \tag{14}$$

calling the straight line LC_0^L *critical line* of T_L (see Fig. 2a) . As we shall see, the line LC_0^L, as well as its images $\mathrm{LC}_i^L = T_L^i(\mathrm{LC}_0^L)$, $i > 0$, (also called critical lines) play an important role in the description of the dynamics of T_L.

The linear map F_3 maps the critical line LC_0^L into itself. Thus, once a point has entered the region R_1, it will be mapped onto LC_0^L and then in a few applications of F_3 it will enter the region R_2 again, where the map F_2 comes to apply.

Moreover, it is easy to check that any trajectory of T_L with initial point in the positive quadrant of the phase plane belongs to the positive quadrant (i.e. it never enters the negative quadrants). In fact, as it was already mentioned, $T_L(R_3) = \{\mathrm{LC}_0^L, x > 0\}$. The trajectory of any point $(x_0, y_0) \in R_2$ spirals away around E_2 in the clockwise direction and, after a finite number of iterations, enters the region R_3. So, it is enough to show that the border lines of R_2, i.e. the segment S_1 and the half line $\{x = 0, y > 0\}$ (see Fig. 2a) are mapped into the positive quadrant: $F_2(S_1) = S_2 \subset \{x = 0, y > 0\}$, $F_2(\{x = 0, y > 0\}) = \{y = ((b+k)x/(1+g)) + A_0, x > 0\} \subset R_2$.

When E_2 is an unstable node then the trajectories are all divergent. When the fixed point E_2 is an unstable focus, using the properties of the critical lines (see [33], and references therein) we can define an invariant closed bounded area $\mathcal{A}$ in the positive quadrant of the phase plane, whose boundary $\partial\mathcal{A}$ is made up by a finite number m of segments of critical lines LC_i^L, $i = 0, \dots, m$. *Invariant* means that $T_L(\mathcal{A}) = \mathcal{A}$. The set $\mathcal{A}$ is called an *absorbing area* which means that any point of the positive quadrant

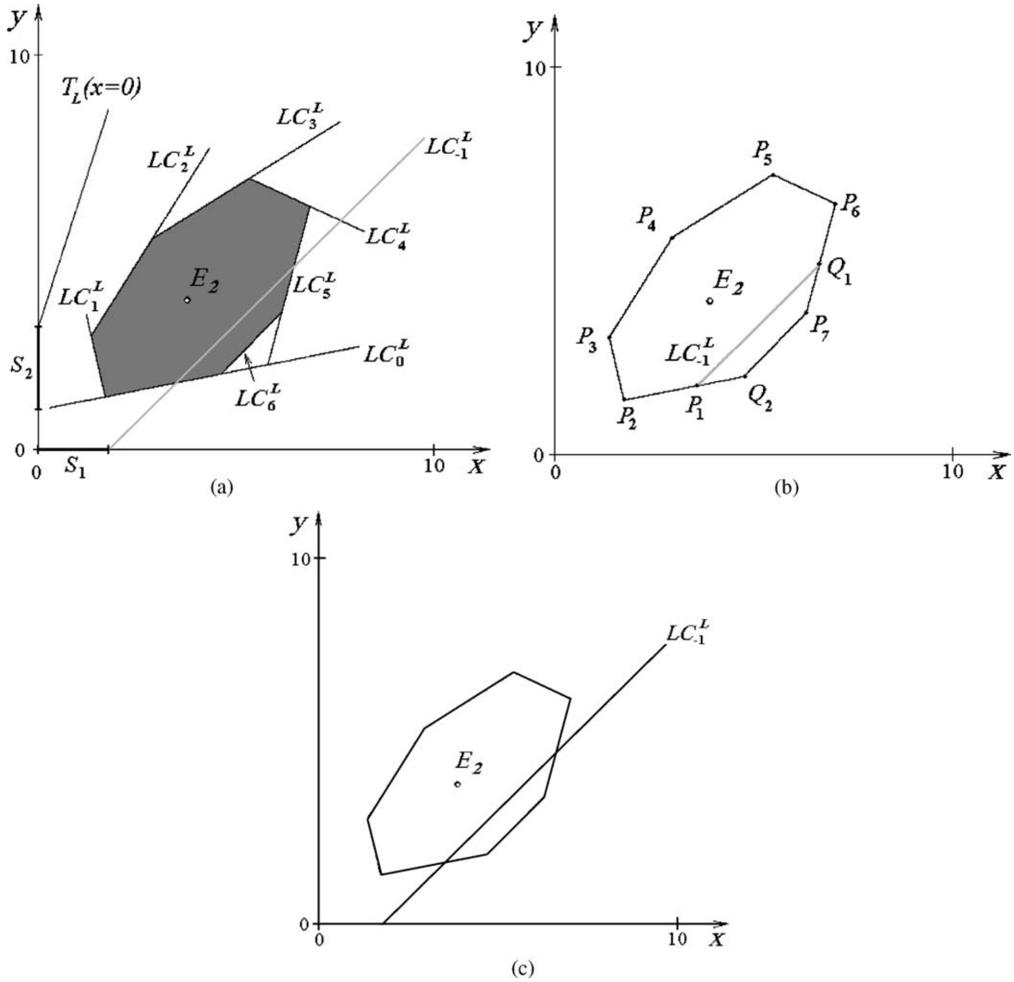
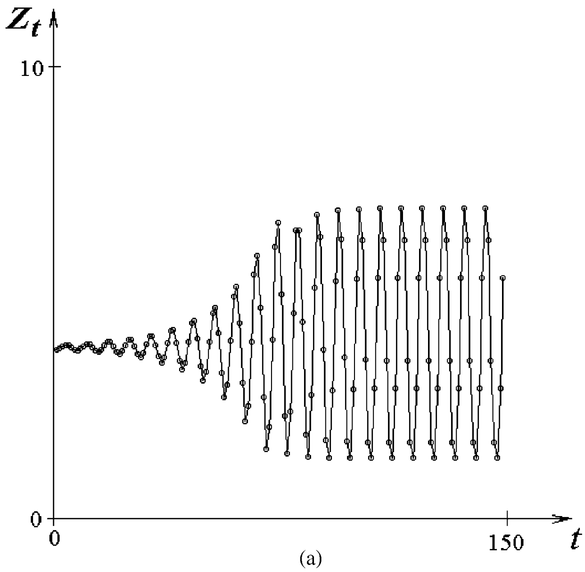


Fig. 2. (x, y)-phase plane of the map T_L for $a = 2$, $A_0 = 3$, $b = 0.2$, $g = 0.01$ and $k = 1.144$: (a) an absorbing area $\mathcal{A}$ (grey region) made up by seven segments of LC_i^L, $i = 0, 1, \ldots, 6$; (b) the boundary $\partial\mathcal{A}$ of $\mathcal{A}$ is $\partial\mathcal{A} = \cup_{i=1}^{7} T_L^i(\overline{P_1 Q_1})$, where $\overline{P_1 Q_1}$ is the segment of LC_{-1}^L belonging to $\mathcal{A}$; (c) an attracting cycle of high period belonging to $\partial\mathcal{A}$.

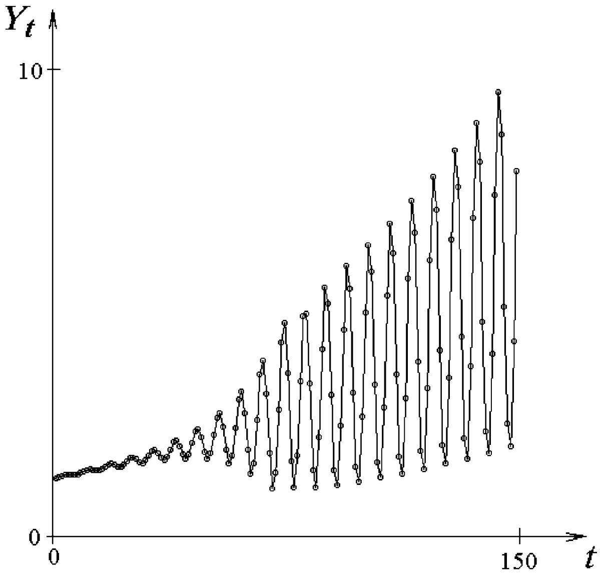
is mapped into $\mathcal{A}$ in a finite number of iterations. Moreover, as we shall see in our particular model, the attracting set of the map T_L belongs to the boundary $\partial\mathcal{A}$ of $\mathcal{A}$.

An example of the determination of such an absorbing area is given in Fig. 2a. Here six images of a segment of LC_0^L are shown so that using seven segments (of LC_i^L, $i = 0, 1, \ldots, 6$) we obtain an absorbing area $\mathcal{A}$. We remark that the area $\mathcal{A}$ intersects the line LC_{-1}^L (the "floor" $y = r_2(x)$) in a segment, say, $\overline{P_1 Q_1}$, and we have that the boundary is $\partial\mathcal{A} = \cup_{i=1}^{7} T_L^i(\overline{P_1 Q_1})$ (see Fig. 2b).

The attracting set of T_L must belong to the boundary of $\mathcal{A}$. This is due to the fact that the map T_L is *linear and invertible* in the region R_2, where the fixed point is a repelling focus and thus no attracting sets can belong to the interior of $\mathcal{A}$. The only points in which the map T_L is noninvertible belong to the critical curve LC_0^L and its images. The trajectory of any initial condition in the positive quadrant rapidly



(a)



(b)

Fig. 3. Temporal variation of Z_t (a) and of Y_t (b) for a trajectory having the initial condition close to the unstable fixed point. Here $a = 2$, $A_0 = 3$, $b = 0.2$, $g = 0.01$ and $k = 1,144$.

converges to a unique attracting set of the map T_L, which belongs to $\partial\mathcal{A}$. In Fig. 2c we show the trajectory, which is a periodic orbit of a high period. It is clear that such a bounded solution for Z_t (see the trajectory in time in Fig. 3a) becomes a growth path in the true variable Y_t (see Fig. 3b), and similar behavior occurs for other values of the parameters.

Let us now describe different kinds of attracting sets (belonging to $\partial\mathcal{A}$) and their bifurcations which can occur in the system depending on the parameters. Note that the most important parameters are k

and b. Variations of other parameters in their specified ranges have only a scaling effect. So, we can fix $A_0 = 3$, $a = 2$ and $g = 0.01$, and study a two-dimensional bifurcation diagram of the map T_L in the (b, k)-parameter plane.

We will show that for $k > k^*$ an attracting set of the map T_L can be either an attracting cycle of some period, or a quasi-periodic trajectory belonging to $\partial\mathcal{A}$. The dynamics of T_L are restricted to a one-dimensional invariant set ($\partial\mathcal{A}$, which is homeomorphic to a circle), so that we can use a *rotation number* for the map T_L similar to the rotation number for the circle maps [32]: It can be rational, say p/q, or irrational, being the average rotation of any initial point on $\partial\mathcal{A}$ around the repelling fixed point E_2.

Let us denote an attracting cycle with rotation number p/q by $\gamma_{p/q}$, where q denotes the period of the cycle and p denotes how many turns must be done around the fixed point in order to have the whole period q.

In Fig. 4 we show in the (b, k)-parameter plane the so-called *tongues of periodicity* in which the piecewise-linear map T_L has attracting cycles $\gamma_{p/q}$, $6 \leq q \leq 45$: each degree of grey corresponds to a different period. The curve $b = \varphi(k)$ (11) is also shown, such that for $b > \varphi(k)$ the fixed point E_2 becomes an unstable node and all the trajectories (except for the fixed point) go to infinity. The white region below the curve $b = \varphi(k)$ corresponds either to attracting cycles of period large then 45, or to quasi-periodic trajectories.

The number of different tongues we may get is infinite (countable). For example, the tongues of period q, $q = 6, 7, \ldots, 12$, are clearly visible in Fig. 4. The tongues of periodicity follow the Farey sequence

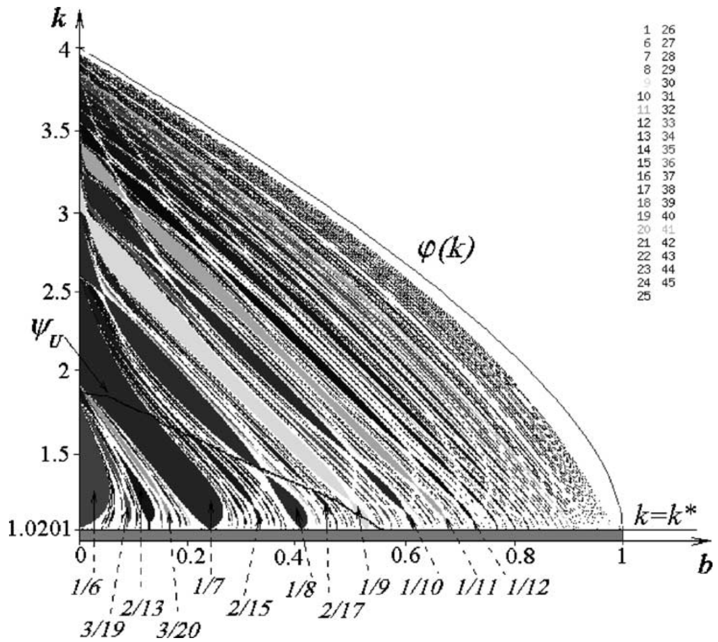


Fig. 4. Two-dimensional bifurcation diagram of the map T_L in the (b, k)-parameter plane for $a = 2$, $A_0 = 3$ and $g = 0.01$. Tongues of different degrees of grey correspond to the attracting cycles of different periods indicated by the numbers. For some tongues the corresponding rotation number of the map T_L is written. The curve ψ_U indicates a contact of the attracting set with the upper bound $y = B_0$; $k = k^*$ is the line of stability loss of the fixed point; $b = \varphi(k)$ is the curve of divergence to infinity.

rule (see [20,34,35]), according to which between a p_1/q_1-tongue and a p_2/q_2-tongue a $(p_1 + p_2)/(q_1 + q_2)$-tongue also exists. Some of such tongues are indicated in Fig. 4, with the corresponding rotation numbers.

Let us now describe how the tongues of periodicity (and, therefore, corresponding attracting cycles) are born. That is, we have to consider what occurs in the phase plane at the bifurcation value $k = k^*$ (10). As it was stated above, the fixed point E_2 is a center, thus we have, locally, around E_2, trajectories which belong to closed invariant curves (ellipses around E_2). Let us first assume that the rotation number of F_2 is rational, say p/q. Then, following the reasoning used in [16], we can prove that in the phase plane there exists a closed invariant polygon $\mathcal{A}$ whose boundary $\partial\mathcal{A}$ is made up by q pieces of critical curves LC_i^L, $i = 0, \dots, q-1$, such that all the orbits inside the area $\mathcal{A}$ are periodic of period q, while any initial condition outside $\mathcal{A}$ converges to a cycle of period q belonging to the boundary $\partial\mathcal{A}$ (see Fig. 5a where $p/q = 1/8$).

When the rotation number of F_2 is irrational, then a closed invariant area $\mathcal{A}$ exists, whose external boundary is an ellipse $\mathcal{E}$ tangent to LC_0^L (and thus, tangent to $T_L^i(\mathrm{LC}_0^L)$ for any $i \geq 1$), such that any trajectory inside $\mathcal{A}$ is quasi-periodic covering densely an ellipse, while any initial condition outside $\mathcal{A}$ converges to a quasi-periodic trajectory belonging to $\mathcal{E}$ (see Fig. 5b).

In particular, for the linear map F_2 it is easy to define the parameter value $b = b_{p/q}$ at which the rotation number of F_2 is rational and associated with the number p/q. As described in [32], to obtain the bifurcation value $b_{p/q}$ it is enough to consider the real part of the eigenvalues of the Jacobian matrix of F_2, which is $\mathrm{Re}\,\lambda_{1,2} = (b+k)/(2(1+g))$. Equating this value to $\cos(2\pi p/q)$ at the bifurcation, that is:

$$\frac{b_{p/q} + k^*}{2(1+g)} = \cos\frac{2\pi p}{q},$$

and substituting the expression of k^* from (10), we get the bifurcation value:

$$b_{p/q} = 2(1+g)\cos\frac{2\pi p}{q} - (1+g)^2. \tag{15}$$

For example, for $g = 0.01$ we have $k^* = 1.0201$, $b_{1/7} \approx 0.23935$, $b_{1/8} \approx 0.40826$, and so on. Thus, at $k = k^*$ and $b = b_{p/q}$ a p/q-tongue is born. To summarize what occurs at the bifurcation value $k = k^*$ we can state the two following propositions:

Proposition 2. *If $k = k^*$ (10) and $b = b_{p/q}$ (15) then in the phase plane there exists a closed invariant polygon $\mathcal{A}$ twhose boundary $\partial\mathcal{A}$ is made up by q segments of the critical lines LC_i^L, $i = 0, \dots, q-1$, such that any point $(x, y) \in \mathcal{A}$ is periodic with rotation number p/q, while any point $(x, y) \notin \mathcal{A}$ is mapped by T_L in finite number of iterations into a point of a p/q-cycle belonging to $\partial\mathcal{A}$.*

Proposition 3. *If $k = k^*$ (10) and $b \not\equiv b_{p/q}$ (15) then in the phase plane there exists a closed invariant area $\mathcal{A}$ bounded by an ellipse $\mathcal{E}$ tangent to LC_0^L such that any point $(x, y) \in \mathcal{A}$ is quasi-periodic on an ellipse, while any point $(x, y) \notin \mathcal{A}$ is mapped by T_L in a finite number of iterations into a quasi-periodic trajectory on $\mathcal{E}$.*

As k increases from the bifurcation value k^* (keeping fixed all the other parameters), several tongues of different periodicity are crossed in the parameter plane. For example, let us fix $b = 0.2$ and increase k. At $k = \widetilde{k}_1 \simeq 1.145$ a tongue of period 7 is crossed (see Fig. 4). At the bifurcation value $k = \widetilde{k}_1$ an attracting

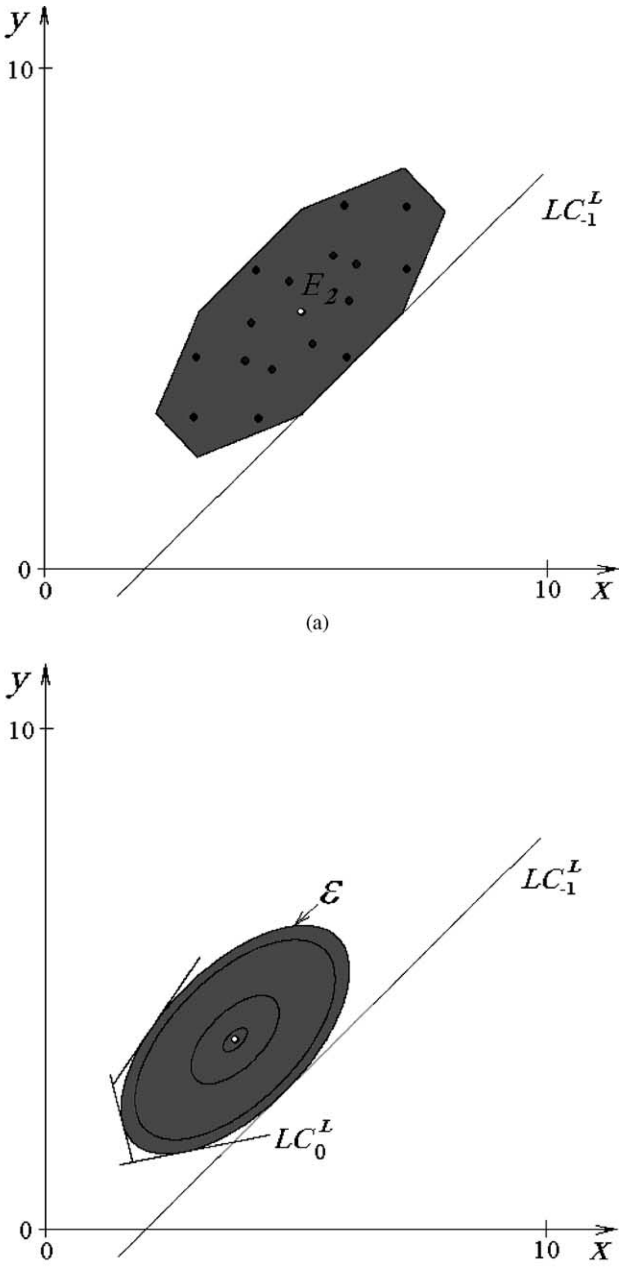


Fig. 5. (a) Closed invariant polygon $\mathcal{A}$ of the map T_L at $a = 2$, $A_0 = 3$, $g = 0.01$, $k = k^* = 1.0201$ and $b = 0.4082557 \approx b_{1/8}$; the boundary $\partial\mathcal{A}$ is made up by eight pieces of critical curves LC_i^L, $i = 0, \ldots, 7$; any point $(x, y) \in \mathcal{A}$ is periodic of period 8, while any $(x, y) \notin \mathcal{A}$ converges to a eight-cycle belonging to $\partial\mathcal{A}$. (b) Closed invariant area $\mathcal{A}$ of the map T_L at $a = 2$, $A_0 = 3$, $g = 0.01$, $k = k^* = 1.0201$ and $b = 0.2$. The external boundary of $\mathcal{A}$ is an ellipse $\mathcal{E}$ tangent to LC_0^L, such that any point $(x, y) \in \mathcal{A}$ is quasi-periodic covering densely an ellipse, while any $(x, y) \notin \mathcal{A}$ converges to a quasi-periodic trajectory belonging to $\mathcal{E}$.

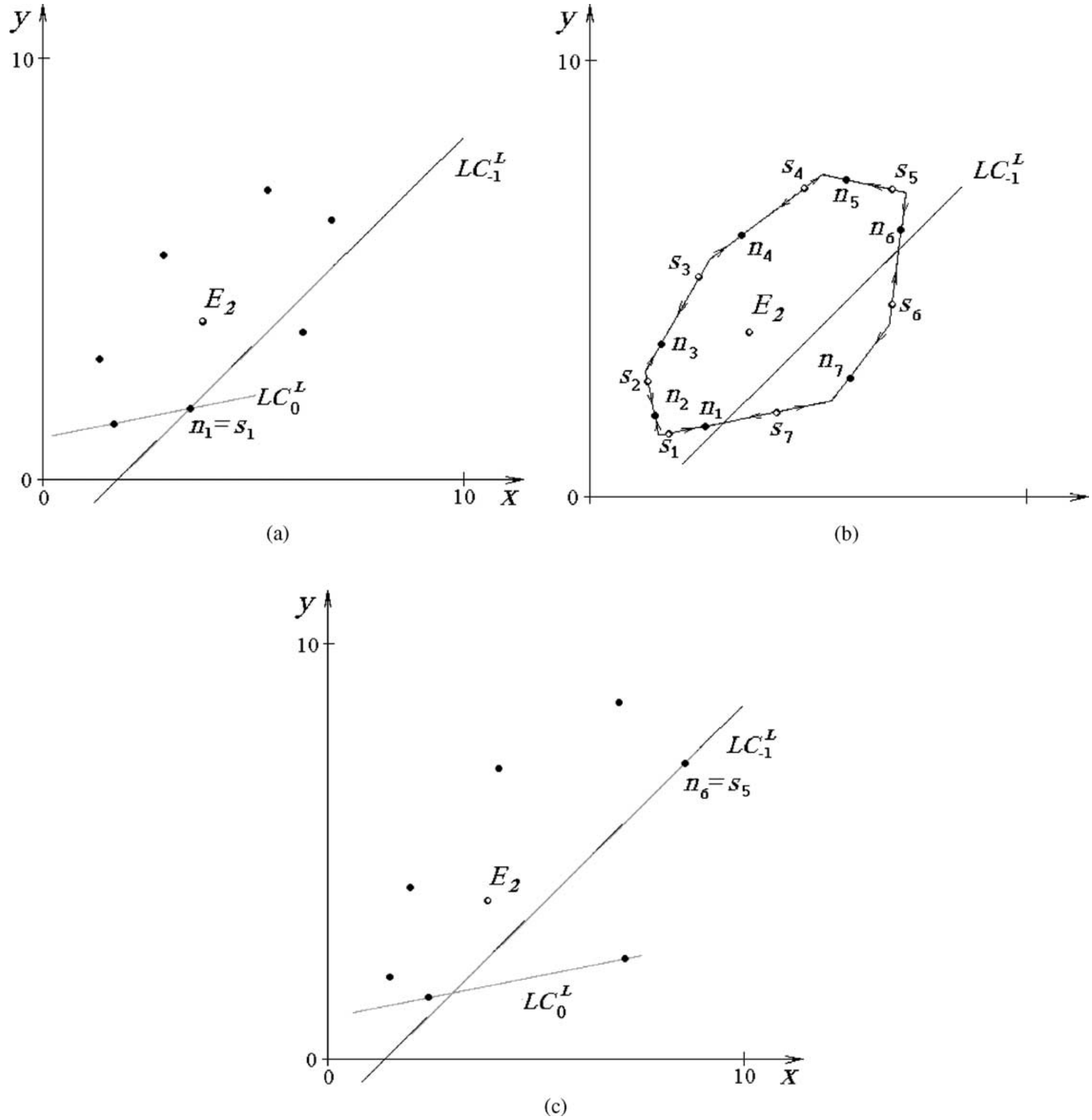


Fig. 6. (a) The first border-collision (and saddle-node) bifurcation of the attracting cycle $\gamma_{1/7} = \{n_1, \dots, n_7\}$ and the saddle cycle $\gamma'_{1/7} = \{s_1, \dots, s_7\}$: $n_i = s_i$, $i = 1, \dots, 7$, at $a = 2$, $A_0 = 3$, $g = 0.01$, $k = 1.145$ and $b = 0.2$. (b) Increasing k ($k = 1.3$), the periodic points of $\gamma_{1/7}$ and $\gamma'_{1/7}$ move on $\partial\mathcal{A}$; the unstable set of $\gamma'_{1/7}$ forms the boundary $\partial\mathcal{A}$ of the absorbing area $\mathcal{A}$. (c) A second border-collision (and saddle-node) bifurcation at $k = 1.4663$.

cycle $\gamma_{1/7} = \{n_1, \dots, n_7\}$ and a saddle cycle $\gamma'_{1/7} = \{s_1, \dots, s_7\}$ appear by a so-called *border-collision bifurcation* (see [24]) which is also a *saddle-node bifurcation*: At the bifurcation (lower boundary of the tongue) we have $n_i = s_i$, $i = 1, \dots, 7$, and, in particular, one periodic point is $n_1 = s_1 = \mathrm{LC}_{-1}^L \cap \mathrm{LC}_0^L$ (see Fig. 6a). As k increases, the periodic points move on $\partial\mathcal{A}$, and at the end of the 1/7-tongue in the parameter plane, for $k = \widetilde{k}_2 \simeq 1.4663$, we have one more border-collision (and saddle-node) bifurcation (see Fig. 6c). For any value of k, $\widetilde{k_1} < k < \widetilde{k_2}$, the unique attractor is the attracting cycle $\gamma_{1/7}$, and the boundary $\partial\mathcal{A}$ is a *saddle-connection*, that is, the unstable set of the saddle cycle $\gamma'_{1/7}$ also belonging to $\partial\mathcal{A}$ (Fig. 6b).

To summarize the results of this section we state the following:

Proposition 4. *For $k > k^*$ and $0 < b < \varphi(k)$, in the phase plane there exists an invariant absorbing area $\mathcal{A}$ whose boundary $\partial\mathcal{A}$ is made up by m segments of critical lines LC_i^L, $i = 0, \dots, m-1$, where m depends on the parameters. The attractor of the map T_L is either a cycle $\gamma_{p/q} \in \partial\mathcal{A}$ of period q (if the rotation number of T_L on $\partial\mathcal{A}$ is rational, p/q), or the invariant set $\partial\mathcal{A}$ with quasi-periodic trajectories on it (if the rotation number of T_L on $\partial\mathcal{A}$ is irrational).*

We remark that as k increases, the width of the invariant area $\mathcal{A}$ increases rapidly, and probably it becomes too wide for an economic interpretation of the states Z_t and Y_t. A criterion to detect when this occurs is the following. Let P_1 be the point of intersection of LC_{-1}^L and LC_0^L, i.e. $\mathrm{LC}_{-1}^L \cap \mathrm{LC}_0^L = P_1$. We need to check if after a finite number i of iterations we have a point $P_i = T_L^i(P_1)$ which is above an upper bound $y = B_0$, in which case we say that the model becomes unrealistic. In Fig. 4 the black line ψ_U indicates such a limit. That is, for (b, k) below the curve ψ_U the attracting set remains below the line $y = B_0$, which means that Z_t never reaches the value B_0, while for (b, k) above the curve ψ_U the attractor has also values $Z_t \geq B_0$. Thus, in order to get values which are also bounded from above, not only from below, it is necessary to introduce a "ceiling", as we shall see in the Section 6.

We end this section noticing that the bifurcation diagram in Fig. 4 shows a particular structure of the bifurcation tongues, which we may call "sausages-structure". This occurrence is typical in piecewise-linear maps, as shown in the two-dimensional parameter plane of a one-dimensional map in Hao [36]. Inside each portion of the sausages-structure the period is the same, but the sequence of linear functions giving the cycle changes. The shrinking points of such a structure correspond to a change of type of the periodic orbit (i.e. cycles of different type but of the same period merge). A similar structure is described in [16] for a piecewise-linear two-dimensional map, showing how the critical lines are involved in the shrinking points. In piecewise-smooth systems a similar structure is described in [28].

5. Piecewise-linear model with an upper bound

In this section we get bounded dynamics for our linear model (4) when the fixed point E_2 is a repelling focus, by adding an upper bound instead of a lower bound, to the values of the state variable, that is a "ceiling" instead of a "floor".

Let us define:

$$Y_t = G(Y_{t-1}, Y_{t-2}), \text{ if } G(Y_{t-1}, Y_{t-2}) < U,$$

otherwise, we have:

$$Y_t = U,$$

where $G(Y_{t-1}, Y_{t-2})$ and U are defined in (2).

With such a definition our model becomes a piecewise-linear continuous system which, by using the variable transformations (5) and (8), is given by:

$$T_U : \begin{cases} (x', y') = F_1(x, y), & \text{if}\,(x, y) \in R_1; \\ (x', y') = F_2(x, y), & \text{if}\,(x, y) \in R_2; \end{cases}$$

where F_2 is given in (9) and

$$F_1 : \begin{cases} x' = y; \\ y' = B_0; \end{cases} \tag{16}$$

$$R_1 = \{(x, y) : x > 0, y > r_1(x)\}; \quad R_2 = \{(x, y) : x > 0, y \leq r_1(x)\};$$
$$r_1(x) = \frac{k}{(1+g)(b+k)}x + \frac{(B_0 - A_0)(1+g)}{b+k}. \tag{17}$$

Let us denote by LC_{-1}^U the straight line $y = r_1(x)$ separating the two regions R_1 and R_2 in which the piecewise-linear map T_U has different definitions. Its image is the critical line:

$$LC_0^U = \{(x, y) \in \mathbb{R}^2 : y = B_0\}. \tag{18}$$

Note that the model is meaningful as long as the fixed point $E_2 = (Z^*, Z^*)$ (7) of the map F_2 belongs to the region R_2. This requires that the following inequality must be fulfilled:

$$Z^* \leq B_0, \tag{19}$$

from which we get a straight line in the (b, k)-parameter plane

$$k = \chi(b) \stackrel{\text{def}}{=} \frac{(B_0 - A_0)(1+g)^2 - B_0 b(1+g)}{B_0 g}, \tag{20}$$

such that for $k \leq \chi(b)$ we have $E_2 \in R_2$ and the model is meaningful. Otherwise, if $Z^* > B_0$, the only fixed point of the map T_U is the fixed point $E_1 = (B_0, B_0)$ of the map F_1 which is globally attracting, but the model becomes economically uninteresting. In Fig. 7 the line $k = \chi(b)$ (10) is shown together with the bifurcation line $k = k^*$ and the tongues of periodicity in different colors. We can see that for k close to the bifurcation value k^* infinitely many tongues exist. Note that for some parameter region a trajectory of T_U can enter the negative quadrants which is unrealistic from an economic point of view. Such a region is also indicated in Fig. 7. Its boundary corresponds to a contact of the attracting set of the map T_U (i.e. of $\partial\mathcal{A}$) with the lines $y = 0$ or $x = 0$. Thus, we have defined a region in the (b, k)-parameter plane where the model with only upper bound is meaningful.

The qualitative behavior of the trajectories in the phase plane is similar to the one already described in the previous section. One should only substitute the critical curves LC_i^L with LC_i^U, $i \geq -1$. That is, after the bifurcation of the fixed point E_2, for $k > k^*$, a closed invariant absorbing area $\mathcal{A}$ is defined, bounded by a finite number m of critical segments of LC_i^U for $i = 0, \dots, m-1$. The attractor of T_U belongs to the boundary $\partial\mathcal{A}$, and is either a periodic orbit (if the parameter values are inside a tongue of periodicity, see Fig. 7), or a quasi-periodic orbit, dense in $\partial\mathcal{A}$, and at the bifurcation value $k = k^*$ the same properties, as described in the previous section, hold (see Propositions 2–4 substituting LC_i^U to LC_i^L).

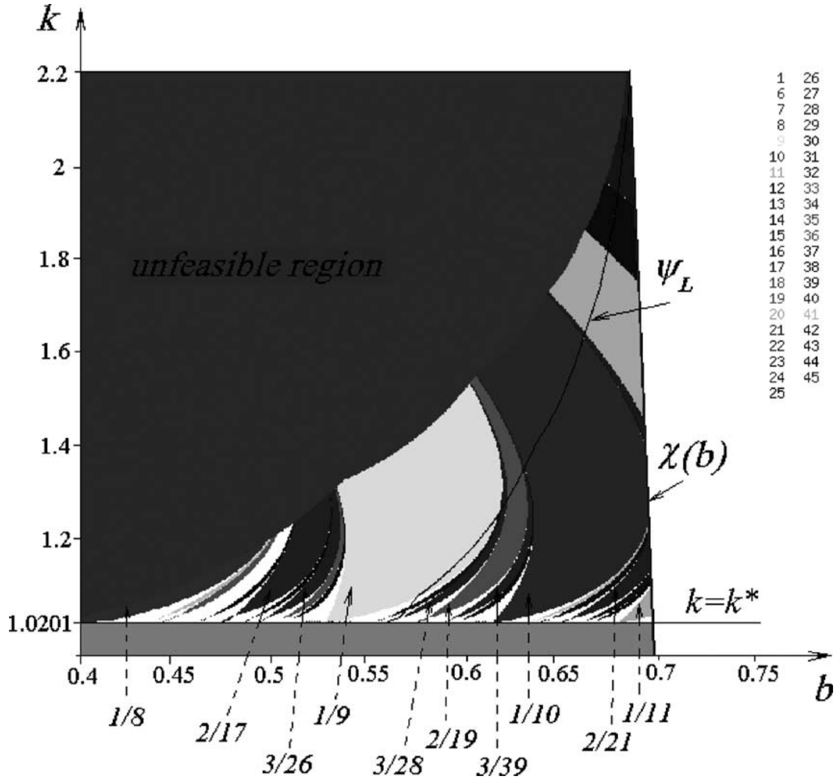


Fig. 7. Two-dimensional bifurcation diagram for the map T_U in the (b, k)-parameter plane at $A_0 = 3$, $B_0 = 10$ and $g = 0.01$. For some tongues the corresponding rotation number of the map T_U is indicated. The dark grey region is unfeasible because some values of Y_t are negative. The curve ψ_L indicates a contact of attracting set with the lower bound $y = r_1(x)$; $k = k^*$ is the line of stability loss of the fixed point; $k = \chi(b)$ indicates a limit for the economic model to be meaningful.

It is worth to note that the Jacobian matrix of the map F_1 has both eigenvalues equal to 0, which means that the region R_1 is mapped in one iteration into a straight line (the critical line LC_0^U), and any point belonging to $\mathrm{LC}_0^U \cap R_1$ is mapped in one iteration into the point (B_0, B_0) (which is not a fixed point of T_U as long as $(B_0, B_0) \in R_2$). This property implies that any trajectory can have at most two consecutive points in the region R_1. Whenever two consecutive points, say (x_n, y_n) and $(x_{n+1}, y_{n+1}) = (x_{n+1}, B_0)$, belong to R_1, then $(x_{n+2}, y_{n+2}) = (B_0, B_0)$ and the trajectory converges to the trajectory of the point (B_0, B_0) (either periodic, or quasi-periodic on the boundary of the invariant absorbing area $\mathcal{A}$). In Fig. 8a we show a quasi-periodic trajectory belonging to $\partial\mathcal{A}$ (made up by 13 segments of the critical curves LC_i^U, $i = 0, \ldots, 12$), and in Fig. 8b an attracting cycle of period 27 belonging to $\partial\mathcal{A}$, which is the trajectory of (B_0, B_0).

As it occurs for the piecewise-linear map T_L, also for the piecewise-linear map T_U as k increases, the invariant area $\mathcal{A}$ growth rapidly, and crosses the positive quadrant, becoming unmeaningful from an economic point of view (see Fig. 7). This occurs when the parameters reach the dark-green region in Fig. 7. However, for our applications, also a lower value of Z_t may be not so good. In order to understand when a low-limit, i.e. a “floor” as we have seen in the previous section, is met and has effects in the

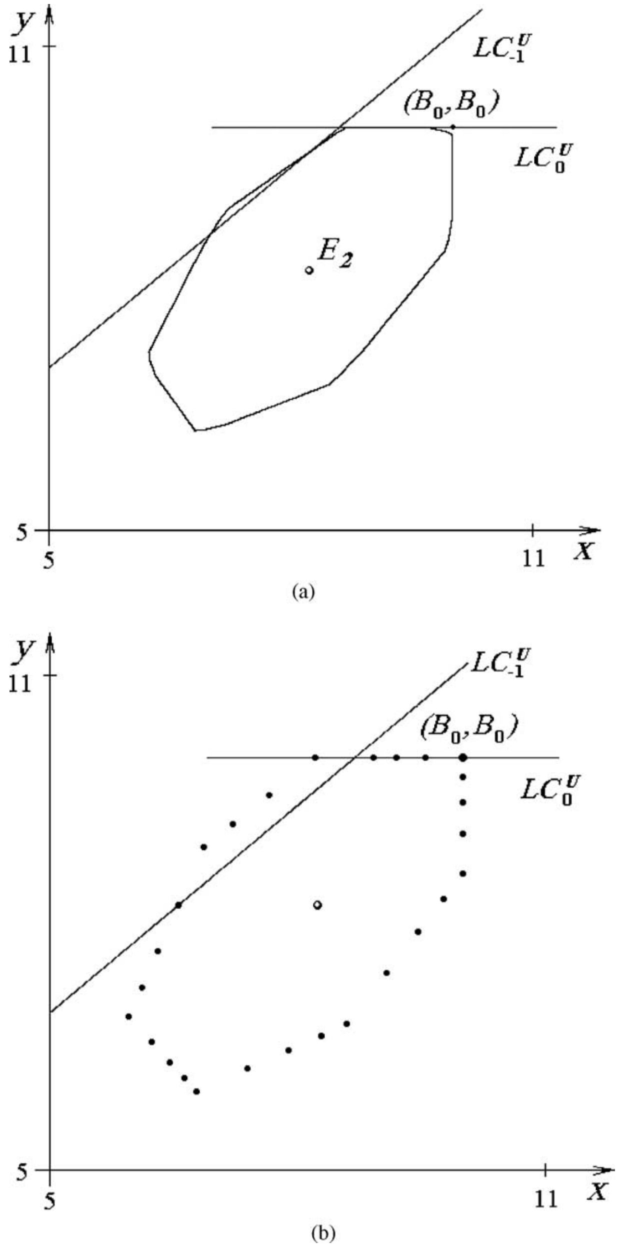


Fig. 8. (a) A quasi-periodic trajectory of the map T_U belonging to $\partial\mathcal{A}$ (made up by 13 segments of the critical curves LC_i^U, $i = 0, \ldots, 12$), at $A_0 = 6.5$, $B_0 = 10$, $a = 2$, $b = 0.2$, $k = 1.03$, $g = 0.01$. (b) An attracting cycle of period 27 which is the trajectory of (B_0, B_0) at $k = 1.1$.

attracting sets, in Fig. 7 we have also shown the curve ψ_L. This curve indicates a contact of the attracting set (i.e. of $\partial\mathcal{A}$) with the straight line $y = r_2(x)$ (13).

Thus, in order to have an invariant area which is always positive and bounded (below and above), we have to introduce both constraints, i.e. a “floor” and a “ceiling”, as we shall do in the next section.

6. Piecewise-linear model with lower and upper bounds

In this section we consider a piecewise-linear map with two constraints, in which we define:

$$Y_t = G(Y_{t-1}, Y_{t-2}), \quad \text{if } L(Y_{t-1}) \le G(Y_{t-1}, Y_{t-2}) \le U;$$

otherwise, the lower and upper bounds apply, i.e.

$$Y_t = U, \quad \text{if } G(Y_{t-1}, Y_{t-2}) > U;$$
$$Y_t = L(Y_{t-1}), \quad \text{if } G(Y_{t-1}, Y_{t-2}) < L(Y_{t-1});$$

where $G(Y_{t-1}, Y_{t-2})$, $L(Y_{t-1})$ and U are defined in (2).

That is, by using the variable transformations (5) and (8), we get a two-dimensional piecewise-linear continuous map F in the form:

$$F : (x', y') = (y', f(x, y)),$$

where

$$f(x, y) = \begin{cases} B_0, & \text{if } y > r_1(x); \\ \dfrac{-xk}{(1+g)^2 + (y(b+k)/(1+g)) + A_0}, & \text{if } r_2(x) \le y \le r_1(x); \\ \dfrac{yb}{(1+g) - a + A_0}, & \text{if } y < r_1(x); \end{cases}$$

the constraints $r_1(x)$ and $r_2(x)$ are given in (17) and (13), respectively.

One can easily check that for the parameter values considered, the slopes of both lines $r_1(x)$ and $r_2(x)$ are positive and less then 1. Moreover, the slope of $r_1(x)$ exceeds the slope of $r_2(x)$. The shift constant of $r_1(x)$ is positive while the shift constant of $r_2(x)$ is negative, so these lines intersect in the positive quadrant of the plane, defining a region which may be considered as the feasible region for the economic model, that is:

$$P = \{(x, y) : 0 < x < x_M, 0 < y < y_M\},$$

where (x_M, y_M) is an intersection point of $r_1(x)$ and $r_2(x)$.

The map F is given by three linear maps F_i, defined, respectively, in the regions R_i, $i = 1, 2, 3$:

$$F : \begin{cases} (x', y') = F_1(x, y), & \text{if } (x, y) \in R_1; \\ (x', y') = F_2(x, y), & \text{if } (x, y) \in R_2; \\ (x', y') = F_3(x, y), & \text{if } (x, y) \in R_3; \end{cases} \tag{21}$$

where the maps F_1 F_2 and F_3 are given in (16), (9) and (12), respectively, and

$$R_1 = \{(x, y) : x > 0, r_1(x) < y < y_M\};$$

$$R_2 = \{(x, y) : x > 0, r_2(x) \le y \le r_1(x), y > 0\};$$

$$R_3 = \{(x, y) : 0 < x < x_M, y < r_2(x)\}.$$

As in the previous section, we have to introduce a parameter constraint

$$k \le \chi(b),$$

where $\chi(b)$ is given in (10), such that for $k \le \chi(b)$ we have $E_2 \in R_2$. Otherwise all the trajectories entering R_1 are converging to $E_1 = (B_0, B_0)$, fixed point of F_1, in at most two iterations, and thus are not economically interesting.

Fig. 9 shows a two-dimensional bifurcation diagram for the map F in the (b, k)-parameter plane, where different colors indicate tongues of periodicity corresponding to attracting cycles of different periods. It is also shown the straight line $k = k^*$, given in (10), corresponding to the stability loss of the fixed point E_2, and the straight line $k = \chi(b)$, given in (10), which indicates the parameter restriction for the economic application of the model, as stated above.

As in the previous sections, we denote the two straight lines $y = r_1(x)$ and $y = r_2(x)$ as LC^U_{-1} (an upper bound) and LC^L_{-1} (a lower bound), respectively. Crossing these lines the map F changes its definition. The images of LC^U_{-1} and LC^L_{-1}, that is the straight lines $LC^U_0 = F(LC^U_{-1})$ and $LC^L_0 = F(LC^L_{-1})$, are called *critical lines* (see (14) and (18)).

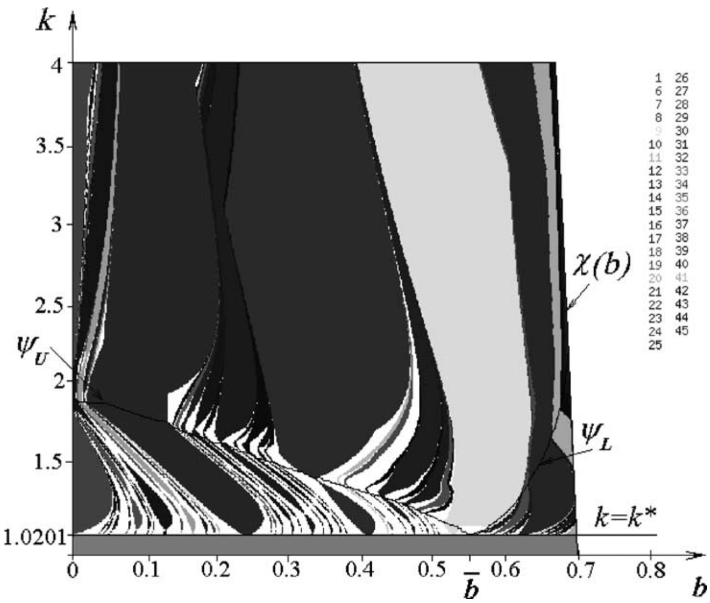


Fig. 9. Two-dimensional bifurcation diagram for the map F at $A_0 = 3$, $a = 2$, $B_0 = 10$ and $g = 0.01$. Tongues of different colors correspond to the attracting cycles of different periods indicated by the numbers. The curve ψ_L (ψ_U) indicates a contact of attracting set with the lower (upper) bound $y = r_1(x)$ ($y = r_2(x)$); $k = k^*$ is the line of stability loss of the fixed point; $k = \chi(b)$ indicates a limit for the economic model to be meaningful.

As we have already seen, the images of these critical lines play an important role in the description of the dynamics of the map F. In fact, in one iteration the whole phase plane is mapped by F into a region bounded by LC_0^U and LC_0^L. Any point of $(x, y) \in R_1$ (respectively, $(x, y) \in R_3$) is mapped in LC_0^U (respectively, in LC_0^L), while the region between LC_{-1}^U and LC_{-1}^L (up to the intersection point (x_M, y_M)) is mapped into the strip between the two critical lines LC_0^U and LC_0^L (up to the point $F(x_M, y_M)$). By taking two more images of the region bounded by the critical lines $\mathrm{LC}_0^{U,L}$ we obtain an absorbing area bounded by six segments of the critical lines $\mathrm{LC}_i^{U,L}$, $i = 0, 1, 2$, (an example is shown in Fig. 10a). Any

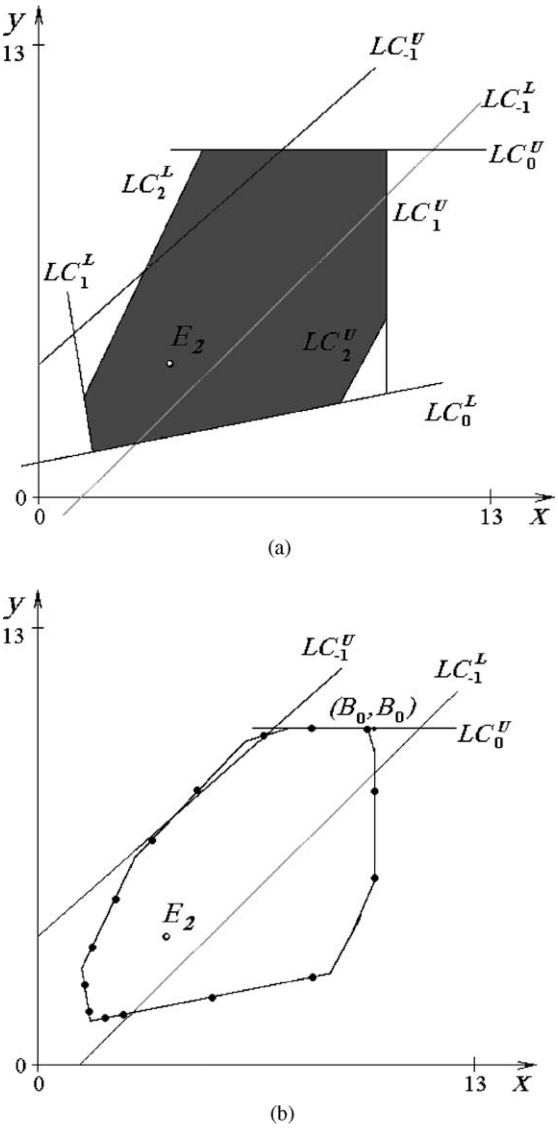


Fig. 10. (a) Example of the absorbing area for the map F bounded by six segments of the critical lines $\mathrm{LC}_i^{U,L}$, $i = 0, 1, 2$, at $A_0 = 3$, $B_0 = 10$, $a = 2$, $b = 0.2$, $k = 1.65$ and $g = 0.01$. The invariant area is bounded by 10 pieces of critical segments, and it is shown in (b), together with the attracting set, which is a periodic orbit of period 15 belonging to $\partial\mathcal{A}$.

initial condition in the positive quadrant of the phase plane is mapped into this area in a finite number of iterations. We remark that this explains why the least period for a periodic orbit of F is 6. By taking further images of the critical curve segments, for $k > k^*$ an invariant area $\mathcal{A}$ is obtained bounded by a finite number of critical segments, $\partial\mathcal{A}$, which also includes the attracting set.

For any point in the (b, k)-parameter plane which is above $k = k^*$ and below the curve ψ_U (see Fig. 9), the dynamics of F are as those already described in Section 4: the invariant absorbing area $\mathcal{A}$ is ultimately bounded only by the segments of LC_i^L, $i \geq 0$ (see the examples shown in Fig. 2). As k is increased so that the parameter point is above the curve ψ_U, then the invariant area has also a segment of the critical line LC_0^U and, thus, both critical lines $\mathrm{LC}_0^{U,L}$ are involved in the boundary $\partial\mathcal{A}$. An example is shown in Fig. 10b (a periodic orbit of period 15 is shown belonging to $\partial\mathcal{A}$ made up by 10 segments of cricial lines belonging to $\mathrm{LC}_i^{U,L}$).

Similarly, for any point in the (b, k)-parameter plane which is above $k = k^*$ and below the curve ψ_L (see Fig. 9) the dynamics of F are as described in Section 5: the invariant absorbing area $\mathcal{A}$ is ultimately bounded only by the segments of LC_i^U, $i \geq 0$ (see the examples shown in Fig. 8). While crossing the curve ψ_L the boundary of $\mathcal{A}$ has also a segment on the critical line LC_0^L.

Clearly, as in the previous sections, as long as the fixed point E_2 is stable (i.e. if $k < k^*$) it is globally attracting. At the bifurcation value $k = k^*$ the fixed point E_2 becomes a center and the rotation number of the map F can be rational or irrational. If it is irrational, then any trajectory either belongs to a closed invariant curve (an ellipse), or it converges to an ellipse tangent to the one of the constraints, that is tangent to LC_{-1}^L (if $b < \bar{b}$), or to LC_{-1}^U (for $b > \bar{b}$), where $\bar{b}$ is the intersection point on $k = k^*$ of the two curves ψ_L and ψ_U shown in Fig. 9. When the rotation number is rational then the invariant area is a polygon, with periodic orbits in it, as already described in Section 3.

To summarize the results obtained for the map F we can state following propositions:

Proposition 5. *For $k = k^*$, if $b < \bar{b}$, then Propositions 2 and 3 hold for the map F given in (21), while if $b > \bar{b}$ then these propositions hold substituting LC_i^U to LC_i^L, $i \geq 0$.*

Proposition 6. *For $k > k^*$, if (b, k) is below ψ_U then for the map F Proposition 4 holds; if (b, k) is below ψ_L then for the map F Proposition 4 holds substituting LC_i^U to LC_i^L, $i \geq 0$; if (b, k) is above ψ_U and above ψ_L then an invariant absorbing area $\mathcal{A}$ exists, whose boundary $\partial\mathcal{A}$ is made up by a finite number of critical segments of $LC_i^{U,L}$, $i \geq 0$. The attractor of the map F is either a cycle $\gamma_{p/q} \in \partial\mathcal{A}$ of period q (if the rotation number of F on $\partial\mathcal{A}$ is rational, p/q), or the invariant set $\partial\mathcal{A}$ with quasi-periodic trajectories on it if the rotation number of F on $\partial\mathcal{A}$ is irrational).*

7. Conclusions

We investigated the mathematical properties of Hicks' piecewise-linear model. Even if in its simplest formulation it does not produce complex dynamics, we have shown that only one bound is needed to have non explosive outcome if the equilibrium point is an unstable focus giving a full description of the structure of the attracting set both in the cases of only one constraint and when two constraints are assumed. We have completely characterized the dinamics occurring at the bifurcation value of the fixed point (which depends on the type of rotation number). We have shown that when the fixed point is unstable the attracting set is always a one-dimensional closed invariant curve, made up of a finite number of linear segments, on which the dynamics are periodic or quasi-periodic. The changes which may occur are due to sequences of border-collision bifurcations, crossing the boundaries of the bifurcation tongues in the (b, k) parameter plane. The bifurcation tongues have a sausages-structure as long as the upper constraint is reached by the invariant attracting set.

While the analysis here performed completely characterizes the simplest form of Hicks' piecewise-linear model, which is two-dimensional, the investigation of more complex systems is still an open problem. For example, the simple introduction of two periods back instead of one leads to a three-dimensional piecewise-linear map, as shown in [14]. In such a case the border-collision bifurcations still continue to characterize the transitions, however more complex routes may appear, which will be the object of further studies.

It is plain that changes of the model towards a smooth formulation are such that the standard Neimark–Hopf bifurcation is involved. This means that attracting closed curves (or endogenous cycles) appear around the fixed point when it becomes unstable. Moreover, depending on the nonlinear terms, routes to complex behaviors may occur (see e.g. [16,37]).

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References

[1] P.A. Samuelson, Interactions between the multiplier analysis and the principle of acceleration, Rev. Econ. Stat. 21 (1939) 75–78.
[2] A. Heertjie, P. Heemeijer, On the origin of Samuelson's multiplier-accelerator model, Hist. Political Econ. 42 (2002) 549–563.
[3] A.D. Knox, On a theory of the Trade Cycle, Economica 17 (1950) 317–327.
[4] J. Duesenberry, Hicks on the Trade Cycle, Q. J. Econ. 64 (1950) 464–476.
[5] S. Alexander, Issues of business cycle theory raised by Mr. Hicks, Am. Econ. Rev. 61 (1951) 123–132.
[6] N. Kaldor, Mr. Hicks on the Trade Cycle, Econ. J. 50 (1951) 78–92.
[7] N. Rau, Trade Cycles: Theory and Evidence, Macmillan, London, 1974.
[8] J.M. Blatt, Dynamic Economic Systems, Sharpe, New York, 1983.
[9] G. Gandolfo, Economic Dynamics, North Holland, Amsterdam, 1985.
[10] W. Brock, Hicksian nonlinearity, in: L. McKenzie, S. Zamagni (Eds.), Value and Capital: Fifty Years Later, Macmillan, London, 1991.
[11] C. Hommes, Periodic, almost periodic and chaotic behaviour in Hicks' non-linear trade cycle model, Econ. Lett. 41 (1993) 391–397.
[12] C. Hommes, A reconsideration of Hicks' non-linear trade cycle model, Struct. Change Econ. Dyn. 6 (1995) 435–459.
[13] D. Saura, F.J. Vazquez, J.M. Vegas, Non-chaotic oscillations in some regularized Hicks models, J. Econ. Dyn. Control 22 (1998) 667–678.
[14] C. Hommes, Chaotic Dynamics in Economic Models—Some Case Studies, Ph.D. Thesis, Wolters-Noordhoff, Groningen, 1991.
[15] T. Puu, I. Sushko, A business cycle model with cubic nonlinearity, Chaos Solitons & Fractals, 2002, in press.

[16] I. Sushko, T. Puu, L. Gardini, The Hicksian Floor–Roof Model for Two Regions Linked by Interregional Trade, Chaos Solitons & Fractals, 2003, in press.
[17] I. Taralova-Roux, D. Fournier-Prunaret, Dynamical study of a second order DCPM transmission system modeled by a piecewise-linear function, IEEE Trans. 49 (11) (2002) 1592–1609.
[18] P. Ashwin, F. Xin-Chu, J. Deane, Properties of the invariant disk packing in a model bandpass sigma-delta modulator, Int. J. Bifurcation Chaos 13 (3) (2003) 631–641.
[19] Y.L. Maistrenko, V.L. Maistrenko, L. Chua, Cycles of chaotic intervals in a time-delay Chua's circuit, Int. J. Bifurcation Chaos 3 (6) (1993) 1557–1572.
[20] Y.L. Maistrenko, V.L. Maistrenko, S.I. Vikul, L. Chua, Bifurcations of attracting cycles from time-delayed Chua's circuit, Int. J. Bifurcation Chaos 5 (3) (1995) 653–671.
[21] Y.L. Maistrenko, V.L. Maistrenko, S.I. Vikul, Bifurcations of attracting cycles of piecewise-linear interval maps, Int. J. Bifurcation Chaos 37 (3) (1996) 270–367.
[22] P. Julian, A. Desages, B. D'Amico, Orthonormal high level canonical PWL functions with applications to model reduction, IEEE Trans. Circuits Syst. I–47 (2000) 702–712.
[23] M. Storace, P. Julian, M. Parodi, Synthesis of nonlinear multiport resistors: a PWL approach, IEEE Trans. Circuits Syst. I-49 (2002) 1138–1149.
[24] H.E. Nusse, J.A. Yorke, Border-collision bifurcations including period-two to period-three for piecewise-smooth systems, Physica D 57 (1992) 39–57.
[25] H.E. Nusse, J.A. Yorke, Border-collision bifurcations for piecewise-smooth one-dimensional maps, Int. J. Bifurcation Chaos 5 (1) (1995) 189–207.
[26] M. di Bernardo, M. Feigin, S. Hogan, M. Homer, Local analysis of C-bifurcations in n-dimensional piecewise-smooth dynamical systems, Chaos Solitons Fractals 10 (11) (1999) 1881–1908.
[27] Zh.T. Zhusubaliyev, E.A. Soukhoterin, E. Mosekilde, Border-collision bifurcations and chaotic oscillations in a piecewise-smooth dynamical system, Int. J. Bifurcation Chaos 11 (12) (2001) 2977–3001.
[28] Zh.T. Zhusubaliyev, E.A. Soukhoterin, E. Mosekilde, Border-collision bifurcations on a two-dimensional torus, Chaos Solitons Fractals 13 (2002) 1889–1915.
[29] Zh.T. Zhusubaliyev, E. Mosekilde, Bifurcations and Chaos in Piecewise-Smooth Dynamical Systems, World Scientific, Singapore, 2003.
[30] R. Frisch, Propagation Problems and Impulse Problems in Dynamic Economics, Economic in Essays in Honour of Gustav Cassel, Allen & Unwin, London, 1933.
[31] R.F. Harrod, Towards a Dynamic Economics, Macmillan, London, 1948.
[32] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, Springer, New York, 1985.
[33] C. Mira, L. Gardini, A. Barugola, J.C. Cathala, Chaotic Dynamics in Two-dimensional Noninvertible Maps, World Scientific, Singapore, 1996.
[34] N.N. Leonov, Map on the line onto itself, Radiophysics 2 (6) (1959) 942–956 (in Russian).
[35] C. Mira, Chaotic Dynamics, World Scientific, Singapore, 1987.
[36] Hao Bai-lin, Elementary Symbolic Dynamics, World Scientific, Singapore, 1989.
[37] T. Puu, Attractors, Bifurcations and Chaos, Springer, London, 2000.
[38] J.R. Hicks, A Contribution to the Theory of the Trade Cycle, Clarendon Press , Oxford, 1950.