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Inertia in binary choices: Continuity breaking and big-bang bifurcation points

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ABSTRACT

In several situations the consequences of an actor's choices are also affected by the actions of other actors. This is one of the aspects which determines the complexity of social systems and make them behave as a whole. Systems characterized by such a trade-off between individual choices and collective behavior are ubiquitous and have been studied extensively in different fields. Schelling, in his seminal papers (1973, 1978), provided an interesting analysis of binary choice games with externalities. In this work we analyze some aspects of actor decisions. Specifically we shall see what are the consequences of assuming that switching decisions may also depend on how close to each other the payoffs are. By making explicit some of these aspects we are able to analyze the dynamics of the population where the actor decision process is made more explicit and also to characterize several interesting mathematical aspects which contribute to the complexity of the resulting dynamics. As we shall see, several kinds of dynamic behaviors may occur, characterized by cyclic behaviors (attracting cycles of any period may occur), also associated with new kinds of bifurcations, called big-bang bifurcation points, leading to the so-called period increment bifurcation structure or to the period adding bifurcation structure.

1. Introduction

One of the aspects which determines the complexity of social systems and makes them behave as a whole is that in several situations the consequences of an actor's choices are also affected by the actions of other actors. The trade-off between individual choices and collective behavior is ubiquitous and has been studied extensively in different fields. For example, Schelling (1973, 1978) provided an interesting analysis of binary choice games with externalities. Although his approach was qualitative, its depth and the number of examples to real life situations made his contributions seminal, and showed how this kind of games can describe several important situations which characterize social dynamics. Recently, Bischi and Merlone (2009) gave a quantitative formalization of this kind of dynamics. The resulting mathematical formalization have been examined by Bischi et al. (2009a,b). Finally, Bischi and Merlone (2010a) provided a generalization of the binary choice

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model in which interactions are considered both in a single large group as in Schelling (1973) and small groups as in Galam (2003); for a review of these model the reader may see Bischi and Merlone (2010b).

In Schelling (1973, 1978) and the other contributions we have cited, agents' behavior depends on the number of people choosing one way or the other. In fact Schelling (1973, p. 383) assumes that "Everybody's payoffs, whichever way he makes his choice, depend only on the number of people who choose one way or the other". In other words each actor's decision process is assumed to be a sort of "black-box" for which input are payoff values and output are decisions.

Although it is possible to reduce decisions simply to payoff comparison, several other aspects are important to determine how decisions are made (see for example Bazerman, 2006).

In this paper we analyze some other aspects of actor decisions. In fact in the model analyzed in Bischi et al. (2009a,b) impulsive agents are considered. Impulsive agents are defined as agents for which the switching decision depends only on the sign of the difference between payoffs no matter how much they differ. There, depending on the switching fractions of agents, the dynamics are analyzed and the bifurcation diagrams are studied. By contrast, in this paper we take into account also the difference between payoffs; in other words, we consider agents whose switch depends not only on the sign of the difference payoffs, but also takes into account the relative difference between payoffs. As we shall see, similarly to the previous case, the dynamics will be characterized by stable cycles of any period.

The structure of the paper is the following. In Section 2 the model we consider is described and the behavior of agents is formalized. In Section 3 the map is analyzed for *linear payoff functions* and depending on the payoff structure two cases are identified. These two cases are analyzed in Sections 4 and 5 where several kinds of dynamic behaviors of the system are studied. In particular, we show that breaking the continuity an attracting fixed point may lead to an attracting cycle of any period, and the period of the cycles depends on the two key parameters of the model. We shall see that changing such parameters, different border collision bifurcation curves are crossed, leading to cycles of different periods. Moreover, new kinds of bifurcation points are evidenced in the two-dimensional parameter plane, characterizing particular regions always associated with cycles of different periods. Following Avrutin and Schanz (2006), Gardini et al. (submitted for publication), and Avrutin et al. (2010b, submitted for publication), they are called *big-bang bifurcation points*, and are associated with particular bifurcation structures, following the so-called *period increment structure* or to the *period adding structure*. These terms are used in Avrutin and Schanz (2006) as well as the recent literature. Roughly speaking, an adding structure means that when we have two cycles of different periods p and q, then for suitable values of the parameters, also a cycle with period $(p+q)$ exists, and this applies iteratively, and bistability cannot occur. While an increment structure means that we may have only an infinite sequence of cycles with increasing periods (incremented by a fixed constant), and they may coexist in pair. Finally, Section 6 is devoted to the conclusions and further research.

2. The model

Several contributions analyze and extend the mathematical formalization of Schelling (1973). In particular, Bischi and Merlone (2009) propose a model where a population of agents is assumed to be engaged in a game where they have to choose between two strategies, say A and B respectively. In their formalization, the set of agents is normalized to the interval $[0, 1]$ and the real variable $x \in [0, 1]$ denotes the fraction of agents choosing strategy A. The payoffs are functions of x, say $A : [0, 1] \to \mathbb{R}$, $B : [0, 1] \to \mathbb{R}$, where $A(x)$ and $B(x)$ represent the payoff associated to strategies A and B respectively. Since binary choices are considered, when fraction x is playing A, then fraction $1 - x$ is playing B. Therefore $x = 0$ means that the whole population of agents is playing B and $x = 1$ means that all the agents are playing A. The basic assumption modeling the dynamic adjustment is that *x will increase whenever $A(x) > B(x)$; on the contrary, it will decrease when the opposite inequality holds*.

This assumption, together with the constraint $x \in [0, 1]$, implies that equilibria are located either in the points $x = x^*$ such that $A(x^*) = B(x^*)$, or in $x = 0$ (provided that $A(0) < B(0)$) or in $x = 1$ (provided that $A(1) > B(1)$). These results are consistent to Schelling (1973). Bischi and Merlone (2009) consider a process of repeated binary choices in which the agents update their binary choice at each time period $t = 0, 1, 2, \ldots$, and x_t represents the number of agents playing strategy A at time period t. At time $(t+1)x_t$ becomes common knowledge, hence each agent is able to either compute or observe payoffs $B(x_t)$ and $A(x_t)$. Agents are homogeneous and myopic, that is, their aim is to increase their own next period payoff.

A discrete-time model is obtained: at time t if x_t agents are playing strategy A and $A(x_t) < B(x_t)$ then a fraction of the x_t agents that are playing A will switch to strategy B in the following turn. Analogously, if $A(x_t) > B(x_t)$ then a fraction of the $(1 - x_t)$ agents that are playing B will switch to strategy A. In Bischi and Merlone (2009), Bischi et al. (2009a,b) agents are assumed to be *impulsive*, that is, agents immediately switch their strategies even when the difference between payoffs is extremely small. In this case the dynamics can be formalized as

$$M : x' = \begin{cases} f(x) = x + \delta_A(x)(1 - x) & \text{if} \quad A(x) > B(x) \\ x & \text{if} \quad A(x) = B(x) \\ g(x) = (1 - \delta_B(x))x & \text{if} \quad A(x) < B(x) \end{cases} \tag{1}$$

where the parameters δ_A and δ_B represent how many agents may switch to A and B respectively. When $\delta_A = \delta_B$, there are no differences in the propensity to switch to either strategies. The case with δ_A and δ_B constant parameters varying in $[0, 1)$ is the one already investigated in the cited literature. Now they are assumed to be functions depending on fraction x of agents.

It is worth noticing that depending on the functions involved in its definition, the map M may be continuous or discontinuous, and even if $f(x)$ and $g(x)$, as well as $B(x)$ and $A(x)$, are smooth functions, the map M in general is not differentiable where the payoff functions intersect.

In this paper we assume that agents are concerned not only about the sign difference in payoffs but also on the relative difference. That is, the smaller the absolute difference between payoffs the less likely are agents to switch choices. We consider two polar behaviors. On one side agents are impulsive and switch to the higher payoff choices whatever is the difference. On the other side, agents' decision depends on the magnitude of payoffs difference and not just on the sign. We consider also agents who decide to switch according to a combination of these behaviors. This can be formalized as follows. Assume that when $x \in]0, 1[$ agents are playing A, the payoffs are $A(x)$ and $B(x)$ with $B(x) < A(x)$. Then let $\delta_A(x)$ be the fraction of the $1 - x$ agents playing B who in the next turn will switch to A; we assume that

$$\delta_A(x) = \min\left\{k_A + (1 - k_A)[A(x) - B(x)], 1\right\} \tag{2}$$

with $k_A \in [0, 1)$. Analogously for $B(x) > A(x)$ we define

$$\delta_B(x) = \min\left\{k_B + (1 - k_B)[B(x) - A(x)], 1\right\} \tag{3}$$

with $k_B \in [0, 1)$.

The function upper bound 1 rules out results in which more than existing agents switch choice.

When $k_{A,B} = 0$, agents decide taking into account the absolute value of the difference between the payoffs. That is, the fraction of those switching decision is proportional to the relative difference between payoffs.

Therefore the resulting map is M, $x' = M(x)$ defined above in (1) with $\delta_A(x)$ and $\delta_B(x)$ defined in (2) and (3), respectively, with k_A and k_B are constants belonging to the interval $[0, 1)$.

3. The analysis of the map with linear payoffs

With impulsive agents as in Bischi and Merlone (2009) and Bischi et al. (2009a,b) the relative difference between payoff is not important, therefore they considered only the number of intersections between payoffs functions. On the contrary, when agents switching choices depend on the relative difference between payoffs, their expression is important. For the sake of simplicity, in the following we consider the case in which $A(x)$ and $B(x)$ are linear functions:

$$\begin{aligned} A(x) &= m_A x + q_A \\ B(x) &= m_B x + q_B \end{aligned} \tag{4}$$

in which the offsets q_A and q_B as well as the slopes m_A and m_B may be of any kind, in sign and modulus.

Let us define as d the solution of the equation

$$A(x) = B(x) \tag{5}$$

that is, assuming $m_B \neq m_A$:

$$d = \frac{q_A - q_B}{m_B - m_A} \tag{6}$$

We are interested in the case with the intersection point $d \in [0, 1]$ which implies the following constraints in the parameters:

$$0 \leq \frac{q_A - q_B}{m_B - m_A} \leq 1 \tag{7}$$

so that we have two cases: either (I) $q_A < q_B$ and $m_B < m_A$ or (II) $q_A > q_B$ and $m_B > m_A$, and this leads to the following conditions, respectively:

Case (I): When $q_A < q_B$ and $m_B < m_A$, then it must be

$$q_A < q_B < q_A + m_A - m_B \tag{8}$$

Case (II): When $q_A > q_B$ and $m_B > m_A$, then it must be

$$q_B < q_A < q_B + m_B - m_A \tag{9}$$

In our specific model. We have $A(x) \geq B(x)$ for $(m_A - m_B)x \geq (q_B - q_A)$. Thus in Case (I) we have $m_A > m_B$, so that $A(x) \geq B(x)$ occurs for $x \geq d$, where d is defined in (6). As a consequence the map M to investigate becomes as follows:

$$M_{\text{I}} : x' = \begin{cases} g(x) = (1 - \delta_B(x))x & \text{if} \quad x < d \\ x & \text{if} \quad x = d \\ f(x) = x + \delta_A(x)(1 - x) & \text{if} \quad x > d \end{cases} \tag{10}$$

By contrast, in Case (II) we have $m_A < m_B$, and $A(x) \geq B(x)$ occurs for $x \leq d$. Thus the map M to investigate becomes as follows:

$$M_{\mathrm{II}} : x' = \begin{cases} f(x) = x + \delta_A(x)(1-x) & \text{if} \quad x < d \\ x & \text{if} \quad x = d \\ g(x) = (1-\delta_B(x))x & \text{if} \quad x > d \end{cases} \tag{11}$$

As it concerns the function $\delta_A(x)$, we have that the constraint $k_A + (1 - k_A)(A(x) - B(x)) < 1$ occurs for $A(x) - B(x) < 1$ that is

$$(m_A - m_B)x < 1 - (q_A - q_B) \tag{12}$$

in which case

$$\delta_A(x) = [k_A + (1-k_A)(q_A - q_B)] - (1-k_A)(m_B - m_A)x \tag{13}$$

otherwise $\delta_A(x) = 1$. Notice that if $\delta_A(x) = 1$ then we have $f(x) = 1$.

For the function $\delta_B(x)$ we have that the constraint $[k_B + (1 - k_B)(B(x) - A(x))] < 1$ occurs for $B(x) - A(x) < 1$ that is for

$$(m_B - m_A)x < 1 + (q_A - q_B) \tag{14}$$

in which case

$$\delta_B(x) = [k_B - (1-k_B)(q_A - q_B)] + (1-k_B)(m_B - m_A)x \tag{15}$$

otherwise $\delta_B(x) = 1$. Notice that for $\delta_B(x) = 1$ then we have $g(x) = 0$.

The properties of the single functions $f(x)$ and $g(x)$ in the two different cases under study, leading to the two different maps M_{I} and M_{II} defined above, are considered in the next sections. Here we only prove the following:

Proposition 1. *In both Cases* (*I*) *and* (*II*) *the map M is continuous in* $x = d$ *for* $k_A = k_B = 0$.

In fact, from $g(d) = (1 - \delta_B(d))d$ *and* $\delta_B(d) = k_B$ *we have* $g(d) = (1 - k_B)d \leq d$ *and* $g(d) \in (0, 1)$. *Thus for* $k_B = 0$ *we have* $g(d) = d$ (*while* $g(d) < d$ *for* $k_B \in (0, 1)$).

Similarly from $f(d) = d + \delta_A(d)(1 - d)$ *and* $\delta_A(d) = k_A$ *we have* $f(d) = d + k_A(1 - d) \geq d$ *and* $f(d) \in (0, 1)$. *Thus for* $k_A = 0$ *we have* $f(d) = d$ (*while* $f(d) > d$ *for* $k_A \in (0, 1)$).

4. Case (I): dynamics of the map M_{I}

In this case we have

$$q_A < q_B < q_A + (m_A - m_B), \quad (m_A - m_B) > 0$$

then for $x < d = (q_B - q_A)/(m_A - m_B)$ the constraint (14) is satisfied for $(m_A - m_B)x > -1 + (q_B - q_A)$. Therefore, for $x > \bar{x} = (-1 + (q_B - q_A))/(m_A - m_B)$ the function $g(x)$ is defined as follows:

$$\begin{aligned} g(x) &= (1-\delta_B(x))x \\ &= x - x[k_B - (1-k_B)(q_A - q_B)] - (1-k_B)(m_B - m_A)x^2 \\ &= x[1 - k_B + (1-k_B)(q_A - q_B)] - (1-k_B)(m_B - m_A)x^2 \\ &= x(1-k_B)[1 - (q_B - q_A)] + (1-k_B)(m_A - m_B)x^2 \end{aligned} \tag{16}$$

so that

$$\begin{aligned} g'(x) &= (1-k_B)[1 - (q_B - q_A)] + 2(1-k_B)(m_A - m_B)x \\ g''(x) &= 2(1-k_B)(m_A - m_B) > 0 \end{aligned} \tag{17}$$

and from $g'(x) = (1 - k_B)[1 - (q_B - q_A) + 2(m_A - m_B)x]$ we have $g'(x) \geq 0$ for $x \geq x_{g,c} = (-1 + (q_B - q_A))/(2(m_A - m_B))$. Assuming that the critical point $x_{g,c}$ of $g(x)$ is negative, which occurs for $0 < (q_B - q_A) < 1$, we have that in the range of its definition the function $g(x)$ *is increasing and convex*, with

$$g'(0) = (1-k_B)[1 - (q_B - q_A)] \in (0, 1) \tag{18}$$

Regarding the function $f(x)$, we have that the constraint (12) is satisfied for $x < (1 + (q_B - q_A))/(m_A - m_B)$ in which case the function $f(x)$ is defined as follows:

$$\begin{aligned} f(x) &= x + \delta_A(x)(1-x) \\ &= x + (1-x)[k_A + (1-k_A)(q_A - q_B)] - (1-k_A)(m_B - m_A)x(1-x) \\ &= -\alpha_1 x^2 + \beta_1 x + \gamma_1 \end{aligned} \tag{19}$$

where

$$\begin{aligned} \alpha_1 &= (1-k_A)(m_A - m_B) \\ \beta_1 &= (1-k_A)[1 + (q_B - q_A) + (m_A - m_B)] \\ \gamma_1 &= k_A + (1-k_A)(q_A - q_B) \end{aligned} \tag{20}$$

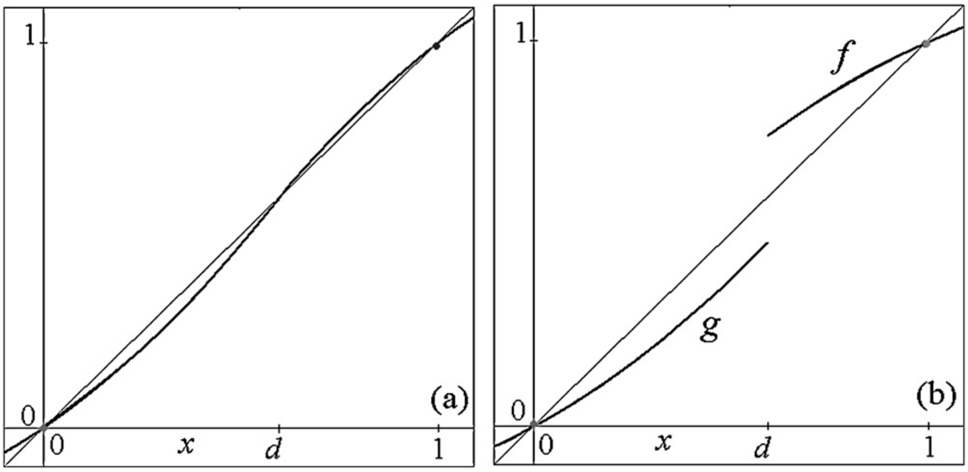

Fig. 1. $q_A = 0.2$, $q_B = 0.5$, $m_A = 0$, $m_B = -0.5$. In (a) $k_A = k_B = 0$. In (b) $k_A = 0.4$, $k_B = 0.2$.

and its first and second derivative are given by:

$$f'(x) = -2(1-k_A)(m_A - m_B)x + (1-k_A)[1 + (q_B - q_A) + (m_A - m_B)]$$
$$f''(x) = -2(1-k_A)(m_A - m_B) < 0 \tag{21}$$

We have $f'(x) \geq 0$ for $x \leq x_{f,c} = (1 + (q_B - q_A) + (m_A - m_B))/(2(m_A - m_B))$. Assuming that the critical point $x_{f,c}$ of $f(x)$ is greater than 1, which occurs for $(m_A - m_B) < 1 + (q_B - q_A)$, we have that in the range of its definition the function $f(x)$ *is increasing and concave*, and (by using the assumption, in (8)) with

$$f'(1) = (1-k_A)[1 + (q_B - q_A) - (m_A - m_B)] \in (0, 1) \tag{22}$$

Summarizing, the explicit expression of the map M_{I} is the following:

$$M_{\text{I}} : x' = \begin{cases} g(x) = x(1-k_B)[1 - (q_B - q_A)] + (1-k_B)(m_A - m_B)x^2 & \text{if} \quad 0 \leq x < d \\ x & \text{if} \quad x = d \\ f(x) = -\alpha_1 x^2 + \beta_1 x + \gamma_1 & \text{if} \quad d < x \leq 1 \end{cases} \tag{23}$$

We remark that the critical points of $f(x)$ and $g(x)$ are independent of the values of the two parameters k_A and k_B.

Then for $k_A = k_B = 0 M_{\text{I}}(x)$ is a continuous function, with three fixed points: $x_0^* = 0$, $x_1^* = 1$ and $x_d^* = d$. The continuity in the end points of the interval is immediate. The continuity in $x = d$ comes from Proposition 1.

The derivatives in the fixed points are given by $g'(0) = [1 - (q_B - q_A)] \in (0, 1)$; $f'(1) = [1 + (q_B - q_A) - (m_A - m_B)] \in (0, 1)$. In the fixed point d we have $g'(d) = 1 + (q_B - q_A) > 1$ and $f'(d) = 1 - (q_B - q_A) + (m_A - m_B) > 1$ so that x_0^* and x_1^* are both locally stable (from the side of interest), while x_d^* is a repelling fixed point, and separates the two basins of attraction: $\mathcal{B}(0) = [0, d)$ and $\mathcal{B}(1) = (d, 1]$. An example is shown in Fig. 1(a). Now considering any value for the parameters k_A and k_B different from 0 and smaller that 1, we have that map F becomes discontinuous in d, and the kind of continuity breaking is, let us say, increasing/increasing with positive jump, being $g(d) = (1-k_B)d < d$ and $f(d) = d + k_A(1-d) > d$. Then the dynamics of the map F persists to be of the same kind. In fact, as noticed above, also now we have $g'(0) \in (0, 1)$ and $f'(1) \in (0, 1)$ thus we still have two coexistent fixed points $x_0^* = 0$ and $x_1^* = 1$, whose basins of attraction are separated by the point $x = d$. That is, as in the continuous case, we still have $\mathcal{B}(0) = [0, d)$ and $\mathcal{B}(1) = (d, 1]$. As example is shown in Fig. 1(b).

Summarizing we have proved the following

Proposition 2. *Let* $0 < (q_B - q_A) < 1$ *and* $0 < (m_A - m_B) < 1 + (q_B - q_A)$. *Then* $g(x)$ *is increasing and convex and* $f(x)$ *is increasing and concave. For any value of the parameters* k_A *and* k_B *in* $[0, 1]$, *the map* M_{I} *has the fixed point* $x_0^* = 0$ *which attracts the points in* $[0, d)$ *and the fixed point* $x_1^* = 1$ *which attracts the points in* $(d, 1]$.

This result confirms and extends the findings of Schelling (1973, p. 403). In fact, in his analysis, Schelling concludes that, depending on the payoff curves position, there may exist two stable equilibria, namely $x = 0$ and $x = 1$, where everybody is respectively choosing one of the choices, whereas the inner equilibrium x_d^* is unstable. While the conclusion by Schelling is based on the qualitative properties of the system, the same results are confirmed by Bischi and Merlone (2009) in their quantitative model. As the model we present in this paper extends the behavior by agents considered in the literature it is important to remark that even when agents consider difference between payoff not just in terms of sign, there are situations in which the population converge to one of the two choices.

As described in the next section, the dynamics occurring in the second case are much different, and leads to attracting cycles of any period.

5. Case (II): dynamics of the map M_{II}

Let us now consider Case (II) and the related map M_{II}. The parameters satisfy

$$q_B < q_A < q_B + m_B - m_A, \quad (m_B - m_A) > 0 \tag{24}$$

so that for the function $\delta_A(x)$ we have that (12) holds for $(m_B - m_A)x > -1 + (q_A - q_B)$. Thus for

$$x > \bar{x} = \frac{-1 + (q_A - q_B)}{(m_B - m_A)} \tag{25}$$

it is $\delta_A(x) = [k_A + (1 - k_A)(q_A - q_B)] - (1 - k_A)(m_B - m_A)x$ and the function $f(x)$ is defined as follows:

$$\begin{aligned} f(x) &= x + \delta_A(x)(1 - x) \\ &= x + (1 - x)[k_A + (1 - k_A)(q_A - q_B)] - (1 - k_A)(m_B - m_A)x(1 - x) \\ &= \alpha_2 x^2 + \beta_2 x + \gamma_2 \end{aligned}$$

where

$$\begin{aligned} \alpha_2 &= (1 - k_A)(m_B - m_A) \\ \beta_2 &= (1 - k_A)[1 - (q_A - q_B) - (m_B - m_A)] \\ \gamma_2 &= k_A + (1 - k_A)(q_A - q_B) \end{aligned} \tag{26}$$

We also have

$$\begin{aligned} f'(x) &= 2(1 - k_A)(m_B - m_A)x + (1 - k_A)[1 - (q_A - q_B) - (m_B - m_A)] \\ f''(x) &= 2(1 - k_A)(m_B - m_A) > 0 \end{aligned} \tag{27}$$

and $f'(x) \geq 0$ for $x \geq x_{f,c} = (-1 + (q_A - q_B) + (m_B - m_A))/(2(m_B - m_A))$.

Two cases may occur, that is, the critical point $x_{f,c}$ of $f(x)$ may be smaller than d or not. As we shall see, these two cases qualify different kinds of dynamic behaviors. We have

$$x_{f,c} < d \quad \text{for} \quad (m_B - m_A) < 1 + (q_A - q_B) \tag{28}$$

otherwise the critical point is $x_{f,c} > d$. Summarizing:

(a) for $(m_B - m_A) < 1 + (q_A - q_B)$ we have $x_{f,c} < d$ and $f(x)$ is *increasing and convex* for x in a left neighborhood of d;
(b) for $(m_B - m_A) > 1 + (q_A - q_B)$ we have $x_{f,c} > d$ and $f(x)$ is *decreasing and convex* for x in a left neighborhood of d.

Regarding the second function $g(x)$, we have that $(m_B - m_A)x < 1 + (q_A - q_B)$ occurs for

$$x < x^* = \frac{1 + (q_A - q_B)}{(m_B - m_A)} \tag{29}$$

so that for $x > x^*$ the function $\delta_B(x) = 1$ leads to $g(x) = 0$, while for $x < x^*$ we have $\delta_B(x) = [k_B - (1 - k_B)(q_A - q_B)] + (1 - k_B)(m_B - m_A)x$ and

$$g(x) = x(1 - k_B)[1 + (q_A - q_B)] - (1 - k_B)(m_B - m_A)x^2 \tag{30}$$

its derivatives are as follows:

$$\begin{aligned} g'(x) &= (1 - k_B)[1 + (q_A - q_B)] - 2(1 - k_B)(m_B - m_A)x \\ g''(x) &= -2(1 - k_B)(m_B - m_A) < 0 \end{aligned} \tag{31}$$

From $g'(x) = (1 - k_B)[1 + (q_A - q_B) - 2(m_B - m_A)x]$ we have $g'(x) \geq 0$ for $x \leq x_{g,c} = (1 + (q_A - q_B))/(2(m_B - m_A))$. Notice that

$$x_{g,c} > d \quad \text{for} \quad (q_A - q_B) < 1 \tag{32}$$

Thus we can distinguish two cases also here:

(a') for $(q_A - q_B) < 1$ we have $x_{g,c} > d$ so that for $d < x < x^*$ the function $g(x)$ defined in (30) is *locally increasing and concave* in a right neighborhood of d;
(b') for $(q_A - q_B) > 1$ we have $x_{g,c} < d$ so that for $d < x < x^*$ the function $g(x)$ defined in (30) is *locally decreasing and concave* in a right neighborhood of d.

Now notice that

$$\bar{x} = \frac{-1 + (q_A - q_B)}{(m_B - m_A)} > 0 \quad \text{for} \quad (q_A - q_B) > 1 \tag{33}$$

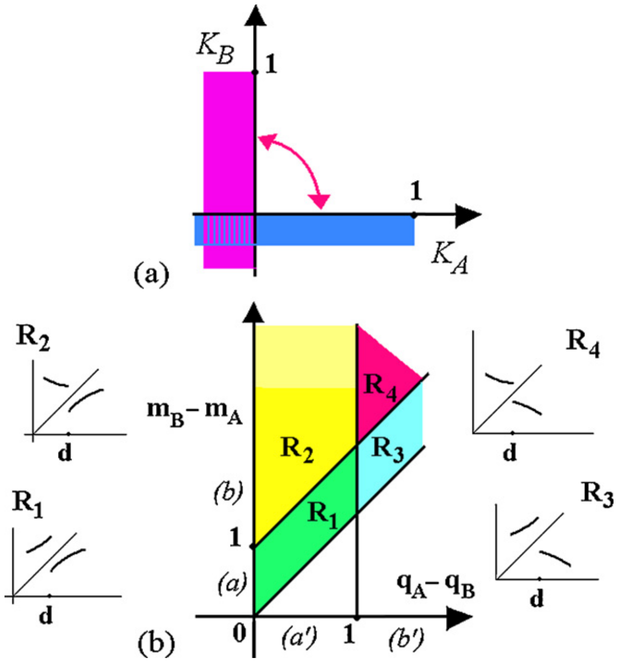

Fig. 2. In (a) the region of interest for parameters k_A and k_B; in (b) regions R_1, R_2, R_3, and R_4.

and that

$$x^* = \frac{1+(q_A-q_B)}{(m_B-m_A)} < 1 \quad \text{for} \quad (m_B-m_A) > 1+(q_A-q_B) \tag{34}$$

Summarizing, the explicit expression of the map M_{II} is the following:

$$M_{\text{II}} : x' = \begin{cases} f(x) = 1 & \text{if} \quad x \le \bar{x} \\ f(x) = \alpha_2 x^2 + \beta_2 x + \gamma_2 & \text{if} \quad \bar{x} < x < d \\ x & \text{if} \quad x = d \\ g(x) = x(1-k_B)[1+(q_A-q_B)] - (1-k_B)(m_B-m_A)x^2 & \text{if} \quad d < x < x^* \\ g(x) = 0 & \text{if} \quad x \ge x^* \end{cases} \tag{35}$$

and it is continuous in the points $x = \bar{x}$ and $x = x^*$.

We remark that the critical points $x_{f,c}$ and $x_{g,c}$ of $f(x)$ and $g(x)$ respectively, do not depend on the parameters k_A and k_B as well as all the other conditions which distinguish between the cases described above.

In all the possible cases, for $k_A = k_B = 0$ the map M_{II} is continuous in $x = d$ (from Property 1) and in $x = d$ the local attractivity or instability is determined from the derivatives of the functions on the two sides of d, where we have

$$f'(d) = (1-k_A)[1+(q_A-q_B)-(m_B-m_A)] \tag{36}$$

which is positive in case (a) and negative in case (b); in both cases its modulus may be smaller or higher than one.

Similarly also

$$g'(d) = (1-k_B)[1-(q_A-q_B)] \tag{37}$$

which is positive in case (a'), negative in case (b'), and its modulus may be smaller or higher than one. However, we notice that when the critical point $x_{f,c}$ is close to d then the derivatives $f'(d)$ is close to zero, and thus the fixed point d is locally attracting on the left side of the discontinuity, and this local stability existing for $k_A = 0$ is the same for any value $k_A \in (0, 1)$. Similarly when the critical point $x_{g,c}$ is close to d then the derivatives $g'(d)$ is close to zero, and thus the fixed point d is locally attracting on the right side of the discontinuity, and this local stability existing for $k_B = 0$ is the same for any value $k_B \in (0, 1)$. In particular, for $|f'(d)|<1$ and $|g'(d)|<1$, at $k_A = k_B = 0$ the continuous map M_{II} has an attracting fixed point in $x = d$.

As already remarked, we can have all the possible combinations (a)–(a'), (b)–(a'), (a)–(b'), (b)–(b'). For the sake of simplicity, they are summarized in Fig. 2(b) in the parameter plane $((q_A - q_B), (m_B - m_A))$ leading to the regions R_1, R_2, R_3, and R_4, respectively. In Fig. 2(a) we show schematically the parameter region of interest. In Fig. 2(b) we also show for $k_{A,B} \in (0, 1)$ the qualitative behavior of the functions in a neighborhood of $x = d$ in the related regions. In fact, for values of the parameters k_A and k_B not both zero, the map M_{II} has a discontinuity in $x = d$, with

$$f(d) = d + k_A(1-d) > d \quad \text{and} \quad g(d) = (1-k_B)d < d \tag{38}$$

This continuity breaking has a strong effect on the dynamic behaviors. In fact, except for the occurrence of cases (b) and (b'), we have that for values of (k_A, k_B) in any neighborhood of (0, 0) we can have attracting cycles of any period, as stated in the following

Proposition 3. (1) *Let* $0<(q_A-q_B)<1$ *and* $(q_A-q_B)<(m_B-m_A)<1+(q_A-q_B)$ (*region* R_1). *Then in the two-dimensional positive parameter plane* (k_A, k_B) *the point* (0, 0) *is a big-bang bifurcation point for the map* $M_{\rm II}$, *from which infinitely many BCB curves are issuing, following the period adding scheme*.

(2) *Let* $0<(q_A-q_B)<1$ *and* $1+(q_A-q_B)<(m_B-m_A)<2+(q_A-q_B)$ (*belonging to region* R_2). *Then in the two-dimensional positive parameter plane* (k_A, k_B) *the point* (0, 0) *is a big-bang bifurcation point for the map* $M_{\rm II}$, *from which infinitely many BCB curves are issuing, following the period increment scheme.*

(3) *Let* $1<(q_A-q_B)<(m_B-m_A)<1+(q_A-q_B)<3$ (*belonging to region* R_3). *Then in the two-dimensional positive parameter plane* (k_A, k_B) *the point* (0, 0) *is a big-bang bifurcation point for the map* $M_{\rm II}$, *from which infinitely many BCB curves are issuing, following the period increment scheme*.

(4) *Let* $2<1+(q_A-q_B)<(m_B-m_A)<2+(q_A-q_B)<4$ (*belonging to region* R_4). *Then for* (k_A, k_B) *in the two-dimensional positive parameter plane close to the point* (0, 0) *there exists an attracting cycle of period 2*.

The proof is reported in Appendix A.

Remark 1. We notice that the conditions given in Proposition 3 are sufficient, but not necessary. In fact, as we shall see in the examples in the next subsections, the condition of local stability for $f'(d)$ and $g'(d)$ are quite strong, and the results of the big-bang bifurcation points can be seen also when these conditions are not satisfied.

Remark 2. The results of the continuity breaking are local, i.e., these hold for values of the parameters (k_A, k_B) close to the point (0, 0), but relevant results also exist in large, for values of the parameters (k_A, k_B) far from the point (0, 0). These are related with the global shapes of the functions $f(x)$ and $g(x)$ (while the previous result is only due to the local shape, close to the discontinuity point $x=d$). These further bifurcations are due to the intersection of BCB curves, whose equations will be given in implicit form. The existence of intersections leads to other big-bang bifurcation points (analogous to the origin (0, 0) of the parameter plane), from which infinitely many curves issue, following the *period adding structure*.

In the next subsections we shall illustrate several examples. In particular, in the last one, we shall see that the result for parameter values in the region R_4 holds in large.

5.1. Increasing/increasing case

Let us consider the parameters of the map $M_{\rm II}$ belonging to the region R_1. As remarked above, we have that when $M_{\rm II}$ is continuous ($k_A=k_B=0$), the fixed point $x_d^*=d$ is attracting on both sides and locally attracting for T. When the continuity is broken and a jump in $x=d$ occurs, the parameters in this region R_1 lead to a discontinuous map with an increasing branch on the left of $x=d$, above the diagonal, and an increasing branch on the right, below the diagonal, so that the fixed point disappears and no fixed points are left in a neighborhood of $x=d$. Locally the map has the qualitative shape shown in Fig. 2 (region R_1). The jump in $x=d$ is determined by $f(d)$ (upper value) and $g(d)$ (lower value). These values determine an absorbing interval $I=[g(d), f(d)]$ from which the dynamics cannot escape. We recall that this increasing/increasing case was already considered by Keener (1980) in a remarkable paper, and we know that, as long as the map is uniquely invertible in I, only stable cycles can exist, and only one at each fixed parameters (i.e., bistability cannot occur). Moreover, the structure of all the possible existing cycles has been recently described in Avrutin et al. (submitted for publication) showing also how to obtain also the BCB curves of the period adding structure. Locally, for (k_A, k_B) in a neighborhood of (0, 0), the functions may be approximated by the linear parts in $x=d$, leading to a piecewise linear map with a discontinuity point. The linear case has been fully described in Gardini et al. (2010a), Avrutin et al. (2010a), for which not only the structure can be explained, but also the bifurcation curves can be determined analytically. We notice that inside the absorbing interval I the only possible bifurcations are due to a collision with a periodic point with the discontinuity point, that is, only *border collision bifurcations* (BCB) can occur.

In Fig. 3 we show an example of continuity breaking in this region. When the map is discontinuous, Fig. 3(b) shows the convergence to a stable 2-cycle. However the period of the attracting cycle existing in the discontinuous map $M_{\rm II}$ depends on the relative values of the parameters k_A and k_B. A picture in the whole parameter plane (k_A, k_B) where both variables range between 0 and 1 is shown in Fig. 4(a), while Fig. 4(b) shows an enlarged part, close to the point (0, 0). There, it can be appreciated the BCB structure related with a piecewise linear discontinuous map, for which the boundaries of the periodicity regions (BCB curves) can also be detected analytically. There are periodicity regions associated with cycles of *first complexity level* (following the notation introduced by Leonov (1959, 1962) and used also in Gardini et al. (2010a), which are also known as *principal cycles* or *maximal cycles*. Below we give the equation in implicit form of the BCB curves bounding the regions. Between any pair of consecutive periodicity regions of the first level, two infinite families of periodicity regions can be found, following the Farey summation rule in the period and rotation number (Hao, 1989), called of *second complexity level*, and the process continues *for any level of complexity*.

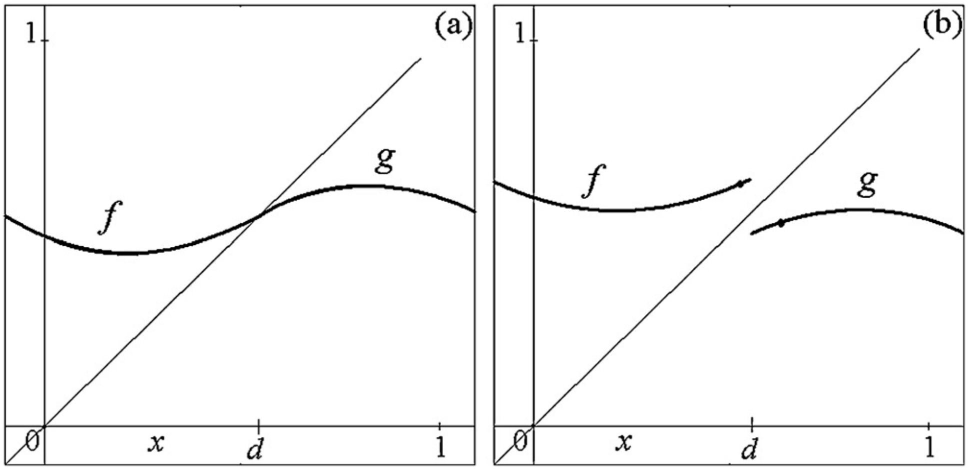

Fig. 3. $q_A = 0.7$, $q_B = 0.2$, $m_A = 0$, $m_B = 0.9$. In (a) $k_A = k_B = 0$. In (b) $k_A = 0.2$, $k_B = 0.1$, a 2-cycle is attracting.

The cycles can be identified using a symbolic sequence, in which we use the letter L (resp. R) to denote a periodic point on the left (resp. right) side of the discontinuity point. We recall that a sequence is cyclically invariant, as it represents periodic points of the same cycle. For example, maximal cycles have the symbolic sequence LR^n or RL^n for any $n \geq 1$.

In our map M_{II}, the periodicity regions of maximal cycles of symbolic sequence LR^n for any $n \geq 1$ have as limit set the axis k_A, of equation $k_B = 0$. The unique periodic point on the left side of $x = d$ of these cycles can be determined by using the equation $g^n \circ f(x) = x$. The boundaries of the periodicity regions in which these cycles exist are given by the BCB curves of implicit equation as follows:

$$g^n \circ f(d) = d, \quad g^{n-1} \circ f \circ g(d) = d \tag{39}$$

On the contrary, the periodicity regions of maximal cycles of symbolic sequence RL^n for any $n \geq 1$ have as limit set the other axis, of equation $k_A = 0$. The unique periodic point on the right side of $x = d$ of these cycles can be determined by using the equation $f^n \circ g(x) = x$. The boundaries of the periodicity regions in which these cycles exist are given by the BCB curves of implicit equation as follows:

$$f^n \circ g(d) = d, \quad f^{n-1} \circ g \circ f(d) = d \tag{40}$$

Similarly we can write the implicit equations of the BCB curves for any level of complexity.

As already remarked, this structure certainly occurs locally, close to (0, 0). On the other hand, when (k_A, k_B) are changed more in large, the qualitative shape of the map is no longer only increasing/increasing. That is, in the absorbing interval inside which the dynamics of the map are confined, the nonlinear functions $f(x)$ and $g(x)$, modify their shape, and other

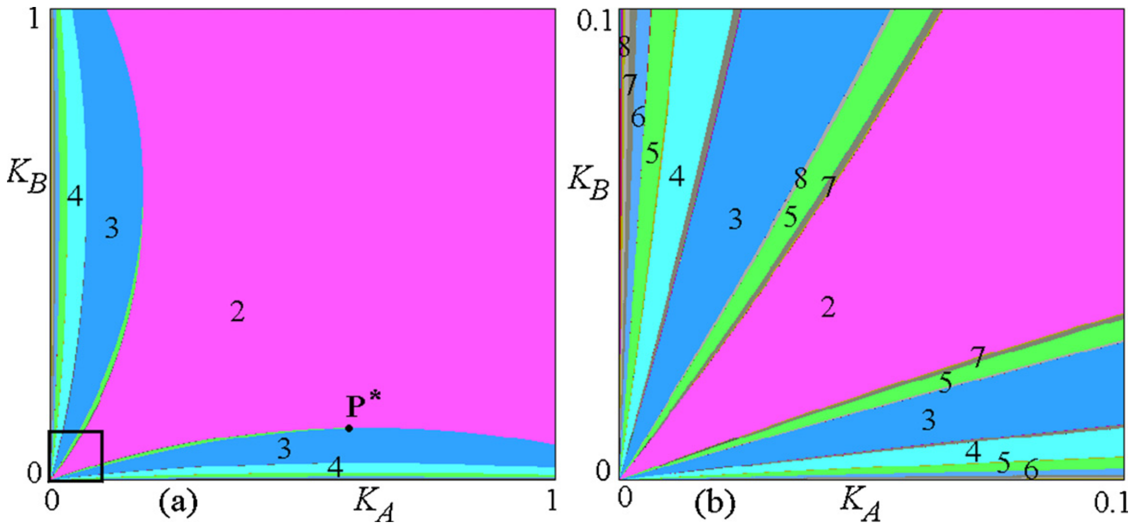

Fig. 4. $q_A = 0.7$, $q_B = 0.2$, $m_A = 0$, $m_B = 0.9$. BCB in the parameter plane (k_A, k_B). In (b) the enlarged part of the small square close to (0, 0).

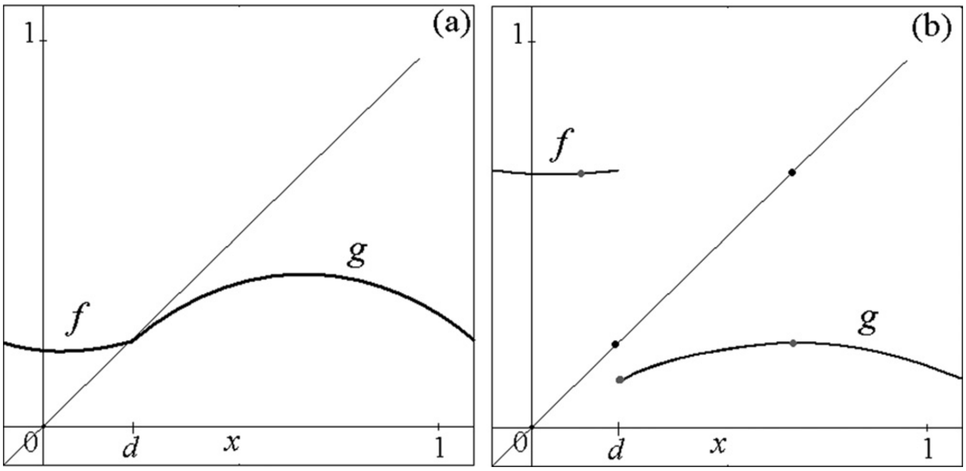

Fig. 5. $q_A = 0.4$, $q_B = 0.2$, $m_A = 0$, $m_B = 0.9$. In (a) $k_A = k_B = 0$. In (b) $k_A = 0.58$, $k_B = 0.4444$; the initial point $x = 0.5$ converges to the 3-cycle, the initial point $x = 0.6$ converges to the 2-cycle.

bifurcations may occur. An example is illustrated in the particular point P^* shown in Fig. 4(a). This particular bifurcation point, as well as the origin (0, 0), is a big-bang bifurcation. In order to provide a better illustration of this bifurcation point we consider a different example, where this particular point is better observable.

In Fig. 5 we show the continuity breaking at other parameters always belonging to the region R_1, while in Fig. 6 we can see that the periodicity regions having as limit set the k_A axis are wider, so that we can better observe that the BCB curves of the 2-cycle and the 3-cycle intersect in a point P_2^* illustrated in Fig. 6(a). Locally the behavior of the periodicity tongues is the one qualitatively drawn in Fig. 6(b). That is, on the right of such a point there is *a region of overlapping, inside which we can see coexistence of a* 2-*cycle and of a* 3-*c* ycle. A numerical example is shown in Fig. 5(b), for a parameter point taken inside this overlapping region: taking an initial condition close to the left of the discontinuity point $x = d$ we have convergence to the 2-cycle, whose periodic points have been drawn on the bisectrix of Fig. 5(b), while taking an initial condition close to the right of the discontinuity point $x = d$ we have convergence to the 3-cycle, whose periodic points have been drawn on the graphs of the functions of Fig. 5(b). It is clear that this big-bang bifurcation point is not unique. In fact, we can see that all the periodicity regions of the maximal cycles with symbolic sequence LR^n for any $n \geq 1$ have a region of bistability issuing from a big-bang bifurcation point in which they are intersecting in pair, that is $P_n^* = \mathrm{BCB}_{LR^n} \cap \mathrm{BCB}_{LR^{n+1}}$ exist for any $n \geq 1$.

5.2. Decreasing/increasing case

In this section we consider parameters in the region R_2. As already remarked, the continuity breaking is characterized by a locally decreasing branch on the left side of the discontinuity point $x = d$, and a locally increasing branch on the right side of it.

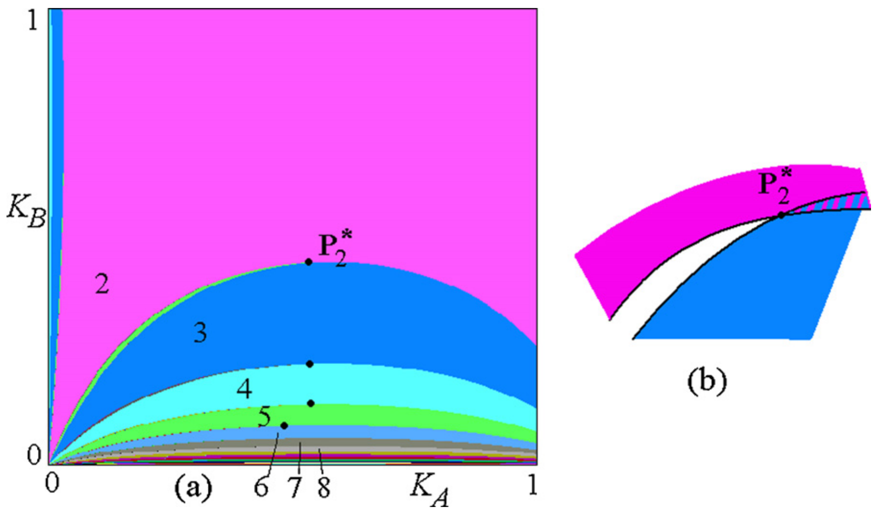

Fig. 6. $q_A = 0.4$, $q_B = 0.2$, $m_A = 0$, $m_B = 0.9$. In (a) BCB in the parameter plane (k_A, k_B). In (b) qualitative picture of the BCB structure leading to a big-bang bifurcation point P^*.

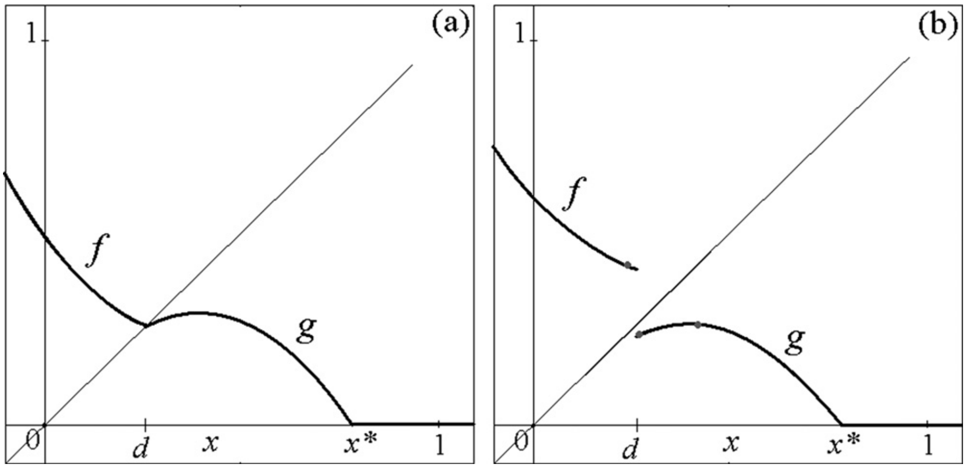

Fig. 7. $q_A = 0.7$, $q_B = 0.2$, $m_A = 0$, $m_B = 1.9$. In (a) $k_A = k_B = 0$. In (b) $k_A = 0.2$, $k_B = 0.1$.

An example is shown in Fig. 7 where, besides the point $x = d$, also the point $x = x^*$ at which $g(x)$ becomes 0 is shown. In Fig. 7(b) the attracting set existing after the continuity breaking is a cycle of period 3, and here it is the only attracting set. However, for different values of the parameters k_A and k_B, it is also possible to find a case of bistability between a 3-cycle and a 2-cycle. In fact, we are in a regime in which the results provided in Gardini et al. (submitted for publication) applies, as shown in Fig. 8. That is, in a neighborhood of $(k_A, k_B) = (0, 0)$ we have the period increment structure, where the maximal cycles of symbolic sequence LR^n exist for any $n \geq 1$. In general, the unique periodic point on the left side of $x = d$ of these cycles can be determined by using the equation $g^n \circ f(x) = x$. The boundaries of the periodicity regions in which these cycles exist are given by the BCB curve of implicit equation already written in (39), that is, $g^n \circ f(d) = d$ and $g^{n-1} \circ f \circ g(d) = d$. Locally, in a neighborhood of $(k_A, k_B) = (0, 0)$, the periodicity regions of the maximal cycles must have a region of bistability (see Gardini and Tramontana, 2010; Avrutin et al., 2010b, submitted for publication; Gardini et al., submitted for publication). In Fig. 8(b) a bistability region is illustrated between the periodicity regions of the 2-cycle and the 3-cycle, as well as between the periodicity regions of the 3-cycle and the 4-cycle. Clearly all the other overlapping regions also exist, although very thin and not observable in Fig. 8(b). All the periodicity regions of the maximal cycles issuing from (0, 0) with symbolic sequence LR^n for any $n \geq 1$ have a region of bistability issuing from (0, 0).

As we know, this structure occurs locally, close to (0, 0). However, when (k_A, k_B) are changed more in large, the qualitative shape of the map is no longer only decreasing/increasing, that is, in the absorbing interval inside which the dynamics of the

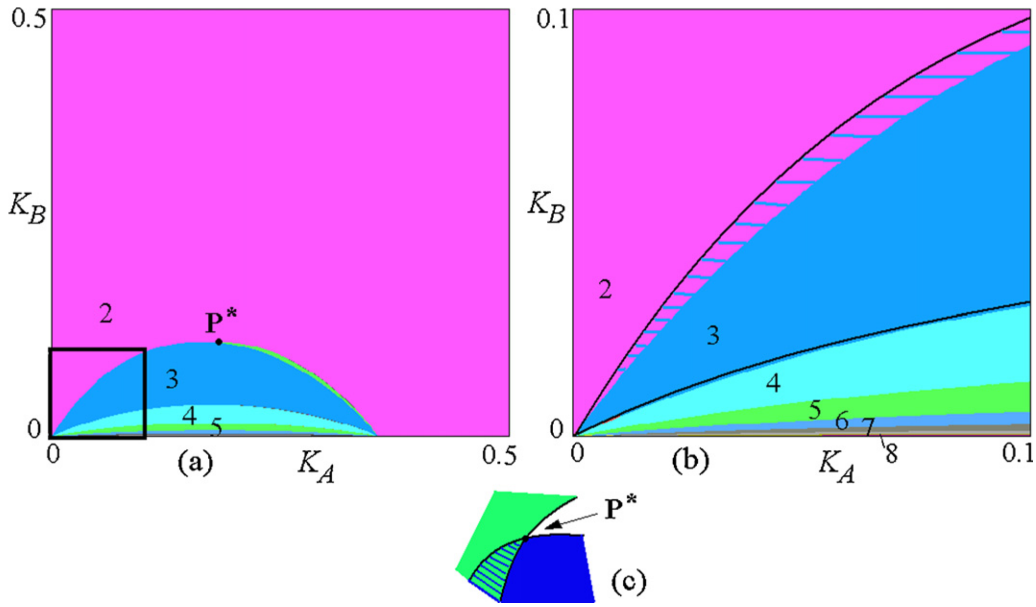

Fig. 8. $q_A = 0.7$, $q_B = 0.2$, $m_A = 0$, $m_B = 1.9$. In (a) BCB in the parameter plane (k_A, k_B), and enlarged part in (b). In (b) a bistability region is emphasized. In (c) the qualitative intersection of two BCB curves leading to a big-bang bifurcation point P^*.

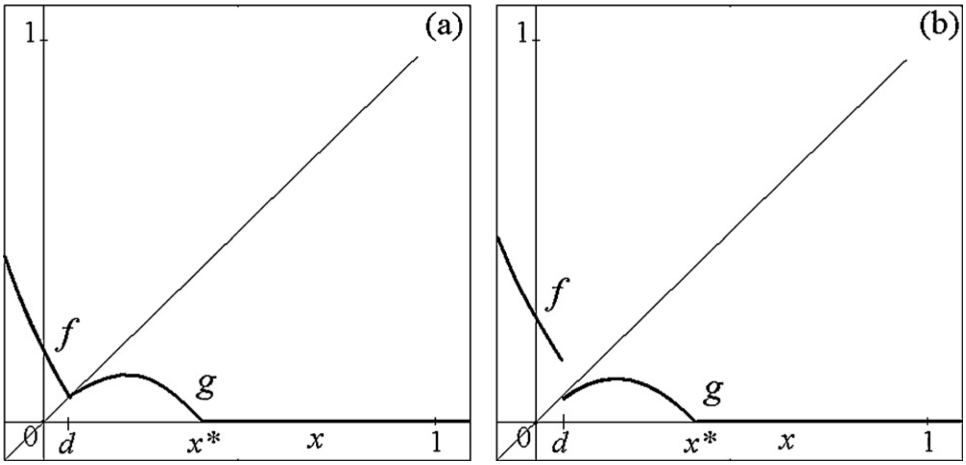

Fig. 9. $q_A = 0.4$, $q_B = 0.2$, $m_A = 0$, $m_B = 2.9$. In (a) $k_A = k_B = 0$. In (b) $k_A = 0.1$, $k_B = 0.05$.

map are confined, the nonlinear functions $f(x)$ and $g(x)$, modify their shape, and other bifurcations may occur. An example is shown in the particular point P^* illustrated in Fig. 8(a). Here, in large, the BCB curves defined by the equations in (39) have one more intersection leading to a big-bang bifurcation point through a different bifurcation mechanism. And this occurs in pair for all the periodicity regions of the maximal cycles issuing from (0, 0). The structure of all the big-bang bifurcation points is clearly similar to the one already met in the previous subsection, and qualitatively shown also in Fig. 8(c). That is, two periodicity regions are overlapping, leading to a portion of bistability (where both cycles exist), two regions where a unique cycle exists, and a region issuing from the intersection point in which these two cycles do not exist. Exactly in this last region the adding mechanism applies and an infinite number of periodicity regions can be found, following the *period adding structure*.

A different example in which the different big-bang bifurcation points $P_n^* = BCB_{LR^n} \cap BCB_{LR^{n+1}}$ for any $n \geq 1$ can be seen in Figs. 9 and 10. In Fig. 10(b) we also show a one-dimensional bifurcation diagram of x as a function of k_B along the path shown as a vertical line in Fig. 10(a), where it is possible to see the quick transition from a period to another one for the attracting set.

5.3. Increasing/decreasing case

When the parameters belong to the region R_3, on the two sides of the discontinuity point the increasing and decreasing parts are exchanged; nevertheless the reasoning is similar. In the parameter plane (k_A, k_B) the BCB curves have a similar shape, although "symmetric" with respect to those described above. That is, in this case we have the periodicity regions of maximal cycles of symbolic sequence RL^n for any $n \geq 1$ having as limit set the k_B axis, of equation $k_A = 0$. The unique periodic point on the right side of $x = d$ of these cycles can be determined by using the equation $f^n \circ g(x) = x$. The boundaries of the

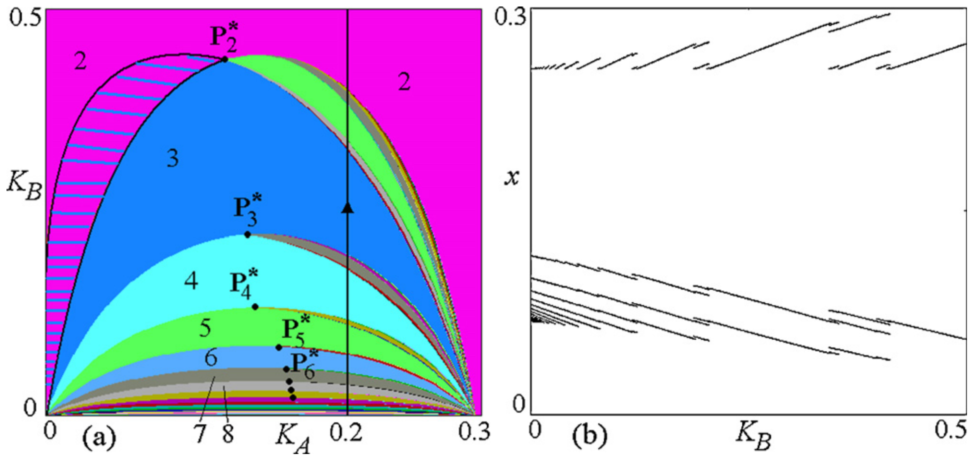

Fig. 10. $q_A = 0.4$, $q_B = 0.2$, $m_A = 0$, $m_B = 2.9$. In (a) BCB in the parameter plane (k_A, k_B), big-bang bifurcation points are emphasized. In (b) one dimensional bifurcation diagram at $k_A = 0.2$ fixed, along the vertical path shown in (a).

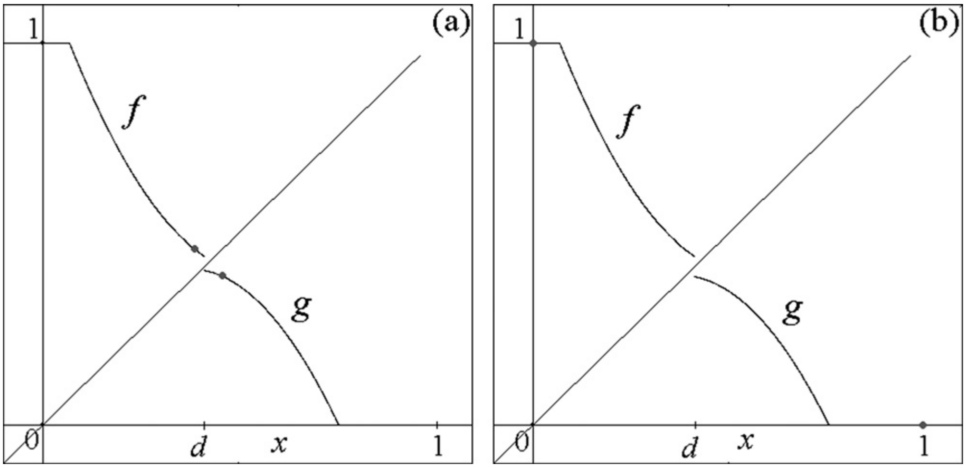

Fig. 11. $q_A = 1.4$, $q_B = 0.2$, $m_A = 0$, $m_B = 2.9$. In (a) at $k_A = 0.05$ and $k_B = 0.02$ a local 2-cycle can be seen. In (b) at $k_B = 0.06$ the 2-cycle with periodic points 0 and 1 is to the result of the constraints.

periodicity regions in which these cycles exist are given by the BCB curve of implicit equation $f^n \circ g(d) = d$ and $f^{n-1} \circ g \circ f(d) = d$, as already given in (40). And similarly we have that in large these periodicity regions intersect in pair leading to infinitely many big-bang bifurcation points and related *period adding structure* between the two BCB curves involved.

5.4. Decreasing/decreasing case

When the parameters belong to region R_4, the dynamic behavior of the map is very simple. Considering the derivatives $f'(d)$ and $g'(d)$ given in (27) and (31), respectively, when we have $f'(d)g'(d) < 1$ then a stable 2-cycle exists after the continuity breaking, as proved (locally) via Proposition 3. However even when this condition is not satisfied, and an absorbing interval including the discontinuity point does not exist, due to the shape of our functions $f(x)$ and $g(x)$ we can only have a stable 2-cycle, as stated in the following

Proposition 4. *Let* $2 < 1 + (q_A - q_B) < (m_B - m_A) < 2 + (q_A - q_B)$ *(region* R_4*). Then for* $(k_A, k_B) \in (0, 1] \times (0, 1]$ *a stable 2-cycle exists.*

Proof. When the parameters belong to region R_4 the minimum of $f(x)$ is above the discontinuity point, thus $f(x)$ is decreasing in its region of definition. Similarly the maximum of $g(x)$ is below the discontinuity point, thus $g(x)$ is decreasing in its region of definition. It follows that a bounded (by construction) decreasing discontinuous function without a fixed point can have at most cycles of period 2. Here we have that a unique 2-cycle can exist, because the functions $f(x)$ and $g(x)$, when not constant, are second degree polynomials. □

Two examples are shown in Fig. 11.

6. Conclusions

Recent literature has considered and examined discrete-time dynamic models of repeated binary choices with externalities, based on the qualitative properties described by Schelling (1973). So far (Bischi et al., 2009a,b), the analysis has been conducted considering impulsive agents, i.e., agents who immediately switch their strategies even when the difference between payoffs is extremely small. In this paper we considered more realistic behaviors. In fact, we assumed that agents may decide to switch choices taking into account the relative difference in terms of payoffs. This way, we were able to model a continuum of behaviors which ranged from agents considering the payoffs in terms of relative differences to impulsive agents as in previous studies. The results of our analysis confirm Schelling's findings about stable equilibria and also the occurrence of cyclic behaviors as described in Bischi and Merlone (2009) and analyzed in Bischi et al. (2009a,b). Nevertheless, in the case of cyclic behaviors, the analysis we provided in this paper shows different kinds of dynamics. First, while with impulsive agents the shape of payoff function is important just in terms of the number of intersections, with non-impulsive agents, the difference between payoffs values is important. Even in the case of linear payoffs the dynamics can be quite different depending on the relative difference between the slopes and intercepts of the payoff functions. In fact, depending on these values, in the origin $(0, 0)$ we can have either a big-bang bifurcation point following the period adding scheme, or a big-bang bifurcation point following the period increment scheme or an attracting cycle of period 2 in its neighborhood. The analytic expression of the border collision bifurcation curves issuing from $(0, 0)$ is given in implicit form. Furthermore, there may exists bifurcation-points different form the origin; since in the case of impulsive agents big-bang bifurcation points following the period adding scheme can occur only in the origin, this shows a remarkable difference in terms of dynamics. The fact that, under some conditions, we may have several big-bang bifurcation points shows how the dynamics, although qualitatively similar, may be quite different from those analyzed with impulsive agents with respect to the period of the attracting set. In Bischi and Merlone (2009) the switching propensity was modeled in terms of the population of agents: given that one choice gave a larger payoff the parameter described the percentage of switching agents. On the contrary, in this paper the switching propensity becomes a function of how the agent considered the difference in payoffs. In this sense the switching propensity becomes a perception parameter instead of a decision parameter. This approach not only allowed us to describe a less simplified behavior of the agents but also provided more interesting dynamics with bifurcation points different from the origin.

The case of nonlinear payoff curves is still to be analyzed, and is left for further research. Other interesting cases to investigate are those with more than a single intersection, as described in Schelling (1973). Finally it will be interesting to explore the boundary between small groups and large groups as in Bischi and Merlone (2010a,b).

Appendix A.

Proof of Proposition 3. At the discontinuity point $x = d$ of the map M_{II} we have $f(d) = d + k_A(1 - d) > d$ and $g(d) = (1 - k_B)d < d$ for ant $k_{A,B} \in (0, 1)$. The conditions given in Proposition 3 are such that $|f'(d)|<1$ and $|g'(d)|<1$, from (36) and (37) at $k_A = k_B = 0$, and thus $|f'(d)|<1$ and $|g'(d)|<1$ for any $k_{A,B} \in (0, 1)$. This proves that for values of the parameters (k_A, k_B) close to the point (0, 0), the map has bounded dynamics close to the discontinuity point. That is, there exists a trapping region close to the discontinuity point, from which the dynamics cannot escape.

When the parameters satisfy the conditions of Case (II) and (a)–(b) given above (region R_1), then we have point (1) of Proposition 3. At these parameters' values we have $f'(d) \in (0, 1)$ as well as $g'(d) \in (0, 1)$, so that close to the discontinuity point, $f(x)$ and $g(x)$ are both increasing functions. Then for values of the parameters (k_A, k_B) close to the point (0, 0) the map M_{II} possesses an absorbing interval given by $I = [f(d), g(d)] = [d + k_A(1 - d), (1 - k_B)d]$ from which the dynamics cannot escape. These conditions are sufficient to state the existence of a big-bang bifurcation point from which periodicity regions following an adding scheme are issuing (Avrutin et al., submitted for publication). Thus in the region R_1 we have a so-called increasing/increasing case with negative jump, in which the breaking of the continuity in the map M_{II} leads to a unique stable cycle. Infinitely many periodicity regions, of any period, are issuing from the point (0, 0) in the two-dimensional parameter plane (k_A, k_B), following the so-called *period adding structure*.

When the parameters satisfy point (2) of Proposition 3 then Case (II) and (a')–(b) given above are satisfied, together with $-1 < f'(d) < 0$, so that close to the discontinuity point $f(x)$ is decreasing and $g(x)$ is an increasing functions (as $g'(d) \in (0, 1)$). Then for values of the parameters (k_A, k_B) close to the point (0, 0) the map M_{II} possesses an absorbing interval given by $I = [g(d), f \circ g(d)]$ from which the dynamics cannot escape. These conditions are sufficient to state the existence of a big-bang bifurcation point. Infinitely many periodicity regions, of any period, are issuing from the point (0, 0) in the two-dimensional parameter plane (k_A, k_B), following the so-called *period increment structure* (with bistability regions) (Avrutin et al., submitted for publication; Gardini et al., submitted for publication).

When the parameters satisfy point (3) of Proposition 3 then Case (II) and (a)–(b') given above are satisfied, together with $-1 < g'(d) < 0$. Thus close to the discontinuity point $f(x)$ is increasing and $f'(d) \in (0, 1)$, while $g(x)$ is a decreasing (and locally stable). Then for values of the parameters (k_A, k_B) close to the point (0, 0) the map M_{II} possesses an absorbing interval given by $I = [g \circ f(d), f(d)]$ from which the dynamics cannot escape. As in the previous case, these conditions are sufficient to state the existence of a big-bang bifurcation point at which periodicity regions following an increment scheme (with bistability regions) exist.

When the parameters satisfy point (4) of Proposition 3 then Case (II) and (a')–(b') given above are satisfied, together with $-1 < f'(d) < 0$ and $-1 < g'(d) < 0$. Thus close to the discontinuity point $f(x)$ and $g(x)$ are both decreasing (and locally stable). Then for values of the parameters (k_A, k_B) close to the point (0, 0) the map M_{II} possesses a stable cycle of period 2, that is, two attracting fixed points of the composite functions $g \circ f(x)$ and $f \circ g(x)$ exist (Avrutin et al., submitted for publication). □

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