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Border Collision and Smooth Bifurcations in a Family of Linear-Power Maps

Laura Gardini
DESP University of Urbino Carlo Bo, Urbino (Italy)
E-mail: laura.gardini@uniurb.it

Roya Makrooni
Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran (Iran)
E-mail: r_makrooni@sbu.ac.ir

Abstract. In this work we describe some properties and bifurcations which occur in a family of linear-power maps typical in Nordmark' systems. The continuous case has been investigated by many authors since a few years, while the discontinuous case has been considered only recently. In particular, having a vertical asymptote, it gives rise to new kinds of bifurcations. Organizing centers related to codimension-two bifurcation points, due to the intersection of a border collision bifurcation and a smooth fold bifurcation of cycles having a different symbolic sequence are evidenced. It is shown the relevant role played by a codimension-two point existing on any border collision bifurcation curve, and related to the smooth fold bifurcation of cycles with the same symbolic sequence. We recall some of the properties proved up to now, evidencing the rich structure which is still to be understood.

1. Introduction

The study of piecewise smooth systems had a wide expansion in the last decade. This is related to the large number of applied models characterized by sharp switching between several states which are ultimately described by nonsmooth systems, continuous or discontinuous. Moreover, many applications in engineering may include nonlinearities in the map, as power functions. In this paper, we consider the well known one-dimensional piecewise smooth (PWS for short) map defined by two functions, $f_L(x)$ and $f_R(x)$, as follows:

$$x \to f_\mu(x) = \begin{cases} f_L(x) = ax + \mu & \text{if} \quad x \le 0 \\ f_R(x) = bx^z + \mu & \text{if} \quad x > 0 \end{cases} \tag{1}$$

where a, b, z are real parameters and $\mu > 0$. The particular case $z = 1/2$ is the one more studied in the literature. It represents a continuous map, related to the square-root nonlinearity typical in Nordmark' systems and grazing bifurcations ([16], [17]). The piecewise linear case with $z = 1$ leads to the continuous skew tent map, whose dynamics are now well known (see e.g. [10], [24]). The power $z = 3/2$ was considered in [7] and in [6] where the normal-form mapping of sliding bifurcations is derived (leading to the map in (1) with power $z = 3/2$, $z = 2$ and $z = 3$, related to different cases of sliding bifurcations). Other examples of grazing and sliding bifurcations with nonlinear leading-order terms occur in power converters and in nonsmooth sliding-mode controls ([4], [5], [1]). System (1) with $z = 2$ is a particular case of the linear-logistic map considered in [22], [23]. A generalization of system (1) in the case $z > 0$ is also considered in

[2], by using the smooth function $f_R(x) = bx^z + cx + \mu$, which introduces a second critical point in the map.

In all the cases mentioned above, the power z takes positive values, which is related to a continuous PWS map, whose characteristic feature is the occurrence of border collision bifurcations (BCB for short), term which entered into use after the works by Nusse and Yorke ([18], [19]). In continuous PWS systems, the use of the skew-tent map as a normal form leads to a powerful analytical tool, which allows to determine the effect of the border collision of cycles of any period (applications can be found in [22], [23]). The same system in the case $z = 1/2$ but with different offsets (and thus a discontinuous system) was considered in [8], and the dynamics do not not differ significantly from the continuous case.

It is worth noticing that the power term z in (1) in the applied context appears through a Taylor series expansion of a nonlinear function, and so it can take only particular values (for example, in hard impact oscillators $z = 1/2$; in soft impact oscillators $z = 3/2$, and so on, as recalled above). However, also the case with real power $z < 0$ has been recently analyzed, and this leads to a discontinuous map, with a vertical asymptote, whose dynamic properties and bifurcations are very much different from those occurring in the continuous case. In discontinuous systems the classification of the possible different results of a BCB is still to be investigated, as well as the use of the PWL map as a normal form, and especially in maps with a vertical asymptote new phenomena arise, which are still to be understood. It was first considered in [20] where, besides the cases $z > 0$, the authors extend the analysis to the discontinuous case with $z < 0$. The particular case with $z = -1/2$ is also considered in [21]. However, the main results on the system in the case $z < 0$ have been shown in some recent papers ([11], [12], [14], [15], [13]), where many open problems are listed and left for further investigations.

The goal of the present work is to describe and evidence the bifurcations which are still to be investigated in this system, considering a generic real value for the power z, positive and negative. Recall that the main point in PWS systems, especially discontinuous, is that besides BCBs also standard bifurcations peculiar of smooth systems are involved. Thus, it is important to investigate the interactions between these two kinds of bifurcations. This is particularly relevant in the case of the system with an hyperbolic branch.

The existence of a vertical asymptote is not new in the engineering applications, and a peculiarity of such systems is that unbounded chaotic attractors can exist (considered also in [3]). However, to our knowledge, the cases known in the literature referring to unbounded chaotic sets are related to structurally unstable situations, and thus not related to true attractors (persistent under parameter perturbation). Differently, in this system with $z < 0$ it can be proved their occurrence and robustness ([11], [12]).

As already remarked, the main role in such systems is the occurrence of both BCBs and smooth bifurcations. However, in the noninvertible cases considered below, the smooth bifurcations are of different type: smooth flip bifurcations for $z > 0$ while smooth fold bifurcations for $z < 0$.

We recall that by using a change of variable we can get rid of one parameter, setting $\mu = 1$, which leads us to consider the following map:

$$x \to f(x) = \begin{cases} f_L(x) = ax + 1 & \text{if} \quad x \leq 0 \\ f_R(x) = bx^z + 1 & \text{if} \quad x > 0 \end{cases} \tag{2}$$

As above, the change of definition occurs at the discontinuity point $x = 0$, and as it is often used in PWS systems, the dynamical properties are studied making use of the symbolic notation based on the letters L and R corresponding to the two disjoint partitions

$$I_L = (-\infty, 0], \ I_R = (0, +\infty) \tag{3}$$

In the following section, as a prototype of the map $f(x)$ to describe the qualitative bifurcations occurring for $b < -1$ in the continuous system we consider examples with $z = 1/2$ and we shall recall some properties in Sec.2.1. While as a prototype of the map $f(x)$ to describe the qualitative bifurcations occurring in the discontinuous system with $z < 0$ we consider the case $z = -1/2$ and in Sec.2.2 we shall recall some properties and bifurcations, as well as open problems. In Sec.3 we show how rich is the bifurcation structure, and still to be understood, in the discontinuous case for $-1 < b < 0$ (when the correspondent continuous system has very simple dynamics).

2. Comparison between the cases $z > 0$ and $z < 0$, $b < -1$

In order to compare the two systems (continuous and discontinuous) let us first consider the parameter $b < -1$. The case with $-1 < b < 0$ will be commented in Sec.3, while $b > 0$ is not considered, as it leads to uninteresting (and trivial) dynamics both for $z > 0$ and $z < 0$.

For any value of z, in the region $b < -1$ we have that for $a \leq 1$ the dynamics cannot be divergent. The qualitative shape of the map is shown in Fig.1. For $z = 1/2$ the fixed point x_R^* on the R side is repelling, while for $z = -1/2$ no fixed point exists. In the continuous case there exists a bounded absorbing interval, $[f_R(1), 1]$, inside which the asymptotic dynamics are confined. In the discontinuous case there exists an unbounded absorbing interval $(-\infty, 1]$ and an unbounded chaotic set always exists (which may be of zero Lebesgue measure or of full measure).

Differently, for $a > 1$ the fixed point $x_L^* = -\frac{1}{a-1}$ on the L side exists and is repelling, and divergent trajectories exist. The immediate basin of ∞ is the interval $(-\infty, x_L^*)$ and its rank-1 preimage on the right side $f_R^{-1}((-\infty, x_L^*)) = (0, x_L^{*-1})$ where $x_L^{*-1} = \left(\frac{-a}{b(a-1)}\right)^{\frac{1}{z}}$. The total basin of divergent trajectories is clearly given by

$$\mathcal{B}_\infty = \bigcup_{k=0}^{\infty} f^{-k}((-\infty, x_L^*)) \tag{4}$$

Different dynamic behaviors occur depending on $x_L^{*-1} \lesseqqgtr 1$ and in any case ($z \lessgtr 0$) when the parameters satisfy the condition $x_L^{*-1} = 1$ it is the homoclinic bifurcation of the fixed point x_L^*, and this leads to the bifurcation curve B_f of equation

$$B_f: \quad b = -\frac{a}{a-1} \tag{5}$$

which is a kind of final bifurcation. It is clearly independent on the value of the parameter z, as it corresponds to $f_R \circ f_L(0) = x_L^*$, that is $f_R(1) = x_L^*$. However, the bifurcation leads to completely different dynamics in the two cases that we are comparing. In fact, for $z > 0$ the fixed point x_L^* is *not homoclinic* for parameter values above the curve B_f (and a bounded chaotic set exists in the absorbing interval $[f_R(1), 1]$). For parameter values on the curve B_f the fixed point becomes homoclinic (with all critical and degenerate homoclinic orbits, see [9]), and for parameter values below the curve B_f the fixed point x_L^* is homoclinic (with non critical and nondegenerate homoclinic orbits), so that an invariant set, chaotic repeller, exists, even if almost all the trajectories are divergent. See the different qualitative shapes of the map in the three different cases as drawn in Fig.1c, corresponding to the colored square points in the two-dimensional bifurcation diagram in Fig.1a.

A very different dynamic behavior occurs for $z < 0$, see the three different qualitative shapes of the map shown in Fig.1d (related to the square points in Fig.1b). The fixed point x_L^* is *not homoclinic* for parameter values below the curve B_f (and all the trajectories are divergent, except for the two points x_L^* and x_L^{*-1}), for parameter values on the curve B_f the fixed point becomes homoclinic (a unique homoclinic orbit exists, see the qualitative picture in red in Fig.1d, and all the other points have divergent trajectories, so that no invariant chaotic set can exist), and for parameter values above the curve B_f the fixed point x_L^* is homoclinic (with non critical and nondegenerate homoclinic orbits), so that an invariant chaotic repeller exists in the interval $]x_L^*, 1[$ and the basin of divergent trajectories has a positive measure (even if the attractor at infinity may coexist with an attracting cycle in the interval $]x_L^*, 1[$).

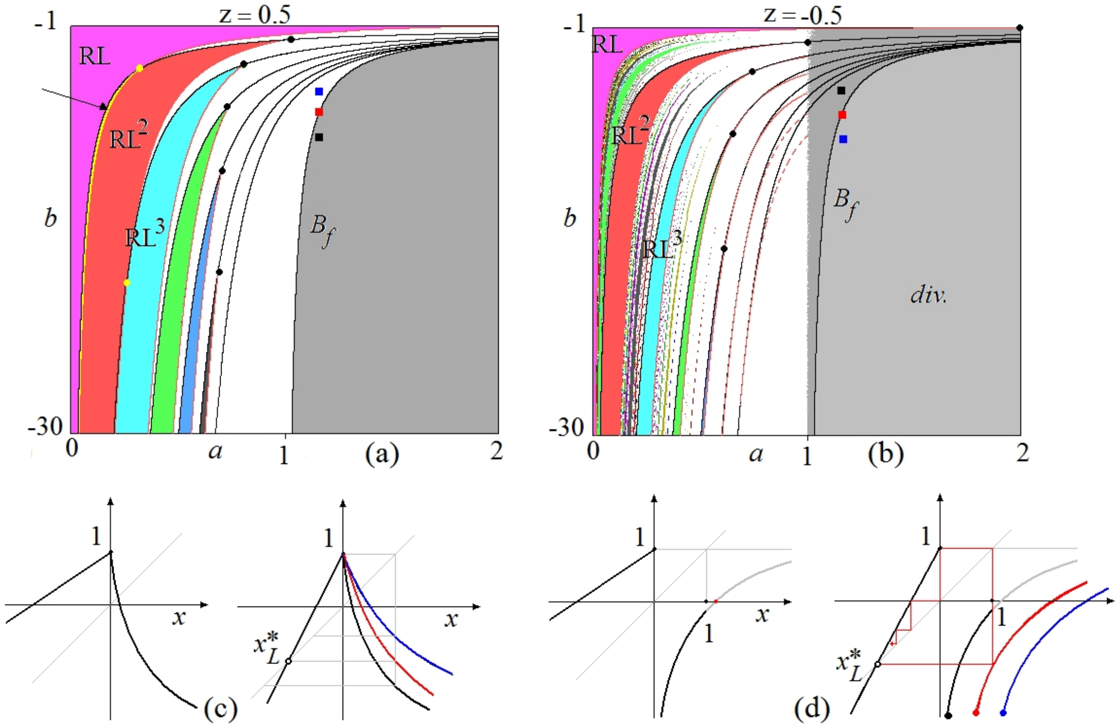

Figure 1. Two-dimensional bifurcation diagrams in the (a, b)-parameter plane, at $z = 0.5$ in (a), $z = -0.5$ in (b). In Fig.1c,d qualitative shape of the map is shown for $z > 0$ in (c) and for $z < 0$ in (d).

As we can see from Fig.1a,b, the main differences between the continuous and discontinuous cases occur in the range $0 < a < 1$. The equation of the BCB curves of the basic cycle with symbolic sequences RL^n comes from the equation

$$f_L^n \circ f_R(1) = 1 \tag{6}$$

and thus it is independent on the parameter z. It is possible to write the equation in explicit form:

$$B_{RL^n}: \ b = -\frac{1-a^n}{a^{n-1}(1-a)} \tag{7}$$

but the dynamic properties crossing the bifurcation curve B_{RL^n} are different depending on z.

2.1. Crossing a BCB curve B_{RL^n}, $z > 0$

In the continuous case ($z > 0$), as it occurs in the skew-tent map (see [?]), the crossing of a bifurcation curve B_{RL^n} corresponds to a fold border collision bifurcation, leading (as the parameter b is decreased) to a pair of cycles, one with symbolic sequence RL^n (which may be attracting or repelling) and a repelling one with symbolic sequence R^2L^{n-1}. If the cycle RL^n is attracting then decreasing the parameter b it becomes repelling via a smooth flip bifurcation. The flip bifurcation curves are given by:

$$\Psi_{RL^n}: \ b = -\frac{1}{za^n}\left(\frac{1-a^{n+1}}{1-a}\frac{z}{z+1}\right)^{1-z} \ , \quad z = \frac{1}{2} \text{ in our example} \tag{8}$$

Each flip bifurcation curve Ψ_{RL^n} intersects the BCB curve B_{RL^n} in a codimension-two point (a_n^ψ, b_n^ψ), $n \geq 1$, which separates the two behaviors: crossing B_{RL^n} at $a < a_n^\psi$ the basic cycle RL^n, decreasing b, appears attracting, while crossing B_{RL^n} at $a \geq a_n^\psi$ also the basic cycle RL^n appears repelling (that is, the border collision is related to a pair of repelling cycles, the flip bifurcation does not occur and the fold bifurcation leads to the existence also of all the subharmonics, as unstable cycles). In Fig.1a some codimension-two points (a_n^{ψ}, b_n^{ψ}) are evidenced by black circles.

As it is typical in the skew-tent map, the basic cycles RL^n are the only ones which may be attracting, and after their supercritical flip bifurcation chaotic attractors in bands are expected: $2(n+1)-$pieces, $(n+1)-$ pieces and $1-$piece, respectively, decreasing b (the qualitative changes occur at the homoclinic bifurcations of the basic cycles appeared at the fold BCB), up to the final bifurcation curve B_f.

However, it is worth to note the difference existing in the continuous map under consideration with respect to the skew-tent map. That is, while in the skew-tent map bistability is not allowed (the flip bifurcations are always supercritical, we cannot have two coexisting attracting cycle, or an attracting cycle and a chaotic attractor), in the present case we can have bistability, even if the map is unimodal. This is related to the fact that the flip bifurcation of a basic cycle may be subcritical. Indeed, we can evidence other codimension-two points $(a_n^{\psi-}, b_n^{\psi-})$, on the BCB curves, due to the intersection, for any $n \geq 2$, of the BCB curve B_{RL^n} with the flip bifurcation curve $\Psi_{RL^{n-1}}$ (see the yellow points in Fig.1a, corresponding to $n = 2$ and $n = 3$, $B_{RL^2} \cap \Psi_{RL}$ and $B_{RL^3} \cap \Psi_{RL^2}$, respectively). The role of such a codimension-two point is the following: for $a < a_n^{\psi-}$ the stability regions of the basic cycles RL^n and RL^{n-1} are overlapping: there exists a region in the $(a, b)-$parameter plane related to their coexistence (in Fig.1a an arrow indicates a yellow strip which is the overlapping region between the stability region of RL and RL^2). For the cycle RL^{n-1} we have that crossing the flip bifurcation curve $\Psi_{RL^{n-1}}$ above the codimension-two point (for $a > a_n^{\psi-}$), decreasing b, *the flip bifurcation is supercritical* while crossing it below the codimension-two point (for $a < a_n^{\psi-}$) *the flip bifurcation is subcritical.*

This implies that for $a < a_n^{\psi-}$ a repelling cycle with symbolic sequence $RL^{n-1}RL^{n-1}$ must have been appeared before (at a larger value of b). Indeed, at the crossing of the BCB curve B_{RL^n} two attractors exist, and thus a pair (by continuity) of repelling cycles must appear before such border collision, via a fold BCB of the map $f^{2(n-1)}$. The basin of attraction of the attracting cycle RL^{n-1} is bounded by the unstable cycle with symbolic sequence $RL^{n-1}RL^{n-1}$, and the basin shrinks to the cycle itself at the subcritical flip bifurcation.

2.2. Crossing a BCB curve for $z < 0$

Turning to the discontinuous case $z < 0$, the crossing of the BCB curve B_{RL^n} given in (7) is now related to a unique basic cycle with symbolic sequence RL^n. Differently from above, it is now suitable to comment the effect of the bifurcations as the parameter b is increased. Increasing b, the crossing of the BCB curve B_{RL^n} leads to the disappearance of a single cycle RL^n which may be attracting or repelling. The difference is related to the crossing (at smaller values of b) of a fold bifurcation curve or not. In fact, in the discontinuous case, since

$$f_R'(x) = \frac{bz}{x^{1-z}} > 0 \ , \ f_R''(x) = -\frac{zb(1-z)}{x^{2-z}} < 0 \tag{9}$$

we have that the slope of map f in (2) (and of any iterate f^n) is positive in all the points, thus no flip bifurcation of cycles can occur. Instead, smooth fold bifurcations can occur. Smooth fold bifurcations of basic cycles RL^n can be detected considering the composite function $F_{RL^n}(x) = f_L^n \circ f_R(x)$ and its tangency with the diagonal in a point $x^*_{RL^n} \in (0, 1]$. The explicit equation of the fold bifurcation curve Φ_{RL^n} is as follows:

$$\Phi_{RL^n} : \ b = +\frac{1}{za^n}\left(\frac{1-a^{n+1}}{1-a}\frac{z}{z-1}\right)^{1-z} \quad , \quad z = -\frac{1}{2} \text{ in our example} \tag{10}$$

The fold bifurcation curve Φ_{RL^n} intersects the BCB curve B_{RL^n} at a codimension-two point (a_n^*, b_n^*), where a_n^* is the solution of the equation $a\frac{1-a^n}{1-a} = -\frac{1}{z}$ and $b_n^* = \frac{1}{za_n^*}$. For any $n \geq 1$ they satisfy the inequalities

$$a_\infty = \frac{1}{1-z} < a_{n+1}^* < a_n^* < a_1^* = -\frac{1}{z} \tag{11}$$

In Fig.1b some codimension-two points (a_n^*, b_n^*) are evidenced by black circles. The role of a codimension-two point (a_n^*, b_n^*) is the following: crossing a BCB curve B_{RL^n} above its codimension-two point (a_n^*, b_n^*), i.e. for $a \geq a_n^*$, the fold bifurcation does not occur (it is virtual) and increasing b, at the BCB curve a unique repelling cycle with symbolic sequence RL^n appears and persist for any larger value of b. While for $a < a_n^*$, the fold bifurcation occurs before the border collision, leading to the existence of a pair of cycles with symbolic sequence RL^n, one attracting and one repelling. Increasing b the attracting one disappears by border collision crossing the curve B_{RL^n}, leaving the repelling cycle with the same symbolic sequence, existing for any larger value of b.

Now coexistence of attracting cycle cannot occur, while infinitely many families of attracting cycles exist, not only those related to basic cycles. Periodicity regions associated with many cycles can be seen in Fig.1b.

In Fig.2 we show an enlargement, and crossing a vertical segment from the BCB of an attracting cycle RL^3 to the BCB of an attracting cycle RL^2 we can prove the existence of infinitely many families of cycles.

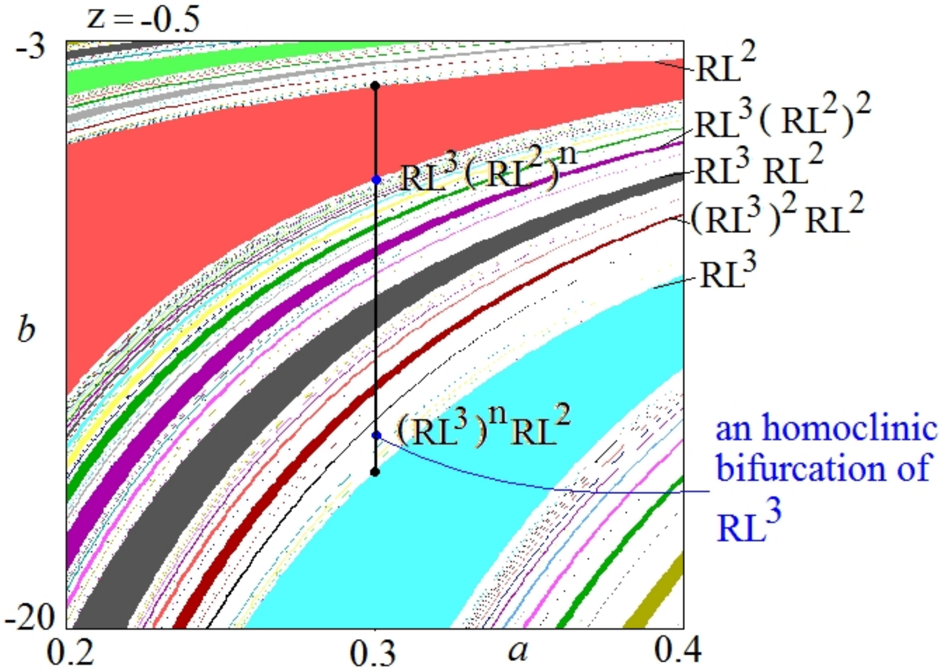

Figure 2. Two-dimensional bifurcation diagrams in the (a, b)-parameter plane, enlargement of a portion of Fig.1b.

Similarly to the description given above for the basic cycles, also for the other families we can prove the existence of the BCB curves, and the cycle which undergoes the collision may be attracting or repelling, depending on the previous crossing of a fold bifurcation curve or not.

In the discontinuous case, the dynamics of map f can be described by using the first return map $F_r(x)$ in the interval $I = [0, 1]$. As shown in [14], $F_r(x)$ is a discontinuous map with infinitely many branches defined as follows:

$$F_r(x) := \begin{cases} F_{RL^n}(x) = f_L^n \circ f_R(x) & \text{if } \xi_{n+1} \leq x \leq 1 \\ F_{RL^{n+1}}(x) = f_L^{n+1} \circ f_R(x) & \text{if } \xi_{n+2} \leq x < \xi_{n+1} \\ \vdots & \vdots \\ F_{RL^{n+j}}(x) = f_L^{n+j} \circ f_R(x) & \text{if } \xi_{n+j+1} \leq x < \xi_{n+j} \\ \vdots & \vdots \end{cases} \tag{12}$$

where $n \geq 0$ is the smallest integer for which

$$f_L^n \circ f_R(1) \in [0,1) \tag{13}$$

with

$$F_{RL^m}(x) = a^m b x^z + \frac{1-a^{m+1}}{1-a} \tag{14}$$

and the discontinuity points are preimages of the origin given by

$$\xi_{m+1} = f_R^{-1} \circ f_L^{-m}(0) = \left(\frac{-b}{\frac{a^m-1}{a^m(a-1)}+1} \right)^{-\frac{1}{z}} \tag{15}$$

which have as limit value, for $m \to \infty$, the point $x = 0$.

For any $m \geq n+1$, $F_{RL^m}(\xi_{m+1}) = 0$ and $F_{RL^m}(\xi_m) = 1$ hold, while the rightmost branch satisfies $F_{RL^n}(\xi_{n+1}) = 0$ and $F_{RL^n}(x) \in (0,1)$ for $\xi_{n+1} < x < 1$. The case

$$f_L^n \circ f_R(1) = 0 \;\; (\; f_L^{n+1} \circ f_R(1) = 1 \;) \tag{16}$$

corresponds to the BCB of a basic cycle with symbolic sequence RL^{n+1}.

An example of first return map is shown in Fig.3

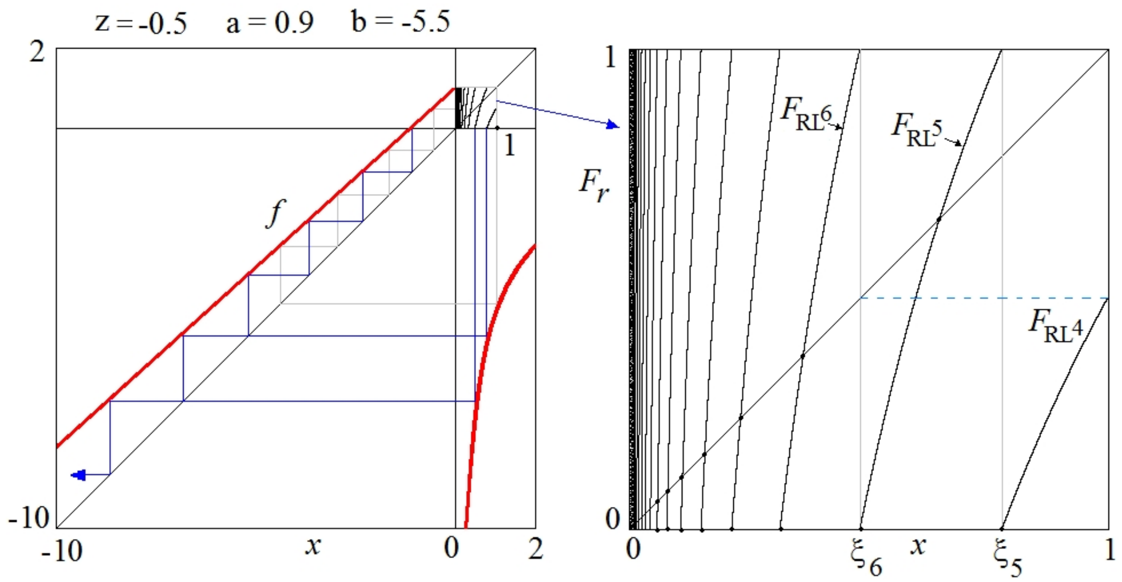

Figure 3. Graph of map f and first return map $F_r(x)$, shown also enlarged.

Only the rightmost branch RL^m can have a fixed point with slope smaller than 1. When this does not occur, and the slope of the first return map is larger than 1 in all the points of continuity, as in the example shown in Fig.3, then the first return map is chaotic in the whole interval $[0,1]$, which means that map f is chaotic in the whole unbounded interval $(-\infty,1]$, as proved in [12].

Increasing b the value in $x = 1$ increases and the diagonal is reached at the BCB value of the cycle RL^m (which appears, unstable, and persists for larger values of b). This is the scenario related to the crossing of a BCB after its codimension-two point, and the robustness of the unbounded chaotic attractor $(-\infty,1]$ is proved for any $a > a_n^*$ and $b < b_n^*$ ([12], [14]).

The dynamic behavior occurring when a BCB is related to a fold bifurcation is illustrated in Fig.4.

When the rightmost branch RL^m has points with slope smaller than 1, then increasing b the fold bifurcation will occur leading to a pair of cycles, one attracting and one repelling (with the

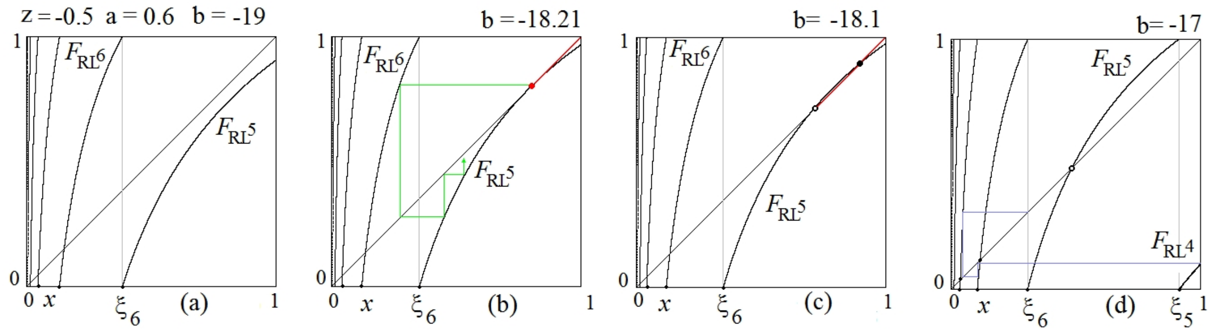

Figure 4. Graph of the first return map in the case related to a fold bifurcation.

same symbolic sequence RL^m). When an attracting cycle exists almost all the points belong to its basin of attraction, but its boundary includes an invariant chaotic set which accumulates to $x = 0$, which means that map f has an unbounded chaotic set in $(-\infty, 1]$. Increasing b the rightmost point of the attracting cycle approaches $x = 1$, merging with it at its BCB. The attracting cycle disappears leaving the repelling cycle, which exists for any larger value of b.

It is easy to prove that:

- all the (infinitely many) repelling fixed points of the first return map $F_r(x)$ are homoclinic, and f has always an unbounded chaotic set in the interval $(-\infty, 1]$;

- let $F_{RL^m}(x)$ be the rightmost branch of $F_r(x)$. If $F'_{RL^m}(1) = ba^m z \geq 1$ then the interval $(-\infty, 1]$ is an unbounded chaotic attractor of f.

Up to now we have considered only the BCB of basic cycles B_{RL^n}, which do not depend on the parameter z, but the BCB of cycles of different symbolic sequence also depend on z. In fact, any BCB involving a not basic cycle of f is represented, for $j \geq (m+1)$ by the condition

$$F_r(1) = \xi_j^{-k} \quad (as \quad F_r^{k+1}(1) = \xi_j \ , \ F_{RL^j} \circ F_r^{k+1}(1) = 1) \tag{17}$$

where $F_{RL^m}(x)$ is the rightmost branch of $F_r(x)$) and $k \geq 0$, leading to the BCB of a cycle of the first return map F_r of period $(k+2)$, and a cycle of f with symbolic sequence related to the involved branches of F_r, say

$$RL^m RL^{n_1} ... RL^{n_k} RL^j \tag{18}$$

On each BCB curve we can reason as for the BCB of basic cycles, determining a related codimension-two point. Let D be the first derivative of the function $F_{RL^j} \circ F_r^{k+1}(x)$ in the point $x = 1$. Then the BCB leads (increasing b) to the appearance of a repelling cycle or to the disappearance of an attracting cycle depending on $D \geq 1$ or $D < 1$, respectively. The case $D = 1$ corresponds to the codimension-two point at which the BCB occurs simultaneously with a fold bifurcation of cycles having the same symbolic sequence.

Although all the repelling cycles of the first return map $F_r(x)$ are homoclinic, a new homoclinic explosion occurs whenever

$$F_r^k(1) = x^* \tag{19}$$

for some $k \geq 0$, where x^* is a point of a repelling cycle of F_r.

For example, in Fig.5a we have $F_r(1) = x_{RL^3}$ where x_{RL^3} is the repelling fixed point of the branch F_{RL^3}, thus for a larger value of b, when $F_r(1) > x_{RL^3}$ holds, there is a new preimage, $F_{RL^2}^{-1}(x_{RL^3})$, which did not exist before (for $F_r(1) < x_{RL^3}$), from which an explosion of infinitely many new homoclinic orbits of x_{RL^3} can be detected.

So we can state that for any BCB of map f, whichever is the symbolic sequence of the colliding cycle, the structure is qualitatively represented in Fig.6, where the codimension-two point (related to the derivative $D = 1$) separates the two different kinds of collision: after that

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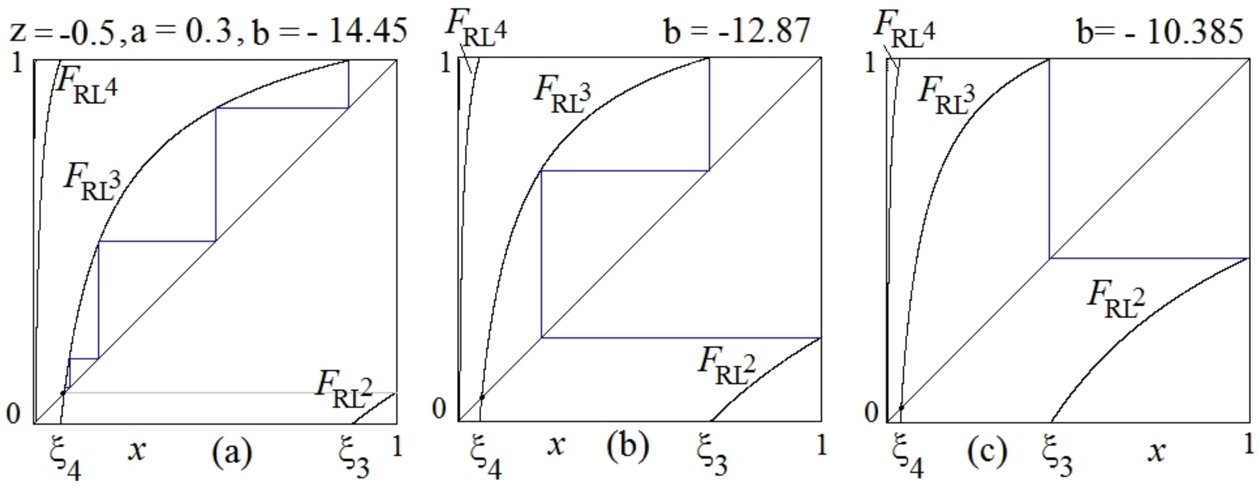

Figure 5. Graph of the first return map in the case related to an homoclinic bifurcation in (a), and to a BCB in (b) and (c).

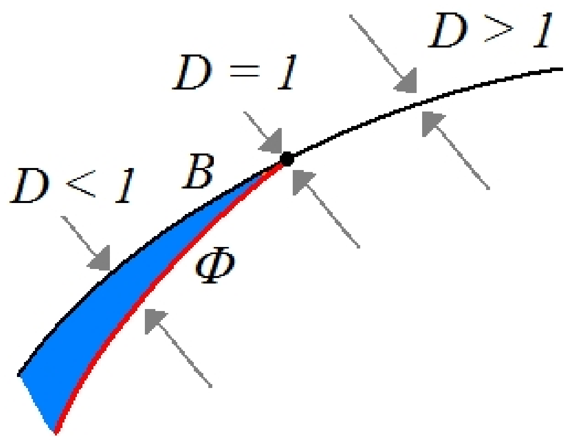

Figure 6. Qualitative representation of a BCB curve and related codimension-two point on it.

point, crossing the curve increasing b a repelling cycle appears, before that point crossing the BCB curve an attracting cycle disappears, leaving the unstable one with the same symbolic sequence, which means that at a smaller value of b the a fold bifurcation must have been occurred. Moreover:

- any BCB curve is a limit set of infinitely many BCB curves, on both sides when related to a repelling cycle, only from above when related to an attracting cycle;
- each fold bifurcation curve is a limit set of infinitely many homoclinic bifurcation curves and BCB curves, only from below:
- each homoclinic bifurcation curve is a limit set of infinitely many BCB curves, both from below and above;
- chaotic bands do not occur.

Referring to the vertical segment in Fig.2, at $a = 0.3$ for $-15.4 = b(B_{RL^3}) < b < b(B_{RL^2}) = -4.3$, let us show that the two families of BCB curves related to cycles with the symbolic sequence obtained by concatenation of RL^2 and RL^3 must exist, that is cycles $RL^2(RL^3)^n$ and $(RL^2)^n RL^3$ for any $n \geq 1$.

Considering the shape of the first return map in Fig.5, a sequence of preimages of the discontinuity point ξ_3 with the branch F_{RL^3} is accumulating to the repelling fixed point x_{RL^3} and after the homoclinic bifurcation shown in Fig.5a, increasing b, the point $F_r(1) = F_{RL^2}(1)$ increases (and it reaches the value 1 at the BCB B_{RL^2}), so that there must exist values of b, say b_k for any $k \geq 0$, such that at $b = b_k$ the BCB curves having the following equations are crossed:

$$F^{k+1}_{RL^3} \circ F_{RL^2}(1) = 1, \;\; k \geq 0$$

which are the BCB curves of cycles of f having symbolic sequence $RL^2(RL^3)^{k+1}$ for any $k \geq 0$. Since for $k \to \infty$ the values b_k tend to $b(x_{RL^3})$ these border collisions have as limit set the homoclinic bifurcation value of this cycle (occurring for $F_{RL^2}(1) = x_{RL^3}$). In Fig.5b,c the last bifurcations related to $k = 1$ and $k = 0$, respectively, are shown.

For the second family, notice that after the last bifurcation commented above, for $k = 0$ (which corresponds to $F_{RL^2}(1) = \xi_3$), we can consider the preimages of the discontinuity point ξ_3 also with the branch F_{RL^2}, as shown in Fig.7. It is clear that for any $n > 1$ the condition

$$F^n_{RL^2}(1) = \xi_3 \tag{20}$$

must occur, leading (from $\xi_3 = F^{-1}_{RL^3}(1)$) to the BCB curves of equation $F_{RL^3} \circ F^n_{RL^2}(1) = 1, \; n \geq 2$ which have as limit set the fold bifurcation curve Φ_{RL^2}.

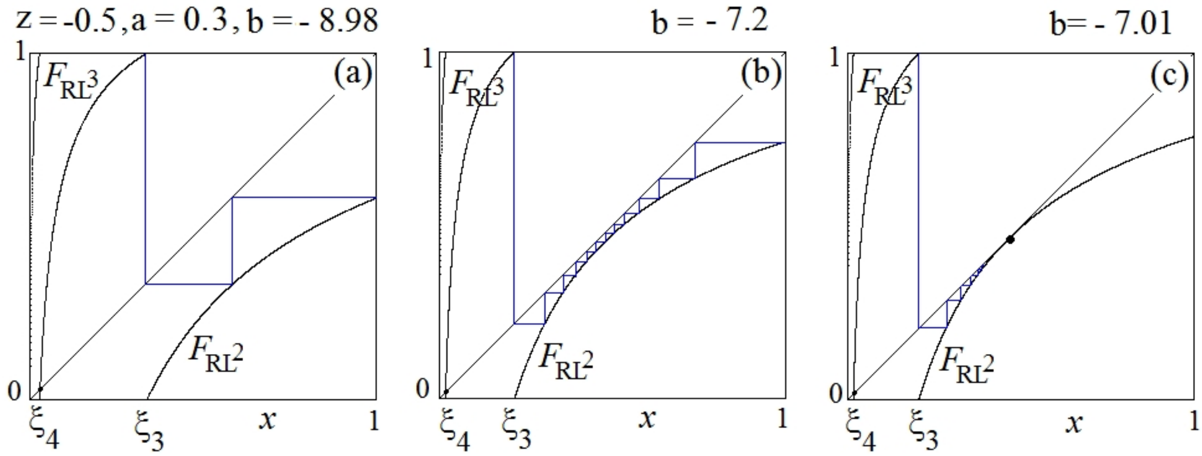

Figure 7. Graph of the first return map in the case related to some BCBs.

For each BCB we have a qualitative shape as that shown in Fig.6. In the considered example in Fig.2 we cannot say precisely which of them are crossed with attracting or repelling cycles for $a = 0.3$. Numerically, it seems that the second family of cycles with symbolic sequence $(RL^2)^n RL^3$ for any $n \geq 1$ is associated with the BCB of attracting cycles (and thus also the related fold bifurcation curves are crossed).

The two families described above are just a simple example. It is clear that many more cycles undergo a BCB in the vertical segment at $a = 0.3$ shown in Fig.2, and especially the structure existing between the lower BCB bifurcation of the cycle RL^3 and the homoclinic bifurcation of x_{RL^3} commented above is still to be understood.

3. Comparison for $-1 < b < 0$

The case with $-1 < b < 0$ is not interesting in the continuous system ($z > 0$), while the discontinuous system is very much interesting and new. In Fig.8a we show the whole range $b < 0$ by using, in the vertical axis, the scaled value $S(b) = arctan(b)$ in the interval $[-\frac{\pi}{2}, 0]$. It can be seen that all the bifurcation curves discussed in the previous section for $b < -1$ are issuing from the point $(a, b) = (0, -\infty)$ which behaves as organizing center. The qualitative shape of the right branch $f_R(x)$ is shown, and increasing b the BCB of the basic cycles $R^n L$ occur for any $n \geq 1$ having as limit set (for $n \to \infty$) the fold bifurcation curve of $f_R(x)$, Φ_R, obtained from (10) for $n = 0$:

$$\Phi_R : \; b = \frac{1}{z}\left(\frac{z}{z-1}\right)^{1-z}, \;\; z = -\frac{1}{2} \text{ in our example} \tag{21}$$

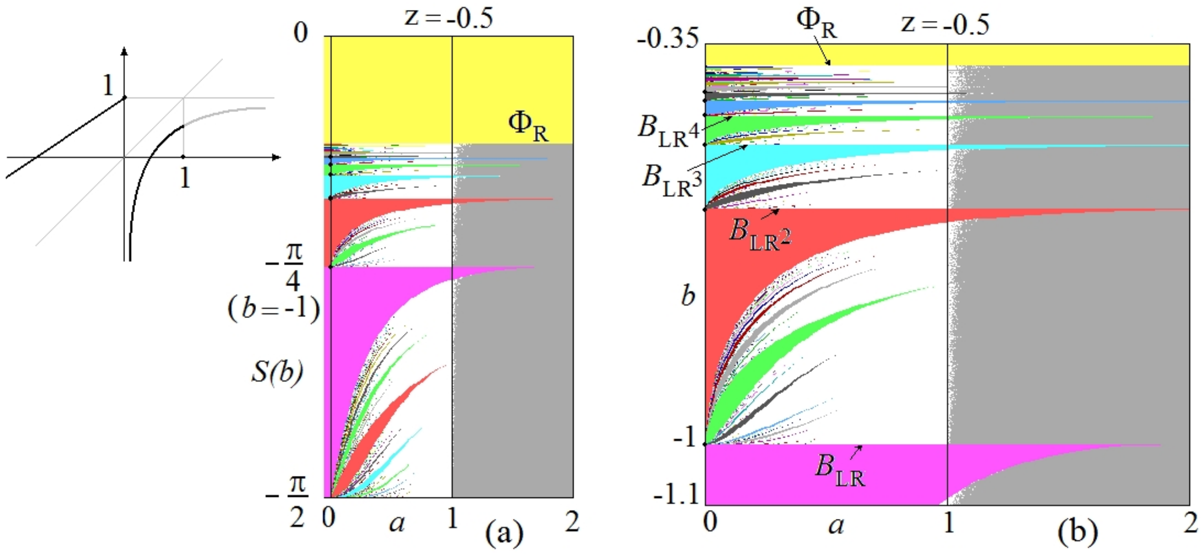

Figure 8. In (a) two-dimensional bifurcation diagram in the $(a, S(b))$-plane where $S(b) = arctan(b)$. In (b) two-dimensional bifurcation diagram enlarged in the (a, b)-plane.

What is evident from Fig.8b (an enlarged portion in the (a, b)-parameter plane) is that besides the organizing center in $(a, b) = (0, -\infty)$, there are infinitely many organizing centers related to codimension-two points which comes from the intersection of bifurcation curves of cycles having different symbolic sequences, that is $(0, b_n^B) = B_{R^nL} \cap \Phi_{R^{n+1}L}$. The organizing centers belong to the line $a = 0$, at which the shape of the map is particular, having a flat branch on the left side (slope 0). In such cases, increasing the parameter b it is known that a pure period incrementing structure (with increment 1) takes place. Moreover, as shown in [13], the vertical asymptote leads to a peculiarity: each border collision is always related to a pair of cycles, one bounded and one, with period +1, having a periodic point at infinity, as shown in the one-dimensional bifurcation diagram in Fig.9 with $S(x) = arctan(x)$ on the vertical axis.

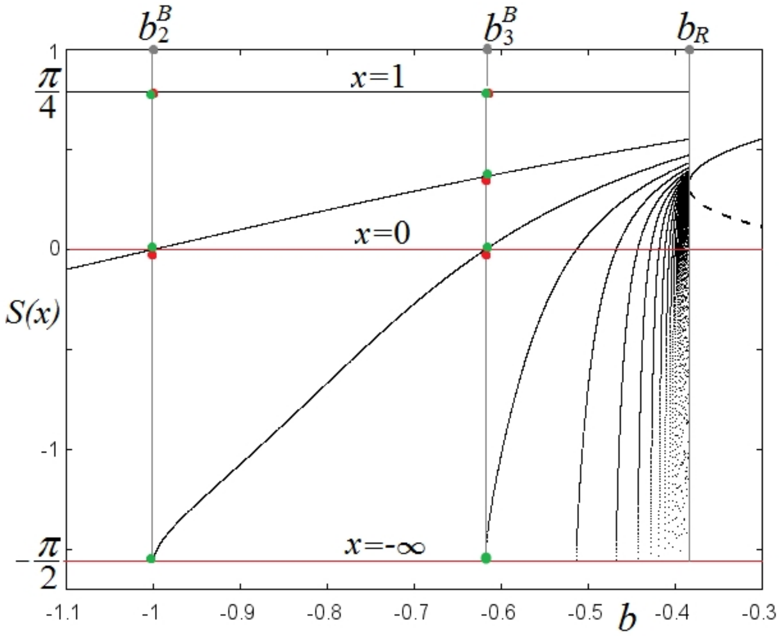

Figure 9. One-dimensional bifurcation diagram at $a = 0$, showing $S(x)$ as a function of b, where $S(x) = arctan(x)$. In red the points of the cycle of period n at its BCB, in green the points of the cycle of period $(n + 1)$, with one point at infinity.

In [15] it is shown that each bifurcation value of B_{R^nL} is independent on a, thus $b = b_n^B$ are the equations of the BCB curves B_{R^nL} (horizontal straight lines in Fig.8). In that work it is shown that also in this case the properties of the map can be studied making use of the return map in the in terval $[0, 1]$, and that the BCB curves have the same property as above, qualitatively expressed in Fig.6

The main point is that the rich structure discussed above for the case $b < -1$ (and related to curves issuing from $(0, -\infty)$) can be repeated with obvious changes from any organizing center $(0, b_n^B)$, and the related structure is still an open problem.

4. Conclusions

In this work we have considered the map f in (2) with power z both for positive and negative values, emphasizing the differences in the bifurcation structures in the (a, b) parameter plane. In particular we have shown how rich is the bifurcation structure in the case $z < 0$, leading to many open problems. We have shown that a a suitable tool for its study is the first return map $F_r(x)$ in the interval $[0, 1]$. We are confident that the existing bifurcation structure may be explained by using the map $F_r(x)$, both in the case $b < -1$ and $-1 < b < 0$. The interactions between two kinds of bifurcations (BCBs and standard smooth fold bifurcations) has been shown, and on each BCB curve (related to any possible cycle) it appears related to a specific codimension-two point which can be determined by the condition $D = 1$ as shown in Sec. 2.2. Moreover, a nice property of the system in the case $z < 0$ is that the interval $(-\infty, 1]$ is an unbounded chaotic attractor of f which is structurally stable in a wide portion of the parameter plane for $b < -1$.

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