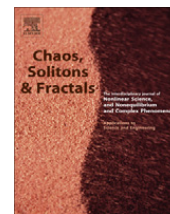




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Critical homoclinic orbits lead to snap-back repellers

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ABSTRACT

When nondegenerate homoclinic orbits to an expanding fixed point of a map $f : X \rightarrow X, X \subseteq \mathbb{R}^n$, exist, the point is called a snap-back repeller. It is known that the relevance of a snap-back repeller (in its original definition) is due to the fact that it implies the existence of an invariant set on which the map is chaotic. However, when does the first homoclinic orbit appear? When can other homoclinic explosions, i.e., appearance of infinitely many new homoclinic orbits, occur? As noticed by many authors, these problems are still open. In this work we characterize these bifurcations, for any kind of map, smooth or piecewise smooth, continuous or discontinuous, defined in a bounded or unbounded closed set. We define a *noncritical homoclinic orbit* and a homoclinic orbit of an expanding fixed point is structurally stable iff it is noncritical. That is, only critical homoclinic orbits are responsible for the homoclinic explosions. The possible kinds of critical homoclinic orbits will be also investigated, as well as their dynamic role.

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1. Introduction

Many real processes in different sciences can be modelled by noninvertible smooth or piecewise smooth systems. In the study of the bifurcation mechanisms which are common to these systems a particular role is played by the homoclinic orbits. It is well known that the homoclinic bifurcations are one of the most important tools to analyze the dynamics properties, both in continuous and in discrete time. The homoclinic bifurcations occurring in continuous time models (see [34–37]) are often studied by use of discrete maps associated with a so-called Poincaré section, and related with invertible maps. No homoclinic orbit can be associated with unstable foci or unstable nodes in a two-dimensional invertible map, or in general with repelling (expanding) cycles in n -dimensional invertible maps, $n \geq 2$. While in *noninvertible maps* we may have homoclinic orbits also associated with

expanding fixed points or cycles. For a smooth map, a repelling cycle p is expanding when all the eigenvalues are higher than 1 in modulus. Marotto was the first to prove in [28] that homoclinic orbits may occur also for such repelling points, and that chaos is associated to the existence of homoclinic orbits. Indeed, his first work included a minor technical mistake, and he himself gave a corrected version in [29], after the appearance of several papers which, trying to correct the mistake, were providing less general proofs (as in Li and Chen [23]).

Recall that a *nondegenerate* homoclinic orbit to an expanding fixed point of a map $f : X \rightarrow X, X \subseteq \mathbb{R}^n, n \geq 1$, is such that in all the points x_j of the orbit the Jacobian $J_f(x_j)$ is defined and $\det(J_f(x_j)) \neq 0$. When nondegenerate homoclinic orbits exist, the point is called, after Marotto ([28,29]), a *snap-back repeller* (SBR for short), and its importance relies on the fact that in any neighborhood of such a homoclinic orbit it is possible to prove the existence of an invariant set on which the map is chaotic. This property holds in the phase space $\mathbb{R}^n, n \geq 1$, or in a generic compact metric space.

An increasing number of papers have been published, dealing with homoclinic bifurcations of expanding cycles

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(we recall [10,9,25,26,20,31,32,11,17,18,15,16]). However, when does the first homoclinic orbit appear? As noticed in [18], as well as in [12,33,27], this is still an open problem, although some investigations in order to answer to this question, have already been published. In [12] the first observations are given for smooth maps of \mathbb{R}^n suggesting to consider critical homoclinic orbits, which are those including a critical point, as responsible for the first homoclinic bifurcation called SBR bifurcation. We recall that according to the definition given by Julia and Fatou, a critical point of a smooth map is a point in which at least two different inverses are merging in a unique point, in which the map is not locally one-to-one (see [12,30]). Recently, other papers appeared considering suitable definitions aimed to characterize the first homoclinic bifurcation of an expanding fixed point. Clearly, to solve the problem one has to analyze what occurs when the condition of nondegeneracy is not satisfied. However, there are several different kinds of degenerate homoclinic orbits. For example, in continuous maps degenerate orbits are those having a homoclinic point q in which $\det(J_f(q)) = 0$, or those having a homoclinic point q belonging to the switching set of a piecewise smooth function in which $\det(J_f(q))$ is not defined. This is the approach followed in a few recent works, among which it is worth to mention the one by Shi and Yu in [33] and by Glendinning in [18]. Indeed, as there are several cases of degeneracy, there are different conditions and approaches. But, as we show in the present paper, no one of these approaches works for all the possible cases. That is, we can easily give an example of homoclinic orbit in which a homoclinic point q has $\det(J_f(q)) = 0$, or in which $\det(J_f(q))$ is not defined, and that is also structurally stable, that is: not associated with a SBR bifurcation or any homoclinic explosion. In fact, we notice that considering a system under parameter variations, the first SBR bifurcation of an expanding fixed point (i.e., transition from no homoclinic orbit to infinitely many nondegenerate homoclinic orbits) is often followed by many other so-called Ω -explosions, i.e. homoclinic bifurcations leading to infinitely many new nondegenerate homoclinic orbits.

As recalled above, homoclinic bifurcations of expanding fixed points or cycles can occur only in noninvertible maps, so it is not surprising to find out that the true condition leading to a SBR bifurcation or any Ω -explosion is associated with the noninvertibility in the neighborhood of some homoclinic point. In the case of a continuous function in a compact set, the definition given above of critical homoclinic orbits leads indeed to the only possible bifurcation cases. Note that a degenerate homoclinic orbit may be critical (as in the case of existence of an extremum point) or not (as in the case of existence of an horizontal inflection point). Then we shall extend the definition and bifurcation condition also to a great variety of noninvertible function, smooth or piecewise smooth, continuous or discontinuous, in a bounded or unbounded closed set $X \subseteq \mathbb{R}^n$.

One more important question is related to the dynamics associated with a critical homoclinic orbit (or a degenerate homoclinic orbit which is also critical): is it possible to find a chaotic set in its neighborhood? We give an answer also to this question: in general nothing can be stated a priori. We give examples in which chaos exists, as well as

examples in which a critical homoclinic orbit is not associated with a chaotic invariant set.

Note that critical homoclinic orbits in noninvertible maps may also be associated with a homoclinic bifurcation similar to a tangent bifurcation between the stable and unstable sets of a saddle in smooth maps (an example is given in Section. 4.2) as well as in piecewise smooth maps (an example can be found in [13]).

The content of the paper is as follows. In Section. 2 we recall the necessary definitions, comparing the related properties. We recall well known theorems and present our main result, namely the definition of critical homoclinic orbit and Theorem 2, which is proved in the Appendix. The dynamic behavior of a map associated with a critical homoclinic orbit is investigated in Section. 3 in the case of one-dimensional (1Dim for short) maps, showing several examples, and illustrating how to proceed in order to see whether a chaotic set exists or not. Section 3.1 is devoted to other kinds of homoclinic bifurcations, of maps defined in unbounded sets. Further results and comments on the structurally stable noncritical homoclinic orbits in $\mathbb{R}^n, n > 1$, are given in Section. 4, where we also discuss several two-dimensional (2Dim for short) examples. Namely, in Section. 4.1 we first analyze a class of maps with separate second iterate, in which several examples of SBR and other homoclinic explosions can be easily constructed, and then, in Section. 4.2 a class of maps in triangular form, with an explicit example showing its peculiar properties and SBR bifurcations. In Section. 4.3 we describe a piecewise smooth 2Dim map, showing how the SBR bifurcation of a focus point can be identified using the critical curves. Section. 5 concludes.

2. Homoclinic theorem

Let us consider a map $f: X \rightarrow X$ where $X \subseteq \mathbb{R}^n$. To simplify the exposition we limit our reasoning to a fixed point p of $f(x)$. Then the same arguments can be applied to a k -cycle of f by considering the fixed points of the map f^k .

The main property used to prove the existence of chaos (see the Appendix) is the existence of two disjoint compact sets, say U_0 and U_1 , such that

$$f^k(U_0) \supset (U_0 \cup U_1) \quad \text{and} \quad f^k(U_1) \supset (U_0 \cup U_1) \quad (1)$$

for a suitable k (in the 1Dim case a map f possessing such a property is called strictly turbulent following [8], see also [22]). We shall see that this property occurs when a homoclinic orbit of an expanding fixed point exists, which satisfies particular conditions.

A homoclinic trajectory of a fixed point is one which tends to this point in the forward process, and in some backward one. For example, in a 1Dim unimodal map it is easy to see when an unstable fixed point p becomes homoclinic (and called snap-back repeller).

Consider the map whose graph is shown in Fig. 1a: the fixed point p is there unstable, but not homoclinic (the pre-images of p different from itself are external to the absorbing interval $I = [c_1, c]$). While in Fig. 1b it is homoclinic, and in any neighborhood U of p two intervals I_0 and I_1 can be

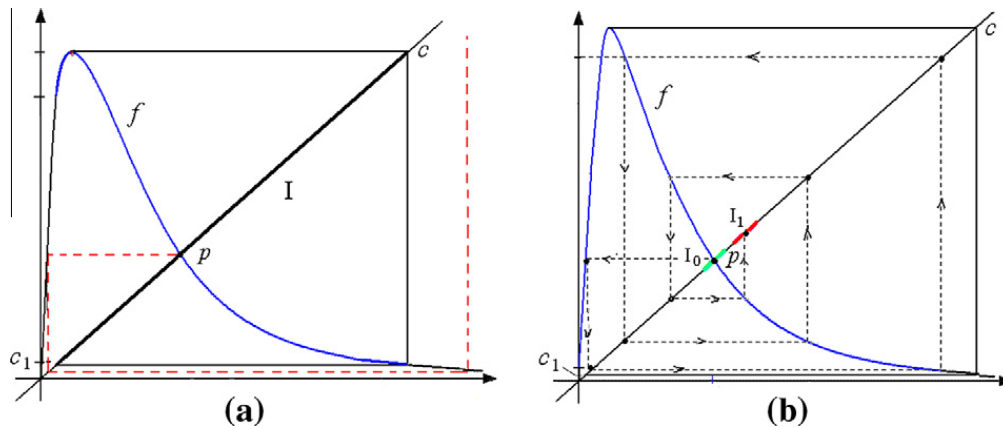


Fig. 1. The fixed point p is expanding. Not homoclinic in (a), homoclinic in (b).

found such that $f^k(I_0) \supset I_0 \cup I_1$ and $f^k(I_1) \supset I_0 \cup I_1$ (there is not a unique pair, an example in Fig. 1b is with $k = 6$).

In the 1Dim case an unstable fixed point in which the map is locally invertible is always expanding, while this is not true in \mathbb{R}^n . So let us recall the definition.

Definition 1. We say that a fixed point p of $f : X \rightarrow X$, $X \subseteq \mathbb{R}^n$, is *expanding* if f is continuous in p and a neighborhood U of p exists such that for any $x \in U \setminus p$ an integer n_x exists for which $f^{n_x}(x) \notin U$ (that is, the trajectory of x leaves U in a finite number of iterations), and a local inverse f_0^{-1} satisfies $\bigcap_{n \geq 0} f_0^{-n}(U) = p$.

Notice that we have not made use of the derivative or Jacobian matrix of f in p , as in fact we shall characterize the homoclinic bifurcations independently on the smoothness of the function in the homoclinic points. It is clear, however, that if f is smooth in p then a sufficient condition for the fixed point p to be expanding is that all the eigenvalues of $J_f(p)$ are larger than 1 in modulus, as the local inverse behaves as a contraction (see [19]).

In the following, considering a neighborhood U of an expanding fixed point p , it is understood that it is a neighborhood as in the definition given above, and we also say that f is locally invertible in p with local inverse f_0^{-1} .

Given an expanding fixed point p of a map f , any point in a neighborhood U of p is repelled away (thus no local stable manifold can exist), but when f is noninvertible then the trajectory of a point may come back in U again. This occurs when p becomes homoclinic. And the fixed point p becomes homoclinic when we can find *preimages of the fixed point itself, arbitrarily close to it*. When this occurs the fixed point is called a snap-back repeller. More precisely, a point q is called a *homoclinic point* of p (or homoclinic to p) if there exists an integer j such that $f^j(q) = p$, and it is possible to find a sequence of preimages of q which tends to p :

$$\mathcal{O}_q(p) = \{p \leftarrow \dots, q_{-i}, \dots, q_{-2}, q_{-1}, q, q_1, q_2, \dots, q_j = p\}, \quad (2)$$

where $f^i(q) = q_i$, $i = 1, \dots, j - 1$, $f^j(q) = p$, and $\{q_{-i}\}_{i=1}^\infty$ is a suitable backward orbit converging to p , i.e. $f(q_{-i}) = q_{-i+1}$ and $q_{-i} \rightarrow p$ as $i \rightarrow \infty$.

Thus, considering a neighborhood $U(p)$ in which the map is locally expansive we can find a point of the homo-

clinic trajectory which belongs to U . So, without loss of generality we can define a homoclinic trajectory via a point $x_0 \in U$, its images $x_j = f^j(x_0)$ and the local inverse $f_0^{-1}(x)$ as follows:

Definition 2. Let p be an expanding fixed point of $f : X \rightarrow X$, $X \subseteq \mathbb{R}^n$. The point p is called a *snap-back repeller* if there exists a point $x_0 \in U \setminus p$ such that $f^m(x_0) = p$ for a suitable integer m . The orbit $\mathcal{O}_{x_0}(p)$ given by

$$\mathcal{O}_{x_0}(p) = \{p \leftarrow \dots, f_0^{-n}(x_0), \dots, f_0^{-1}(x_0), x_0, x_1, \dots, x_m = p\} \quad (3)$$

is the related homoclinic orbit, where $x_i = f^i(x_0)$.

Notice that according to this definition the fixed point may be a point of non smoothness of $f(x)$, differently from the usual definition given, for example, in [28] or in [10]. We recall that Marotto (see [28] and [29]) gave the definition of snap-back repeller for a differentiable map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, under the assumption that all the eigenvalues of $J_f(p)$ exceed 1 in magnitude, and $\det(J_f(x_j)) \neq 0$ in all the homoclinic points. Smooth maps were considered also by Devaney in [10], and he gave the definition of nondegenerate homoclinic orbit as follows:

Definition 3. A homoclinic orbit $\mathcal{O}(p)$ of an expanding fixed point p of a smooth map $f : X \rightarrow X$, $X \subseteq \mathbb{R}^n$ is called *nondegenerate* if $\det(J_f(x_j)) \neq 0$ in all the points x_j of the orbit.

Recall that in [28] (as well as in [10]) for a smooth map the following theorem is proved:

Theorem 1. (Marotto [28,29]). *If a smooth map $f : X \rightarrow X$, $X \subseteq \mathbb{R}^n$, has a snap-back repeller p such that all the eigenvalues of $J_f(p)$ exceed 1 in magnitude, and $\mathcal{O}(p)$ is a nondegenerate homoclinic orbit, then in any neighborhood of $\mathcal{O}(p)$ f is chaotic in some invariant set.*

Here the set is chaotic in the sense of Li and Yorke [24], and of Devaney [10]. This result holds also for a wider class of maps, which includes also discontinuous maps. In fact, the theorem can be easily formulated for nondegenerate homoclinic orbits of a *piecewise smooth map*, $f : X \rightarrow X$, $X \subseteq \mathbb{R}^n$, when p is an expanding fixed point of f in which all the eigenvalues of $J_f(p)$ exceed 1 in magnitude and $\mathcal{O}(p)$

is a *nondegenerate* homoclinic orbit of p . Then in any neighborhood of $\mathcal{O}(p)$ there exists an invariant Cantor set \mathcal{A} on which f is chaotic.

In [15] Theorem 1 has been extended to piecewise smooth maps in \mathbb{R}^n for some *degenerate homoclinic orbits* under some assumptions on the Jacobian $J_f(p)$. We do not write here the statement because below we shall improve the result in Theorem 2.

Notice, however, that from the above theorem nothing can be stated about the bifurcation which leads an expanding fixed point to become a *snap-back repeller*, called SBR bifurcation, as well as other homoclinic bifurcations leading to a new explosion of infinitely many nondegenerate homoclinic orbits, called Ω -explosions. This was indeed still an open problem, both for smooth maps and for generic piecewise smooth ones.

It is clear that to identify such bifurcations one must look for homoclinic orbits which are *degenerate*, i.e. not satisfying the definition of nondegenerate given above (in Definition 3). There are, however, several kinds of degeneracy, also for a smooth map, and the relevant ones are only the degenerate homoclinic orbits which are structurally unstable (with respect to the existence of the homoclinic orbit itself). For example, a homoclinic orbit having a point q such that $\det(J_f(q)) = 0$, is degenerate, but it is not necessarily related to something particular. Under suitable assumptions (given below in Theorem 2) it may also be a structurally stable orbit, not involved in any Ω -explosion. For example, in the 1Dim case a local extremum may be particular, but not a point q of the homoclinic orbit which is a horizontal inflection point (as shown in Fig. 2a), where the homoclinic orbit is persistent. In the case of piecewise smooth maps, the derivative or Jacobian may be not defined in a homoclinic point q , but this does not necessarily mean that the orbit is structurally unstable, as shown in Fig. 2b with a 1Dim example where the homoclinic orbit is persistent as well. The function may be discontinuous in a homoclinic point q (see Fig. 2c), and here the homoclinic orbit is not persistent (if the point q moves on its right side the homoclinic orbit disappears, differently occurs if it moves on the left).

Notice that if a point q of a homoclinic orbit is a point of non differentiability or is such that $\det(J_f(q)) = 0$, then Theorem 1 (and the related generalizations) cannot be applied. However, in the particular situations shown in (Fig. (2)a, b) Theorem 2 given below applies and states the existence of chaos. In fact, what matters in order to classify the first

homoclinic orbit to an expanding fixed point p , as well as to characterize further homoclinic explosions, is a homoclinic orbit in which the local invertibility is lost in some homoclinic point (which thus is not structurally stable). The important property that we have in a structurally stable homoclinic orbit $\mathcal{O}(p)$ in the case of 1Dim maps is that in each point x_k of the homoclinic orbit *the function is continuous and locally monotone (either increasing or decreasing)* and thus locally invertible (or locally one-to-one) in a neighborhood of each point of the homoclinic orbit. Similarly in the phase space \mathbb{R}^n this property corresponds to the fact that in each point x_k of a structurally stable homoclinic orbit f is *continuous in x_k* and a neighborhood $V(x_k)$ exists such that f is *one-to-one in $V(x_k)$* (and clearly onto the set $f(V(x_k))$). When this occurs we say that f is locally invertible in each point of $\mathcal{O}(p)$. Then we give the following:

Definition 4. A homoclinic orbit $\mathcal{O}(p)$ of an expanding fixed point p of a map $f : X \rightarrow X, X \subseteq \mathbb{R}^n$, is called *noncritical* if f is locally invertible in each point of $\mathcal{O}(p)$ (i.e. if in each point x_k of the homoclinic orbit f is continuous, and a neighborhood $V(x_k)$ exists in which f is one-to-one).

We can also state the following property:

Property 1. A homoclinic orbit $\mathcal{O}(p)$ is *critical* if

- (i) it includes a point in which f is continuous but not locally invertible,
- (ii) or it includes the limit value at a discontinuity point,
- (iii) or it is unbounded.

A homoclinic orbit of p is *noncritical* when none of the three conditions above are satisfied, i.e. it is bounded, without any critical point, and without any discontinuity point. When the function f is defined in a *compact set* X of \mathbb{R}^n then if f is continuous only (i) can occur, while if f is discontinuous then also (ii) may occur. In part (ii) we do not write “it includes a discontinuity point” to emphasize that it does not matter how the function is defined, all the limit values may be involved in a critical homoclinic orbit, but it is clear that a critical homoclinic orbit with a limit value as defined in (ii) also includes a discontinuity point. The last condition in (iii) can occur only if X is a *closed unbounded set* of \mathbb{R}^n , in which case the space \mathbb{R}^n is considered compactified (that is, we include the points at infinity). In fact, the main theorem (Theorem 2 below), characterizing the

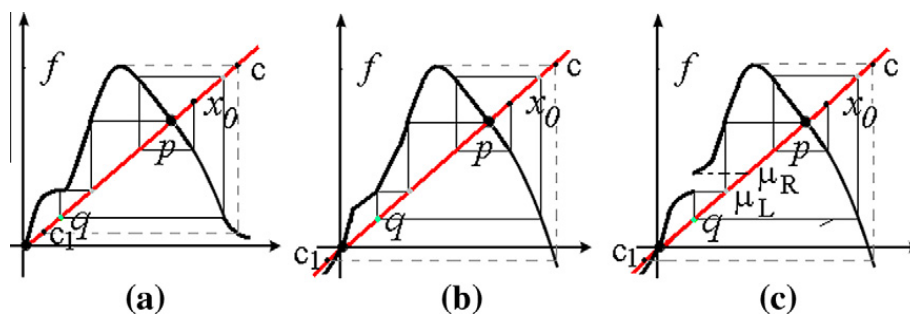


Fig. 2. Degenerate homoclinic orbits. In (a) q is a point having vanishing derivative. In (b) q is a point of non differentiability in which f is continuous. In (c) q is a point of discontinuity.

structurally stable homoclinic orbits, only considers non-critical (and thus bounded) homoclinic orbits of p , and its proof is independent on the structure of the set X . While a critical homoclinic orbit, which characterizes all the structurally unstable homoclinic orbits leading to SBR bifurcations or homoclinic explosions, also depends on the definition of f in X . It is clear that if X is a closed unbounded set of \mathbb{R}^n we still may have a critical homoclinic orbit of type (i) or (ii), but also a new kind of homoclinic bifurcation mechanism can occur: unbounded homoclinic trajectories (as described in Section. 3.1).

The existence of a *noncritical* homoclinic orbit is important because it is possible to prove that in any neighborhood of a noncritical homoclinic orbit there exists an invariant set on which the restriction of the map is chaotic. We remark that when we have a noncritical homoclinic orbit, we also have a sequence of well defined inverse functions associated with each point of the homoclinic trajectory. In fact, starting from the expanding fixed point $p = x_m$ we must have $x_{m-1} = f_{j_1}^{-1}(x_m)$, (where the inverse $f_{j_1}^{-1}$ is necessarily different from f_0^{-1}), then $x_{m-2} = f_{j_2}^{-1}(x_{m-1})$, $x_{m-3} = f_{j_3}^{-1}(x_{m-2})$, and so on, where each index j_k detects the suitable unique inverse which has to be applied in order to get the point of the homoclinic orbit under consideration. We can now state our main theorem, which generalizes Theorem 1 (we do not require the homoclinic orbit to be nondegenerate¹):

Theorem 2. *Let f be a piecewise smooth noninvertible map, $f : X \rightarrow X, X \subseteq \mathbb{R}^n$. Let p be an expanding fixed point of f and $\mathcal{O}(p)$ a noncritical homoclinic orbit of p . Then in any neighborhood of $\mathcal{O}(p)$ there exists an invariant Cantor like set A on which f is chaotic.*

As mentioned above, this theorem improves a similar one given in [15]. In fact in the previous version the function was assumed differentiable in the fixed point, while here we make no assumptions on the derivatives or Jacobians, and f may be not differentiable in the fixed point p as well as in other homoclinic points. The proof is given in the Appendix.

Note that the difference between Cantor set and Cantor like set (the definition of which is given in Property 3 in the Appendix) is that a Cantor set is proved to be a set of points, while a Cantor like set is a set of compact elements which (similarly to the points of a Cantor set) are in 1-1 correspondence with the elements of the space Σ_2 of one sided infinite sequences of two symbols $\{0,1\}$, and the set includes infinitely many cycles.

In the next section we shall focalize our attention and comments to the 1Dim case, coming back in Section 4 to the space \mathbb{R}^n .

3. Critical homoclinic orbits in \mathbb{R}^1

As noticed above, Theorem 2 holds for noncritical homoclinic orbits. Thus the homoclinic orbits which are particular are only the critical homoclinic orbits in our definition, and are the candidates to characterize the

homoclinic bifurcations (not only the first one, the SBR bifurcation, but any other as well). Here we shall investigate the properties of critical homoclinic orbits in \mathbb{R}^1 .

A first result is that the dynamic behavior in a neighborhood of a critical homoclinic orbit cannot be uniquely characterized a priori (as chaotic or not). In fact, as we will see below, in a neighborhood of a critical homoclinic orbit the dynamic behavior depends on the particular system. That is, both in smooth and piecewise smooth maps, in a neighborhood of a critical homoclinic orbit a chaotic set may exist or not exist.

Fig. 3 shows a discontinuous map in which we have the left limit value μ_L as maximum, and the map is invariant in an absorbing interval $I = [f(\mu_L), \mu_L]$. As long as $f^2(\mu_L) > p$ (Fig. 3a) the fixed point p has no homoclinic orbit. A neighborhood of p exists which cannot include other periodic points. It can be easily seen in Fig. 3a that periodic points cannot exist in the interval $[f^3(\mu_L), f^2(\mu_L)]$. Therefore, no invariant chaotic set can include p . This is true as long as the only rank-1 preimage of p different from itself is outside the interval I (as in Fig. 3a). For $f^2(\mu_L) = p$, or $f(\mu_L) = p_{-1}$ (Fig. 3b), the fixed point p has a critical homoclinic orbit (SBR bifurcation of p) and, as we shall see below, in any neighborhood of the critical homoclinic orbit we can find an invariant set in which the map is chaotic. For $f^2(\mu_L) < p$ (Fig. 3c) the fixed point p has infinitely many noncritical and nondegenerate homoclinic orbits (structurally stable).

Moreover, in the applied context, when a map is investigated as a function of some parameters, it is usually observed the first SBR bifurcation followed by infinitely many further homoclinic explosions (Ω -explosions), and all of them are associated with critical homoclinic orbits. In fact, whenever a preimage of the fixed point from outside the invariant absorbing interval enters inside, a new critical homoclinic orbit is created. For example, the point p_{-2} is a rank-2 preimage of p outside the interval I in Fig. 3c: if the parameters are changed so that $f(\mu_L) = p_{-2}$, then other new critical homoclinic orbit appear (followed by infinitely many other noncritical and nondegenerate homoclinic orbits), again associated with chaotic behaviors and with a new explosion of unstable periodic orbits.

Similarly, in Fig. 4 we show the first SBR bifurcation of a fixed point p , but now before the bifurcation, see (Fig. 4a), the fixed point does not belong to an invariant absorbing interval $I = [f(\mu_L), \mu_L]$ (in (Fig. 4a), p and its rank-1 preimage bound the basin of attraction of I). The difference is that in the example shown in (Fig. 3c) the nondegenerate homoclinic orbits appearing due to the critical homoclinic orbit belong to a chaotic set inside the invariant attracting interval I , while in the example shown in (Fig. 4c) these belong only to a chaotic repeller (in fact, the interval $I = [f(\mu_L), \mu_L]$ is invariant but not attracting in (Fig. 4b) and no longer invariant in Fig. 4c). Notice that the other fixed point x^* of the map shown in Fig. 4, belonging to I , is homoclinic, and as the parameters are changed from (Fig. 4a) to (Fig. 4b) infinitely many critical homoclinic orbits of x^* occur.

A critical homoclinic orbit, however, is not necessarily associated with a chaotic invariant set. Fig. 5 shows a discontinuous map in which we are varying the right offset

¹ Clearly a noncritical homoclinic orbit may be degenerate or nondegenerate, while a nondegenerate homoclinic orbit is also noncritical.

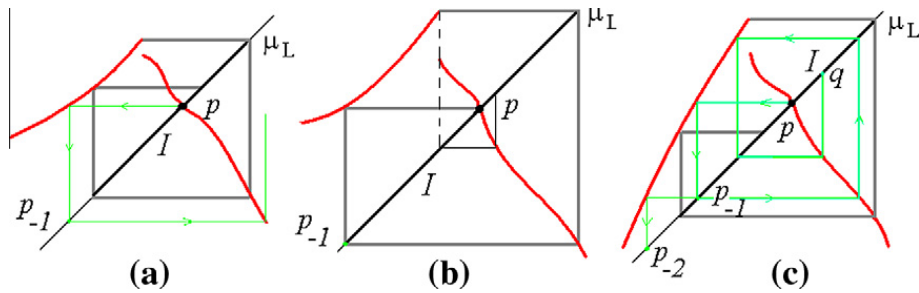


Fig. 3. The fixed point p is not homoclinic in (a). In (b) critical homoclinic orbits of p exist. In (c) noncritical homoclinic orbits of p exist.

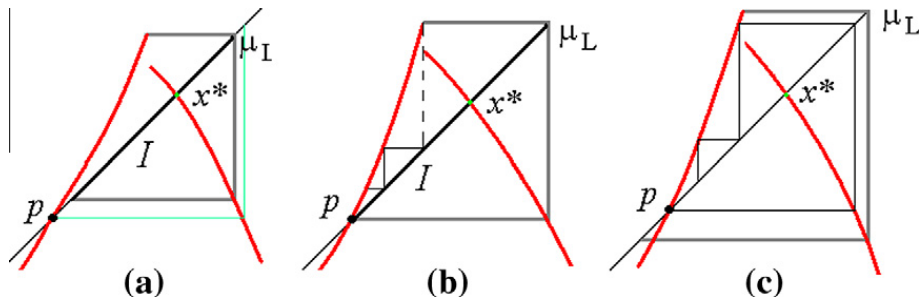


Fig. 4. The fixed point p is not homoclinic in (a). In (b) critical homoclinic orbits of p exist. In (c) noncritical homoclinic orbits of p exist.

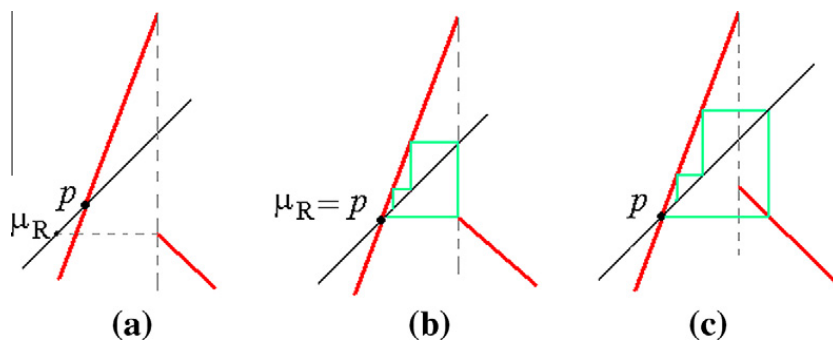


Fig. 5. The fixed point p is not homoclinic in (a). In (b) a unique critical homoclinic orbit of p exists. In (c) noncritical homoclinic orbits of p exist.

μ_R . For $\mu_R < p$ (Fig. 5a) the expanding fixed point p has no homoclinic orbit. For $\mu_R = p$ (Fig. 5b) the fixed point p has a *unique critical homoclinic orbit* (snap-back repeller bifurcation of p) but nevertheless in any neighborhood of the critical homoclinic orbit we cannot find an invariant set in which the map is chaotic. In fact, in Fig. 5b the unstable fixed point p and this critical homoclinic orbit are the unique nondivergent trajectories. For $\mu_R > p$ (Fig. 5c) we have an explosion of infinitely many different noncritical homoclinic orbits of p (structurally stable) and a chaotic repeller exists.

To prove the existence of chaos associated with a critical homoclinic orbit, to which the Theorems given above cannot be applied, it is enough to show that we can find two disjoint intervals, I_0 and I_1 , such that $f^k(I_0) \supset I_0 \cup I_1$ and $f^k(I_1) \supset I_0 \cup I_1$, for a suitable k . That this not always occurs has been shown with the example in Fig. 5b and one more example is given below (in Fig. 8b), showing a different case in which it is not possible to find two intervals

which behave as required above, although other critical homoclinic orbits with a chaotic invariant set can be found (and a third example is given in Fig. 9d).

In the process described below to prove the existence of chaos, continuous and discontinuous maps differ because of the different properties in the critical points. We follow the definition used by Julia and Fatou for *smooth maps*: a critical point c is a local extremum, so that at least two distinct inverses exist, say f_l^{-1} and f_r^{-1} , whose preimages of c are merging in one point, say $f_l^{-1}(c) = f_r^{-1}(c) = c_{-1}$ ($f_l^{-1}(x)$ and $f_r^{-1}(x)$ giving points on opposite sides with respect to c_{-1}), and the function has a vanishing derivative in c_{-1} . In [30] it is extended to nondifferentiable maps (in the 1Dim case a local extremum), and also to discontinuous maps, defining as critical points the two extrema of a function at a discontinuity point. The images of a critical point are also relevant in the description of the dynamics, so that c is called *critical point of rank-1* and $c_i = f^i(c)$, $i = 1, 2, \dots$ are called *critical points of rank $(i + 1)$* .

When a critical homoclinic orbit involves a critical point c which is a local extremum (case (i) in Property 1), then at least two inverses exist, which are defined in c and merging in a point c_{-1} , and both are suitable. While when a discontinuity point is involved (case (ii) in Property 1), and a limit value of f at the discontinuity point, μ_L or μ_R , then we have a unique inverse which is of interest in that point.

Let us start with the case of a critical homoclinic orbit involving a critical point c which is a local maximum, to fix the reasoning (considerations similar to those given below clearly apply in the case of a local minimum). Consider a critical homoclinic orbit $\mathcal{O}_k(p)$ including a critical point c representing a local maximum, for which $f^k(c) = p$. At least two merging preimages exist giving $f_l^{-1}(c) = f_r^{-1}(c) = c_{-1}$ (see the qualitative picture in Fig. 6). Then problems may arise from the fact that in any neighborhood of c_{-1} f is not 1-1, so that locally a compact interval including c_{-1} is mapped by f in a compact interval having $c = f(c_{-1})$ on its boundary and thus in a finite number of iterations p is also on the boundary. However, this is not a problem if we consider only the topological property, without considering the derivatives of the function in the homoclinic points (and indeed the map may also be not differentiable in some homoclinic points). Thus, let us consider a compact interval $W = W_l \cup W_r$ where $W_l = [\alpha, c_{-1}]$ and $W_r = [c_{-1}, \beta]$ for which $f(W) = [\gamma, c]$. That is, at least two inverses are defined in $f(W)$, say $f_l^{-1}([\gamma, c]) = W_l$ and $f_r^{-1}([\gamma, c]) = W_r$. Then $f^{k+1}(W)$ must include a compact interval bounded by p . Let $U \subseteq f^{k+1}(W)$ be a compact interval bounded by p . The local inverse $f_r^{-1}(U)$ is disjoint from the inverse $f_l^{-1}(U)$ where $f_l^{-1}(p)$ gives the homoclinic point x_{k-1} (as the preimages are on opposite sides with respect to c_{-1}). Then considering

the preimages of U following the same functions which give the homoclinic orbit by backward iterations, we get an interval $f^{-k}(U) \subset [\gamma, c]$ and due to the existence of the critical value c we are at a point in which at least two different inverses can be chosen. Then for any choice (for f_l^{-1} as well as for f_r^{-1}) we have $f^{-(k+1)}(U) \subset W_l$ or $f^{-(k+1)}(U) \subset W_r$. In any case, continuing to follow the points of the given homoclinic orbit we can detect a suitable n such that $I_1 = f^{-(k+n)}(U) \subset U$ (not including the point p). When the set obtained with the repeated applications of the local inverse $I_0 = f_r^{-(k+n)}(U)$ (which includes p and is disjoint from I_1) satisfies $I_0 = f_r^{-(k+n)}(U) \subset U$, the proof is done. As, defining $F = f^{k+n}$, we have constructed two disjoint intervals I_0 and I_1 such that $F(I_0) \supset I_0 \cup I_1$ and $F(I_1) \supset I_0 \cup I_1$, and this ends the proof. Moreover, we can construct two different invariant sets, once considering f_l^{-1} and once f_r^{-1} in the above construction.

For example, let us consider the SBR bifurcation (the first homoclinic explosion) of p in the map qualitatively shown in Fig. 7. This bifurcation occurs when $f^2(c) = p$. Consider the critical homoclinic orbit, say $\mathcal{O}_2(p)$, given by

$$\mathcal{O}_2(p) = \{p \leftarrow f_r^{-n}(c_{-1}), \dots, f_r^{-1}(c_{-1}), c_{-1}, c, c_1, c_2 = p\}. \quad (4)$$

Here we have a chaotic set in any neighborhood of $\mathcal{O}_2(p)$, no matter if c_{-1} is obtained via $f_l^{-1}(c)$ or $f_r^{-1}(c)$. Clearly, changing the possible sequence of preimages of c_{-1} we can find infinitely many other critical homoclinic orbits, different from $\mathcal{O}_2(p)$, but all of them end with the points $(c_{-1}, c, c_1, c_2 = p)$, and all of them are associated with a chaotic set.

However, this is not the case for the critical homoclinic orbit

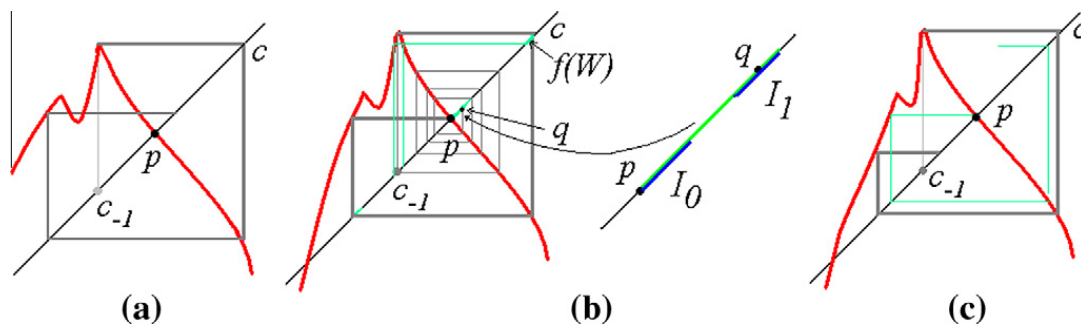


Fig. 6. The fixed point p is not homoclinic in (a). In (b) critical homoclinic orbits of p exist, associated with a chaotic set. In (c) noncritical homoclinic orbits of p exist.

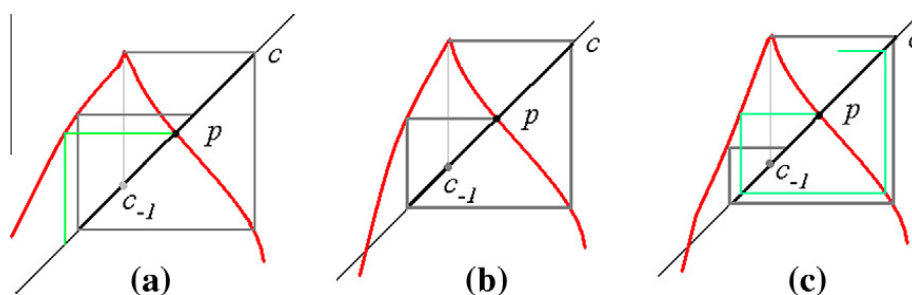


Fig. 7. The fixed point p is not homoclinic in (a). In (b) SBR bifurcation of p and the critical homoclinic orbits are associated with chaotic sets. In (c) noncritical homoclinic orbits of p exist.

$$\mathcal{O}_{3a}(p) = \{p \leftarrow f_r^{-n}(c_{-1}), \dots, f_r^{-1}(c_{-1}), c_{-1}, c, c_1, c_2, c_3 = p\} \quad (5)$$

shown in (Fig. 8a).

This orbit $\mathcal{O}_{3a}(p)$ corresponds to the second homoclinic explosion of the fixed point p , occurring when $f^3(c) = p$. Then consider the point $q = f_r^{-1}(c_{-1})$ and note that for any interval including this point q we need 5 iterations to return to obtain an interval including the point q again (see Fig. 8b). Even if we consider as q any other homoclinic point of the same homoclinic orbit, closer to p , the number of iterations needed to an interval I_1 including q to return on the right side including q and I_1 again, is an odd number. By contrast, any right neighborhood I_0 of p needs an even number of iterations to be on the right side again. It follows that we can find two integers k_1 and k_2 satisfying Lemma 4 in [18], one odd and one even, and indeed a homoclinic orbit exists, but this homoclinic orbit is not associated with a chaotic set because we cannot find a unique integer k such that $f^k(I_0) \supset I_0 \cup I_1$ and $f^k(I_1) \supset I_0 \cup I_1$.

Nevertheless, there are infinitely many critical homoclinic orbits, all ending with the same critical points $(c_{-2}, c_{-1}, c, c_1, c_2, c_3 = p)$, which are associated with chaos, as for example the critical homoclinic orbit

$$\mathcal{O}_{3b}(p) : \{p \leftarrow f_r^{-n}(c_{-2}) \dots f_r^{-1}(c_{-2}), c_{-2} = f_r^{-1}(c_{-1}), c_{-1}, c, c_1, c_2, c_3 = p\} \quad (6)$$

(see Fig. 8c). In fact, let $q = f_r^{-1}(c_{-2})$ as in Fig. 8d and consider the point ξ on the left of q such that $f^3(\xi) = q$, then let $I_1 = [\xi, q]$: we have $f^6(I_1) = [p, c]$. Consider now $I_0 = f_r^{-6}[p, c]$ (see Fig. 8e). So we have found two disjoint intervals I_0 and I_1 such that $f^6(I_0) \supset I_0 \cup I_1$ and $f^6(I_1) \supset I_0 \cup I_1$.

Notice that the critical orbit involved in the first homoclinic bifurcation of a fixed point p is not necessarily the one associated with the critical point bounding the invariant absorbing interval of interest, as shown in Fig. 9.

And also in such a case the critical homoclinic orbit in Fig. 9b (when $c = p$) is associated with a chaotic set and nothing changes in the construction of the intervals described above except for the inverses which are used: here the local inverses f_l^{-1} and f_r^{-1} associated with the critical point $c = p$, giving c_{-1} , are different from the local inverse f_0^{-1} acting on the unstable fixed point p , and we can have different kinds of homoclinic orbits (by using both inverses, f_l^{-1} and f_r^{-1} , as shown in Fig. 9c).

Similarly we can reason in the case of a discontinuous map. Let us consider a critical homoclinic orbit $\mathcal{O}_k(p)$ including a point μ_L or μ_R . Then the critical homoclinic orbit must include the discontinuity point d and for some integer k we have $f^k(\mu_L) = p$ or $f^k(\mu_R) = p$. In these cases, problems may arise from the fact that a compact interval including d is mapped by f in two disjoint intervals, one bounded by μ_L and one by μ_R . Clearly we have to consider only one of them, the compact interval bounded by μ_L if it is $f^k(\mu_L) = p$, or the one bounded by μ_R if it is $f^k(\mu_R) = p$. Thus in the discontinuous case, in the construction of the suitable intervals we have not an alternative associated with the left/right side of the discontinuity point, but a unique choice.

From Fig. 10 we can see that it is easy to find particular cases in which a single critical homoclinic orbit involves two conditions in Property 1: twice (i) (Fig. 10a), or both (i) and (ii) (Fig. 10b and c). However this does not introduce more complicated reasoning with respect to those given below. In other words, in Fig. 10a for each critical point we can reason in a similar way, thus proving the existence of different invariant chaotic sets associated with the same critical homoclinic orbit. In the cases of (Fig. 10b and c) a particular attention must be paid in order to detect whether it is the critical point or the discontinuity point or both, to be associated with complex dynamics.

Another critical homoclinic orbit not associated with a chaotic set is shown in Fig. 10d. In this example, any left

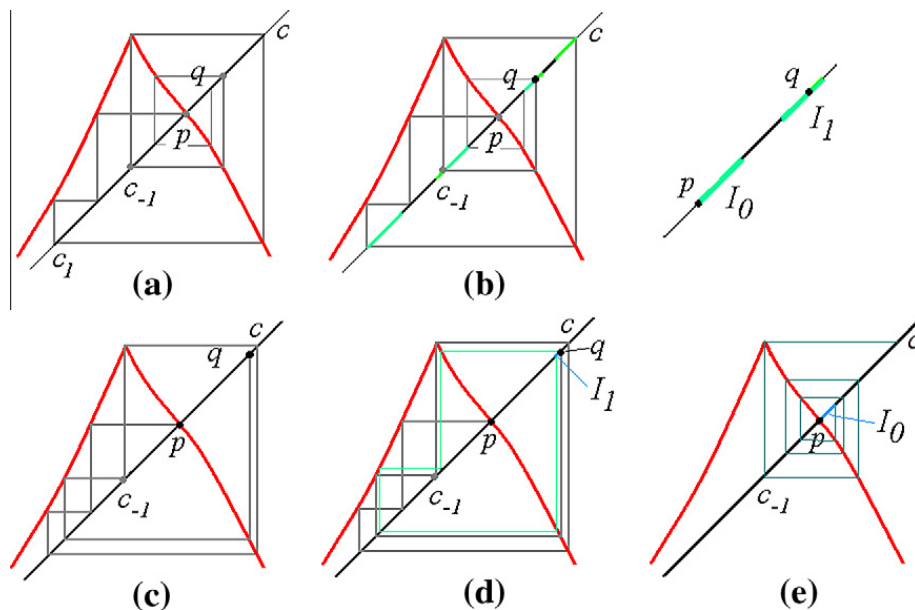


Fig. 8. Second homoclinic explosion of the fixed point p . In (a) the critical homoclinic orbit is not associated with a chaotic set. In (b) a neighborhood of q takes 5 iterations to include q again. In (c) a different critical homoclinic orbits of p is considered, associated with chaos. In (d) a suitable interval I_1 is shown. In (e) a suitable interval I_0 is shown.

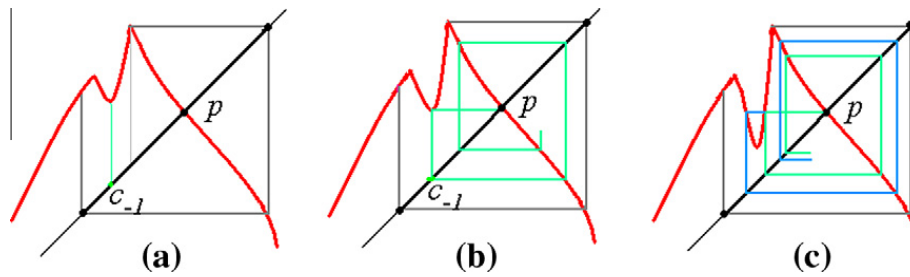


Fig. 9. The fixed point p is not homoclinic in (a). In (b) critical homoclinic orbits of p exist. The critical point involved in the homoclinic bifurcation is not associated with the absorbing interval. In (c) noncritical homoclinic orbits of p exist.

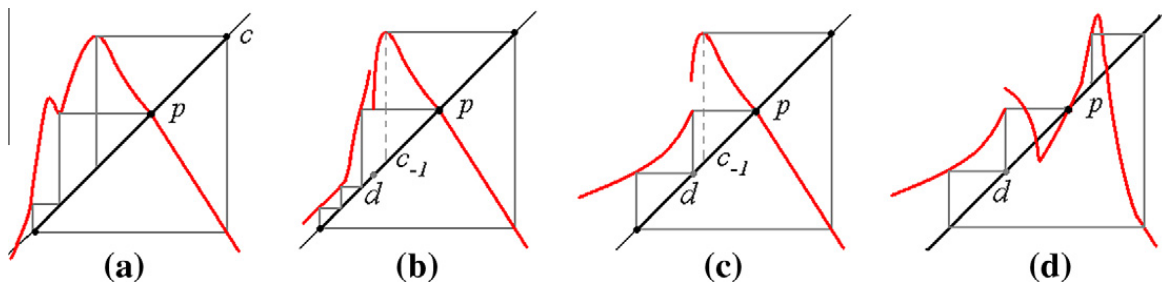


Fig. 10. In (a) a critical homoclinic orbit involves two critical points, a local maximum and a local minimum. In (b) a critical homoclinic orbit involves two critical points, a local maximum and a value (right limiting value) at a discontinuity point. In (c) a critical homoclinic orbit involves two critical points, a local maximum and a value (left limiting value) at a discontinuity point. In (d) a critical homoclinic orbit is not associated with a chaotic set.

neighborhood of $\mu_L = p$, following backward the homoclinic orbit, has preimages only on the right side of p .

3.1. Homoclinic orbits in unbounded sets $X \subseteq \mathbb{R}^1$

Up to now we have considered examples of critical homoclinic orbits for a map $f: X \rightarrow X$ in which X is a compact (closed and bounded) subset of \mathbb{R}^1 . Let us now examine via an example what may occur when we consider a closed and unbounded set X of \mathbb{R}^1 .

It is clear that we still may have expanding fixed points which undergo their SBR bifurcation and other Ω -explosions via critical homoclinic orbits of type (i) or (ii) in Property 1. However, when the map has an unbounded domain and range, a new kind of homoclinic bifurcation mechanism can occur, which leads to part (iii) in Property 1. In fact, we have to introduce a new concept, which in the 1Dim case is associated with the occurrence of a horizontal asymptote (which means that the rank-1 preimage of a real point may be at infinity, $+\infty$ or $-\infty$), and that of a vertical asymptote as well. In such cases we consider the real space closed, including the points at infinity.

To better understand this new kind of critical homoclinic orbit let us describe an example. Consider the map whose shape is shown in Fig. 11. We can see that $f: X \rightarrow X$ where $X = [-\infty, c]$ is unbounded. There is a vertical asymptote in $x = v$ ($f(v) = -\infty$, and let $f_0^{-1}(-\infty) = v$) and an horizontal asymptote in h ($f(-\infty) = h$, and $f_1^{-1}(h) = -\infty$). Here the critical point $x = c$ is not involved in the homoclinic bifurcation of the fixed point $x = p$. The fixed point p in Fig. 11a is below the horizontal asymptote h ($h > p$), and has no other rank-1 preimage different from itself in the interval X , i.e. in p only the local inverse is defined, say $f_0^{-1}(p) = p$. As long as it is $h > p$ the fixed point is

not homoclinic. A new kind of homoclinic bifurcation (here SBR bifurcation of p) occurs when $h = p$ (see Fig. 11b), as after this bifurcation, when $h < p$ (see Fig. 11c) there is an explosion of nondegenerate homoclinic orbits of p , involving another inverse function. That is, when $p > h$ one more rank-1 preimage of the fixed point p appears, say $p_{-1} = f_1^{-1}(p)$, creating homoclinic orbits, and at the bifurcation, when $p = h$, this new rank-1 preimage of the fixed point appears at $-\infty$. Indeed at the bifurcation we can see that both the horizontal and vertical asymptotes are involved: we can consider the unbounded critical homoclinic orbit (see Fig. 11b)

$$\mathcal{O}_\infty(p) : \{ \dots, f_0^{-n}(v), \dots, f_0^{-1}(v), v = f_0^{-1}(-\infty), -\infty = f_1^{-1}(h), p = h \}. \quad (7)$$

Also notice that an unbounded critical homoclinic orbit in general is not the unique existing one. In Fig. 11c we show a second one. These new unbounded critical homoclinic orbits lead to infinitely many noncritical (and nondegenerate) bounded homoclinic orbits of p when $p > h$. The bounded homoclinic orbit shown (in black) in Fig. 11d is given by

$$\mathcal{O}(p) : \{ \dots, f_0^{-n}(p_{-1}), \dots, f_0^{-1}(p_{-1}), p_{-1} = f_1^{-1}(p), p \}. \quad (8)$$

After this first homoclinic explosion (SBR bifurcation of p) other Ω -explosions may occur due to the crossing of the same horizontal asymptote h of other homoclinic points. For example, in Fig. 11d the homoclinic point $q = f_0^{-2}(p_{-1})$ is below h . We can assume a change in the function so that to have $q = h$ (and thus a new unbounded critical homoclinic orbit), leading, for $q > h$, to a new explosion of nondegenerate homoclinic orbits of p .

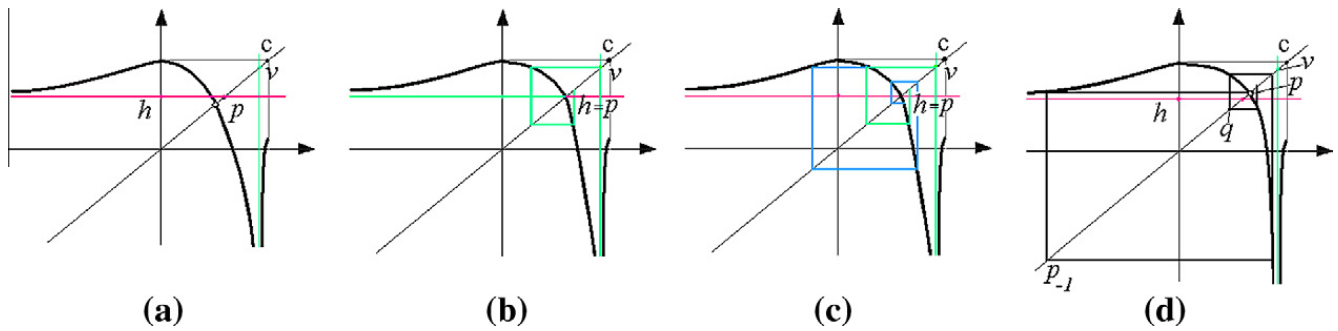


Fig. 11. The fixed point p is not homoclinic in (a). In (b) and (c) unbounded critical homoclinic orbit of p are shown. In (d) noncritical homoclinic orbits of p exist.

3.2. Different homoclinic bifurcations

In this work we are characterizing the homoclinic bifurcations which lead an existent fixed point p , expanding, to become SBR. We also recall that in the 1Dim case, if the eigenvalue of an expanding fixed point p is negative then its SBR bifurcation leads to homoclinic points on both sides of p . While if the eigenvalue of p is positive then its SBR bifurcation usually leads to homoclinic points on one side only. Thus it may occur, after, that another Ω -explosion creates homoclinic points on the other side. In this case we may distinguish between two SBR bifurcations associated with the two sides of p (as both are global bifurcations which may have relevant dynamic effects).

However, it is worth to mention that the appearance of noncritical and nondegenerate homoclinic orbits of a fixed point may also occur via different kind of bifurcations, both local and global, as shown in the following examples.

Example 1. A stable fixed point may become homoclinic due to its flip bifurcation. That is, we may have a fixed point p attracting for $\xi < \xi_f$, at $\xi = \xi_f$ its flip bifurcation occurs (eigenvalue equal to -1), and for $\xi > \xi_f$ nondegenerate homoclinic orbits exist. An example is shown in Fig. 12. In Fig. 12a the fixed point p is stable, and via a subcritical flip bifurcation it becomes unstable and immediately with nondegenerate and noncritical homoclinic orbits (Fig. 12b).

Example 2. A fixed point or cycle may appear via border collision bifurcation, and soon after its appearance it may have homoclinic orbits. As an example consider the transition from invertibility to noninvertibility in the 1Dim piecewise linear map with one discontinuity point d , and

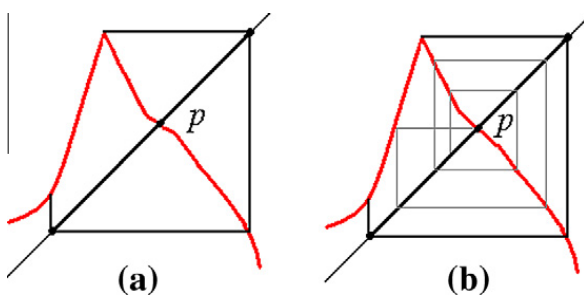


Fig. 12. The fixed point p is stable in (a), and homoclinic in (b).

two branches f_L and f_R as shown in Fig. 13. In Fig. 13a it is $f_L \circ f_R(d) > f_R \circ f_L(d)$ so that the map is invertible in $I = [\mu_R, \mu_L]$, where $\mu_R = f_R(d)$ and $\mu_L = f_L(d)$, and there can be only a stable regime (the structurally stable attractor can only be a stable cycle, or quasiperiodic trajectories can exist, but no homoclinic orbit). In Fig. 13b it is $f_L \circ f_R(d) = f_R \circ f_L(d)$ (bifurcation value) and the map is topologically conjugate to a linear rotation (only periodic or quasiperiodic orbits can exist). In Fig. 13c it is $f_L \circ f_R(d) < f_R \circ f_L(d)$ so that the map is noninvertible in I , inside which there can be only a chaotic regime (no stable cycles can exist, only unstable and homoclinic). That is, after the bifurcation value infinitely many unstable cycles appear via border collision bifurcations and these are homoclinic. For the details we refer to [21] and [14].

Example 3. A saddle node bifurcation in a smooth map may give rise to a cycle which is already homoclinic at its appearance, and with homoclinic points (on one side only) also at the saddle node bifurcation value. For example this occurs in the logistic map (or any map topologically conjugate to it) after the Feigenbaum point, at any saddle node bifurcation value opening a periodic window of a k -cycle (and infinitely many of them occur, also many for the same value of k).

4. Critical homoclinic orbits in \mathbb{R}^n

Let us now turn to the general case of homoclinic orbits of maps in \mathbb{R}^n and consider an expanding fixed point p of a map $f: X \rightarrow X$ with $X \subseteq \mathbb{R}^n$. The property of a homoclinic orbit to be *noncritical* is structurally stable, that is, under parameter variation the homoclinic orbit persists and is noncritical (and the function f and \tilde{f} after the perturbation are conjugated), thus Theorem 2 applies to structurally stable homoclinic orbits. In other words, the *critical* homoclinic orbits are structurally unstable, as in fact, a small variation of the parameters brings a critical homoclinic orbit on one side to become noncritical and on the other side to disappear. We exclude particular cases in which a critical homoclinic orbit persists under some parameter variation, that is, we assume that a kind of transversality condition is satisfied. Let a critical homoclinic bifurcation occur at a parameter value $\xi = \xi_c$. Then on one side of the bifurcation value, say for $\xi < \xi_c$, this homoclinic orbit does not exist, while if the transversality condition is satisfied,

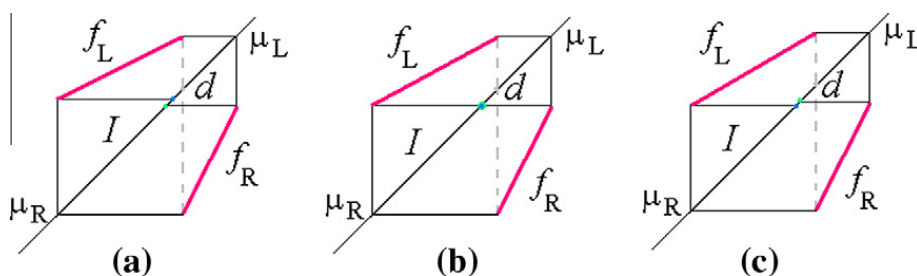


Fig. 13. In (a) the map is invertible in I . In (b) the map is at a bifurcation. In (c) the map is noninvertible in I .

for $\xi > \xi_c$, close to the bifurcation value, this homoclinic orbit exists and is noncritical. The first SBR bifurcation and further homoclinic explosions of an expanding fixed point p must involve a critical homoclinic orbit.

We notice that our approach generalizes the definition given by Glendinning in [18], where a particular set of points W for a hybrid continuous map is considered, given by the set of points in which the Jacobian is not defined. In that paper it is noticed that a homoclinic orbit having a point in the set W is not structurally stable. However, this is not necessarily always true. We can easily find examples of n -dimensional maps in which a point of the homoclinic trajectory belongs to the set W but keeping the property of being locally invertible, and thus noncritical in our definition (generalization of the 1Dim case shown in Fig. 2b), so that such a homoclinic orbit is also structurally stable, and Theorem 2 works perfectly.

A similar notion is introduced also by Shi and Yu in [33] with respect to points in which the determinant of the Jacobian is vanishing. A homoclinic orbit with a point in which the Jacobian determinant is vanishing is called “not regular”, and the authors show that it may be not structurally stable. However, also this classification of “not regular” homoclinic orbit may lead to a structurally stable orbit, noncritical in our definition, and thus not related to a bifurcation value. We can easily find examples of n -dimensional maps in which a point of the homoclinic trajectory belongs to the set W_0 of vanishing Jacobian, but satisfying in the meantime the property of being locally invertible and, thus, noncritical (generalization of the 1Dim case shown in Fig. 2a), so that Theorem 2 can be applied.

As we have seen in Section 2, the true property leading to a homoclinic orbit which is not structurally stable (and thus leading to the SBR bifurcation and other homoclinic explosions), is the one here described. We can so state corollaries of the main result given in Theorem 2:

Corollary 1. Let $f : X \rightarrow X, X \subseteq \mathbb{R}^n$, be a piecewise smooth noninvertible map, p an expanding fixed point of f and $\mathcal{O}(p)$ a critical homoclinic orbit of p , then $\mathcal{O}(p)$ determines either the SBR bifurcation or another homoclinic explosion of p .

So, assuming a kind of transversality condition, we can state that the critical homoclinic orbits are structurally unstable. In other words, we can state the following:

Corollary 2. Let $f : X \rightarrow X, X \subseteq \mathbb{R}^n$, be a piecewise smooth noninvertible map, p an expanding fixed point of f and $\mathcal{O}(p)$ a homoclinic orbit of p . The homoclinic orbit $\mathcal{O}(p)$ is structurally stable iff it is noncritical.

In section 3 we have described what is a critical point c for a 1Dim map. Let us now characterize a critical point of maps in \mathbb{R}^n . As for the 1Dim case, we can have several kinds of critical points: with vanishing Jacobian, or not defined Jacobian (and as in the 1Dim case different one-side Jacobians can be defined). Associated with points in which the map is continuous, all the points in which f is not locally invertible (satisfying part (i) of Property 1) lead to the so-called Critical Set denoted as CS , or, in the case of a 2Dim map, to the Critical Line² traditionally denoted as LC (from the french *Ligne Critique*, as in [30] and references therein). The critical line LC is the set of points of the plane having at least two merging preimages in points in which the map is not locally 1 – 1. The set of merging preimages is denoted by LC_{-1} . Similarly in a higher dimensional space, the Critical Set CS is the set of points of the space having at least two merging rank-1 preimages in points in which the map is not locally 1-1, and the set of merging preimages is denoted by CS_{-1} . That is, let $x \in CS$, then in any neighborhood of x we can find at least two inverse functions, say $f_r^{-1}(x)$ and $f_l^{-1}(x)$ such that $f_r^{-1}(x) = f_l^{-1}(x) = x_{-1} \in CS_{-1}$ (f_l^{-1} and f_r^{-1} giving points on opposite sides with respect to CS_{-1}). When the map f is smooth, the set CS_{-1} is a subset of the locus defined by $\det(J_f(x)) = 0$ (it may be equal or strictly included in it). When the map is piecewise smooth and continuous, the set CS_{-1} may include (not necessarily) a set in which the map changes definition (also called switching manifolds).

Considering discontinuous maps, we include in the set CS_{-1} also the points of discontinuity, and clearly in the set CS the limit values at the discontinuity points are included. That is, considering a point of discontinuity $x_{-1} \in CS_{-1}$ then $\lim_{y \rightarrow x_{-1}} f(y) \in CS$. Thus in discontinuous maps part (ii) of Property 1 may occur, in which case the critical homoclinic orbit includes also a critical point associated with the discontinuity.

Considering part (iii) of Property 1, we have seen that in the 1Dim case the critical homoclinic orbit must necessarily include both a horizontal and a vertical asymptote. Similarly we can reason in \mathbb{R}^n . Such an unbounded critical homoclinic orbit must necessarily include both a real point whose inverse is at infinity, and a real point in which the map takes infinite value (i.e. the rank-1 preimage of a point at infinity must be real). Several examples of maps of the plane having a vanishing denominator or inverse functions

² With the meaning of Critical curve, i.e. not necessarily a straight line.

with a vanishing denominators have been considered in [2–5].

Consider now a critical homoclinic orbit $\mathcal{O}(p)$ of an expanding fixed point p , so that f does not satisfy the property of being locally invertible in each point of $\mathcal{O}(p)$. One more problem is whether such an orbit is associated with an invariant chaotic set or not. As we have shown in the previous section, for maps in \mathbb{R}^1 this may occur or not. The same result holds in general for maps in \mathbb{R}^n . In [18] it was shown that, depending on the signs of the eigenvalues of a repelling node in \mathbb{R}^2 , a critical homoclinic orbit may be associated with chaos or not.

In the case of a repelling focus in \mathbb{R}^2 a critical homoclinic orbit is more likely associated with chaos. Examples of critical homoclinic orbits to an expanding focus have been given for a smooth map in [15], for a piecewise smooth continuous map and for a discontinuous one in [16]. Other 2Dim examples are given in the following subsections.

4.1. Maps with separate second iterate

A first example of 2Dim maps from which several examples can be obtained and whose properties are strictly related to those of a 1Dim map is the class of maps given by

$$\Phi : \begin{cases} x' = g(y) \\ y' = f(x) \end{cases} \quad (9)$$

having separate second iterate (see [6]) in which the 1Dim function

$$F : z' = f \circ g(z) \quad (10)$$

determines the bifurcations of Φ . Here the SBR bifurcation and other homoclinic explosions of cycles of the 1Dim map F in attracting invariant sets also correspond to SBR and homoclinic explosions of cycles of the 2Dim map Φ in attracting invariant 2Dim sets. In particular, defining

$$T : \begin{cases} x' = y \\ y' = f(x) \end{cases} \quad (11)$$

all the qualitative maps $z' = f(z)$ considered in the 1Dim examples of Section 3 lead to analogues in the 2Dim case. That is, all the SBR and homoclinic bifurcations there described for fixed points also occur for the related fixed points in the 2Dim map T defined in (11).

For example, a 2Dim analogue of the 1Dim example shown in Section 3.1 defined in a closed and unbounded set $X \subseteq \mathbb{R}^2$ can be easily constructed via the map T given in (11), and the function $f(x)$ of Fig. 11.

4.2. Triangular maps

One more class of maps in the phase space \mathbb{R}^2 which can be obtained making use of any 1Dim map f already considered in Section 3 is related to triangular maps. A map T of the plane into itself is called triangular when it has the following structure:

$$T : \begin{cases} x' = f(x) \\ y' = g(x, y) \end{cases} \quad (12)$$

for which the cycles and critical points are associated with the related cycles and critical points of the 1Dim map $x' = f(x)$. A peculiarity of this class of maps is that the eigenvalues associated with any cycle are always real, so that it is not possible to have repelling foci. The unstable cycles are only saddles and repelling nodes, as the Jacobian matrix has a triangular structure with diagonal elements leading to the eigenvalues: $J_{11} = f'(x)$ and $J_{22} = \partial_y g(x, y)$. As an example let us consider the standard logistic function $f_a(x) = ax(1 - x)$ coupled with a linear function $g(x, y) = x + by$:

$$T(x, y) : \begin{cases} x' = f_a(x) = ax(1 - x) \\ y' = g_b(x, y) = x + by \end{cases} \quad (13)$$

It is immediate to see that T in (13) is of so-called $Z_0 - Z_2$ type of noninvertibility. The critical sets LC_{-1} and LC are given by the straight lines $x = c_{-1} = 0.5$ and $x = c = a/4$, respectively. Any point of the half-plane on the right side of LC (region Z_0) has no rank-1 preimages. By contrast, any point of the half-plane on the left side of LC (region Z_2) has two distinct preimages, one on the right and one on the left of LC_{-1} , given by:

$$T_l^{-1}(x', y') : \begin{cases} x = \frac{a - \sqrt{a^2 - 4ax'}}{2a} \\ y = \frac{y' - x}{b} \end{cases}, \quad T_r^{-1}(x', y') : \begin{cases} x = \frac{a + \sqrt{a^2 - 4ax'}}{2a} \\ y = \frac{y' - x}{b} \end{cases} \quad (14)$$

Clearly any point $(x', y') \in LC$ has two merging preimages $T_l^{-1}(x', y') = T_r^{-1}(x', y')$ on LC_{-1} .

The two fixed points $x^* = \frac{a-1}{a}$ and $x = 0$ of $f_a(x)$ lead to the two fixed points of T : $P^* = (x^*, y^*)$ where $y^* = \frac{x^*}{1-b}$ and $O = (0, 0)$. As one eigenvalue is constant and equal to the parameter b we have that for $|b| < 1$ no cycle can be expanding, and T can only have attracting cycles or saddles. Moreover, the stable eigenvector associated with the eigenvalue b belongs to vertical lines through the periodic points. The initial conditions with $x < 0$ and $x > 1$ lead to divergent trajectories.

Differently, for $|b| > 1$ no attracting cycle can exist, and the cycles of T can only be saddles or repelling nodes (i.e. expanding), and the unstable eigenvector associated with the eigenvalue b belongs to vertical lines through the periodic points. Thus almost all the trajectories are divergent in this case, and the invariant set in the phase space is a so-called chaotic repeller.

The SBR bifurcation of the fixed point x^* of the logistic map occurs when the parameter is $a = a^* \approx 3.67857351$, such that $f_a^2(c) = x^*$, and at this bifurcation value we have that all the homoclinic orbits of x^* are critical (and also associated with chaotic sets).

For the 2Dim map T in (13) the result of this bifurcation depends on the parameter b . For $|b| < 1$ it corresponds to the first homoclinic bifurcation of the saddle fixed point P^* (the unstable set of P^* becomes tangent to the stable one) whose dynamic effect is that an attracting set made up of two disjoint pieces for $a < a^*$ becomes in one unique piece for $a > a^*$ (as for the 1Dim case).

Similarly, at $a = 4$ the SBR bifurcation of the origin for the logistic map becomes, for T in (13), the first tangency between the stable and unstable sets of the saddle O , and for $a > 4$ all the cycles belonging to a chaotic repeller in

the phase plane are saddles. Thus no SBR bifurcation of P^* (or of O) can occur in these cases. An example is shown in (Fig. 14a and b) at $b = 0.9$ before and after the homoclinic tangency at $a = a^*$, and in (Fig. 14c and d) at $b = -0.9$ before and after the homoclinic tangency at $a = a^*$. The boundaries of the regions shown in Fig. 14 are made up by lateral segments of critical curves LC_i , while upper and lower boundaries are given by the unstable sets of saddle cycles. In [30] the boundaries of these areas are so-called of mixed type, different from those occurring in other cases, where all the boundaries are made up of critical segments, an example of which will be given in the next subsection. We remark that in this case also the homoclinic bifurcations of saddle cycles (via tangency of the stable and unstable sets) occur via a homoclinic orbit which includes a point belonging to the critical set LC .

For $|b| > 1$ the SBR bifurcation of the fixed point x^* of the logistic map occurring at $a = a^*$ corresponds to the SBR bifurcation of the expanding node P^* . At $a = a^*$ all the homoclinic orbits of P are critical (an example is shown in Fig. 15a at $b = 1.1$ and in Fig. 15b at $b = -1.1$). While for $a > a^*$, close to the bifurcation, the homoclinic orbits of P^* are noncritical and nondegenerate. This situation persists up to the second homoclinic explosion of P^* occurring at the parameter \bar{a} such that $f_{\bar{a}}^3(c) = x^*$ when other critical

homoclinic orbits emerge. Regarding the existence of a chaotic set associated with the critical homoclinic orbits at the bifurcation values, it is most likely that these sets exist, because the eigenvalues of P^* are either both negative or one positive and one negative.

The situation at $a = 4$ is similar: the SBR bifurcation of the origin for the logistic map is also for T the SBR bifurcation of the fixed point O . At $a = 4$ all the homoclinic orbits of O are critical, while noncritical and nondegenerate for $a > 4$.

4.3. A smooth map of type $Z_1 - Z_3 - Z_1$

The last example that we propose is a smooth map considered in [7]. We refer to that work for the details regarding the critical sets recalled below, and other properties of this map. It is a 2Dim noninvertible map of type $Z_1 - Z_3 - Z_1$ defined by:

$$T : \begin{cases} x' = x + y \\ y' = ax + bx^2 + cx^3 + dy \end{cases} \quad (15)$$

The map T in (15) has three fixed points $O = (0,0)$, $P = (x_p, y_p)$ and $Q = (x_q, y_q)$, where

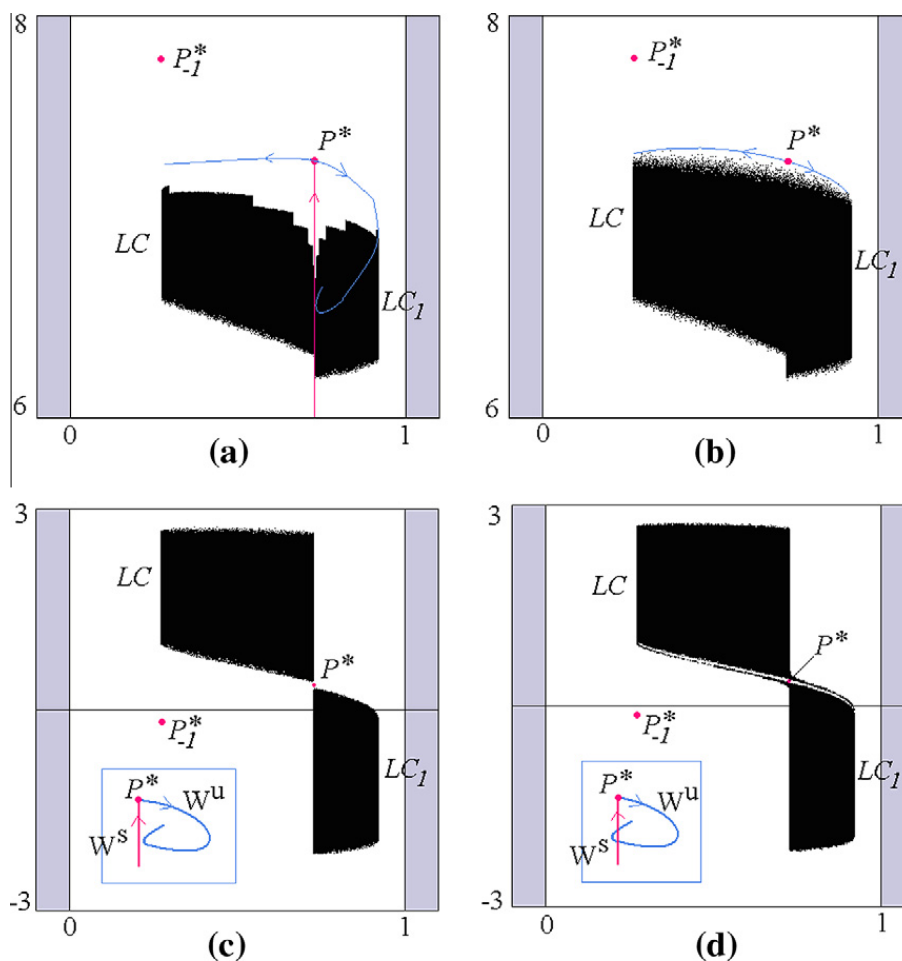


Fig. 14. Attracting sets of map T in (13). Initial conditions in the gray region give divergent trajectories. LC_{-1} is the vertical line at $x = 0.5$. The vertical line through P^* is its local stable set. In (a) $b = 0.9$ and $a < a^*$. In (b) $b = 0.9$ and $a > a^*$. In (c) $b = -0.9$ and $a < a^*$. In (d) $b = -0.9$ and $a > a^*$.

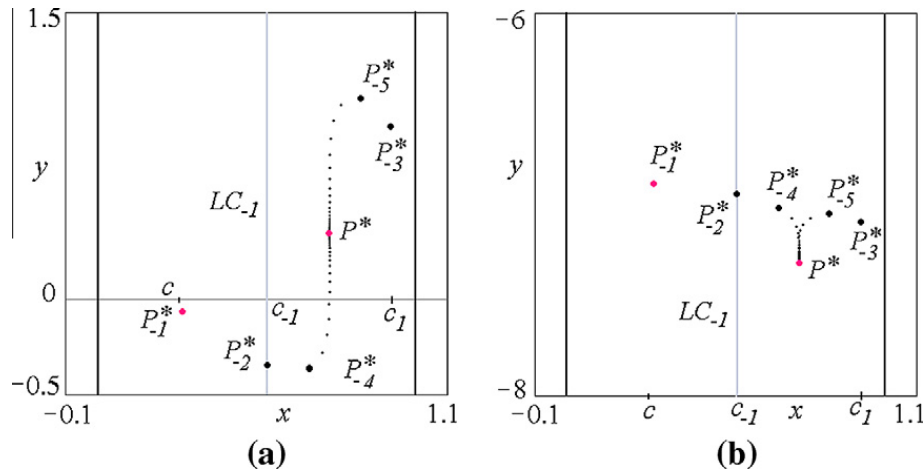


Fig. 15. Critical homoclinic orbits of the fixed point P of the map T in (13) at $a = a^*$. In (a) at $b = 1.1$ and in (b) at $b = -1.1$.

$$\begin{aligned}
 x_P &= \frac{-b + \sqrt{b^2 - 4ac}}{2c}, & y_P &= 0 \\
 x_Q &= \frac{-b - \sqrt{b^2 - 4ac}}{2c}, & y_Q &= 0
 \end{aligned}
 \tag{16}$$

The inverses of T in (15) are obtained via the solutions of a cubic equation. The rank-1 preimages of a point (x', y') are the points (x, y) which are solutions of the following equations:

$$cx^3 + bx^2 + (a - d)x + dx' - y' = 0, \quad y = x' - x \tag{17}$$

The phase space of map T in (15) is divided into three regions $Z_k, k = 1, 2, 3$, characterized by points having k distinct rank-1 preimages. The boundaries of the regions $Z_k, k = 1, 3$, are made up of two parallel straight lines L, L' , defining the critical curve $LC = L \cup L'$. The critical set LC is the locus of points having at least two merging rank-1 preimages (obtained by two different determinations of the inverse map T^{-1}) and the set of merging points gives the critical set $LC_{-1} = L_{-1} \cup L'_{-1}$. So we have $LC = T(LC_{-1}) = T(L_{-1}) \cup T(L'_{-1})$ and here the two branches of LC_{-1} are given by the condition $\det J_T(x, y) = 0$. Therefore we obtain

$$\begin{aligned}
 L_{-1} : x &= \frac{-b - \sqrt{b^2 - 3c(a - d)}}{3c}, \\
 L'_{-1} : x &= \frac{-b + \sqrt{b^2 - 3c(a - d)}}{3c}
 \end{aligned}
 \tag{18}$$

Considering the parameters $a = 0.25, c = -0.5, d = -1.08$, then for b in a suitable range the fixed point Q is a stable focus, which becomes for increasing b an unstable focus, and sequences of further bifurcations lead to an annular chaotic area (see [30]) surrounding the unstable focus. No homoclinic orbit of Q can exist as long as the points in a neighborhood of Q are repelled and can never come back, as it occurs when an annular area exists. At $b = 1.01$ the fixed point Q is still surrounded by an annular chaotic area, but the hole around it is now very small (see Fig. 16a), and at $b \simeq 1.017$ the SBR bifurcation of Q occurs. This bifurcation can be easily detected considering the critical curves which bound the annular chaotic area A . In fact, the boundary of an invariant annular area A is obtained by a finite number of critical segments $T^i(g), i = 1, \dots, n$,

where $g = A \cap LC_{-1}$ is called the “generating arc” (see [13,30]). In our example shown in Fig. 16 the boundary of A belongs to the set $\cup_{i=1}^8 T^i(g)$, where $g = A \cap L_{-1}$. Fig. 16b illustrates the 8 images of the generating arc and in Fig. 16a we show the critical segments on the annular area A .

Let Q_{-1} be the rank-1 preimage of Q different from itself and belonging to the region between the two lines L_{-1} and L'_{-1} .³ As long as Q_{-1} is located outside the invariant area A no homoclinic orbit of Q can exist. The SBR bifurcation occurs when the preimage Q_{-1} belongs to the boundary of A . At this bifurcation value infinitely many homoclinic orbits appear, whereby all of them are critical. In fact, as Q_{-1} belongs to the boundary of A , made up of critical segments, we have that a point $q_{-1} \in g(\subset L_{-1})$ must exist, which is mapped in Q_{-1} in a finite number of iterations, and preimages of q_{-1} spiraling towards Q can be easily found. In our example at the SBR bifurcation of Q we have that Q_{-1} belongs to an arc of LC_3 , so that $T^4(q_{-1}) = Q_{-1}$ and thus $T^5(q_{-1}) = Q$. Fig. 17 shows the invariant area at the SBR bifurcation value, which is now obtained via $\cup_{i=1}^7 T^i(g)$ (where $g = A \cap L_{-1}$). Thus all the homoclinic orbits must be of the following kind:

$$\begin{aligned}
 \mathcal{O}(Q) : \{Q \leftarrow \dots q_{-1} \in L_{-1}, q_0 \in L, q_1 \in L_1, q_2 \in L_2, q_3 \\
 = Q_{-1} \in L_3, q_4 = Q \in L_4\}
 \end{aligned}
 \tag{19}$$

In particular, denoting by T_0^{-1} the inverse of T for which Q is fixed, i.e. $T_0^{-1}(Q) = Q$, we obtain that

$$\mathcal{O}(Q) : \{Q \leftarrow T_0^{-n}(q_{-1}), \dots, q_{-1} \in g, q_0, q_1, q_2, q_3 = Q_{-1}, q_4 = Q\}
 \tag{20}$$

is a critical homoclinic orbit of Q (the critical point q_0 belongs to LC and q_{-1} belongs to LC_{-1}).

In our example, it is most likely true that all the critical homoclinic orbits at the SBR bifurcation value are associated with chaotic sets. This comes from the following arguments. We know that a neighborhood W of q_{-1} is folded along L inside the invariant area A and the image $T^5(W)$ reaches the fixed point Q . Thus a portion of area bounded

³ We notice that a third preimage also exists, it is a so called extra-preimage (see [30]) which plays no dynamic role.

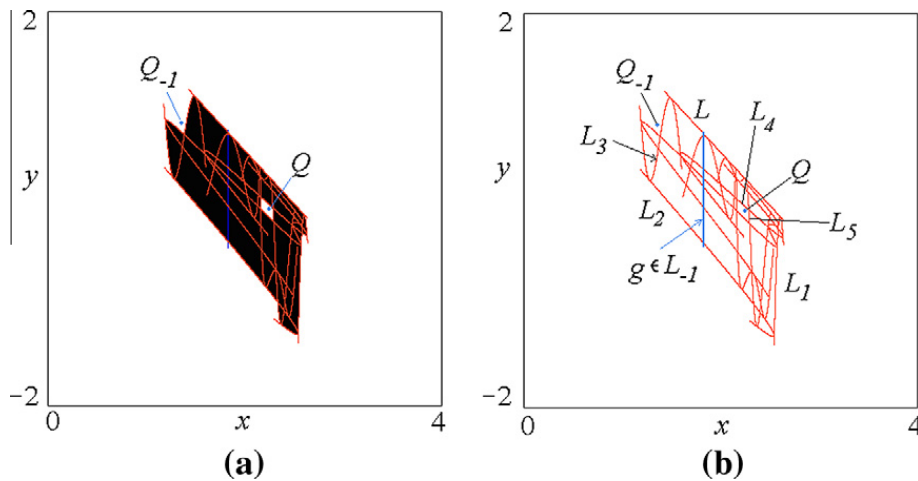


Fig. 16. Invariant annular area as attracting set of the map T in (15) at $a = 0.25$, $c = -0.5$, $d = -1.08$ and $b = 1.01$. In (b) only the critical segments are drawn, giving the boundary of the annular area. The rank-1 preimage Q_{-1} of Q is external to the invariant area and Q is not homoclinic.

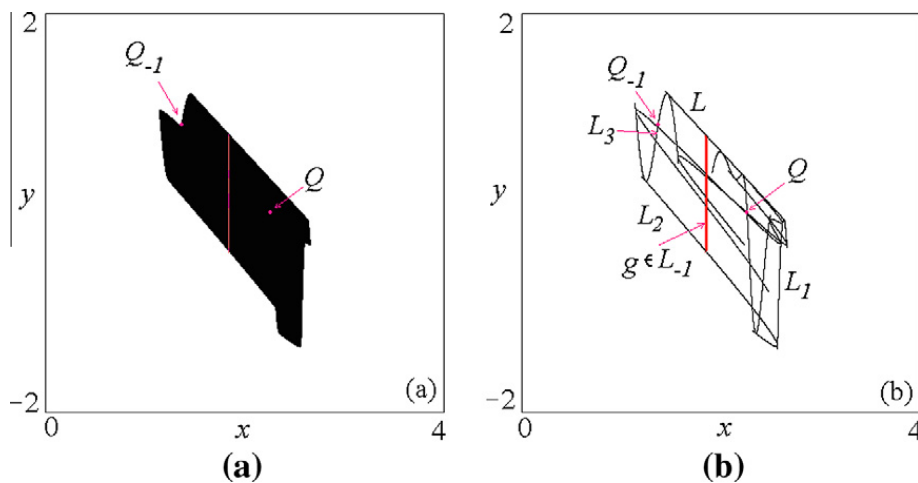


Fig. 17. Attracting set in (a) and invariant area in (b) of the map T in (15) at $b = 1.017$. The rank-1 preimage Q_{-1} of Q is on the boundary of the invariant area, and critical homoclinic orbits of Q exist.

by an arc of critical curve L_4 has the fixed point Q on the boundary. Then all the images of $T^j(W)$ are also areas with a boundary of critical curve L_j for any $j > 4$, and through the fixed point Q . It follows that if we take this area $U = T^2(W)$, most likely we can have $T_0^{-n}(U) \subset U$ for several integers n (such a set plays the role of the set U_0), while taking the preimages of U following the homoclinic trajectory an area U_1 belonging to U may also be found. The most difficult thing to prove is that we can find an integer k such that $T^k(U_0) \supset U_0 \cup U_1$ and $T^k(U_1) \supset U_0 \cup U_1$.

In any case, independently on what occurs at the critical homoclinic orbits, the important fact is that for $b > 1.017$ the preimage Q_{-1} is internal to the invariant area A and infinitely many noncritical and nondegenerate homoclinic orbits of Q exist (for which the proof of the existence of the chaotic set is standard).

We notice that similarly to what occurs in 1Dim maps, after the SBR bifurcation other homoclinic explosions of Q occur. This happens whenever preimages of Q from outside the invariant area A enter inside. At the bifurcation, when the preimage belongs to the boundary of A , critical

homoclinic orbits are created, followed by an explosion of noncritical homoclinic orbits.

5. Conclusions

In this work we have characterized the occurrence of the first SBR bifurcation of an expanding fixed point p (or of a cycle) of a map $f : X \rightarrow X, X \subseteq \mathbb{R}^n$, as well as all the possible homoclinic explosions of p . The most important notion leading to the classification of homoclinic orbits as structurally stable or not is the notion of *noncritical homoclinic orbit* given in this work (in Definition 4). In fact, when the homoclinic orbit is noncritical then we have proved in Theorem 2 that, whichever is the map (smooth or piecewise smooth, continuous or discontinuous), in a neighborhood of the noncritical homoclinic orbit an invariant Cantor like set \mathcal{A} exists, and this property (of being noncritical) is persistent under small variation of the parameters, so that a noncritical homoclinic orbit is structurally stable. Using several 1Dim examples we have shown that in general nothing can be stated about a critical homoclinic

orbit a priori. That is, in a neighborhood of it we can find an invariant Cantor like set or not. By using similar examples with triangular maps it is clear that the same conclusion holds in 2Dim or higher dimensional maps. While critical homoclinic orbits of expanding 2Dim foci are perhaps associated with chaos. However, in our opinion, the dynamic behavior at these bifurcation values with critical homoclinic orbits is less important with respect the occurrence of infinitely many noncritical homoclinic orbits after the SBR and the others homoclinic explosions, which are robust under parameter variation.

Appendix A

Let us first recall some properties whose proof is nowadays well known (see [19,10,1]).

Property 2. Let σ be the shift map acting on the space Σ_2 of one sided infinite sequences of two symbols $\{0,1\}$, then σ is chaotic.⁴ Any map which is topologically conjugated with the shift map is also chaotic.

Property 3. Let (X,d) be a metric space and $F: X \rightarrow X$ a map. If there exist two compact sets $U_0 \subset X$ and $U_1 \subset X$ with $U_0 \cap U_1 = \emptyset$ such that $F(U_0) \supset U_0 \cup U_1$ and $F(U_1) \supset U_0 \cup U_1$ then the set $U_0 \cup U_1$ includes a closed invariant Cantor like set \mathcal{A} , that is a set of closed compact elements χ which are in one-to-one correspondence with the elements S_χ of Σ_2 and $S_{F(\chi)} = \sigma(S_\chi)$.

Property 4. Let (X,d) be a metric space and consider a map $f: X \rightarrow X$. Let $F = f^n$ for some positive integer n . If F is chaotic on some invariant set $\mathcal{A} \subset X$ then also f is chaotic on $\cup_{k=1}^n f^k(\mathcal{A})$.

For convenience let us recall the proof of Property 3 when a set $U \supset U_0 \cup U_1$ exists such that⁵ $F(U_0) = F(U_1) = U$. Then two suitable inverses exist such that $F_0^{-1}(U) = U_0, F_1^{-1}(U) = U_1$ and defining $F^{-1} = F_0^{-1} \cup F_1^{-1}$ we have

$$\begin{aligned} F^{-1}(U) &= U_0 \cup U_1 \\ F^{-2}(U) &= F^{-1}(U_0) \cup F^{-1}(U_1) = U_{00} \cup U_{01} \cup U_{10} \cup U_{11} \subset F^{-1}(U) \\ \dots \end{aligned} \tag{21}$$

It is clear that $F^{-k}(U)$ includes 2^k disjoint sets and

$$\mathcal{A} = \lim_{k \rightarrow \infty} F^{-k}(U) = \bigcap_{k \geq 0} F^{-k}(U) \tag{22}$$

(where $F^{-0} = F^0 = I$ is the identity function). Any element $\chi \in \mathcal{A}$ is either a point or a compact set. Moreover, to any element $\chi \in \mathcal{A}$ we can associate by construction a symbolic sequence, called *itinerary* or address of χ , $S_\chi = (s_0 s_1 s_2 s_3 \dots)$ with $s_i \in \{0,1\}$, i.e. S_χ belongs to the set of all one-sided infinite sequences of two symbols Σ_2 . S_χ is constructed via the symbols we put as indices to the compact sets in

the construction process, and there exists a one-to-one correspondence between the points $\chi \in \mathcal{A}$ and the elements $S_\chi \in \Sigma_2$. From the construction process we have that if χ belongs to the set $U_{s_0 s_1 \dots s_n}$ then $F(\chi)$ belongs to $U_{s_1 \dots s_n}$. Thus the action of the function F on the elements of \mathcal{A} corresponds to the application of the shift map σ to the itinerary S_χ in the code space Σ_2 , as $S_{F(\chi)} = \sigma(S_\chi)$. In fact

$$\begin{aligned} \text{if } \chi \in \mathcal{A} \text{ has } S_\chi &= (s_0 s_1 s_2 s_3 \dots) \\ \text{then } F(\chi) \in \mathcal{A} \text{ has } S_{F(\chi)} &= (s_1 s_2 s_3 \dots) = \sigma(s_0 s_1 s_2 s_3 \dots) = \sigma(S_\chi) \end{aligned} \tag{23}$$

that is, $S_\chi \circ F = \sigma \circ S_\chi$. Given an element $\chi \in \mathcal{A}$ we construct its itinerary S_χ in the natural way: we put $s_0 = 0$ if $\chi \in U_0$ or $s_0 = 1$ if $\chi \in U_1$, then we consider $F(\chi)$ and we put $s_1 = 0$ if $F(\chi) \in U_0$ or $s_1 = 1$ if $F(\chi) \in U_1$, and so on. It follows that \mathcal{A} is a Cantor like set and F can be considered chaotic in the Cantor like set \mathcal{A} .

Notice that we have not used any assumption on the functions F_0^{-1} and F_1^{-1} , and also without any assumption, by using the fixed point theorem, we can say that the sets associated with a periodic symbolic sequence must include a periodic point, i.e. a cycle with that symbolic sequence must exist.

Assuming that F_0^{-1} and F_1^{-1} are contraction mappings it follows that the elements of \mathcal{A} are single points, and thus F is conjugated with the shift map σ . However, as already commented in [1], a Cantor set of points \mathcal{A} in the above process can be obtained also with less strong assumptions in these functions (an example of class of functions satisfying less strict conditions is given in [15]).

Proof of Theorem 2. To prove the statement it is enough to show that we can find a compact neighborhood U of p , and two disjoint compact sets U_0 and U_1 , $U_0 \cup U_1 \subset U$ such that $F(U_0) = F(U_1) = U$. We do this constructing the needed sets starting from a compact neighborhood U of p , and following the homoclinic orbit in a backward way, as the function is, by assumption, locally invertible in all the points of the homoclinic orbit. By definition of expanding point p we know that a compact neighborhood U of p exists and a local inverse f_0^{-1} is such that $\cap_{n \geq 0} f_0^{-n}(U) = p$. Let x_0 be a homoclinic point of p belonging to U such that $f^m(x_0) = p$, and $\mathcal{O}_{x_0}(p)$ the considered homoclinic orbit:

$$\mathcal{O}_{x_0}(p) : \{p \leftarrow f_0^{-n}(x_0), \dots, f_0^{-1}(x_0), x_0, x_1, \dots, x_m = p\} \tag{24}$$

Let us define f_1^{-1} the inverse of f which satisfies $f_1^{-1}(p) = x_{m-1}$, and it is always possible to choose U such that $f_1^{-1}(U)$ and $f_0^{-1}(U)$ are disjoint. Let us consider the local inverses which give the homoclinic points by backward iterations, say $f_2^{-1}(x_{m-1}) = x_{m-2}, f_3^{-1}(x_{m-2}) = x_{m-3}, \dots, f_m^{-1}(x_1) = x_0$. Then consider the set $f_m^{-1} \circ \dots \circ f_2^{-1} \circ f_1^{-1}(U)$, if it belongs to U then we are done, otherwise let k be a suitable integer (which necessarily exists) such that $U_1 = f_0^{-k} \circ f_m^{-1} \circ \dots \circ f_2^{-1} \circ f_1^{-1}(U) \subset U$. Let $n = m + k$ and consider $U_0 = f_0^{-n}(U)$. By construction, U_0 and U_1 are disjoint, and it is always possible to choose k such that $U_0 \cup U_1 \subset U$. This ends the proof as defining $F = f^n$ we have $F(U_0) = F(U_1) = U$ and the desired inverses are given explicitly by $F_0^{-1} = f_0^{-n}, F_1^{-1} = f_0^{-k} \circ f_m^{-1} \circ \dots \circ f_2^{-1} \circ f_1^{-1}$. \square

⁴ That is, σ has a positive topological entropy, is topologically transitive, and is chaotic in the sense of Li-Yorke [24] as well as in the sense of Devaney [10].

⁵ We remark that the property is true also when $F(U_0) \cap F(U_1) \supset U_0 \cup U_1$ and $F(U_0)$ is different from $F(U_1)$, but the proof is slightly different.

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