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Role of the virtual fixed point in the Center Bifurcations in a family of Piecewise Linear Maps

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Abstract

In this work we consider a continuous two-dimensional piecewise linear family of maps in which the fixed point and other cycles undergo a center bifurcation, the analogue of a Neimark-Sacker bifurcation, which has a particular structure when occurring in piecewise linear maps. Our goal is to determine and characterize the occurrence of a center bifurcation of supercritical or subcritical type. We show that for the fixed point a determinant role is played by the virtual fixed point of the system. Also for other k-cycles the virtual fixed point plays a determinant role, as well as the saddle cycles related to the cycle which undergoes a center bifurcation. A particular bifurcation point separating the two transitions, supercritical and subcritical, is related to a conservative system (the analogue of the Chenciner point in smooth maps).

Keywords: subcritical center bifurcation, supercritical center bifurcation, two-dimensional piecewise linear map, border collision bifurcation

1 Introduction

Many dynamic behaviors and bifurcations have been considered in two-dimensional maps, smooth and piecewise smooth (PWS for short). The simplest cases that have been studied are clearly continuous two-dimensional piecewise linear maps (PWL for short) with one border line at which the system changes definition. The most famous studies within this type of maps are the well known Lozi map [Lozi, 1978] and the Gingerbread-man map considered in [Devaney 1984, Aharonov et al. 1987] in the conservative (area preserving) case, which exhibit infinitely many elliptic periodic orbits and chaos related to homoclinic bifurcations of saddle cycles [Gonchenko & Shilnikov, 2000].

Several families of PWL maps have been considered by many authors, see [Grove & Ladas 2005, Tikjha et al. 2010, 2017] mainly with the goal to classify possible families having a global attracting fixed point, suitable for applications in biology [Cannings et al., 2005, Cull 2006]. However, two-dimensional PWL systems rarely succeed in such a simple dynamical behavior. On the contrary, there are several works published in the last decade referring to PWS systems and showing multistability [Simpson 2010, Zhusubaliyev et al. 2006a,b,c, 2008], also infinitely many coexisting attractors [Simpson 2014a,b], which is a result similar to the one already known for diffeomorphisms: smooth maps may exhibit infinitely many attractors on dense sets of parameter values, known as a Newhouse regions [Newhouse, 1974].

Among the reasons leading to several works dealing with PWS and PWL maps we mention their relevance in the applied context. In fact, PWS maps typically arise as discrete time models (obtained for example via Poincaré return maps) of dynamical systems in continuous time when the system is subject to impacts or discrete switching events that modify the equations describing the system. Examples of such systems include power electronic converters and switching circuits (see e.g. [Banerjee & Verghese, 2001, Zhusubaliyev & Mosekilde, 2003]), mechanical systems with impacts and friction (see e.g. [Brogliato, 1999, di Bernardo et al., 2008, Ma et al. 2006, Ing. et al. 2010]. Clearly, situations where several attractors coexist simultaneously may lead to a fundamental source of uncertainty with respect to the applied meaning of the map (given an initial condition to which attractor will converge the trajectory?). This raised the interest of many researchers, and with respect to smooth systems new kind of bifurcations became relevant, such as the so-called *border collision bifurcations* (BCB for short), due to the collision of an invariant set (typically a periodic point) with the border of definition of the map. The term BCB is due to [Nusse & York, 1992, 1995] and since then it is used to describe this particular kind of bifurcations. Moreover, when some parameter is varied through a bifurcation leading an attracting set to collide with the border (and, as recalled in the references cited above, phenomena of this type are really observed in physical and engineering systems), what will then occur to the system trajectories? The study of this kind of bifurcations in PWL systems lead the researchers to consider a particular simple piecewise linear map in which the parameters are only trace and determinant of the linear functions in the two different partitions of the plane, and many results and properties have been described for this map, called two-dimensional border collision normal form map, see e.g. [Banerjee et al. 1998, Banerjee & Grebogi 1999, Glendinning & Wong, 2011, Glendinning 2016] for the transition to chaotic dynamics, and in particular [Simpson & Meiss, 2008, Sushko & Gardini 2008] related to the bifurcations involving cycles with complex eigenvalues (a focus and a center). Recall that the well known Neimark–Sacker bifurcation occurring in smooth maps is related to a continuous crossing of a pair of complex conjugate eigenvalues of a periodic orbit through the unit circle, in the supercritical case is related to the existence of periodicity regions of cycles (also known as Arnold tongues) whose periods follow the Farey summation rule in a period adding bifurcation structure. Differently, the crossing of a pair of complex eigenvalues through the unit circle has a peculiar structure in the case of PWL maps, since the cycle is a center at the bifurcation value and the center bifurcation studied in [Sushko & Gardini 2008] describes the structure of the dynamics at the bifurcation value. As for a border collision, the question is what shall occur after, when the cycle becomes a repelling focus?.

Other kinds of bifurcations peculiar of PWL maps are those occurring at infinity, that is, periodic points which disappear reaching the boundary of the plane. This occurs when a cycle tends to the Poincaré Equator of the real phase plane, which we denote as P.E. for short, or appears from infinity. Properties of this kind of bifurcations have been considered in [Avrutin et al. 2010, Avrutin et al. 2016]. In particular, the degenerate transcritical bifurcation is described in [Sushko & Gardini, 2010].

Both the last two aspects mentioned above are of interest in the present work. In fact, the center bifurcations studied up to now (or Neimark-Sacker bifurcations for PWL maps) have been considered mainly related to the appearance of attracting closed invariant curves, which consist in a saddle-node connection (also called modelocked periodic orbit or phase-locked invariant curve) or in the closure of quasiperiodic trajectories, which are related to a rational rotation number or an irrational one, respectively, called supercritical center bifurcation. The exact dynamic behavior occurring at a center bifurcation value does not say which kind of bifurcation occurs, and which kind of dynamics will appear after (when a cycle becomes a repelling focus). So the bifurcation may be supercritical, leading to new attracting sets, or subcritical, or degenerate: the conservative case, area preserving. To our knowledge, the reason why a subcritical/supercritical center bifurcation occurs in this class of maps has not yet been studied. In this work we consider a PWL system in which fixed points and cycles undergo a center bifurcation, and our goal is to show why and when it occurs subcritical or supercritical. Clearly, it is well known that the existence of attracting sets in PWL maps is associated with the kind of eigenvalues and eigenvectors related to the two linear functions involved in the definition of the map. Indeed, we shall see that the character of the center bifurcation of a fixed point or cycle is also related to the properties of the virtual fixed point of the system (recall that a fixed point is called virtual when it does not belong to the proper partition) and the existence of repelling cycles. To be precise, for both a fixed point and for a 4-cycle we shall prove that the center bifurcation curve is separated in two parts (supercritical and subcritical) by a particular point at which the map is conservative, and the center bifurcation is supercritical when the P.E. is repelling, subcritical when the P.E. is attracting. In all these cases the virtual fixed point is either attracting or a repelling focus. A similar result also applies to a 3-cycle, but not in all the cases of a center bifurcation. In fact, those bifurcations which can be characterized do not cover all the possible situations, since it may occur that the P.E. is neither attracting nor repelling. However, as we shall see in one example, also in such cases the character of the center bifurcation may be determined considering the global dynamic behavior of the map.

The two-dimensional PWL continuos map that we consider is given by

$$T:\begin{cases} x' = |x| - ay \\ y' = x - cy + d \end{cases} \tag{1}$$

where a, c, and $d \neq 0$ are real parameters, which is a particular case of a family of maps proposed in [Tikjha et al. 2017].

After this Introduction, the structure of the paper is as follows. In Sec.2 we describe some properties of the map, we classify the system depending on the property of invertibility/noninvertibility, determining the fixed points, commenting the related bifurcations which may occur crossing the stability region of the unique real fixed point. We also describe the existence regions of a pair of 3-cycles and of 4-cycles, commenting some bifurcations related to their existence and stability. Since the stability regions of both cycles are in part overlapped with that of the attracting fixed point, these regions will be used in the description of the bifurcation sequences associated with the center bifurcations of supercritical and subcritical type. In Sec.3 we prove (in Proposition 1) the conditions leading to the type of center bifurcation for the existing fixed point, showing the relevance of the virtual fixed point. The separator between subcritical and supercritical center bifurcation is the conservative case. In Sec.4 we first prove that, as for the fixed point, also for the attracting 4-cycle the kind of center bifurcation is determined (in Proposition 2) by using the virtual fixed point and the unstable cycles. In Sec.5 we consider the two branches of center bifurcation which may occur to the attracting 3-cycle. In one case we can prove (in Proposition 3) that the center bifurcation must be subcritical. Differently from all the other cases, in the second branch *the virtual fixed point is a saddle*, and this leads to different way of reasoning, which involve the noninvertibility of the map and the existence of absorbing areas. Sec.6 concludes.

2 Properties of the two-dimensional PWL map

Consider the two-dimensional PWL map given by $(x', y') = T(x, y)$ defined in (1). It is immediate to see that the parameter $d \neq 0$ has a role of scaling factor. Here we consider the case $d > 0$ and via the change of variable $X = \frac{x}{d}$, $Y = \frac{y}{d}$, after simplification, and renaming the state variables as (x, y), we get the system (1) with $d = 1$. Thus, we consider the system as a function of the two parameters (a, c). This PWL continuous map has a critical line in $x = 0$, denoted LC_{-1} (following [Mira et al., 1996]), and it separates the regions where the map has different definitions. Let us rewrite the system as follows:

$$T \ : \ = \begin{cases} T_L(x,y) \text{ if } \ x \leq 0 \\ T_R(x,y) \text{ if } \ x \geq 0 \end{cases} \quad \text{where} \tag{2}$$

$$T_L(x,y) \ : \ = \begin{cases} x' = -x - ay \\ y' = x - cy + 1 \end{cases}, \ T_R(x,y) := \begin{cases} x' = x - ay \\ y' = x - cy + 1 \end{cases} \tag{3}$$

since we shall use the properties of the two different linear maps defined in the two partitions.

A property of map T to emphasize refers to its invertibility, since the dynamic behaviors of the map depend on it, and this comes soon from the inverse(s) of a point of the plane, or, equivalently, considering the images of the half-planes under the action of the two linear maps. Due to continuity, both functions T_L and T_R map the critical line $x = 0$ into the same line given, for $a \neq 0$, by

$$LC: \quad y = \frac{c}{a}x + 1 \tag{4}$$

Considering the preimages of a point belonging to the right/left side of the LC curve and looking for their position with respect to the boundary $x = 0$ (or considering the images of points belonging to the different partitions and looking for the position with respect to LC), we can state the following

Property-1.

For $c > 0$ map T is invertible for $a > c$, otherwise it is noninvertible of $Z_0 - Z_2$ type;

for $c = a$ map T is degenerate, the half-plane $x > 0$ is mapped into the critical line LC ($y = x + 1$), thus it is noninvertible of $Z_0 - Z_1 - Z_\infty$ type;

for $c < 0$ map T is invertible for $a > |c|$, otherwise it is noninvertible of $Z_0 - Z_2$ type;

for $c = -a$ map T is degenerate, the half-plane $x < 0$ is mapped into the critical line LC ($y = -x + 1$), thus it is noninvertible of $Z_0 - Z_1 - Z_\infty$ type.

In the degenerate cases each point of LC has infinitely many preimages: all the points of an half-line. Recall the main differences between invertible and noninvertible maps with respect the dynamics: In invertible maps homoclinic orbits can be related only to saddle cycles (intersection between the stable and unstable sets), and the basins of attracting sets are always simply connected. Differently, in noninvertible maps the homoclinic orbits can occur also for repelling nodes or foci (related to snap-back repellers) and the basins of attraction can also be multiply connected or disconnected.

2.1 Fixed points

We can easily write the two fixed points P_L^* and P_R^* of the linear functions T_L and T_R, respectively, which may be real fixed points or virtual ones (i.e., not belonging to the proper partition). The regions $x > 0$ and $x < 0$ are also called left/right partitions and, as usual in PWS systems, the symbols R and L are used to denote the itinerary of a point and, in particular, the symbolic sequence of cycles. Notice that even if a fixed point is virtual, it influences the dynamics in the related partition of the phase plane (x, y), thus it is relevant for the dynamic behavior of the map. In our system we have

$$P_R^* = (x_R^*, y_R^*) = (-1, 0) \quad , \quad P_L^* = (x_L^*, y_L^*) = (-\frac{a}{2+a+2c}, \frac{2}{2+a+2c}) \tag{5}$$

Since P_R^* belongs to the left partition, it is always a virtual fixed point for our system. Differently, P_L^* is a real fixed point of map T for $a > 0$ and $c > -1 - \frac{a}{2}$ or $a < 0$, and $c < -1 - \frac{a}{2}$, while for $a = 0$ it is $P_L^* \in LC_{-1}$ and thus this corresponds to a border collision.

The stability analysis depends on the eigenvalues of the Jacobian matrix J of the two linear maps. Considering T_R we have

$$tr(J_R) = (1 - c) \ , \ D_R = det(J_R) = (a - c) \tag{6}$$

so that the characteristic polynomial $\mathcal{P}_R(\lambda)$ leads to $\mathcal{P}_R(1) = a$, $\mathcal{P}_R(-1) = 2+a-2c$ and the region in the (a, c) parameter plane in which it is a virtual attracting fixed point, given by $\mathcal{P}_R(1) > 0$, $\mathcal{P}_R(-1) > 0$ and $D_R < 1$, is the following:

$$S_R^* := \left\{ a > 0, \ c < \frac{a}{2} + 1 \ , \ c > a - 1 \right\} \tag{7}$$

Clearly, the borders of the regions correspond to bifurcations of the fixed point. Explicitly the eigenvalues are given by

$$\lambda_{1,2}(P_R^*) = \frac{1}{2} \left((1 - c) \pm \sqrt{(1 + c)^2 - 4a)} \right) \tag{8}$$

and are complex conjugate for $|1 + c| < 2\sqrt{a}$.

For the linear map T_L we have

$$tr(J_L) = -(1 + c) \ , \ D_L = det(J_L) = (a + c) \tag{9}$$

so that the characteristic polynomial $\mathcal{P}_L(\lambda)$ leads to $\mathcal{P}_L(-1) = a$, $\mathcal{P}_L(1) = 2+a+2c$ and the region in the (a, c) parameter plane in which P_L^* is an attracting fixed point, given by $\mathcal{P}_L(1) > 0$, $\mathcal{P}_L(-1) > 0$ and $D_L < 1$, corresponds to

$$S_L^* := \left\{ a > 0, \ c > -\frac{a}{2} - 1 \ , \ c < -a + 1 \right\} \tag{10}$$

in explicit form the eigenvalues are as follows:

$$\lambda_{1,2}(P_L^*) = \frac{1}{2} \left(-(1 + c) \pm \sqrt{(1 - c)^2 - 4a} \right) \tag{11}$$

The existence region of the real fixed point P_L^* is shown colored in the (a, c) parameter plane in Fig.1(a), and the bright yellow triangle is the region S_L^* in which P_L^* is attracting. In the same figure we have also reported the boundaries of the region S_R^* to evidence the region in which the virtual fixed point P_R^* is attracting.

It is worth noting that for a parameter point (a, c) belonging to the white region of Fig.1(a), we have both fixed points which are virtual and repelling, and map T has divergent orbits. Some boundaries of the stability region of P_L^* can also be commented soon. Crossing the lower boundary $\mathcal{P}_L(1) = 2 + a + 2c = 0$ from inside, one eigenvalue approaches 1 and also the denominator of the expression of the fixed point in both components tends to zero, that is, the fixed point P_L^* tends to infinity, to the P.E.. Thus, it corresponds to a degenerate transcritical bifurcation [Sushko & Gardini, 2010], after which the fixed point disappears (i.e. it becomes virtual). Considering the first coordinate of P_L^* we can see that it crosses the border $x = 0$ when the parameter a changes sign, from negative to positive or *vice versa*. Moreover, from the expression of the eigenvalues given in (11) we also have that for $a = 0$ the eigenvalues become $\lambda_1 = -1$ and $\lambda_2 = -c$. Thus, crossing the boundary $a = 0$ of the stability region from inside we have that the fixed point becomes virtual exactly when one eigenvalue crosses -1. The last boundary of the stability region, for $0 < a < 4$, corresponds to complex eigenvalues (since $D_L = a + c = 1$ and $\lambda_{1,2}(P_L^*) = \frac{1}{2}\left(-(2-a) \pm i\sqrt{a(4-a)}\right)$) leading to a center bifurcation, that will be investigated in Sec.3.

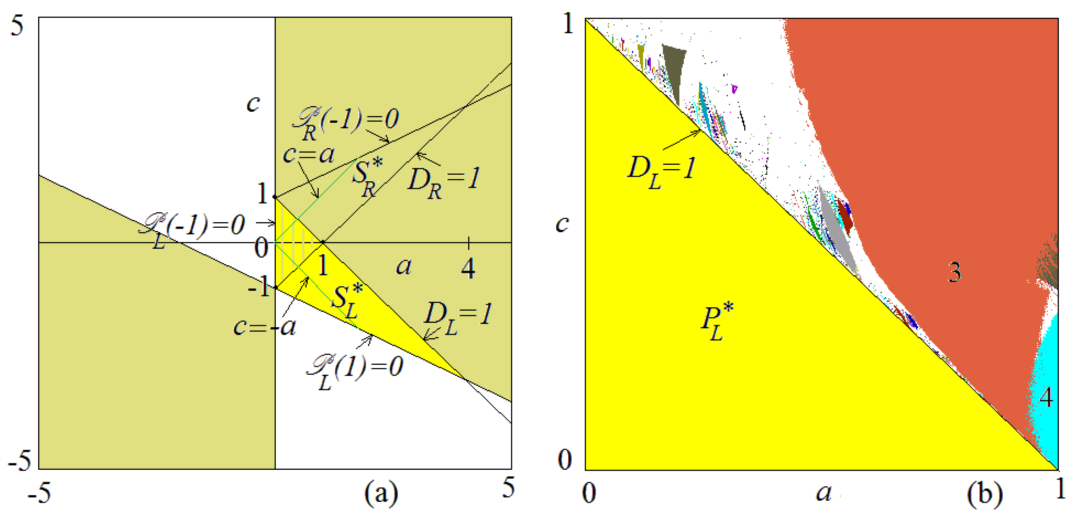

Fig.1 In (a) the stability regions of the fixed points are evidenced. In the yellow triangular region the real fixed point P_L^* is attracting, in the portion belonging to $c > 0$ also the virtual fixed point P_R^* is attracting. In (b) a two-dimensional bifurcation diagram in the (a, c) parameter plane, the initial condition is close to the fixed point P_L^*.

Besides the properties related to the center bifurcation that we shall consider in Sec.3, in the small triangle with $0 < a < 1$ and $0 < c < 1$ (hatched in the yellow region of Fig.1(a)) both fixed points satisfy the stability conditions, so that map T

has bounded trajectories, but not all converging to the attracting fixed point P_L^*. Indeed, for parameters in that region, the trajectory of a point in the right partition of the phase plane (x, y) is necessarily mapped into the left partition in a finite number of iterations, and the orbit then follows the curves related to the attracting fixed point P_L^* but may cross again the right partition. This creates the possibility to have attracting cycles with periodic points in both partitions, coexisting with the attracting fixed point P_L^*. Recall that in a PWL continuous map, cycles of period $n > 1$ appearing by fold-BCB must have a periodic point of the cycle on the critical line $x = 0$, from which a pair of cycles appear/disappear with symbolic sequence which differ by 1 symbol only. This means that 2-cycles cannot appear by fold-BCB, and the minimum period for a cycle of map T related to a fold-BCB is 3. The 2D bifurcation diagram in Fig.1(b) shows attracting cycles of period 3 and 4 existing after the center bifurcation of P_L^* when the parameter a approaches the value 1, but from the figure we can also argue that these regions are not issuing from the bifurcation curve related to the (supercritical) center bifurcation. Indeed, investigating the existence of these cycles we show that they appear by fold-BCB when the fixed point P_L^* is still attracting, and at the center bifurcation value of P_L^* we have coexistence between one or both of these cycles with the invariant area related to the center bifurcation of P_L^*. The investigation of some of these cycles is the goal of the next subsections.

2.2 Cycles of period 3

We determine the existence of a 3-cycle of map T looking for the solutions of the equation $T^3(x, y) = (x, y)$ noting that the symbolic sequence of the two cycles are RLR and RLL and when they are merging we have RLC denoting with C a point on the critical line LC_{-1}. Considering

$$T_R \circ T_L \circ T_R(x, y) = \begin{bmatrix} (ac - 2a - 1) & (ac + a - ac^2 + a^2) \\ (-a - c + c^2 + 1) & (a + 2ac - c^3) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2a + ac \\ -a - c + c^2 + 1 \end{bmatrix} \tag{12}$$

and solving for $T_R \circ T_L \circ T_R(x, y) = (x, y)$ we obtain a solution $(x_{3,1}^s, y_{3,1}^s)$ given by

$$(x_{3,1}^s, y_{3,1}^s) = \left(\frac{a(-c^2 - a^2 + ac + 2a + c - 1)}{2c^3 + a^3 - a^2c - c^2a - 3ac + a + 2}, \frac{2(c^2 - c - ac + a + 1)}{2c^3 + a^3 - a^2c - c^2a - 3ac + a + 2} \right) \tag{13}$$

which is a periodic point of a real 3-cycle $\mathcal{C}_3^s$ when all the periodic points belong to the proper partition, that is $x_{3,1}^s > 0$, $x_{3,2}^s < 0$ and $x_{3,3}^s > 0$, and we have denoted it with "s" because it corresponds to the 3-cycle which may be attracting. Considering

$$T_L \circ T_R \circ T_L(x, y) = \begin{bmatrix} 2a + 1 + ac & -c^2a - ac + a^2 + a \\ c - a + c^2 - 1 & 2ac - a - c^3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} ac \\ c^2 - a - c + 1 \end{bmatrix} \tag{14}$$

and solving for $T_L \circ T_R \circ T_L(x,y) = (x,y)$ we obtain a solution $(x^u_{3,1}, y^u_{3,1})$ given by

$$(x^u_{3,1}, y^u_{3,1}) = \left(\frac{-(a^2 - ac + c - c^2 - 1)}{a^2 + ac - 3c - c^2 - 1}, \frac{2(a-1)}{a^2 + ac - 3c - c^2 - 1} \right) \tag{15}$$

which represents a periodic point of a real 3-cycle $\mathcal{C}^u_3$ when all the periodic points belong to the proper partition, that is $x^u_{3,1} < 0$, $x^u_{3,2} > 0$ and $x^u_{3,3} < 0$, and we have denoted it with "u" because it corresponds to the 3-cycle which is repelling.

We look for the existence of this pair of cycles by considering the possible BCB, that is, a parameter point (a,c) for which it is $x^s_{3,1} = 0$, and for $a \neq 0$ this leads to the necessary condition

$$fold - BCB_{3,1}: \quad c^2 + a^2 - ac - 2a - c + 1 = 0 \tag{16}$$

Crossing this curve a pair of 3-cycles may appear/disappear. It is know that cycles appeared by fold-BCB as a saddle and a node (attracting or repelling) may also disappear by fold-BCB when another pair of periodic points merge on the critical line $x = 0$. For the pair of 3-cycles this corresponds to the merging of the periodic point with symbolic sequence RRL (fixed point of $T_L \circ T_R \circ T_R(x,y) = (x,y)$) for the 3-cycle $\mathcal{C}^s_3$ with the periodic point with symbolic sequence LRL (fixed point of $T_L \circ T_R \circ T_L(x,y) = (x,y)$) for the 3-cycle $\mathcal{C}^u_3$. This leads to the equation of the second fold-BCB, considering the numerator of $x^u_{3,1}$ in (15), we get

$$SN - BCB_{3,2}: \quad a^2 - ac + c - c^2 - 1 = 0. \tag{17}$$

which is called saddle-node BCB because the pair of merging cycles are a saddle and an attracting node. The relevant portions of these curves are shown in Fig.2(a). Moreover, the existence region of the pair of 3-cycles is bounded by a third curve, related to a degenerate transcritical bifurcation occurring when one eigenvalue becomes 1 and the periodic points of the cycles tend to infinity. For the 3-cycle $\mathcal{C}^s_3$ this occurs when the parameter point (a,c) belongs to the curve denoted by $\tau(\mathcal{C}^s_3)$:

$$\tau(\mathcal{C}^s_3) \;:\; 2c^3 + a^3 - a^2c - c^2a - 3ac + a + 2 = 0 \tag{18}$$

while for the other 3-cycle saddle $\mathcal{C}^u_3$ the degenerate transcritical bifurcation occurs for:

$$\tau(\mathcal{C}^u_3) \;:\; a^2 + ac - 3c - c^2 - 1 = 0 \tag{19}$$

bounding the existence regions, which are thus different for the two cycles, as shown in Fig.2(a).

As mentioned above, when a parameter point belongs to the degenerate transcritical bifurcation curves we also have the related Jacobian matrix with one eigenvalue +1. Other bifurcations of the cycles are related to the eigenvalues of the map T^3. Let us consider the Jacobian matrix J_{RLR} from eq.(12) for the 3-cycle $\mathcal{C}^s_3$, we have

$$tr(J_{RLR}) = 3ac - c^3 - a - 1 \;,\; \det(J_{RLR}) = (a-c)^2(a+c) \tag{20}$$

The stability conditons are $\mathcal{P}_{J_{RLR}}(1) = 2c^3 + a^3 - a^2c - c^2a - 3ac + a + 2 > 0$, $\mathcal{P}_{J_{RLR}}(-1) = a(a^2 - c^2 - ac + 3c - 1) > 0$ and $\det(J_{RLR}) < 1$ leading to the red region shown in Fig.2(a). Notice that only two small arcs of the fold-BCB curve in (16) are related to the appearance of an attracting node and a saddle, the other two arcs are related to the appearance of two unstable 3-cycles.

The bifurcations curves obtained from the stability conditions $\mathcal{P}_{J_{RLR}}(-1) = 0$ and $\det(J_{RLR}) = 1$ bound the stability region in two different parts. A numerical investigation has shown that the portion of $\mathcal{P}_{J_{RLR}}(-1) = 0$ in the lower region (associated with an attracting fixed point) is related to a degenerate flip-BCB of subcritical type (an attracting 6-cycles merges with LC_{-1} at the bifurcation of a saddle 3-cycle with an eigenvalue equal to -1, leading to an attracting 3-cycle), while the portion in the upper part is related to a supercritical one (an attracting 3-cycle undergoes a bifurcation with an eigenvalue equal to -1, becoming a saddle and leading to an attracting 6-cycle). Similarly for the center bifurcation curve ($\det(J_{RLR}) = 1$), there are two portions belonging to the existence region. The lower one is in a region in which the map is invertible and, as we shall see in Sec.5, related to a center bifurcation of subcritical type, while the upper one in a region of noninvertibility and the center bifurcation is supercritical.

Differently, for the 3-cycle $\mathcal{C}_3^u$, from the Jacobian matrix J_{LRL} in eq.(14) we have $\mathcal{P}_{J_{LRL}}(1) = \ a^2 + ac - 3c - c^2 - 1$ and $\mathcal{P}_{J_{LRL}}(1) > 0$ is the region below the curve $\tau(\mathcal{C}_3^u)$, so that for any (a, c) belonging to the existence region of the cycle $\mathcal{C}_3^u$ one real eigenvalue is always greater than 1 (being $\mathcal{P}_{J_{LRL}}(1) < 0$).

It is worth noting that many bifurcation curves related to the bifurcations of the 3-cycles are issuing from the particular point $(a, c) = (1, 0)$.

As it can be seen from Fig.2(a), there is an overlapping between the stability region of the fixed point P_L^* with the existence region of the three cycles, in particular with a region in which the 3-cycle $\mathcal{C}_3^s$ is attracting. In Sec.3 we shall describe some dynamic behaviors related to parameters belonging to this region, crossing the center bifurcation curve of P_L^*. Moreover, within the stability regions of the fixed point and the 3-cycle, there is also a region in which both coexist also with an attracting 4-cycle, as described in the next subsection.

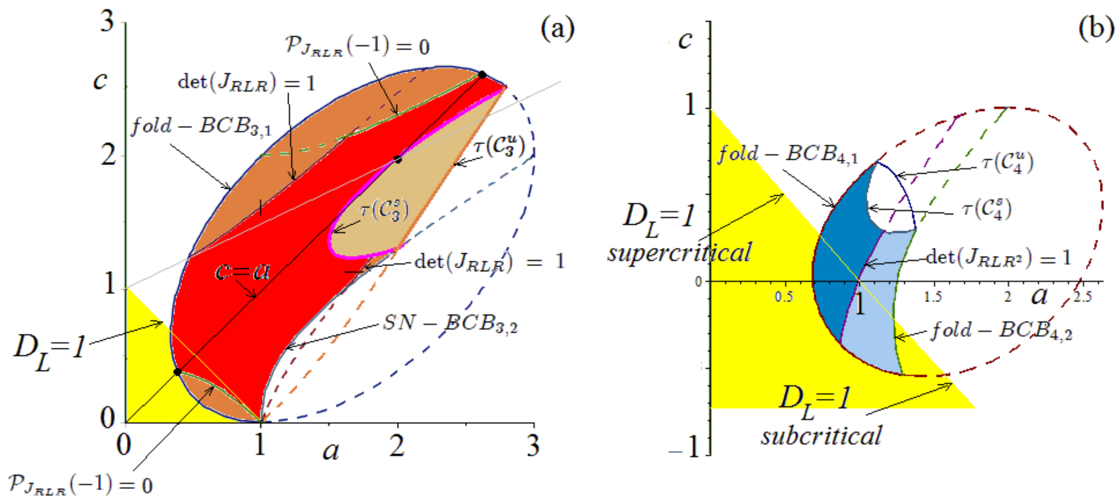

Fig.2 In (a) Existence region of 3-cycles in the (a, c) parameter plane. In the red colored area the 3-cycle C_3^s is attracting. In (b) the existence region of 4-cycles in the (a, c) parameter plane. In the blue colored area the 4-cycle C_4^s is attracting. In the azure area the two 4-cycles are both unstable.

2.3 Cycles of period 4

For the appearence via fold-BCB of a pair of 4-cycles we consider the function $T_R^2 \circ T_L \circ T_R(x, y)$:

$$\begin{bmatrix} 2ac - ac^2 + a^2 - 1 - a & -2a^2c + ac + ac^3 - ac^2 + a \\ 2ac + c^2 - c^3 - 2a + c - 1 & a^2 - 3ac^2 + a + c^4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -ac^2 + 2ac - 3a + a^2 \\ c^2 - c + 1 - 2a + 2ac - c^3 \end{bmatrix}$$

and looking for its fixed point (a periodic point with symbolic sequence $RLRR$), we get

$$x_{4,1}^s = \frac{a(-a^3 + 2c^3 + 2a^2c - 2ac^2 + 2a^2 - 2c^2 - 4ac + 2a + 2c - 2)}{a^4 - 2c^4 - 2a^3c + 2ac^3 + 4ac^2 - 2a^2 - 2ac + 2} \tag{21}$$

$$y_{4,1}^s = \frac{2(1-c)((a-c)^2 + 1)}{a^4 - 2c^4 - 2a^3c + 2ac^3 + 4ac^2 - 2a^2 - 2ac + 2}$$

which results in the periodic point of a 4-cycle $\mathcal{C}_4^s$ which can be attracting, appearing via fold-BCB crossing through an arc of the curve of equation

$$fold - BCB_{4,1} : \quad \frac{1}{2}a^3 - c^3 - a^2c + ac^2 - a^2 + c^2 + 2ac - c - a + 1 = 0. \tag{22}$$

obtained from $x_{4,1}^s = 0$, for $a \neq 0$, and bounding the blue colored existence region, related to a pair of 4-cycles, which we have found as a saddle and an attracting node. The repelling 4-cycle saddle, say $\mathcal{C}_4^u$, appearing at the fold-BCB has symbolic sequence $LLRR$, and a point of this cycle is obtained considering the fixed point of the function $T_R^2 \circ T_L^2(x,y)$, leading to

$$x^u_{4,1} = \frac{-(a^3 - 2c^3 - 2a^2c + 2ac^2 - 2a^2 + 2c^2 + 4ac - 2a - 2c + 2)}{a^3 - 2ac^2 + 4c^2 - 2a}$$
$$y^u_{4,1} = \frac{2((a-c)^2 + 2c - a - 1)}{a^3 - 2ac^2 + 4c^2 - 2a}$$

The stability of the 4-cycle $\mathcal{C}_4^s$ can be determined considering the Jacobian matrix J_{RLR^2} and its trace and determinant, that is:

$$tr(J_{RLR^2}) = 2ac - 4ac^2 + 2a^2 - 1 + c^4 \ , \ \det(J_{RLR^2}) = (a-c)^3(a+c) \qquad (23)$$

The stability region in the parameter plane is the one in which we have satisfied the three conditons $\mathcal{P}_{J_{RLR^2}}(1) = a^4 - 2c^4 - 2a^3c + 2ac^3 + 4ac^2 - 2a^2 - 2ac + 2 > 0$, $\mathcal{P}_{J_{RLR^2}}(-1) = 2ac - 4ac^2 + 2a^2 - 2a^3c + 2ac^3 + a^4 > 0$ and $\det(J_{RLR^2}) = (a-c)^3(a+c) < 1$.

The stability region is colored in Fig.2(b) in dark blue, while the azure color denotes 4-cycles both repelling. Since the attracting 4-cycle node becomes a focus and then becomes repelling via a center bifurcation, the separation curve is a center bifurcation occurring at $\det(J_{RLR^2}) = 1$.

The lower boundary of the azure region in Fig.2(b) (bounding the existence region) is another fold-BCB in which the periodic point with symbolic sequence R^3L (of a repelling node) and the periodic point with symbolic sequence LR^2L (of a saddle) are merging. Considering the function $T_L \circ T_R^3(x,y)$ and looking for its fixed point, we get

$$x^n_{4,1} = \frac{a(-a^3 - 2c^3 + 2a^2c + 2c^2 - 2c + 2)}{a^4 - 2c^4 - 2a^3c + 2ac^3 + 4ac^2 - 2a^2 - 2ac + 2}$$
$$y^n_{4,1} = \frac{2(1-c)(c^2 - 2a + 1)}{a^4 - 2c^4 - 2a^3c + 2ac^3 + 4ac^2 - 2a^2 - 2ac + 2}$$

and the fold-BCB (in which a 4-cycle saddle and a repelling node are merging on $x = 0$) occurs at parameter points belonging to the curve obtained from $x^n_{4,1} = 0$, for $a \neq 0$:

$$fold - BCB_{4,2} : \quad a^3 + 2c^3 - 2a^2c - 2c^2 + 2c - 2 = 0 \qquad (24)$$

The existence region of both 4-cycles $\mathcal{C}_4^s$ and $\mathcal{C}_4^u$ is bounded by degenerate transcritical bifurcations of the two cycles, occurring for the attracting cycle when

$$\tau(\mathcal{C}_4^s) : \quad \mathcal{P}_{J_{RLR^2}}(1) = a^4 - 2c^4 - 2a^3c + 2ac^3 + 4ac^2 - 2a^2 - 2ac + 2 = 0 \qquad (25)$$

and for the unstable one when

$$\tau(\mathcal{C}_4^u) : \quad \mathcal{P}_{J_{L^2R^2}}(1) = a^3 - 2ac^2 + 4c^2 - 2a = 0 \qquad (26)$$

whose interesting arcs are shown in Fig.2(b).

As for the pair of 3-cycles, we observe that also the existence region of the pair of 4-cycles overlaps with the stability triangle of P_L^* (in which also the virtual fixed point P_R^* is attracting).

In the next section we describe which kind of center bifurcation occurs to the fixed point P_L^*.

3 Super- and sub-critical center bifurcation of P_L^*

We have already mentioned that the structure of the dynamics at a center bifurcation depends on the rotation number, as described in [Sushko & Gardini, 2008]. However, the result of the center bifurcation of P_L^*, crossing the boundary of S_L^* from inside to outside the stability region, depends on the global structure of the phase plane, since the dynamics of the map on the right partition depends also from the virtual fixed point P_R^*. As we have seen in Sec.2.1, inside the stability region of P_L^* there is a region in which the virtual fixed point P_R^* is also attracting. This leads to a difference of behavior when crossing the bifurcation curve $D_L = 1$ for $c > 0$ or $c < 0$. In the first case, the crossing occurs when both fixed points are attracting, and we shall see that the result is a *supercritical center bifurcation.* Differently, in the second case, the crossing of the bifurcation curve occurs when the virtual fixed point P_R^* is a repelling focus, and we have a *subcritical center bifurcation* for the fixed point P_L^*. The switching occurs for $(a, c) = (1, 0)$ which is related to a particular condition: the determinants of both Jacobian matrices are equal to one, $D_L = 1$ and $D_R = 1$, which means that the map is conservative. We prove the following

Proposition 1. *The center bifurcation of P_L^* is supercritical for $c > 0$, subcritical for $c < 0$. Map T is conservative for $c = 0$ with rotation number 1/3 at the center P_L^*.*

Proof. It is easy to see that in our system the fixed point P_L^* is always below the straight line LC. At the bifurcation (when $D_L = a+c = 1$ with complex eigenvalues, for $0 < a < 4$) P_L^* is a center, and we know that depending on the rotation number, rational or irrational, the existing invariant area, belonging to the left side, $x \leq 0$, is either a polygon bounded by a finite number of segments of critical curves or it is the envelop curve of critical segments, an invariant ellipse tangent to LC_{-1}. When it is bounded by segments, these are images of a segment on LC_{-1} between $(0, 0)$ and $T(0, 0) = (0, 1)$, which is mapped on LC ($y = \frac{c}{a}x + 1$) on the segment between $(0, 1)$ and $T(0, 1) = (-a, -c+1)$ and the further images are spiraling around P_L^* (in particular for $c = 1-a$, LC_1 intersects LC_{-1} in $(0, \frac{1-a}{2-a})$). The invariant area is filled with periodic orbits of the same period when the rotation numer is rational, while it is filled with invariant ellipses when it is irrational, with quasiperiodic orbits dense on the curves.

When the parameter point (a, c) belongs to the center bifurcation curve $D_L = 1$ the invariant area exists (a polygon or an ellipse) which means, by continuity, that for parameters just before the bifurcation (for $a+c < 1$) a closed area exists, mapped into itself, bounded by a finite number of critical segments, and it belongs to the basin of attraction of the focus P_L^*.

Clearly this is true crossing the bifurcation curve ($a + c = 1$) at any value of the parameter c. Thus, what makes the difference between the different kinds of center bifurcation, is the behavior of the points of the phase plane close to the pre-existing closed area bounded by critical segments.

For $c > 0$ and parameters inside the stability region of P_L^* (which means $0 < a < 1$), the virtual fixed point P_R^* is attracting (with real or complex eigenvalues), and close to the bifurcation the point P_L^* is an attracting focus. Thus, the P.E. of the phase plane is repelling, that is, the map cannot have divergent trajectories, and it is $0 < D_L < 1$, $|D_R| < 1$. In particular, map T cannot have any repelling expanding cycle (node or focus), since this would imply a determinant larger than 1 in modulus, while the Jacobian determinant related to a cycle of period $(n + m)$ having a symbolic sequence with n points in the left partition and m points in the right partition is given by ${D_L}^n {D_R}^m$, and it is necessarily smaller than 1 in modulus. Notice that clearly map T can have saddle cycles, with related Jacobian determinant smaller than 1, but due to the fact that the two linear functions are contractions the two branches of the unstable set of a saddle are converging to some sets at finite distance, while the two branches of the stable set, preimages of the local eigenvector, extend up to infinity, since the inverse on the right side leads to points in the left partition where the inverse is an expanding rotation.

At the bifurcation value P_L^* belongs to an invariant area (filled with periodic or quasiperiodic orbits) in the left partition, the P.E. does not attract any trajectory (as $|det(J_T)| \leq 1$ in any point of the phase plane, with $D_L = 1$ for $x \leq 0$ and $|D_R| < 1$ for $x \geq 0$), the system is not conservative and the boundary of the invariant set related to the center P_L^* cannot be a closed curve repelling from outside. That is, in case of a subcritical bifurcation we ought to have, before the center bifurcation, a repelling closed curve bounding the set of points converging to the fixed point, but a repelling closed curve is, in the generic case, a saddle-repelling node connection, which cannot exist. Thus a neighborhood of the former basin of attraction of P_L^* must exist, which is the set of points converging to the invariant area existing at the bifurcation, and thus the result of the bifurcation is an attracting closed curve, which crosses LC_{-1}.

Reasoning in a similar way, crossing the bifurcation curve $D_L = 1$ for $c < 0$ (which occurs for $a > 1$) the virtual fixed point P_R^* is a repelling focus, so $D_R > 1$ in any point of the right partition of the phase plane (any point of the R side is mapped into the L side in a finite number of iterations). In this case, before the bifurcation, when P_L^* is an attracting focus ($D_L < 1$), the P.E. of the phase plane may attract several orbits and invariant repelling closed curves may exist. At the center bifurcation we have two fixed points *of focus type* (one real and one virtual) and $det(J_T) \geq 1$ in any point of the phase plane (it is $D_L = 1$ for $x \leq 0$ and $D_R > 1$ for $x \geq 0$), so that an attracting cycle cannot exist. After the bifurcation, when $det(J_T) > 1$ in any point of the phase plane, no attracting node can exist, which means that an attracting invariant curve given in the generic case by a saddle-attracting node connection, is not allowed. Thus, at the bifurcation, the external boundary of the invariant set related to the center P_L^* (tangent to $x = 0$ or with a segment on it) cannot be a closed curve attracting from outside, and the bifurcation must be of subcritical type (before the center bifurcation, repelling closed curves connection saddle-repelling nodes, are allowed).

Notice that this means that a neighborhood of the former basin of attraction of P_L^* cannot exist any longer, and that the boundary of the former basins of attraction corresponds, at the bifurcation, to the external closed curve of the existing invariant set, and it is repelling from outside.

For $c = 0$, at the point $(a, c) = (1, 0)$ the map is conservative (area-preserving in the whole phase plane) since $det(J_T) = 1$ in any point (x, y) of the plane, with P_L^* as a real center while P_R^* is a virtual center. The fixed point $P_L^* = (-\frac{1}{3}, \frac{2}{3})$ is a center with rational rotation number $\frac{1}{3}$ (since the eigenvalues are $\lambda_{1,2}(P_L^*) = \frac{1}{2}\left(-1 \pm i\sqrt{3}\right)$). The invariant polygon bounded by the critical segments connecting the points $(0, 0)$, $(0, 1)$ and $(-1, 1)$ is filled with cycles of period 3. In fact, let (x, y) be a point with $x \leq 0$, then $T_L(x, y) = (-x-y, x+1)$, $T_L^2(x, y) = (y-1, x+y-1)$, $T_L^3(x, y) = (x, y)$ and the only points to which we can apply consecutively the map in the left partition are those of the invariant polygon bounded by the critical segments connecting the 3-cycle $\{(0, 0), (0, 1), (-1, 1)\}$. The trajectory of any other point crosses also the right partition. □

A two-dimensional bifurcation diagram in the (a, c) parameter plane when the center bifurcation is supercritical is shown in Fig.1(b), and periodicity regions of attracting cycles of different periods issuing from the bifurcation curves can be seen (different periods are evidenced with different colors), and it is known that the periods follow the standard Farey summation rule [Boyland 1986].

Let us illustrate the bifurcation mechanisms with some examples. In the supercritical case, for $c > 0$, we can distinguish between the noninvertible/invertible map. In the noninvertible case, for $0 < a \leq 0.5$, considering a suitable segment J_0 on LC_{-1} we know that the halfplane $x > 0$ is mapped below or onto LC, and the images of J_0 are spiraling around P_L^* so that an absorbing area bounded by a finite number of critical segments, mapped into itself, exists, which must belong to the basin of the attracting focus P_L^* or to the stable set of the invariant polygon at the center bifurcation. In the example shown in Fig.3(a) it is considered a small segment J_0 on LC_{-1} starting in $y = \frac{1-a}{2-a}$ and ending in $y = y_L^* + 0.1$ and in five iterations a closed area is obtained, which is mapped into itself. In that figure the values of the parameters are very close to the center bifurcation value, and the figure does not change at the bifurcation value. That is, the basin of attraction of P_L^* existing before the center bifurcation persits and becomes the set of points which are mapped into the invariant area around the center.

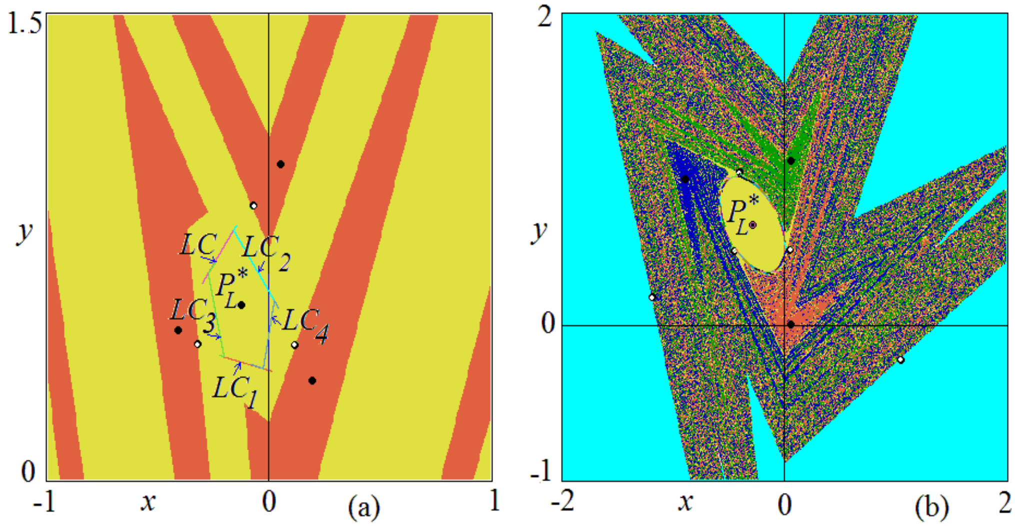

Fig.3 In (a) $a = 0.45$, $c = 0.549$, close to the center bifurcation value. In yellow the basin of attraction of P_L^* bounded by the stable set of the saddle C_3^u, in red the basin of attraction of C_3^s. Segments of critical curves bound an area mapped into itself. In (b) $a = 0.899$, $c = 0.1$, close to the center bifurcation value. In yellow the basin of attraction of P_L^* bounded by the stable set of the saddle C_3^u, three different colors denote the basin of attraction of C_3^s for map T^3. In azure the basin of attraction of C_3^4 bounded by the stable set of C_4^u. Segments of critical curves bound an area around P_L^* mapped into itself.

We know that in this range ($0 < a \leq 0.5$), at the bifurcation value an attracting 3-cycle may coexist, and Fig.3(a) shows such an example, in which the basin of attraction of P_L^* is bounded by the stable set of the 3-cycle saddle. In the other case, when the pair of 3-cycles does not exist, the invariant area is also simpler to construct, since there are no other attracting cycles.

In the invertible case, for $0.5 < a < 1$, we know from Sec.2 that the attracting fixed point always coexists with an attracting 3-cycle. Then the basin of attraction of P_L^* is bounded by the stable set of the 3-cycle saddle which, as remarked above, cannot end at finite distance (i.e. it is issuing from the repelling P.E.). Also in this case we can find, considering the images of a small segment J_0 on LC_{-1} a closed area which is mapped into itself and, as long as P_L^* is attracting, completely included inside its basin of attraction. As an example, in the case shown in Fig.3(b), considering the segment J_0 on LC_{-1} starting in $y = 0.5$ and ending in $y = 0.6$ in a finite number of iterations a closed area around P_L^* mapped into itself is obtained. Also here the parameters are just before the bifurcation value, and the figure is similar to that at the center bifurcation value. We can notice that, as we know from Sec.2.3, at the parameters considered in this figure we also have the coexistence of an attracting 4-cycle. The basin of attraction of the 3-cycle is shown for map T^3 to emphasize the fractal structure of the basin, since one branch of the 3-cycle saddle is homoclinic, while the other unstable branch of the 3-saddle converges to the attracting fixed point (or to the related invariant set). In both the examples shown in Fig.3, after the center bifurcation an attracting closed invariant curve Γ_+ exists, close to the boundary of the invariant area, and the basin of P_L^* before the bifurcation becomes the basin of Γ_+.

A different behavior occurs in the subcritical the case, for $c < 0$ $(1 < a < 4)$, and the map is always in the invertible range.

Close to the bifurcation, with the images of a suitable segment J_0 on LC_{-1}, a closed area which is mapped into itself can be obtained. As long as P_L^* is attracting it is completely included inside its basin of attraction. In the example shown in Fig.4(a), considering the segment J_0 on LC_{-1} shown in color, in a finite number of iterations a closed area around P_L^* is obtained. The parameters are just before the bifurcation value, and the basin of attraction of P_L^* is close to the area mapped into itself, bounded by a repelling closed curve, and outside this area the trajectories are divergent. As the parameters approach the bifurcation value, the basin of P_L^* shrinks and at the center bifurcation (see Fig.4(b)) the repelling closed curve of the basin boundary becomes the boundary of the invariant area around the center P_L^*.

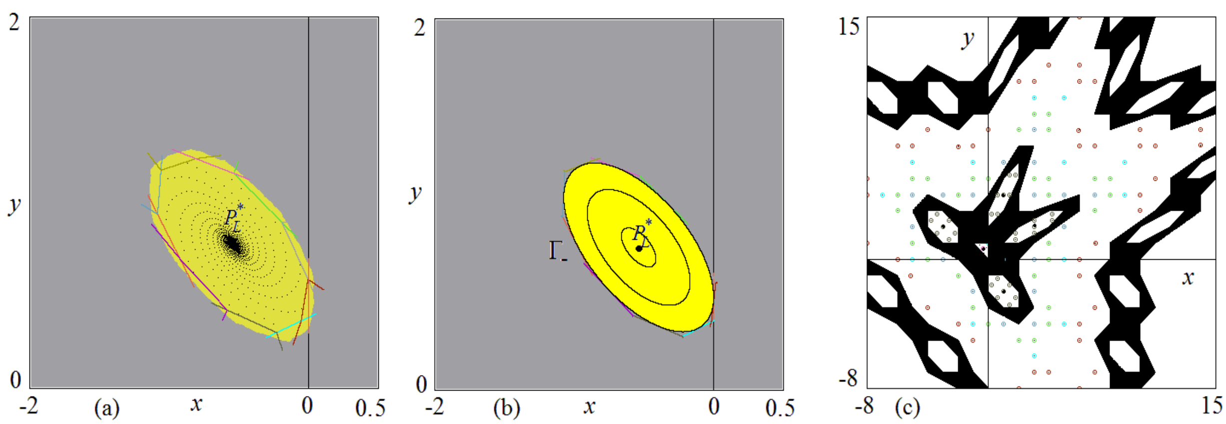

Fig.4 Fig.super In (a) $a = 1.4$, $c = 0.41$, close to the center bifurcation value. In yellow the basin of attraction of P_L^* bounded by a closed repelling curve. In gray divergent trajectories. Segments of critical curves bound an area mapped into itself. In (b) $a = 1.4$, $c = 0.4$, at the center bifurcation value, the external closed curve is the envelope of critical segments. In (c) $a = 1$, $c = 0$, phase space in the conservative (area preserving) case.

On the center bifurcation curve $D_L = 1$, the transition supercritical/subcritical, that is P.E. repelling/attracting, clearly occurs via a conservative case (at which the P.E. is neither attracting nor repelling). In Fig.4(c) we show the case of map T conservative at the particular point $(a, c) = (1, 0)$ of the parameter plane. This point is also a codimension-two point (as we have seen in Sec.2, many bifurcation curves of the parameter plane are issuing from this point), and the fact that the map is conservative leads to a particular structure of orbits in the phase plane (x, y). That is, all the orbits are bounded, the existing cycles are either centers (also called elliptic periodic points) or saddles with reciprocal eigenvalues. The phase plane presents infinitely many islands formed by centers of different periods surrounded by invariant sets related to saddles, bounding other elliptic cycles, with chaotic rings due to the transverse intersection of the stable and unstable sets of the saddles, giving rise to Smale horseshoes and chaotic sets [Gonchenko & Shilnikov, 2000]. The existence of a triangular region around the fixed point P_L^* filled with cycles of period 3 has been proved in Proposition 1, while the existence of 4 hexagonal regions around a 4-cycle center filled with cycles of period 24 will be proved in the next section (Proposition 2). We notice that the dynamics of this conservative map are the same as those of the Gingerbread-man map considered by [Devaney 1984, Aharonov et al. 1987] and also the same of the Lozi map in the particular conservative case considered in [Botella-Soler et al. 2011, Elhadj, 2014].

4 Super- and sub-critical center bifurcations of $\mathcal{C}_4^s$

We have already seen the relevance of the virtual fixed point in determining the kind of center bifurcation of the fixed point P_L^*, and here we show that the virtual fixed point is relevant also in determining the type of center bifurcation of the 4-cycle. Whenever the center bifurcation curve is crossed for $c < 0$, when the fixed point P_L^* undergoes a subcritical center bifurcation and the virtual fixed point P_R^* is a repelling focus, we have a supercritical center bifurcations of the 4-cycle C_4^s. Differently, for $c > 0$, when the fixed point P_L^* undergoes a supercritical center bifurcation and the virtual fixed point P_R^* is an attracting focus, we have a subcritical center bifurcation of the 4-cycle C_4^s. Notice that the values of center bifurcation curve of C_4^s belong to the parameter region in which the map is uniquely invertible and also in this case, as already remarked, the difference of dynamic behaviors can be explained by using the virtual fixed point. We prove the following

Proposition 2. *The center bifurcation of the 4-cycle* C_4^s *is supercritical for* $c < 0$*, subcritical for* $c > 0$*, critical (conservative) for* $c = 0$ *with rational rotation number* $\frac{1}{6}$ *for map* $T^4 = T_R^2 \circ T_L \circ T_R$.

Proof. Let us consider the region in which the center bifurcation curve of equation $\det(J_{RLR^2}) = 1$, that is $(a^2 - c^2)(a - c)^2 = 1$, is crossed starting from the stability region (where there is an attracting 4-cycle focus), which is evidenced in dark blue and indicated by an arrow in Fig.5(a).

For $c > 0$ it occurs for $a > 1$ and $a > c$, in a region in which we have $D_L = (a+c) > 1$ and $D_R = (a-c) \in (0, 1)$. However, the product is $D_L D_R = (a^2-c^2) > 1$ (in fact, since $(a - c)^2 < 1$ in order to have $(a^2 - c^2)(a - c)^2 = 1$ it must necessarily be $(a^2 - c^2) > 1$).

Differently, for $c < 0$ $\det(J_{RLR^2}) = 1$ occurs for $-c < a < 1$ in a region in which we have $D_L = (a + c) \in (0, 1)$ and $D_R = (a - c) > 1$, but the product is $D_L D_R = (a^2 - c^2) < 1$ (since $(a - c)^2 > 1$ to have $(a^2 - c^2)(a - c)^2 = 1$ it must necessarily be $(a^2 - c^2) < 1$).

Recall that the symbolic sequence of the repelling 4-cycle is $LLRR$, thus the determinant of the Jacobian of map T^4 in the regions of the fixed points of the saddle 4-cycle is given by $(D_L D_R)^2$ and thus it is greater than 1 for $c > 0$, smaller than 1 for $c < 0$. This makes the difference between the two portions of the center bifurcation curve.

Considering $c < 0$, in the stability region of the 4-cycle close to the center bifurcation where $D_R = (a - c) > 1$ we have (besides $0 < D_L < 1$ and $0 < D_L D_R < 1$) $\det(J_{RLR^2}) \in (0, 1)$ and $\det(J_{L^2R^2}) \in (0, 1)$ which means that no cycle repelling node can exist, and the P.E. is repelling, no divergent trajectories can exist for map T^4. Saddle cycles can exist and the stable set of the saddle 4-cycle comes from the P.E.. At the center bifurcation value, $\det(J_{RLR^2}) = 1$ and $\det(J_{L^2R^2}) \in (0, 1)$, for map T^4, so that the invariant closed curve or the invariant polygon bounding $\mathcal{C}_4^s$ and tangent to LC_{-1} cannot be a closed curve repelling from outside, since before the bifurcation a repelling closed curve, which for map T^4 is generically a saddle-repelling node connection, cannot exist. The bifurcation is not at a conservative case, being $\det(J_{L^2R^2}) \in (0, 1)$. Thus the center bifurcation is supercritical. The existing closed curve bounding the invariant area for T^4 belongs to an invariant area (of the basin) whose boundary is given the stable set of the saddle 4-cycle $\mathcal{C}_4^u$.

Vice versa, considering $c > 0$, in the stability region of the 4-cycle, for $a > 1$, we have $\det(J_{RLR^2}) \in (0, 1)$ and $\det(J_{L^2R^2}) > 1$ which means that the P.E. may be attracting, and at the bifurcation value it is $\det(J_{RLR^2}) = 1$ and $\det(J_{L^2R^2}) > 1$ which means that map T^4 cannot have any attracting cycle. The invariant closed curve or the invariant polygon bounding $\mathcal{C}_4^s$ cannot be a closed curve attracting from outside (and after the center bifurcation a connection saddle-attracting node cannot exist). Thus, the center bifurcation cannot be of supercritical type, nor a conservative case, being $\det(J_{L^2R^2}) > 1$, but necessarily subcritical.

In both cases $c > 0$ and $c < 0$ we cannot have a conservative map, but for $c = 0$ the center bifurcation curve of $\mathcal{C}_4^s$ is crossed at $a = 1$ and we already know that map T is conservative in that point of the parameter space. In particular, for the 4-cycle it is $tr(J_{RLR^2}) = 1$ and $\det(J_{RLR^2}) = 1$ with eigenvalues $\lambda_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, which means a rational rotation number $\frac{1}{6}$ for the map $T^4 = T_R^2 \circ T_L \circ T_R$. $\square$

Considering the stable 4-cycle $\mathcal{C}_4^s$ for $c = 0$ and increasing the value of a, it can be observed the progressive increase of coexisting cycles of map T: all those existing for $a = 1$ must be created by fold-BCB increasing a in the range $0 < a < 1$. An example of the phase plane when a is close to 1 is shown in Fig.5(b). The four colors represent the 4 basins of the fixed points of the 4-cycle $\mathcal{C}_4^s$ for map T^4, the 4-cycle saddle $\mathcal{C}_4^u$ is homoclinic on both sides, and the white regions represent areas including attracting cycles of different periods (besides P_L^*, attracting cycles of period 13, 17 and 35 are evidenced). Notice that in this range it is $D_L \in (0, 1)$ and $D_R \in (0, 1)$ so that the P.E. is repelling (no repelling node nor divergent trajectories can exist).

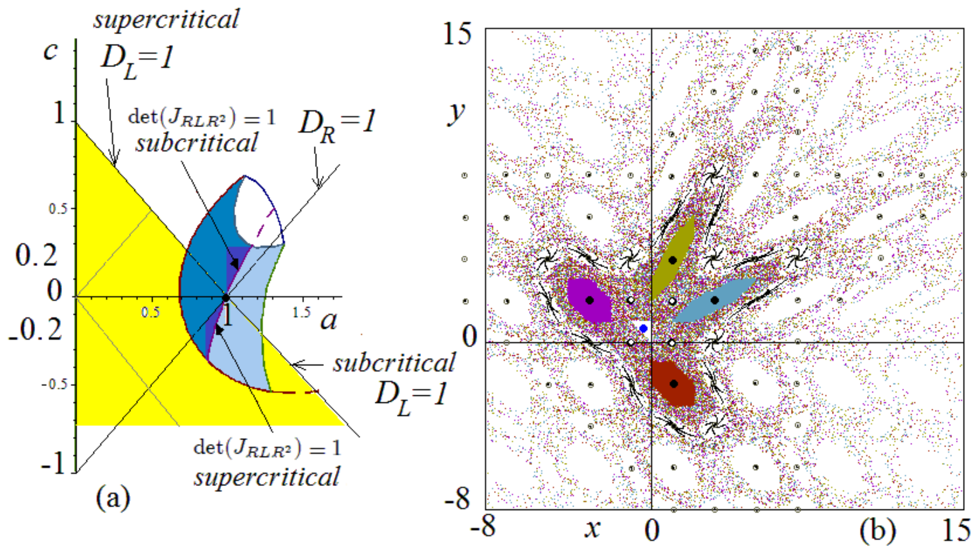

Fig.5 In (a) existence region of 4-cycles in the (a, c) parameter plane. In the dark blue colored area for $c > 0$ and $c < 0$ the center bifurcation of the 4-cycle C_4^s changes from subcritical to supercritical, respectively. In (b) phase space of map T at $a = 0.9999$ and $c = 0$, close to the area preserving case, when several attracting cycles of different periods coexist.

Both in the center bifurcation of the fixed point P_L^* and of the 4-cycle $\mathcal{C}_4^s$ the bifurcation point $(a, c) = (1, 0)$ separates the two arcs of subcritial and supercritical type, and plays the role of the Chenciner point in the Neimark-Sacker bifurcation for smooth systems, and also here for parameters in a neighbourhhod of this point, homoclinic bifurcations occur [Kuznetsov, 2004].

5 Center bifurcations related to the 3-cycle $\mathcal{C}_3^s$

In this section we complete our investigation of the center bifurcation of the cycles determined in Section 2, where we have obtained also a pair of 3-cycles, and we have shown two portions of bifurcation curves in which the attracting 3-cycle $\mathcal{C}_3^s$ undergoes a center bifurcation.

In the cases considered up to now, in Sec.3 and Sec.4, the virtual fixed point P_R^* was either an attracting fixed point or a repelling focus, and this determines a particular behavior of the points of the phase plane. A similar behavior occurs also for the lower branch of the center bifurcation of the 3-cycle $\mathcal{C}_3^s$. In fact, the lower branch belongs to a region in which the map is invertible, the real fixed point P_L^* is a repelling focus ($D_L = (a + c) > 1$) while the virtual fixed point P_R^* is an attracting focus ($D_R = (a - c) \in (0, 1)$). Thus, all the trajectories from the left side rotate to the right side and from here the trajectory rotate again to the left side. Inside the stability region close to the bifurcation we have a 3-cycle attracting focus, $D_{RLR} = (a - c)^2(a + c) \in (0, 1)$, while the repelling 3-cycle $\mathcal{C}_3^u$ is a saddle with determinant $D_{LLR} = (a-c)(a+c)^2 >> 1$. Thus, at the center bifurcation of the 3-cycle, in a neighborhood of the invariant area, for map T^3, it is $D_{RLR} = 1$ and $D_{LLR} >> 1$ which implies that map T^3 cannot have attracting cycles, neither after the center bifurcation (when it is also $D_{RLR} > 1$) a saddle-attracting node connection cannot exist. Thus, at the bifurcation the border of the invariant areas around $\mathcal{C}_3^s$ of map T^3 cannot be a closed curve attracting from outside. It is not a conservative system, and a subcritical center bifurcation must occur. We have so proved the following

Proposition 3. *The center bifurcation of the 3-cycle C_3^s occurring in the lower branch of the center bifurcation curve is of subcritical type.*

An example is shown in Fig.6. In Fig.6(a) before the bifurcation, one branch of the unstable set of the 3-cycle saddle $\mathcal{C}_3^u$ is homoclinic while the other branch tends to the attracting 3-cycle, and the stable set of the saddle bounds the basin of attraction of $\mathcal{C}_3^s$. In Fig.6(b), close to the center bifurcation, the saddle $\mathcal{C}_3^u$ is no longer on the basin boundary of $\mathcal{C}_3^s$, and both branches of the unstable set of the 3-cycle saddle are homoclinic. The transition is similar to the one commented in the previous section related to the 4-cycle.

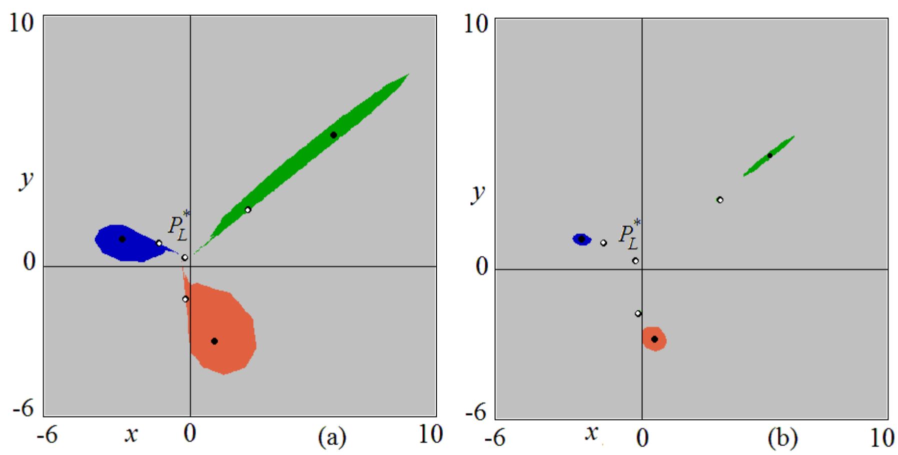

Fig.6 Phase space around P_L^*, with the basins of the three fixed points of C_3^s for map T^3 in three different colors. In (a) $a = 1.6451$, $c = 1.1$ the boundaries are the stable set of the saddle C_3^s, having one branch already homoclinic. In (b) $a = 1.69$, $c = 1.1$ the boundaries are closed repelling invariant curves. Very close to the subcritical center bifurcation of C_3^s.

Differently, the center bifurcation in the upper branch of the 3-cycle $\mathcal{C}_3^s$ occurs in a parameter region with $a > 0.5$ and $c > 1 + \frac{a}{2}$ in which map T is noninvertible, P_L^* is a repelling focus and *the virtual fixed point P_R^* is a saddle.* We show that the P.E. is neither attracting nor repelling.

In fact, from the eigenvalues of P_R^* given in (8) for $c > 1 + \frac{a}{2}$ we have $\lambda_+ \in (0,1)$ and $\lambda_- < -1$. The eigenvector $v_R(\lambda_+)$ related to the positive eigenvalue λ_+ has a positive slope ($v_R(\lambda_+)$ is given by the straight line $y = \frac{1-\lambda_+}{a}(x+1)$), which means that necessarily an invariant half-line issuing from the P.E. exists in $x > 0$, so that the P.E. has a repelling branch.

Moreover, below the curve of the center bifurcation, it can be shown the existence of a bifurcation curve related to the appearance of a 2-cycle repelling node via transcritical bifurcation. In fact, looking for the solutions of the equation $T_L \circ T_R(x, y) = (x, y)$ considering

$$T_L \circ T_R(x, y) = \begin{bmatrix} -(1+a) & a(1+c) \\ (1-c) & (c^2-a) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -a \\ 1-c \end{bmatrix} \tag{27}$$

we obtain a solution $\mathcal{C}_2^u = (x_{2,R}^u, y_{2,R}^u)$ given by

$$(x_{2,R}^u, y_{2,R}^u) = \left(\frac{a^2}{2c^2 - (a^2+2a+2)}, \frac{2(c-1)}{2c^2 - (a^2+2a+2)} \right) \tag{28}$$

which belongs to the right partition for $x_{2,R}^u > 0$, which holds for $c > \sqrt{1 + a + \frac{a^2}{2}}$. In such a case, its image is

$$(x_{2,L}^u, y_{2,L}^u) = \left(\frac{a^2 - 2a(c-1)}{2c^2 - (a^2+2a+2)}, \frac{2(c-a-1)}{2c^2 - (a^2+2a+2)} \right) \tag{29}$$

and $x_{2,L}^u < 0$ is always satisfied. From the Jacobian matrix, having trace $Tr_{RL} = c^2 - 2a - 1$ and determinant $D_{RL} = -(c^2 - a^2)$, the characteristic polynomial $\mathcal{P}_{RL}(\lambda)$ leads to $\mathcal{P}_{RL}(1) = a^2 + 2a + 2 - 2c^2$, $\mathcal{P}_{RL}(-1) = a(a-2)$. Thus the cycle appears via degenerate transcritical bifurcation when $\mathcal{P}_{RL}(1) = 0$ (the denominator in (28) becomes zero) at

$$\tau(\mathcal{C}_2^u) \ : \ c^2 = 1 + a + \frac{a^2}{2} \tag{30}$$

For $c > \sqrt{1 + a + \frac{a^2}{2}}$ the 2-cycle $\mathcal{C}_2^u$ exists, and it is $\mathcal{P}_{RL}(1) < 0$ (increasing c, $\mathcal{P}_{RL}(1)$ decreases) so that, of the two eigenvalues $\lambda_\pm = \frac{1}{2}((c^2 - 2a - 1) \pm \sqrt{(c^2 - 2a - 1)^2 + 4(c^2 - a^2)})$, it is always $\lambda_+ > 1$. Moreover, from $\mathcal{P}_{RL}(-1) = a(a-2)$ we have that for $a < 2$ the 2-cycle is a repelling node ($\lambda_- < -1$), while for $a > 2$ it is a saddle as long as $-1 < \lambda_- < 1$. The bifurcation value $a = 2$ will be commented below, since it is a degenerate flip bifurcation of subcritical type.

The eigenvector $v_{RL}(\lambda_+)$ related to the eigenvalue $\lambda_+ > 1$ (which always exists after the appearance of the 2-cycle) is given by

$$y = y_{2,1}^u + m(\lambda_+)(x - x_{2,1}^u) \quad , \quad m(\lambda_+) = \frac{\lambda_+ + 1 + a}{a(1+c)} > \frac{1}{a} \tag{31}$$

where $\frac{1}{a}$ is the slope of the critical line in the region $x > 0$ associated with the second iterate T^2. Recall that the critical lines of T^2 are given by LC_{-1} ($x = 0$) and its rank-1 preimages with the inverses T_R^{-1} and T_L^{-1}, which are the straight lines of equations $y = \frac{1}{a}x$ and $y = -\frac{1}{a}x$, respectively. Thus the eigenvector $v_{RL}(\lambda_+)$ is an half-line (whose points are diverging) issuing from the periodic point $(x^u_{2,R}, y^u_{2,R})$ of $\mathcal{C}^u_2$ and completely included in the region in which it is a fixed point of map $T_L \circ T_R(x, y)$. Clearly, the image by T of the eigenvector $v_{RL}(\lambda_+)$ gives the unstable branch issuing from the other point of the 2-cycle (in the left partition). That is, on these two half-lines invariant for map T^2 the P.E. is attracting. While the eigenvector $v_R(\lambda_+)$ has a slope $\frac{1-\lambda_+}{a}$ smaller than $\frac{1}{a}$ and in the region in which $T_R \circ T_R(x, y)$ applies, the P.E. is repelling on this half-line for map T^2.

The phase plane of the map in this case is shown in Fig.7(a). In the same figure we show the branch which is issuing from the P.E. eigenvector $v_R(\lambda_+)$ of the virtual fixed point, and the eigenvector $v_{RL}(\lambda_+)$ issuing from the right point of the 2-cycle (here repelling node). In Fig.7(b) it is shown the set of nondivergent trajectories of map T^3 showing the three basins of the attracting 3-cycle focus $\mathcal{C}^s_3$ with three different colors. The points in the gray region have divergent trajectories.

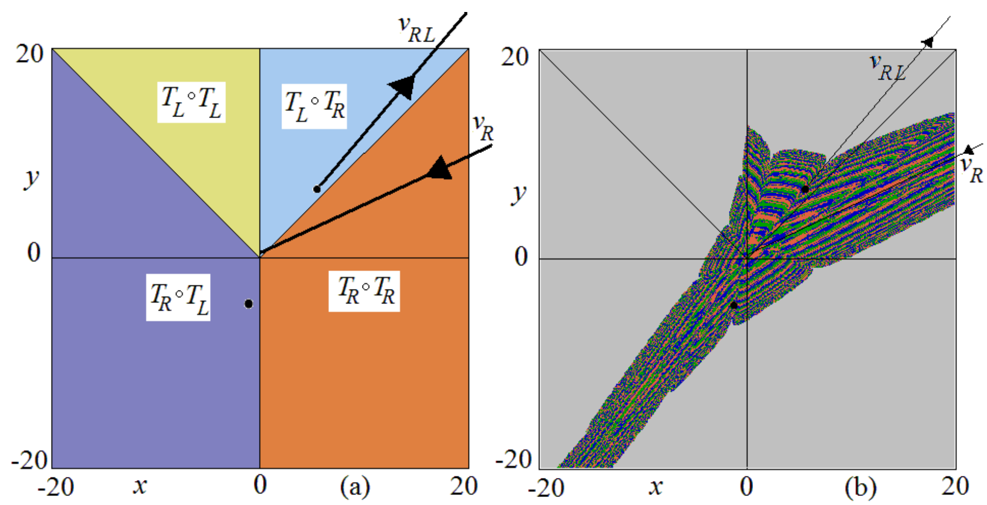

Fig.7 In (a) phase plane with the critical lines of map T^2, evidencing the four regions in which the different composite functions apply. The eigenvector $v_R(\lambda_+)$ is a stable branch of the virtual fixed point, the eigenvector $v_{RL}(\lambda_+)$, is an unstable branch of the 2-cycle repelling node. In (b), $a = 1$, $c = 1.61$, phase space showing in gray divergent trajectories, and the basins of the three fixed points of C^s_3 for map T^3 in three different colors. The 2-cycle repelling node belongs to the boundary of divergent trajectories.

It follows that the P.E. is neither attracting nor repelling, and the arguments used up to now cannot be applied in this case. We cannot give a general proof, but we have evidence that in the considered upper branch the center bifurcation is supercritical, and we can illustrate the mechanism via an example.

Let us fix the value $a = 1$, at which it can be shown that for $c = 1$ the 3-cycle is almost globally attracting and the P.E. is repelling. As c is increased, other cycles may appear and indeed we observe divergent trajectories (i.e. the P.E. may become partially attracting) also before the appearance via transcritical bifurcation of the 2-cycle detected above, and the unstable 3-cycle from saddle becomes a repelling node. Before the center bifurcation the phase plane has the 3-cycle focus as unique attractor, with a complex structure in the frontier of the three basins for the third iterate of the map, T^3, as shown in Fig.7(b). An enlargement is shown in Fig.8(a) to evidence the points of the 3-cycle attracting focus and the 3-cycle repelling node.

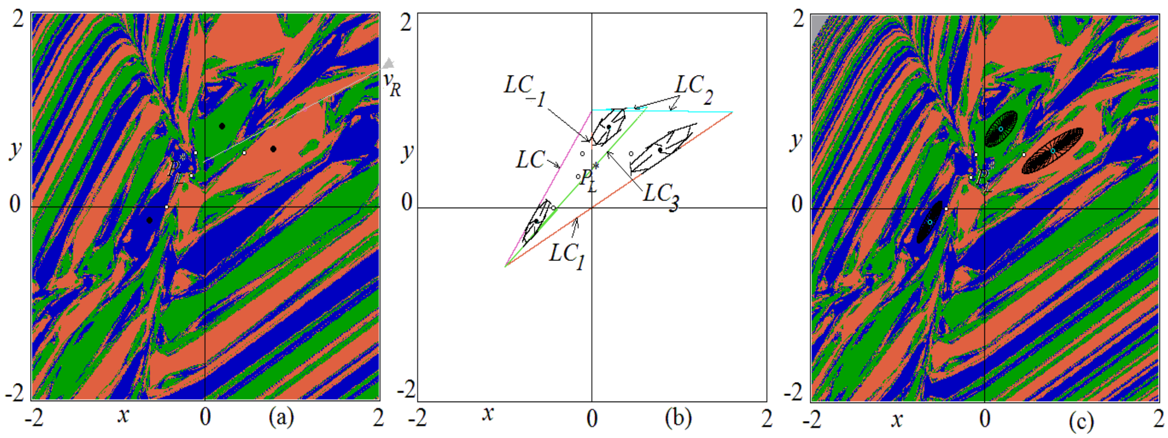

Fig.8 In (a) enlargement of the phase plane around P_L^* at $a = 1$, $c = 1.61$, close to the center bifurcation of the 3-cycle, showing the basins of the three fixed points of C_3^s for map T^3 in three different colors. The existence of a closed area bounded by four critical segments is shown in (b), together with three closed areas around C_3^s bounded by a finite number of critical segments. In (c) $a = 1$, $c = 1.6181$, soon after the supercritical center bifurcation of the 3-cycle, showing in three different colors the basins of the three attracting closed invariant curves for map T^3 (around the repelling focus C_3^s).

The noninvertibility of the map is clearly visible from the disconnected and multiply connected structure of the three basins of attraction. From the noninvertibility we have that a closed area mapped into itself and bounded by segments of critical curves can be found. This holds both for map T and for map T^3. In Fig.8(b) we show four images of the segment $(0,0)-(0,1)$ of LC_{-1}, which bound an area invariant for map T. Inside this area, with a finite number of images of the smaller segment of LC_{-1} there evidenced, three areas (each one mapped into itself for map T^3) around the attracting 3-cycle focus $\mathcal{C}_3^s$ are obtained. These areas are internal to the three basins of $\mathcal{C}_3^s$ for map T^3, which have on their boundaries the repelling node $\mathcal{C}_3^u$ and the fixed point P_L^* (repelling focus). The value of the parameter c is very close to the center bifurcation, so that the invariant areas existing at the center bifurcation must be internal to that basin, which still exists at the center bifurcation. This implies that the bifurcation must be supercritical, as shown in Fig.8(c): Soon after the center bifurcation the trajectory of a point close to the 3-cycle repelling focus converges to three cyclical closed curves and the basins of map T^3 do not change significantly.

Similar arguments hold when crossing the upper center bifurcation curve also in other points.

6 Conclusions

In the present paper we have considered some bifurcations occurring in the two-dimensional continuous piecewise linear map $(x', y') = T(x, y)$ defined in (2). This subject is nowadays one of the richest in terms of bifurcations to be investigated and open problems. In this work we have mainly faced the problem related to the center bifurcation of fixed points and cycles. In section 2 we have determined the real fixed point P_L^* and related stability region, showing its coexistence with attracting cycles of period 3 and 4. But also the virtual fixed point P_R^* is relevant in qualifying the center bifurcation of the real fixed point P_L^*, as we have shown in section 3: when the virtual fixed point is attracting then the center bifurcation is supercritical, when the virtual fixed point is a repelling focus then the center bifurcation of P_L^* is subcritical. The conservative case separates the two different kind of bifurcations, and is similar to Chenciner point. A similar result holds for the center bifurcation of the 4-cycle determined attracting in section 2: The conservative case separates the two different kind of bifurcations, subcritical and supercritical. As for the fixed point, when the Poincaré Equator is attracting/repelling the center bifurcation of the 4-cycle is subcritical/supercritical, as shown in section 4. For the center bifurcation of the 3-cycle we have two different situations, as shown in section 5. In one case the Poincaré Equator is attracting and the center bifurcation of the 3-cycle can be shown be of subcritical type. Differently, in the upper branch of the center bifurcation curve the Poincaré Equator is neither attracting nor repelling, and we cannot apply the previous arguments to determine which kind of center bifurcation occurs. Numerical simulations evidence that it occurs of supercritical type.

Clearly, our arguments are not conclusive and cannot be applied in all the possible situations. Thus, further investigations are necessary to characterize the center bifurcations, as well as other bifurcations occurring in these classes of maps, which are left for future works.

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