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BORDER-COLLISION BIFURCATIONS IN 1D PIECEWISE-LINEAR MAPS AND LEONOV'S APPROACH

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50 years ago (1959) in a series of publications by Leonov, a detailed analytical study of the nested period adding bifurcation structure occurring in piecewise-linear discontinuous 1D maps was presented. The results obtained by Leonov are barely known, although they allow the analytical calculation of border-collision bifurcation subspaces in an elegant and much more efficient way than it is usually done. In this work we recall Leonov's approach and explain why it works. Furthermore, we slightly improve the approach by avoiding an unnecessary coordinate transformation, and also demonstrate that the approach can be used not only for the calculation of border-collision bifurcation curves.

Keywords: Discontinuous piecewise-linear 1D maps; border collision bifurcations; adding mechanism; Farey structure.

1. Introduction

Bifurcations occurring in piecewise-smooth systems are quite different from those occurring in smooth ones. It is nowadays well known that in smooth systems the dynamics may evolve from a regular dynamic behavior to a complex one via a sequence of bifurcations (for example, routes to chaos via Feigenbaum cascades of period doubling bifurcations), whereas in piecewise-smooth systems *Border-Collision Bifurcations* (BCB for short) may occur, leading to a sharp transition from regular dynamics to chaos.

In piecewise-linear systems, which we are considering in this paper, mainly BCB and contact bifurcations[1] occur. It is classified as *border-collision* any contact between an invariant set of a map with the border of its region of definition, and this may, or may not, produce a bifurcation.[2] The term *border-collision bifurcation* was introduced for the first time in [Nusse & Yorke, 1992], (see also [Nusse & Yorke, 1995]) and it is now widely used in this context (i.e. for piecewise-smooth maps). Recently, these bifurcations have been studied mainly because of their relevant applications in

[1]Following [Mira *et al.*, 1996] and [Fournier-Prunaret *et al.*, 1997], a *contact bifurcation* occurs when two invariant sets of different nature have a contact in one or more points. The dynamic effect of a *contact bifurcation* may be of several different kinds, and depends on the nature of the invariant sets and on the map.

[2]For example, when a stable fixed point undergoes a border-collision crossing a boundary, it may persist, stable, also after the crossing, and in this case no bifurcation takes place due to the border-collision.

engineering (electrical and mechanical), and also in economics and social sciences. In fact, several publications were motivated by models describing particular circuits or models for the transmission of signals (see for example [di Bernardo *et al.*, 1999; Banerjee & Grebogi, 1999; Banerjee *et al.*, 2000a, 2000b; Avrutin & Schanz, 2006; Avrutin *et al.*, 2006; Zhusubaliyev *et al.*, 2006, 2007]). Before the work by Nusse and Yorke, the bifurcations associated with piecewise smooth maps were also studied in some papers (even if the bifurcations were not called border-collision). We recall, for example, [Mira, 1978, 1987; Maistrenko *et al.*, 1993, 1995, 1998] and others. In particular, in [di Bernardo *et al.*, 1999] some results, already printed several years ago by Feigin in 1978,[3] are republished. We may also go further back, citing the work by Leonov in the end of 50th, [Leonov, 1959, 1962]. In his work, Leonov described several bifurcations, giving a recurrence relation to find analytic expressions of the family of bifurcations occurring in a one-dimensional piecewise-linear map with one point of discontinuity, which is still mainly unknown. The aim of this work is to give a new interpretation and improvements of some of his results, associated with the map

$$x' = f(x) = \begin{cases} f_L(x) = a_L x + \mu_L & \text{if } x < 0 \\ f_R(x) = a_R x + \mu_R & \text{if } x > 0 \end{cases} \tag{1}$$

in the case of positive slopes. That is, we restrict our analysis to the ranges

$$a_L > 0, \quad a_R > 0, \quad \mu_R < 0 < \mu_L \tag{2}$$

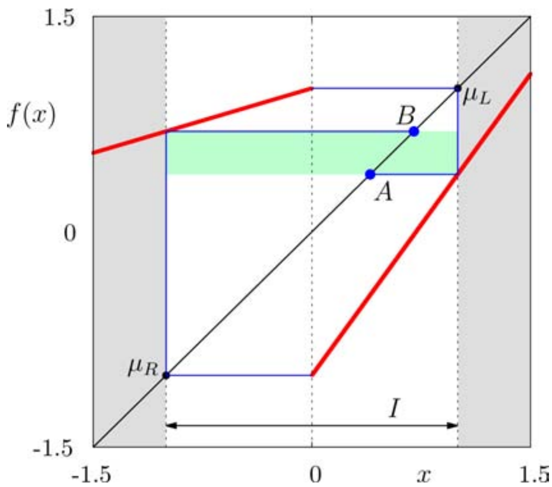
Note that the symmetry in the two functions f_L and f_R with respect to the parameters, will turn out to be quite useful in the computations. Considering the map given in (1) and the parameters as given in (2), we can immediately say that no stable fixed point can exist, but we may have one or two unstable fixed points: the unstable fixed point $P_L^* = \mu_L/(1-a_L) < 0$ exists when $a_L > 1$ while $P_R^* = -\mu_R/(a_R - 1) > 0$ exists when $a_R > 1$. It is easy to see that we can limit our study to the interval $I = [f_R(0), f_L(0)] = [\mu_R, \mu_L]$. It is obvious that almost all (except for a set of zero Lebesgue measure) the trajectories are divergent when $f_R(0) < P_L^*$ or when $P_R^* < f_L(0)$. Thus we are interested in the bounded dynamics existing when the interval $I = [\mu_R, \mu_L]$ is *absorbing*, which means that points in a suitable neighborhood of I are mapped into I from which the trajectories can never escape. This occurs when either the fixed points do not exist (both slopes are less than one), or $f_R(0) > P_L^*$ (when $a_L > 1$ and P_L^* exists) or $P_R^* > f_L(0)$ (when $a_R > 1$ and P_R^* exists).

The basin of attraction of the interval I can be determined easily. Note that if P_L^* exists and is unstable then all points $x_0 < P_L^*$ lead to divergent trajectories. Similarly, if P_R^* exists and is unstable then all points $x_0 > P_L^*$ lead to divergent trajectories. So, when the interval I is absorbing, then its basin of attraction $\mathcal{B}(I)$ is determined by the fixed points of the map:

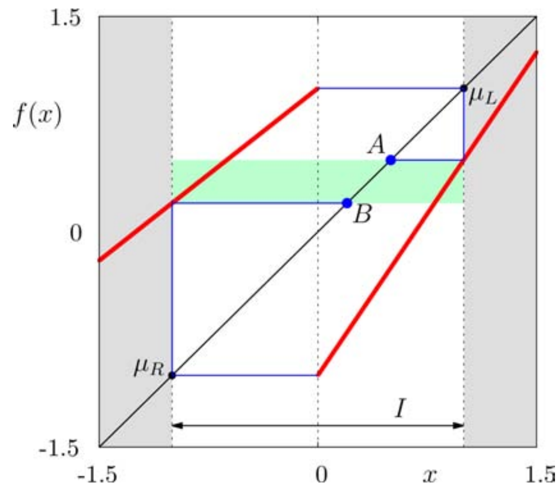
$$\mathcal{B}(I) = \begin{cases} \mathbb{R} & \text{if no fixed points exist} \\ (P_L^*, +\infty) & \text{if only the unstable fixed point } P_L^* \text{ exists} \\ (-\infty, P_R^*) & \text{if only the unstable fixed point } P_R^* \text{ exists} \\ (P_L^*, P_R^*) & \text{if both unstable fixed points } P_L^* \text{ and } P_R^* \text{ exist} \end{cases} \tag{3}$$

As long as both slopes are less than one, then *no unstable* cycle of any period can exist, while when both the slopes are larger than one, then *no stable cycle* can exist and thus only chaos in suitable intervals or invariant Cantor sets, or divergent trajectories can occur. But when the slopes are one less and one larger than one, it may result in an uncertainty, and we may expect both regular dynamics (with attracting cycles or quasiperiodic trajectories) as well as chaotic dynamics. The most obvious expectation in this case is a kind of progressive destabilization of the possible cycles. However, this expectation turns out to be wrong and there is no smooth transition to chaos. Instead, only a sharp transition can occur from a regular regime with no unstable cycles and no chaos in the absorbing interval I to a chaotic regime with only chaos and no stable cycles inside I. The bifurcation value is determined by the condition that the map in its absorbing interval I changes from invertible to noninvertible (see Fig. 1). This — our case — is in fact just a particular one of the discontinuous piecewise-smooth problems already considered in [Keener, 1980], where the function is smooth and increasing, with one point of discontinuity. In that

[3] It is worth noticing that the clear and simple analysis performed by Feigin in 1978 is the first one for *n-dimensional piecewise linear continuous maps*, with $n > 1$.



(a)



(b)

Fig. 1. The system function is invertible on the absorbing interval I (the mapping is *into* and 1-1) as long as the point $A = f_R(f_L(0))$ is located on the left side of the point $B = f_L(f_R(0))$ (a). Otherwise the map is noninvertible on the interval I (the mapping is *onto* but not 1-1) (b). Parameter settings: $a_L = 0.3$, $a_R = 1.4$ (a), $a_L = 0.8$, $a_R = 1.5$ (b), $\mu_L = 1$, $\mu_R = -1$.

paper, the author proves that as long as the map is invertible [as in Fig 1(a)] then the map is regular. As we will see in this work, depending on the values of the two slopes, periodic orbits of any period can exist, which are globally attracting in I, or, for a set of parameter values of zero Lebesgue measure, the dynamic behavior in I is bounded and nonchaotic.

In our notation, the map is invertible as long as $f_R(f_L(0)) < f_L(f_R(0))$, that is, as long as $\mu_L(1 - a_R) - \mu_R(1 - a_L) > 0$. Otherwise the map is nonuniquely invertible in I, and, as already proved in [Keener, 1980], the nondivergent dynamic behavior is chaotic in some invariant set. The set of parameters (S) at which the bifurcation occurs is given by:

$$(S)\colon\ \mu_L(1 - a_R) - \mu_R(1 - a_L) = 0 \qquad (4)$$

We will describe in detail also the dynamics occurring at these particular bifurcation values, showing that (and when) either all the points in the absorbing interval I are periodic, or all are quasiperiodic (i.e. the trajectory of any point is quasiperiodic and dense in the absorbing interval I).

It is already well-known how extremely rich the structure of the regions associated with stable cycles (also called periodicity regions) is, and that the rotation numbers form in this case the self-similar *devil's staircase structure*. Also the analytical equations of the BCB curves associated with some cycles, called *maximal* or *principal cycles* in the recent literature, are known. By contrast, the analytical formulation of the BCB curves of more complex cycles, in a general setting, is a quite new subject (although already considered by Leonov), and this motivates the present work. Due to Leonov's technique we will describe how the analytical expressions of all the bifurcations curves, bounding the periodicity regions (see Fig. 2), can be detected by using an iterative process which leads to a significant simplification of the calculations. As we will see, the proof given by Leonov represents a kind of mapping of the coefficients of the system function. Moreover, we will see that the technique used by Leonov requires a specific change of variable. We improve this technique showing how this change of variable may be avoided.

The plan of the work is as follows. In Sec. 2 we describe Leonov's approach for the calculation of BCB curves of periodic orbits. First, in Sec. 2.1 we recall some known results for the two infinite families of maximal cycles, which we denote (following Leonov's notation) as cycles of complexity level one. Then, in Sec. 2.2 we describe the recursive mode of operation which represents the key point of Leonov's approach and obtain the analytical expressions for all cycles of the four infinite families of the complexity level two. In Sec. 2.3 an improvement of Leonov's technique is given. Section 2.4 describes the calculation for the complexity level three and for the general case. Additionally, Sec. 2.5 deals with the particular case, not considered in the previous sections, in which both slopes of the map are equal to one. As a next step, in Sec. 3 we demonstrate

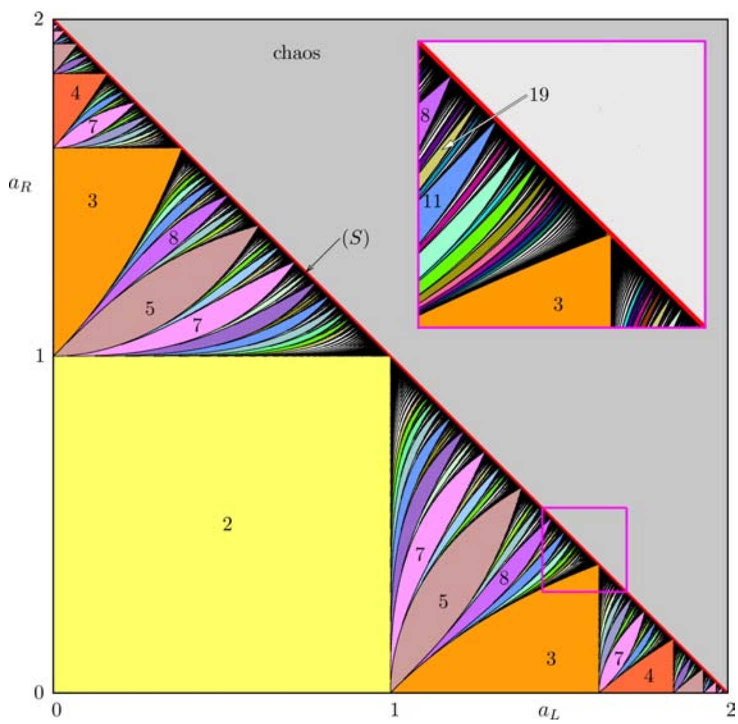


Fig. 2. Numerically detected nested period adding bifurcation structure in map (1). The inset shows the marked rectangle enlarged. Numbers correspond to periods in specific regions. Parameter setting: $\mu_L = 1$, $\mu_R = -1$.

that the recursive mode of operation suggested by Leonov is useful not only for analytical calculation of BCB curves. First, in Sec. 3.1 we recall some results on the devil's staircase structure formed by rotation numbers. The change of stability due to the crossing of the set (S) is discussed in Sec. 3.2 while in Sec. 3.3 we consider the dynamics occurring when the parameters belong to (S).

In this paper, we do not consider in detail the chaotic regime. The bifurcation structure formed by crises bifurcation in the regime of robust chaos is quite complicated and in some cases self-similar as reported in [Avrutin & Schanz, 2008; Avrutin *et al.*, 2008a, 2008b, 2009]. It is further shown in [Avrutin & Schanz, 2009] that an extended Leonov approach can be used for a significantly simplified calculation of the crises bifurcations structuring the chaotic regime.

2. Leonov's Approach

2.1. *Complexity level one*

We consider the map in the generic form given in (1) and the parameters as given in (2). In this and the following sections we will see how it is possible to determine analytically the infinitely many bifurcation curves which give the boundary of the periodicity regions in the considered parameter space. For simplicity reasons let us start considering the case $0 < a_L < 1$ and $0 < a_R < 1$ so that there are no unstable fixed points and the invariant interval $I = [\mu_R, \mu_L]$ is globally absorbing. Firstly, we will characterize the asymptotic behavior inside I and later we will generalize the reasoning to the whole regular parameter region. When a stable cycle exists, then it is globally attracting in I, and we are interested in determining the parameter regions associated with all the possible stable cycles. We will do this by using Leonov's approach. To find the possible stable cycles we look for fixed points of the iterated map, whose main property is that to be *again piecewise-linear*, with pieces separated by the discontinuity point $x = 0$ and its preimages, and each piece is obtained by composition of the different components f_L and f_R in a suitable way.

The simplest cycles to analyze are those called by Leonov as *first level of complexity*. The cycles of *first level of complexity* (also called *maximal* or *principal cycles*) are characterized by only one point in one region, say L, and the other points in the other region R, which leads to the corresponding symbolic sequence LR^{n_1}, for $n_1 \geq 1$. For such a cycle of period $(n_1 + 1)$ we can order the periodic points: let us define $x_0^* < 0$ and $x_1^* > \cdots > x_{n_1}^* > 0$. Then the x_i^* with $i = 0, \ldots, n_1$ represent n_1+1 fixed points of the map $f^{n_1+1}(x)$ and only the point x_0^*

is in the negative side. This point is a fixed point of the linear function $f_R^{n_1} \circ f_L(x)$, and its range of existence as a fixed point of f^{n_1+1} in I is the following:

$$f_R(0) = \mu_R \leq x_0^* \leq 0 \tag{5}$$

The two equations associated with (5) denote the BCBs leading to emergence and disappearance of the cycle. The reason why the situation $x_0^* = 0$ corresponds to a BCB is clear: the periodic point is merging with the discontinuity point from the left side. Then the cycle exists until its last periodic point $x_{n_1}^*$ is merging with the discontinuity point $x = 0$ from the right side. It is worth noticing that this merging of the *last* periodic point with the border $x = 0$ is equivalent to the merging of the *first* periodic point x_0^* with $f_R(0) = \mu_R$. Notice also that a periodic point cannot exit from the absorbing interval $I = [\mu_R, \mu_L]$ and thus μ_R is the lowest possible value. This cycle is *simple* because having only one point on one side and n_1 points on the other, the computation of the power n_1 of a linear map is quite easy, and the formulas (in geometric progression) can be simplified. We proceed as follows: consider a point $\mu_R \leq x_0 \leq 0$ and apply the maps in the given order $LR \cdots R$. Thus we obtain[4]:

$$\begin{aligned}
x_1 &= f_L(x_0) = a_L x_0 + \mu_L \qquad (6)\\
x_2 &= f_R \circ f_L(x_0) = a_R[a_L x_0 + \mu_L] + \mu_R\\
&\cdots\\
x_{n_1+1} &= f_R^{n_1} \circ f_L(x_0) = a_R^{n_1}[a_L x_0 + \mu_L]\\
&\quad + \mu_R(a_R^{n_1-1} + \cdots + a_R + 1)\\
&= a_R^{n_1} a_L x_0 + \mu_L a_R^{n_1} + \mu_R \frac{1 - a_R^{n_1}}{1 - a_R}\\
&\qquad \text{if } a_R \neq 1 \qquad (7)
\end{aligned}$$

Consequently, we have the function

$$\begin{aligned}
f_R^{n_1} \circ f_L(x) &= (a_R^{n_1} a_L)x + \left(\mu_L a_R^{n_1} + \mu_R \frac{1 - a_R^{n_1}}{1 - a_R}\right)\\
&=: (Ax + M)
\end{aligned} \tag{8}$$

and by using the equality $x_0^* = f_R^{n_1} \circ f_L(x_0^*)$ we obtain its fixed point, which is the periodic point x_0^* for f. This point exists for:

$$\begin{aligned}
f_R(0) &= \mu_R \leq x_0^*\\
&= \frac{1}{1 - a_R^{n_1} a_L}\left(\mu_L a_R^{n_1} + \mu_R \frac{1 - a_R^{n_1}}{1 - a_R}\right) \leq 0.
\end{aligned} \tag{9}$$

The two BCB curves bounding the region of existence of the cycle are denoted by $\xi_{LR^{n_1}}^l$ (respectively, $\xi_{LR^{n_1}}^r$) as they are associated with the situations that the boundary $x = 0$ is collided by a periodic point from the left (respectively, from the right) side. Both can be deduced from the two equations associated with (9). Moreover, we know the inequality which clarifies on which side of the bifurcation curve the periodicity region is located. So, noticing that $a_R^{n_1} a_L$ is the eigenvalue of the cycle and in the regular regime the cycle is stable, which means $(1 - a_R^{n_1} a_L) > 0$, we get from $x_0^* \leq 0$:

$$\xi_{LR^{n_1}}^l\colon \mu_L a_R^{n_1} + \mu_R \frac{1 - a_R^{n_1}}{1 - a_R} \leq 0 \tag{10}$$

or:

$$\xi_{LR^{n_1}}^l\colon \mu_L \leq -\mu_R \varphi_{n_1}^R \quad \text{with } \varphi_{n_1}^R = \frac{1 - a_R^{n_1}}{(1 - a_R)a_R^{n_1}} \tag{11}$$

Similarly, from $\mu_R \leq x_0^*$ we get:

$$\xi_{LR^{n_1}}^r\colon \mu_L a_R^{n_1} + \mu_R \frac{1 - a_R^{n_1}}{1 - a_R} - \mu_R\left(1 - a_R^{n_1} a_L\right) \geq 0 \tag{12}$$

or:

$$\xi_{LR^{n_1}}^r\colon \mu_L \geq -\mu_R(a_L + \varphi_{n_1-1}^R). \tag{13}$$

Clearly, the equalities associated with (11) and (13) represent the corresponding BCB curves.

There exists a particular case in which $a_R = 1$ comes simply from the formulas given above when substituting the term $(1 - a_R^{n_1})/(1 - a_R)$ with n_1 (and thus $\varphi_{n_1}^R = n_1$).

Now, instead of reasoning with a unique periodic point on the L side we can reason in some sense "symmetrically" (with respect to the L and R sides) looking for cycles having the symbolic sequence RL^{n_1}. In this case, we can order the periodic points as $x_0^* > 0$, $x_1^* < \cdots < x_{n_1}^* < 0$. Then such a cycle corresponds again to fixed points of the map f^{n_1+1}, with only one point in the positive side, which is the fixed point of the function $f_L^{n_1} \circ f_R(x)$, and as a periodic point for f, it exists as long as:

$$0 \leq x_0^* \leq \mu_L = f_L(0) \tag{14}$$

Due to the structure of the initial map with its symmetry in the parameters with respect to the indexes L and R it is clear that, to get the equations of the BCB curves associated with these cycles, it is enough to exchange the symbols L and R in the

[4]Note that the particular cases $a_L = 1$ and $a_R = 1$ are considered in Sec. 2.5.

above calculations (L into R and R into L), and reverse the related inequalities.

Notice, in fact, that changing sign in the previous sequence $x_0^* < 0$, $x_1^* > \cdots > x_{n_1}^* > 0$ we get the new one: $x_0^* > 0$, $x_1^* < \cdots < x_{n_1}^* < 0$. Changing the letters and the inequalities in the previous constraint $\mu_R \leq x_0^* \leq 0$ we get the correct new one: $\mu_L \geq x_0^* \geq 0$. Thus the new periodic point x_0^* on the R side and the new BCB curves (which are obtained from the associated equalities in the conditions) are obtained from (9) when $a_L \neq 1$:

$$\mu_L \geq x_0^* = \frac{1}{1 - a_L^{n_1} a_R}\left(\mu_R a_L^{n_1} + \mu_L \frac{1 - a_L^{n_1}}{1 - a_L}\right) \geq 0 \tag{15}$$

as $(1 - a_L^{n_1} a_R) > 0$ we get from (11):

$$\xi^r_{RL^{n_1}}\colon \mu_R \geq -\mu_L \varphi^L_{n_1} \quad \text{with } \varphi^L_{n_1} = \frac{1 - a_L^{n_1}}{(1 - a_L)a_L^{n_1}} \tag{16}$$

and from (13):

$$\xi^l_{RL^{n_1}}\colon \mu_R \leq -\mu_L(a_R + \varphi^L_{n_1 - 1}) \tag{17}$$

Of course, the particular case exists in which $a_L = 1$ comes simply from the formulas given above when substituting the term $(1 - a_L^{n_1})/(1 - a_L)$ with n_1 (and thus $\varphi^L_{n_1} = n_1$).

In Fig. 3 we can recognize the BCB curves bounding the periodicity regions of the first level of complexity. Notice that when we plot Eqs. (11) and (13) for $n_1 = 1, 2, \ldots$ we obtain the regions of the cycles of period $2, 3, \ldots$ and similarly when we plot the Eqs. (16) and (17) for $n_1 = 1, 2, \ldots$ we obtain the regions of the cycles of period $2, 3, \ldots$. Thus, the region and the borders of the 2-cycle are obtained from both the families LR^{n_1} and RL^{n_1} for $n_1 = 1$. A similar property holds for the BCB curves of other periodicity families of complexity levels larger than one.

2.2. *Complexity level two*

Before we turn to the calculation of the conditions for the BCB curves for orbits of complexity level two, let us first recall the discussion given in [Leonov, 1959; Mira, 1978] to prove that the periodicity regions in this stability regime are disjoint, and that between any two of them we have infinitely many other periodicity regions. Considering Eqs. (11) and (13), we can see that the range in the parameter $(\mu_L/-\mu_R)$ for the cycle of period

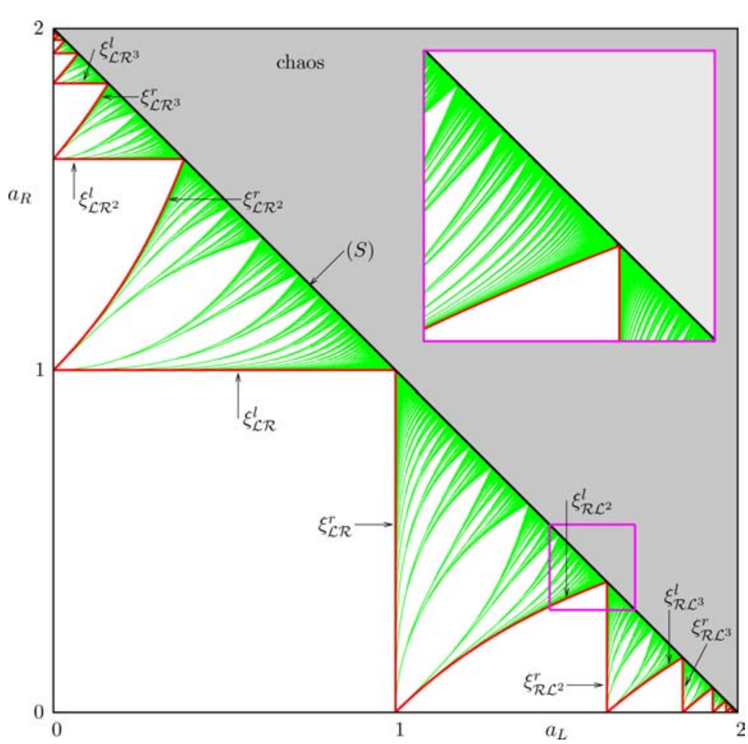


Fig. 3. BCB curves of the first complexity level calculated analytically (red). In the background the numerically calculated nested period adding bifurcation structure is shown (green). The inset shows the marked rectangle enlarged. Parameter setting: $\mu_L = 1$, $\mu_R = -1$.

(n_1+1) for any $n_1 \geq 1$ is given by an interval. For the periodicity regions of cycles of type LR^{n_1}, we have the intervals

$$\varphi_{n_1-1}^R + a_L \leq \frac{\mu_L}{-\mu_R} \leq \varphi_{n_1}^R \tag{18}$$

where

$$\varphi_{n_1}^R = \frac{1-a_R^{n_1}}{(1-a_R)a_R^{n_1}} \quad \text{if } a_R \neq 1, \quad \varphi_{n_1}^R = n_1 \quad \text{if } a_R = 1$$

whose width $(\varphi_{n_1}^R - \varphi_{n_1-1}^R - a_L) = (1/a_R)^{n_1} - a_L$ increases with n_1 when $a_R < 1$ and tends to infinity. Two consecutive intervals are separated by a fixed amount, given by a_L, which leads to disjoint periodicity regions (a few of which can be seen in Figs. 2 and 3).

For the other family of periodicity regions of type RL^{n_1} the symmetry may be seen in the parameter $-\mu_R/\mu_L$ for which the intervals are

$$\varphi_{n_1-1}^L + a_R \leq \frac{-\mu_R}{\mu_L} \leq \varphi_{n_1}^L \tag{19}$$

where

$$\begin{aligned} \varphi_{n_1}^L &= \frac{1-a_L^{n_1}}{(1-a_L)a_L^{n_1}} \quad \text{if } a_L \neq 1 \\ \varphi_{n_1}^L &= n_1 \quad \text{if } a_L = 1 \end{aligned} \tag{20}$$

so that the range in the same parameter $\mu_L/-\mu_R$ considered above is given by

$$\frac{1}{\varphi_{n_1}^L} \leq \frac{\mu_L}{-\mu_R} \leq \frac{1}{\varphi_{n_1-1}^L + a_R} \tag{21}$$

and are always disjoint intervals but now their width tends to zero.

Thus, considering for example, the region in the parameter space between the periodicity regions of the cycles with symbol sequences LR^{n_1} and LR^{n_1+1}, in the interval $\varphi_{n_1}^R < (\mu_L/-\mu_R) \leq \varphi_{n_1}^R + a_L$ we can find numerous regions corresponding to orbits with complexity level larger than one, among them the orbits of complexity level two.

To obtain the conditions for the BCB curves of these orbits, the key point is an operator which leads from one complexity level to the consecutive one. In order to obtain the operator, i.e. the mapping of the coefficients, the main point is to observe that for some suitable combinations of iterated system functions, we are in a situation similar to that previously considered (when computing the families of the first level of complexity).

This situation is illustrated in Fig. 4(a). As one can see, the two pieces of the second iterate f^2 labeled as LR and RL (separated by the origin O) are located above the diagonal and do not have fixed points any longer. Regarding the original function f that means that the period-2 orbit corresponding to the symbolic sequence LR is already destroyed. Similarly, the three pieces of the function f^3 labeled as LRR, RLR and RRL are located below the diagonal. That means, the function f^3 still does not have fixed points, or in other words, the period-3 orbit of the original system function f corresponding to the symbolic sequence LRR is still not emerged. Note also that the origin O separates the pieces LRR and RLR while its preimage $O_{-1} = f_R^{-1}(0)$ is the discontinuity point between the pieces RLR and RRL.

The reasoning in Leonov's paper is as follows: if we consider the preimage of the origin with the map f^2, see the point $O' = (f^2)^{-1}(O) = f_L^{-1} \circ f_R^{-1}(0)$ in Fig. 4, and consider the change of coordinate $y = x - O'$ then, with respect to the y-axes (see the enlargement in Fig. 4(b)), the branches LR and LRR are similar to the original map f. That is, the slopes are both less than one and no fixed point exists in an absorbing interval, thus the trajectories are probably converging to some cycle (for the parameters used in Fig. 4 we can see that a periodic orbit of period-2 exists with one point on the branch LR and one point on the branch LRR). And indeed, for the map constructed in this way, with one branch of f^2 on the left of O' and one branch of f^3 on the right side, we can repeat the same reasoning that we have done above for the BCB curves of cycles of the first level of complexity. We can also use the same formulas already calculated, we only have to take into account that the coefficients of the functions on the right and on the left side of the discontinuity point are changed: *instead of the coefficients of the function* f_L *we have to use the coefficients of the function* $f_{LR} := f_R \circ f_L$, *and instead of the coefficients of the function* f_R *we have to use the coefficients of the function* $f_{LRR} := f_R \circ f_R \circ f_L$. Clearly, the periodic points that we observe in Fig. 4(b) of a 2-cycle for this composite map

$$y' = \begin{cases} f_{LR}(y) & \text{if } y < 0 \\ f_{LRR}(y) & \text{if } y > 0 \end{cases} \tag{22}$$

represent periodic points of a 5-cycle for the original map f, as they are fixed points for the function $f^3 \circ f^2$ and $f^2 \circ f^3$. Indeed, we may look for cycles of the map f associated with fixed points of the functions $(f^3)^{n_2} \circ f^2$ and $(f^2)^{n_2} \circ f^3$ for any n_2.

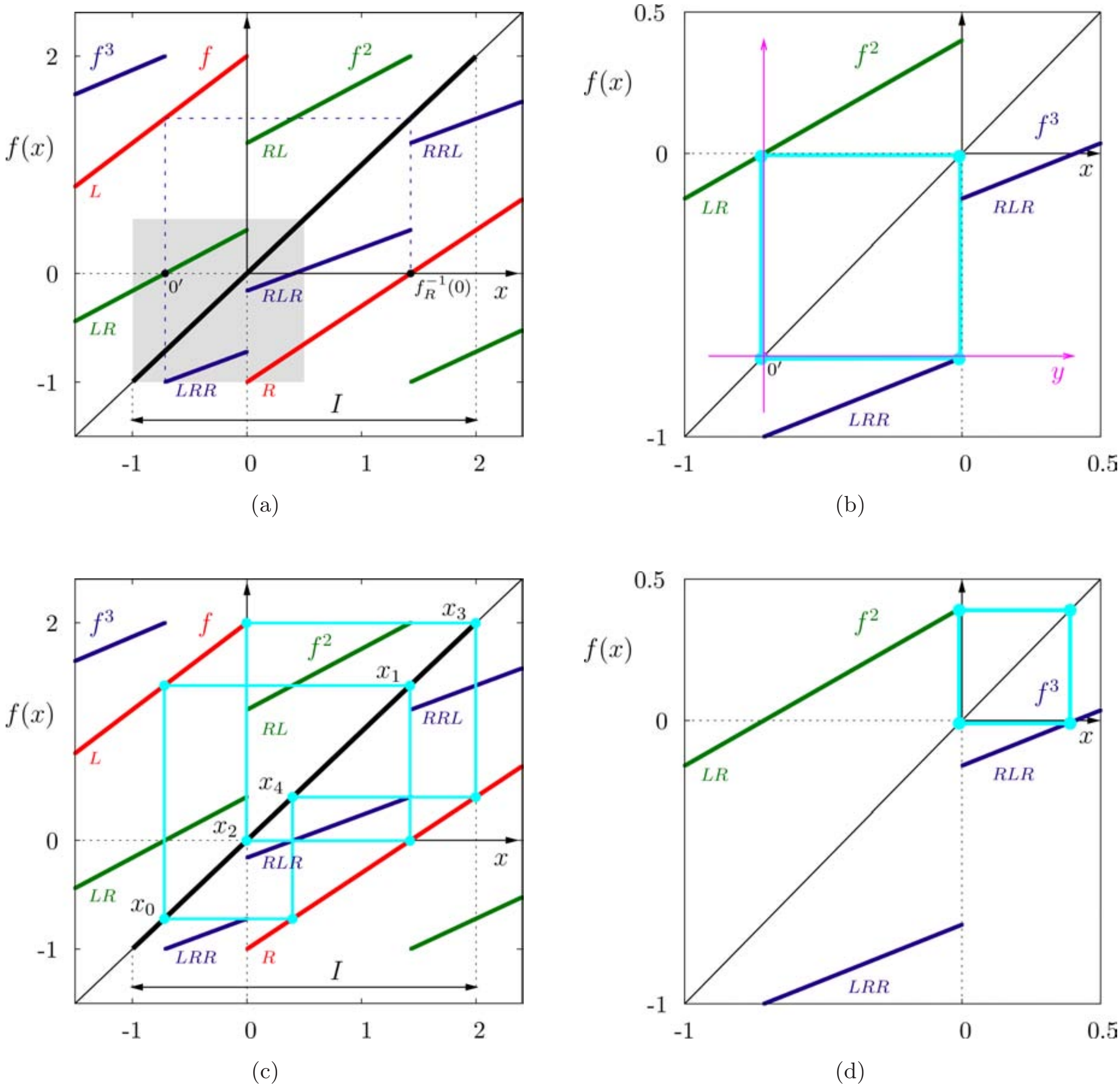


Fig. 4. In (a) and (c) the system function (red), its second (green) and third (blue) iterated functions are shown at the parameter values $a_L = 0.8$, $a_R = 0.7$, $\mu_L = 2$, $\mu_R = -1$. The rectangle marked gray in (a) is shown enlarged in (b) and (d). As one can see in (b), at the used parameter values the function composed from the piece LR of the second iterate and the piece LRR of the third iterate has in the translated coordinate system (shown magenta) with the new origin O' a period-2 orbit (shown cyan) colliding with the boundary $y = 0$, that means $x = O'$, from the left side. This orbit represents the period-5 orbit corresponding to the symbolic sequence $LRLRR$ of the original map, as shown in (c). The change of coordinates is not necessary if one considers the function composed from the piece LR of the second iterate and the piece RLR of the third iterate, as shown in (d).

Such kind of cycles are called the *second level of complexity*. Notice that these cycles have two periodic points on the L side of $x = 0$, but due to the change of coordinate only one periodic point exists on the L side of $y = 0$, so that we are in the same situation as previously analyzed.

In general, when considering the functions associated with the symbolic sequence LR^{n_1} and LR^{n_1+1}, the reasoning is the same: if we perform the change of coordinate shifting the origin in the preimage of order $(n_1 + 1)$ $O' = (f_R^{n_1} \circ f_L)^{-1}(0) = f_L^{-1} \circ f_R^{-n_1}(0)$, then we can determine the leftmost periodic point, and the related BCB curves, which we already know explicitly. In formulae, from

$$\begin{aligned} 0 &= f_R^{n_1} \circ f_L(x) \\ &= (a_R^{n_1} a_L)x + \left(\mu_L a_R^{n_1} + \mu_R \frac{1 - a_R^{n_1}}{1 - a_R}\right) \\ &=: Ax + M \end{aligned} \tag{23}$$

we have

$$O' = -\frac{M}{A} = -\frac{\mu_L a_R^{n_1} + \mu_R \dfrac{1-a_R^{n_1}}{1-a_R}}{a_R^{n_1} a_L} = -\frac{\mu_L}{a_L} - \mu_R \frac{\varphi_{n_1}^R}{a_L}. \tag{24}$$

Then we perform a change of coordinate which shifts the origin in O':

$$y = x - O' \tag{25}$$

Considering the function $f_R^{n_1} \circ f_L(x)$ on the left side and $f_R^{n_1+1} \circ f_L(x)$ on the right side of the new discontinuity point, with $x = y + O'$, we obtain the composite map

$$y' = T(y) = \begin{cases} T_L(y) = A_L y + M_L, & \text{if } y < 0 \\ T_R(y) = A_R y + M_R, & \text{if } y > 0 \end{cases} \tag{26}$$

The coefficients

$$\begin{aligned} A_L &= a_L a_R^{n_1} \\ A_R &= a_L a_R^{n_1+1} \\ M_L &= \frac{\mu_L}{a_L} + \mu_R \frac{\varphi_{n_1}^R}{a_L} \\ M_R &= \frac{\mu_L}{a_L} + \mu_R \left(1 + \frac{\varphi_{n_1}^R}{a_L}\right) \end{aligned} \tag{27}$$

define the required operator (mapping of the coefficients) leading from the bifurcation curves of the first complexity level to the bifurcation curves of the second complexity level.

Now it is only a matter of applications of the previous results. Given any two consecutive cycles of the first level of complexity for the map f, and considering the function $f_R^{n_1} \circ f_L$ on the left side of $y = 0$ and $f_R^{n_1+1} \circ f_L$ on the right side, we can immediately have the periodic points and the BCB curves of the cycles obtained as fixed points of the functions in the form $T_R^{n_2} \circ T_L(y)$ and $T_L^{n_2} \circ T_R(y)$. This first complexity level for the composite map T corresponds to the second complexity level for the map f. So we substitute the coefficients given by Eqs. (27) into the expressions for the bifurcation curves corresponding to the sequences LR^{n_2} and RL^{n_2} and obtain the bifurcation curves for the two families of cycles having the symbolic sequences $LR^{n_1}(LR^{n_1+1})^{n_2}$ and $LR^{n_1+1}(LR^{n_1})^{n_2}$.

For the function $T_R^{n_2} \circ T_L = (f_R^{n_1+1} \circ f_L)^{n_2} \circ (f_R^{n_1} \circ f_L)$ we have from (9):

$$T_R(0) = M_R \le y_0^* = \frac{1}{1 - A_R^{n_2} A_L}\left(M_L A_R^{n_2} + M_R \frac{1 - A_R^{n_2}}{1 - A_R}\right) \le 0 \tag{28}$$

for the periodic point y_0 of f satisfying $y_0 = T_R^{n_2} \circ T_L(y_0)$. Similarly, from (11) and (13) we get the BCB curves

$$\xi^l_{LR^{n_1}(LR^{n_1+1})^{n_2}}\colon M_L \le -M_R \frac{1 - A_R^{n_2}}{(1 - A_R)A_R^{n_2}} =: -M_R \Phi_{n_2}^R \tag{29}$$

$$\xi^r_{LR^{n_1}(LR^{n_1+1})^{n_2}}\colon M_L \ge -M_R \left(A_L + \frac{1 - A_R^{n_2-1}}{(1 - A_R)A_R^{n_2-1}}\right) =: -M_R(A_L + \Phi_{n_2-1}^R) \tag{30}$$

Using the expressions for the composite coefficients given in (27), we obtain the expressions for these bifurcation curves in terms of the original parameters:

$$\xi^l_{LR^{n_1}(LR^{n_1+1})^{n_2}}\colon \mu_L \le -\mu_R \left(\frac{\Phi_{n_2}^R a_L}{1 + \Phi_{n_2}^R} + \varphi_{n_1}^R\right)$$

$$= -\mu_R \left(\frac{\dfrac{1 - (a_L a_R^{n_1+1})^{n_2}}{(1 - a_L a_R^{n_1+1})(a_L a_R^{n_1+1})^{n_2}} a_L}{1 + \dfrac{1 - (a_L a_R^{n_1+1})^{n_2}}{(1 - a_L a_R^{n_1+1})(a_L a_R^{n_1+1})^{n_2}}} + \frac{1 - a_R^{n_1}}{(1 - a_R)a_R^{n_1}}\right)$$

$$\xi^r_{LR^{n_1}(LR^{n_1+1})^{n_2}}\colon \mu_L \geq -\mu_R\left(\frac{(A_L+\Phi^R_{n_2-1})a_L}{1+(A_L+\Phi^R_{n_2-1})}+\varphi^R_{n_1}\right)$$
$$= -\mu_R\left(\frac{\left(a_La_R^{n_1}+\dfrac{1-(a_La_R^{n_1+1})^{n_2-1}}{(1-a_La_R^{n_1+1})(a_La_R^{n_1+1})^{n_2-1}}\right)a_L}{1+a_La_R^{n_1}+\dfrac{1-(a_La_R^{n_1+1})^{n_2-1}}{(1-a_La_R^{n_1+1})(a_La_R^{n_1+1})^{n_2-1}}}+\frac{1-a_R^{n_1}}{(1-a_R)a_R^{n_1}}\right) \tag{31}$$

We remark that for the cycles of complexity level larger than one, we assume always $A_R \neq 1$ and $A_L \neq 1$ as such cases are considered separately in Secs. 2.5 and 3.3, related with the particular case $a_L = a_R = 1$ and the set (S), respectively.

To find the fixed point of the function

$$T_L^{n_2} \circ T_R = (f_R^{n_1} \circ f_L)^{n_2} \circ (f_R^{n_1+1} \circ f_L) \tag{32}$$

we start either from (16), (17) repeating the mechanism of substitution or, more simply, we take the formulas (29) and (30), exchange L and R and invert the inequalities to get the "symmetric" (in the symbols) case. So we have:

$$T_L(0) = M_L \geq y_0^*$$
$$= \frac{1}{1-A_L^{n_2}A_R}\left(M_RA_L^{n_2}+M_L\frac{1-A_L^{n_2}}{1-A_L}\right)$$
$$\geq 0 \tag{33}$$

and

$\xi^r_{LR^{n_1+1}(LR^{n_1})^{n_2}}$:

$$M_R \geq -M_L\frac{1-A_L^{n_2}}{(1-A_L)A_L^{n_2}} =: -M_L\Phi^L_{n_2} \tag{34}$$

$\xi^l_{LR^{n_1+1}(LR^{n_1})^{n_2}}$:

$$M_R \leq -M_L\left(A_R+\frac{1-A_L^{n_2-1}}{(1-A_L)A_L^{n_2-1}}\right)$$
$$=: -M_L(A_R+\Phi^L_{n_2-1}) \tag{35}$$

Eventually, using the expressions for the composite coefficients given in (27), we can obtain the bifurcation curves with respect to the original parameters. The resulting bifurcation curves for all four families of cycles with the second complexity level are shown in Fig. 5.

Note that the periodic point y_0^* detected in (28) as the fixed point of the map $T_R^{n_2} \circ T_L$, represents for the map f the fixed point of the function $(f_R^{n_1+1} \circ f_L)^{n_2} \circ (f_R^{n_1} \circ f_L)$, that means the leftmost periodic point of the cycle of period $(n_1+1)+n_2(n_1+2)$. In order to get its expression in the x-coordinate we have to consider the point $x_0^* = y_0^* + O'$ (in the example shown in Fig. 4 the periodic point x_0^* is denoted x_0).

Now, as already done at the beginning of Sec. 2.2, we can perform the analysis of the ranges of existence, we have:

$$\Phi^R_{n_2-1}+A_L \leq \frac{M_L}{-M_R} \leq \Phi^R_{n_2} \tag{36}$$

which gives disjoint intervals for any $n_2 \geq 1$ and the relation between this and the previous parameter is as follows: from the definition in (27) we have $a_LM_L = \mu_L + \mu_R\varphi^R_{n_1}$ so that

$$\frac{\mu_L}{-\mu_R} = \varphi^R_{n_1} + a_L\frac{\dfrac{M_L}{-M_R}}{1+\dfrac{M_L}{-M_R}} \tag{37}$$

and similarly for the second family we have intervals in the parameter

$$\Phi^L_{n_2-1}+A_R \leq \frac{-M_R}{M_L} \leq \Phi^L_{n_2} \tag{38}$$

that is

$$\frac{1}{\Phi^L_{n_2}} \leq \frac{M_L}{-M_R} \leq \frac{1}{\Phi^L_{n_2-1}+A_R} \tag{39}$$

and thus the two families of the second level constructed above give cycles in disjoint boxes, which are cumulating on the BCB curves at the boundaries of the two cycles considered initially, which means that inside the starting interval $]\varphi^R_{n_1}, \varphi^R_{n_1}+a_L[$ we have the first family of disjoint intervals which are approaching the right side as n_2 tends to ∞, and the second family of disjoint intervals which are approaching the left side as n_2 tends to ∞.

The BCB curves of the remaining two families of the second level of complexity can be calculated analogously. Instead of starting from two consecutive cycles of the first level with symbolic sequences LR^{n_1} and LR^{n_1+1} we start now

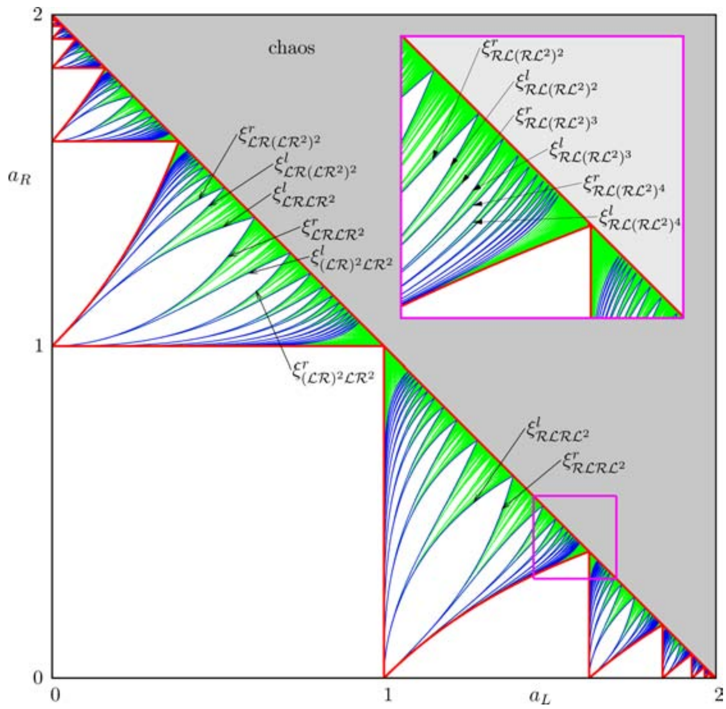


Fig. 5. BCB curves of complexity level two calculated analytically using Leonov's approach (blue). The inset shows the marked rectangle enlarged. Numerically calculated nested period adding bifurcation structures are shown in background (green) as well as the curves of the analytical curves of complexity level one (red). Parameter setting: $\mu_L = 1$, $\mu_R = -1$.

with the other two consecutive cycles of the first level with symbolic sequences RL^{n_1} and RL^{n_1+1} with parameter $\mu_L/-\mu_R$ belonging to the interval $]1/(\varphi^L_{n_1} + a_R), 1/\varphi^L_{n_1}[$.

Well, here we do not have to do much work, only notice that instead of reasoning on the map operator with the function $f_R^{n_1} \circ f_L$ on its L side and $f_R^{n_1+1} \circ f_L$ on the R side as previously done, we can consider as T_L the function $f_L^{n_1} \circ f_R(x)$ and as T_R the function $f_L^{n_1+1} \circ f_R(x)$ *which means that on the right side of the coefficients of the operator we have to exchange L and R.* Then we have:

$$\begin{aligned} A_L &= a_R a_L^{n_1} \\ A_R &= a_R a_L^{n_1+1} \\ M_L &= \frac{\mu_R}{a_R} + \mu_L \frac{\varphi^L_{n_1}}{a_R} \\ M_R &= \mu_L + M_L = \frac{\mu_R}{a_R} + \mu_L\left(1 + \frac{\varphi^L_{n_1}}{a_R}\right) \end{aligned} \tag{40}$$

Now it is only a matter of applications of the previous results, given the new composition of the coefficients. For the function $T_R^{n_2} \circ T_L = (f_L^{n_1+1} \circ f_R)^{n_2} \circ (f_L^{n_1} \circ f_R)$ which now represents the family of cycles with symbolic sequence $RL^{n_1}(RL^{n_1+1})^{n_2}$ we have formally the same equations as given above in (29) and (30):

$\xi^l_{RL^{n_1}(RL^{n_1+1})^{n_2}}$:

$$M_L \le -M_R \frac{1 - A_R^{n_2}}{(1 - A_R)A_R^{n_2}} =: -M_R \Phi^R_{n_2}$$

$\xi^r_{RL^{n_1}(RL^{n_1+1})^{n_2}}$:

$$\begin{aligned} M_L &\ge -M_R\left(A_L + \frac{1 - A_R^{n_2-1}}{(1 - A_R)A_R^{n_2-1}}\right) \\ &=: -M_R(A_L + \Phi^R_{n_2-1}) \end{aligned} \tag{41}$$

but now with the coefficients given in (40). These BCB curves can be rewritten as:

$\xi^l_{RL^{n_1}(RL^{n_1+1})^{n_2}}$:

$$\mu_L \le \frac{-\mu_R(1 + \Phi^R_{n_2})}{a_R \Phi^R_{n_2} + (1 + \Phi^R_{n_2})\varphi^L_{n_1}}$$

$\xi^r_{RL^{n_1}(RL^{n_1+1})^{n_2}}$:

$$\mu_L \ge \frac{-\mu_R(1 + (A_L + \Phi^R_{n_2-1}))}{a_R(A_L + \Phi^R_{n_2-1}) + (1 + (A_L + \Phi^R_{n_2-1}))\varphi^L_{n_1}} \tag{42}$$

For the function $T_L^{n_2} \circ T_R = (f_L^{n_1} \circ f_R)^{n_2} \circ (f_L^{n_1+1} \circ f_R)$ which now represents the family of cycles

$RL^{n_1+1}(RL^{n_1})^{n_2}$ we have formally the same equations as given above in (34) and (35):

$$\begin{aligned}
&\xi^r_{RL^{n_1+1}(RL^{n_1})^{n_2}}:\\
&\qquad M_R \geq -M_L \frac{1-A_L^{n_2}}{(1-A_L)A_L^{n_2}} =: -M_L \Phi^L_{n_2}\\
&\xi^l_{RL^{n_1+1}(RL^{n_1})^{n_2}}:\\
&\qquad M_R \leq -M_L\left(A_R + \frac{1-A_L^{n_2-1}}{(1-A_L)A_L^{n_2-1}}\right)\\
&\qquad\quad =: -M_L(A_R + \Phi^L_{n_2-1})
\end{aligned} \tag{43}$$

but now with the coefficients given in (40). These BCB curves can be rewritten as:

$$\begin{aligned}
&\xi^r_{RL^{n_1+1}(RL^{n_1})^{n_2}}: \mu_L \geq \frac{-\mu_R(1+\Phi^L_{n_2})}{a_R + (1+\Phi^L_{n_2})\varphi^L_{n_1}}\\
&\xi^l_{RL^{n_1+1}(RL^{n_1})^{n_2}}: \mu_L \leq \frac{-\mu_R(1+A_R+\Phi^L_{n_2-1})}{a_R + (1+A_R+\Phi^L_{n_2-1})\varphi^L_{n_1}}
\end{aligned} \tag{44}$$

Moreover, for any $n_2 \geq 1$, from the inequality

$$\Phi^R_{n_2-1} + A_L \leq \frac{M_L}{-M_R} \leq \Phi^R_{n_2} \tag{45}$$

and from (40) we get:

$$\frac{1}{\dfrac{a_R(A_L+\Phi^R_{n_2-1})}{1+(A_L+\Phi^R_{n_2-1})} + \varphi^L_{n_1}} \leq \frac{\mu_L}{-\mu_R} \leq \frac{1}{\dfrac{a_R\Phi^R_{n_2}}{1+\Phi^R_{n_2}} + \varphi^L_{n_1}} \tag{46}$$

and for the second one, from (44) we get:

$$\begin{aligned}
\frac{1}{\dfrac{a_R}{(1+\Phi^L_{n_2})} + \varphi^L_{n_1}} &\leq \frac{\mu_L}{-\mu_R}\\
&\leq \frac{1}{\dfrac{a_R}{(1+A_R+\Phi^L_{n_2-1})} + \varphi^L_{n_1}}
\end{aligned} \tag{47}$$

so that they belong to disjoint intervals inside the starting range $]1/(\varphi^L_{n_1} + a_R), 1/\varphi^L_{n_1}[$, such that the first family approaches the left boundary as n_2 tends to ∞, while the second family approaches the right boundary.

2.3. *Improvement of Leonov's technique*

We have reported above the proof given by Leonov for historical reasons. However, we can improve this technique and avoid the steps associated with the change of the coordinate system. In fact, the idea of this simplification is clearly visible in Fig. 4.

It is clear that the pieces of the function f^2 and f^3 close to the origin are in a situation similar to that of the original function, and indeed, for the parameter values used in Fig. 4 we can see that a periodic orbit of period-2 exists not only with points on the branches LR and LRR, but there also exists a periodic orbit of period-2 with one point on the branch LR and one point on the branch RLR (and clearly a periodic orbit of period-2 exists also with one point on the branch RL and one point on the branch RRL). Therefore, without changing the coordinate system, we can consider the two branches of functions which are immediately on the left and on the right of the origin. In general, considering any two consecutive cycles with symbolic sequence LR^{n_1} and LR^{n_1+1}, or equivalently with symbolic sequence LR^{n_1} and RLR^{n_1}, we consider the composite function $\tilde{T}_L(x) = f_R^{n_1} \circ f_L(x)$ on the left side of $x = 0$ and $\tilde{T}_R(x) = f_R^{n_1} \circ f_L \circ f_R(x)$ on the right side of $x = 0$:

$$x' = \tilde{T}(x) = \begin{cases} \tilde{T}_L(x) = A_L x + \tilde{M}_L, & \text{if } x < 0\\ \tilde{T}_R(x) = A_R x + \tilde{M}_R, & \text{if } x > 0 \end{cases} \tag{48}$$

with

$$\begin{aligned}
A_L &= a_L a_R^{n_1}\\
A_R &= a_L a_R^{n_1+1}\\
\tilde{M}_L &= \mu_L a_R^{n_1} + \mu_R \frac{1-a_R^{n_1}}{1-a_R}\\
\tilde{M}_R &= A_L \mu_R + \tilde{M}_L
\end{aligned} \tag{49}$$

As in the previous case, for $a_R = 1$ the term $(1-a_R^{n_1})/(1-a_R)$ must be replaced by n_1. Now, it is only a matter of applications of the results of the first level. By using the equality $x^* = \tilde{T}_R^{n_2} \circ \tilde{T}_L(x^*)$ we obtain the periodic point of f which is the first one on the left of the origin:

$$\begin{aligned}
\tilde{M}_R \leq x^* &= \frac{1}{1-A_R^{n_2}A_L}\left(\tilde{M}_L A_R^{n_2} + \tilde{M}_R \frac{1-A_R^{n_2}}{1-A_R}\right)\\
&\leq 0
\end{aligned} \tag{50}$$

(in the example shown in Fig. 4 this periodic point corresponds to x_2) and the BCB curves of the cycles obtained by using $\tilde{T}_R^{n_2} \circ \tilde{T}_L(x)$ with this first level for the map $\tilde{T}$ correspond to cycles of the second level for the map f. From (11) and (13) with obvious changes we obtain:

$$\begin{aligned}
&\xi^l_{LR^{n_1}(RLR^{n_1})^{n_2}}:\\
&\qquad \tilde{M}_L \le -\tilde{M}_R \frac{1-A_R^{n_2}}{(1-A_R)A_R^{n_2}} =: -\tilde{M}_R \Phi^R_{n_2}\\
&\xi^r_{LR^{n_1}(RLR^{n_1})^{n_2}}:\\
&\qquad \tilde{M}_L \ge -\tilde{M}_R \left(A_L + \frac{1-A_R^{n_2-1}}{(1-A_R)A_R^{n_2-1}}\right)\\
&\qquad\quad =: -\tilde{M}_R \left(A_L + \Phi^R_{n_2-1}\right)
\end{aligned} \tag{51}$$

which can be rewritten as:

$$\begin{aligned}
&\xi^l_{LR^{n_1}(RLR^{n_1})^{n_2}}:\\
&\qquad \mu_L \le -\mu_R \left(\frac{\Phi^R_{n_2} a_L}{1+\Phi^R_{n_2}} + \varphi^R_{n_1}\right)\\
&\xi^r_{LR^{n_1}(RLR^{n_1})^{n_2}}:\\
&\qquad \mu_L \ge -\mu_R \left(\frac{a_L(A_L+\Phi^R_{n_2-1})}{1+A_L+\Phi^R_{n_2-1}} + \varphi^R_{n_1}\right)
\end{aligned} \tag{52}$$

and the BCB curves of the second complexity level previously determined in (31) are the same as those determined now in (52) (without the change of variable). That is, the analytic expressions of $\xi^r_{LR^{n_1}(RLR^{n_1})^{n_2}}$ (respectively, $\xi^l_{LR^{n_1}(RLR^{n_1})^{n_2}}$) are the same as that of $\xi^r_{LR^{n_1}(LR^{n_1+1})^{n_2}}$ (respectively, $\xi^l_{LR^{n_1}(LR^{n_1+1})^{n_2}}$) representing the border-collision bifurcation of the same cycle of f, but computed referring to a different periodic point (x^* instead of y^*_0).

Similarly we can reason for all the other BCB curves of the second complexity level, avoiding the change of coordinate, and we know that the analytic expressions of the BCB curves will be the same as those given in (42) and (44).

As the BCB curves analytically determined are the same, also the same are comments related to the intervals of existence of such cycles.

2.4. *Level of complexity larger than two*

It is clear that the process described above can be repeated recursively, considering any pair of consecutive intervals of existence of cycles of the second complexity level. Between these intervals there is an empty space inside which one can repeat the same reasoning, obtaining two families of intervals of existence of cycles of the third complexity level, and so on.

So the third level of complexity comes in a similar way: given two consecutive functions involved in the second level, for example $\tilde{T}_L\tilde{T}_R^{n_2}$, and $\tilde{T}_L\tilde{T}_R^{n_2+1}$ (i.e. corresponding to the symbolic sequences $LR^{n_1}(RLR^{n_1})^{n_2}$ and $LR^{n_1}(RLR^{n_1})^{n_2+1}$) we consider the functions $T'_L = \tilde{T}_R^{n_2} \circ \tilde{T}_L$ on the left side of $x = 0$ and $T'_R = \tilde{T}_R^{n_2} \circ \tilde{T}_L \circ \tilde{T}_R$ on the right side. Assuming that $\tilde{T}_L$ has coefficients A_L and M_L and $\tilde{T}_R$ has coefficients A_R and M_R, we apply the same operator in (48)–(49), which we denote as T', substituting the coefficients A_L, M_L, A_R, M_R for a_L, μ_L, a_R, μ_R to obtain the coefficients A'_L, M'_L, A'_R, M'_R of the new composite map T':

$$x' = T'(x) = \begin{cases} T'_L(x) = A'_L x + M'_L, & \text{if } x < 0\\ T'_R(x) = A'_R x + M'_R, & \text{if } x > 0 \end{cases} \tag{53}$$

with

$$\begin{aligned}
A'_L &= A_L A_R^{n_2}\\
A'_R &= A_L A_R^{n_2+1}\\
M'_L &= M_L A_R^{n_2} + M_R \frac{1-A_R^{n_2}}{1-A_R}\\
M'_R &= A'_L M_R + M'_L\\
&= A_L A_R^{n_2} M_R + M_L A_R^{n_2} + M_R \frac{1-A_R^{n_2}}{1-A_R}
\end{aligned} \tag{54}$$

Then from the function

$$T'^{n_3}_R \circ T'_L(x) = A'^{n_3}_R A'_L x + M'_L A'^{n_3}_R + M'_R \frac{1-A'^{n_3}_R}{1-A'_R} \tag{55}$$

we obtain the periodic point of f which is the first on the left of the origin:

$$M'_R \le x^* = \frac{1}{1-A'^{n_3}_R A'_L} \times \left(M'_L A'^{n_3}_R + M'_R \frac{1-A'^{n_3}_R}{1-A'_R}\right) \le 0 \tag{56}$$

Eventually, we express the coefficients A'_L, M'_L, A'_R, M'_R in terms of a_L, μ_L, a_R, μ_R, n_3 and n_1 and obtain the expressions for the BCB curves of the family of cycles of the third level of complexity obtained from the function $T'^{n_3}_R \circ T'_L(x)$

for any $n_3 \geq 1$:

$$- M'_R(A'_L + \varphi^R_{n_3-1}) \leq M'_L \leq -M'_R\varphi^R_{n_3}$$
$$\text{with } \varphi^R_{n_3} = \frac{1 - A'^{n_3}_R}{(1 - A'_R)A'^{n_3}_R} \quad (57)$$

This family of orbits with complexity level three corresponds to symbolic sequences $LR^{n_3}(RLR^{n_3})^{n_2}(RLR^{n_3}LR^{n_3}(RLR^{n_3})^{n_2})^{n_1}$ or, written in a more compact way (which is equivalent to the previous form up to a cyclical shift) $LR^{n_3}(LR^{n_3+1})^{n_2}(LR^{n_3}(LR^{n_3+1})^{n_2+1})^{n_1}$. The second family of complexity level three is obtained exchanging L and R so that the symbolic sequences of this family are $(RL^{n_3})(RL^{n_3+1})^{n_2}((RL^{n_3})(RL^{n_3+1})^{n_2+1})^{n_1}$. For the calculation of the regions of existence the inequalities must be inverted, that is, the periodic point and the BCB curves of this family of cycles of complexity level three, obtained in the form $T'^{n_3}_L \circ T'_R(x)$ for any integer $n_3 \geq 1$, are given by:

$$M'_L \geq x^* = \frac{1}{1 - A'^{n_3}_L A'_R} \times \left(M'_R A'^{n_3}_L + M'_L \frac{1 - A'^{n_3}_L}{1 - A'_L} \right) \geq 0 \quad (58)$$

$$-M'_L(A'_R + \varphi^L_{n_3-1}) \geq M'_R \geq -M'_L\varphi^L_{n_3}$$
$$\text{with } \varphi^L_{n_3} = \frac{1 - A'^{n_3}_L}{(1 - A'_L)A'^{n_3}_L} \quad (59)$$

Similarly, the other families of the complexity level three can be obtained, as for example, the families with symbolic sequences $LR^{n_3}(LR^{n_3+1})^{n_2+1}(LR^{n_3}(LR^{n_3+1})^{n_2})^{n_1}$ and $LR^{n_3+1}(LR^{n_3})^{n_2+1}(LR^{n_3+1}(LR^{n_3})^{n_2})^{n_1}$. So from the four families of the second level obtained in the previous sections, we obtain eight families of the third level of complexity. As one can see, this process can be continued iteratively. It is clear that the number of families is doubled from one level to the consecutive one.

As we have already noticed in the Introduction, as long as we are in the regular regime, we can have stable cycles of any period. Some of them may even have the same period. However, it is not true that for any given set of parameter values we are inside a periodicity region or on its boundary, and thus, that a stable cycle always exists. In fact, each periodicity region is a limit set of other periodicity regions, but also the union of all the existing periodicity regions does not cover the regular parameter region completely. Some points are left (the complementary set, which is a set of zero measure in the parameters space) at which we have no cycles, but quasiperiodic trajectories.

2.5. *Particular case $a_R = a_L = 1$*

In this section, we consider the map (1) in the particular case $a_R = a_L = 1$. The results are already stated in the previous sections, so we only summarize the dynamic behavior of the simple map:

$$x' = f(x) = \begin{cases} f_L(x) = x + \mu_L, & \text{if } x < 0 \\ f_R(x) = x + \mu_R, & \text{if } x > 0 \end{cases} \quad (60)$$

It can be easily shown by simply looking at the shape of the system function that in the following three cases: (i) $\mu_L < 0, \mu_R < 0$, (ii) $\mu_L < 0, \mu_R > 0$, (iii) $\mu_L > 0, \mu_R > 0$ the only possible asymptotic behavior is divergent. Obviously, in the degenerate case $\mu_L = 0$ and $\mu_R = 0$ each point x is a fixed point. For $\mu_L = 0$, $\mu_R \neq 0$ each point $x < 0$ is a fixed point and for the initial values $x > 0$ there are two possibilities: for $\mu_R < 0$ each initial value $x > 0$ converges in a finite number of steps to one of the fixed points on the interval $(\mu_R, 0)$, and for $\mu_R > 0$ each initial value $x > 0$ diverges. In the case $\mu_R = 0$, $\mu_L \neq 0$ the situation is vice versa. Hence, the only region of periodic dynamics is

$$\mu_L > 0, \quad \mu_R < 0 \quad (61)$$

Obviously no fixed point exists in this case (as both the slopes are equal to one) and the invariant interval $I = (\mu_R, \mu_L)$ is globally absorbing.

Then considering the periodic orbits corresponding to the symbolic sequences LR^{n_1} of complexity level one we have

$$f^{n_1}_R \circ f_L(x) = x + \mu_L + n_1\mu_R \quad (62)$$

and all the points inside I are periodic of period $n_1 + 1$ (and eventually periodic outside I) when

$$\mu_L + n_1\mu_R = 0 \quad (63)$$

More generally, given a periodic orbit of any complexity level with $p \geq 1$ distinct points on the left side and $q \geq 1$ distinct points on the right side, we have

$$f^{p+q}(x) = f^p_R \circ f^q_L(x) = x + p\mu_L + q\mu_R \quad (64)$$

thus all the points inside I are of period $p + q$ (and eventually periodic outside I) when $p\mu_L + q\mu_R = 0$, that is when we have

$$\frac{\mu_L}{-\mu_R} = \frac{p}{q} \quad (65)$$

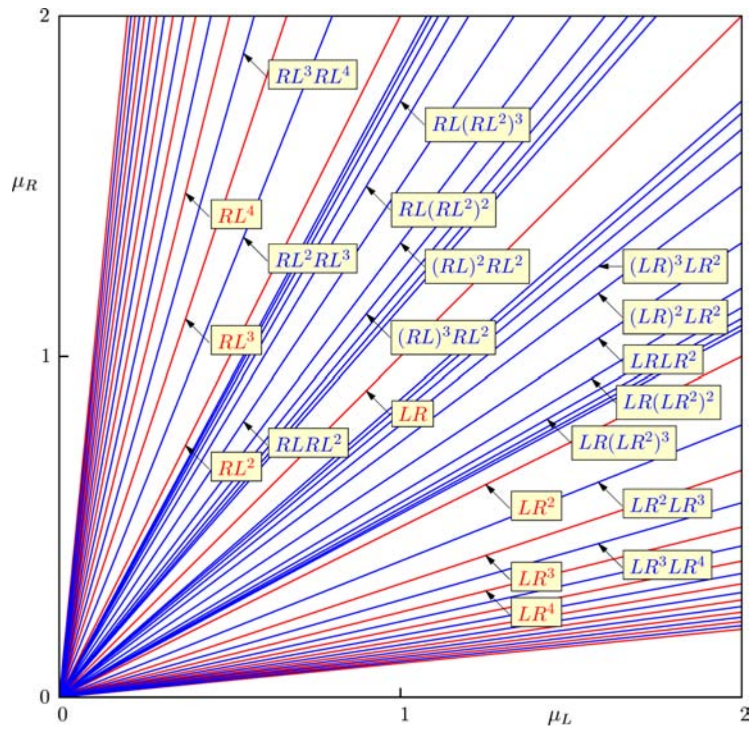


Fig. 6. Straight lines in the plane (μ_L, μ_R) corresponding to some orbits of the complexity levels one and two for $a_L = a_R = 1$.

This leads us to the following

Proposition 1. *Let $a_R = a_L = 1$, $\mu_L > 0$, $\mu_R < 0$ in (1) and $\rho = \mu_L/-\mu_R$. Then*

- *if ρ is a rational number $\rho = p/q$ where p and q are integers with no common divisors, then each orbit starting in the globally absorbing invariant interval $I = (-\mu_R, \mu_L)$ is periodic with the period $(p+q)$, whereby p points of the orbit are located on the left side and q points on the right side.*
- *if ρ is an irrational number then each orbit in I is quasiperiodic and dense in I.*

Thus in the parameter plane $(-\mu_R, \mu_L)$ the first quadrant (see Fig. 6) is filled with a cone of straight lines $\mu_L = -\rho\mu_R$. To each of these lines corresponds an absorbing interval I in the state space filled with periodic orbits when ρ is rational, or quasiperiodic orbits when ρ is irrational.

3. Further Applications

3.1. *Rotation numbers*

To find the boundaries of the BCB curves we have followed a recursive rule. Between any pair of consecutive cycles associated with the complexity level n there are two families of infinitely many (countable) periodicity regions of the complexity level $n+1$. In order to distinguish between different cycles of the same period we can associate a number to each periodic orbit, called *rotation number*. Following this idea a periodic orbit of period k is characterized not only by the period but also by the number of points in the two partitions separated by the discontinuity point, here $x = 0$, denoted by L and R, respectively. So, we say that a cycle of period $k = (p+q)$ has a rotation number p/q if it has p points on the left side (represented by the symbol L) and the other q points on the right side (represented by the symbol R). Obviously, the eigenvalue of such a cycle is $\lambda = a_L^p a_R^q$. As one can see, the cycles of complexity level one corresponding to the symbolic sequences LR^{n_1} and RL^{n_1} have the rotation numbers

$$\rho(LR^{n_1}) = \frac{1}{n_1} \quad \text{and} \quad \rho(RL^{n_1}) = \frac{n_1}{1} \tag{66}$$

Then between any pair of adjacent regions of first level, associated for example with the symbolic sequences LR^{n_1}, LR^{n_1+1} and hence with the rotation numbers $1/n_1$ and $1/(n_1+1)$, we have constructed two infinite families of regions of the complexity level two. The rotation numbers for the cycles of the family of level two with the symbolic sequence $LR^{n_1}(LR^{n_1+1})^{n_2}$ are given by

$$\rho(LR^{n_1}(LR^{n_1+1})^{n_2}) = \frac{n_2+1}{n_1n_2+n_1+n_2} \tag{67}$$

and the rotation numbers of the cycles with the symbolic sequence $LR^{n_1+1}(LR^{n_1})^{n_2}$ are given by

$$\rho(LR^{n_1+1}(LR^{n_1})^{n_2}) = \frac{n_2+1}{n_1n_2+n_1+1} \tag{68}$$

The values given by Eqs. (67) and (68) can be calculated easily by counting the letters L and R in the corresponding symbolic sequences. However, it is also possible to obtain these values in a more general way, by expressing them in terms of the rotation numbers of the previous level of complexity. Using the well-known Farey-addition rule

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d} \tag{69}$$

we can write

$$\begin{aligned}
&\rho(LR^{n_1}) \oplus \rho(LR^{n_1+1})\\
&\quad = \frac{1}{n_1} \oplus \frac{1}{n_1+1}\\
&\quad = \frac{2}{2n_1+1} = \rho(LR^{n_1}LR^{n_1+1})\\
&\underbrace{\rho\left(LR^{n_1}\right) \oplus \rho\left(LR^{n_1+1}\right)}_{\rho\left(LR^{n_1}LR^{n_1+1}\right)} \oplus \rho(LR^{n_1+1})\\
&\quad = \frac{2}{2n_1+1} \oplus \frac{1}{n_1+1}\\
&\quad = \frac{3}{3n_1+2} = \rho(LR^{n_1}(LR^{n_1+1})^2)
\end{aligned} \tag{70}$$

and so on, so that in general we have

$$\begin{aligned}
&\rho(LR^{n_1}(LR^{n_1+1})^{n_2})\\
&\quad = \rho(LR^{n_1}) \oplus (\rho(LR^{n_1+1}) \otimes n_2)\\
&\rho(LR^{n_1+1}(LR^{n_1})^{n_2})\\
&\quad = \rho(LR^{n_1+1}) \oplus (\rho(LR^{n_1}) \otimes n_2)
\end{aligned} \tag{71}$$

where $\otimes$ denotes a repeated application of the Farey-addition (similar to multiplication with a natural number) defined as follows:

$$\frac{a}{b} \otimes n = \underbrace{\frac{a}{b} \oplus \cdots \oplus \frac{a}{b}}_{n \text{ times}} = \frac{a \cdot n}{b \cdot n} \tag{72}$$

Note also that the result of the application of the operator $\otimes$ represents only an auxiliary value which is not associated with any periodic orbit and cannot be considered as a usual fraction (especially, it must not be reduced). However, applied in Eq. (71) in a combination with $\oplus$, it leads to irreducible fractions:

$$\begin{aligned}
&\rho(LR^{n_1}) \oplus (\rho(LR^{n_1+1}) \otimes n_2)\\
&\quad = \frac{1}{n_1} \oplus \left(\frac{1}{n_1+1} \otimes n_2\right)\\
&\quad = \frac{1}{n_1} \oplus \frac{n_2}{(n_1+1)n_2}\\
&\quad = \frac{n_2+1}{n_1n_2+n_1+n_2}\\
&\quad = \rho(LR^{n_1}(LR^{n_1+1})^{n_2})
\end{aligned} \tag{73}$$

$$\begin{aligned}
&\rho(LR^{n_1+1}) \oplus (\rho(LR^{n_1}) \otimes n_2)\\
&\quad = \frac{1}{n_1+1} \oplus \left(\frac{1}{n_1} \otimes n_2\right)\\
&\quad = \frac{1}{n_1} \oplus \frac{n_2}{n_1\, n_2} = \frac{n_2+1}{n_1n_2+n_1+1}\\
&\quad = \rho(LR^{n_1+1}(LR^{n_1})^{n_2})
\end{aligned} \tag{74}$$

It is not difficult to see that the resulting values are organized as follows

$$\begin{aligned}
\rho(LR^{n_1+1}) &< \rho(LR^{n_1}(LR^{n_1+1})^{n_2+1})\\
&< \rho(LR^{n_1}(LR^{n_1+1})^{n_2})\\
&< \cdots < \rho(LR^{n_1+1}LR^{n_1})\\
&< \cdots < \rho(LR^{n_1+1}(LR^{n_1})^{n_2})\\
&< \rho(LR^{n_1+1}(LR^{n_1})^{n_2+1})\\
&< \rho(LR^{n_1})
\end{aligned} \tag{75}$$

that means, for each fixed n_2 the families of rotation numbers given by Eq. (71) are bounded in the interval $[\rho(LR^{n_1+1}), \rho(LR^{n_1})]$ and also that the two sequences of rotation numbers are accumulating at the boundaries of the interval, since

$$\begin{aligned}
&\lim_{n_2\to\infty} \rho(LR^{n_1}(LR^{n_1+1})^{n_2}) = \frac{1}{n_1+1} = \rho(LR^{n_1+1})\\
&\lim_{n_2\to\infty} \rho(LR^{n_1+1}(LR^{n_1})^{n_2}) = \frac{1}{n_1} = \rho\left(LR^{n_1}\right)
\end{aligned} \tag{76}$$

Clearly, this reasoning can be repeated for further complexity levels. Especially, between any pair of adjacent regions of complexity level two, for example, having the rotation numbers $\rho' = (n_2+1)/(n_1n_2+n_1+n_2)$ and $\rho'' = (n_2+2)/(n_1(n_2+1)+n_1+n_2+1)$ there exist two infinite families of periodicity regions of complexity level three with the rotation numbers $\rho' \oplus (\rho'' \otimes n_3)$ and $\rho'' \oplus (\rho' \otimes n_3)$ located between ρ' and ρ''. This self-similar process can be continued *ad infinitum* and results in the well-known devil's staircase structure of the rotation numbers.

3.2. *Change of stability*

As we can see in Figs. 3 and 5, all the BCB curves bounding the existence regions of the cycles intersect at points which belong to the set (S) defined by Eq. (4). For parameter values on one side of (S) all the cycles are stable (globally attracting) and their periodicity regions are nonoverlapping. On the other side of (S) the regions are overlapping and all the cycles are unstable. This region represents the domain of robust chaos: for any point in the parameters space outside of (S) there exist infinitely many unstable cycles and if the trajectories are bounded, then they belong to a chaotic attractor consisting of one or more intervals (bands).

Now we prove the following

Proposition 2. *For a periodic orbit of any complexity level the following holds*:

(1) *One of the intersection points of two BCB curves confining the existence region of this orbit belongs to the set* (S).
(2) *At this intersection point, the orbit changes its stability.*

Proof. First let us consider the bifurcation curves for the first family of orbits of complexity level one, with symbolic sequence LR^{n_1}. From (11) and (13) for each $n_1 \geq 1$, the two BCB curves:

$$\xi^l_{LR^{n_1}}\colon \mu_L = -\mu_R \varphi^R_{n_1} \tag{77}$$

$$\xi^r_{LR^{n_1}}\colon \mu_L = -\mu_R(a_L + \varphi^R_{n_1-1}) \tag{78}$$

intersect at the parameter values where the eigenvalue $\lambda_{1,n_1} = a_L a^n_R$ of the cycle LR^{n_1} is equal to one, and hence the cycle changes its stability. In fact, the equation

$$a_L + \varphi^R_{n_1-1} = \varphi^R_{n_1} \tag{79}$$

leads to

$$a_L = \frac{1-a^{n_1}_R}{(1-a_R)a^{n_1}_R} - \frac{1-a^{n_1-1}_R}{(1-a_R)a^{n_1-1}_R} = \frac{1}{a^{n_1}_R} \tag{80}$$

that means

$$a_L a^{n_1}_R = 1 \tag{81}$$

Moreover, it is easy to see that for any $n_1 > 0$, the intersection point of each pair of BCB curves (77) and (78), given by

$$\begin{aligned}(a_L, \mu_L) &= (a_L, -\mu_R \varphi^R_{n_1})\\ &= \left(\frac{1}{a^n_R}, -\mu_R \frac{1-a^{n_1}_R}{(1-a_R)a^{n_1}_R}\right)\end{aligned} \tag{82}$$

belongs to the set (S), i.e. satisfies Eq. (4). In fact, using (81) we get from (82)

$$\begin{aligned}\mu_L &= -\mu_R \frac{1-a^{n_1}_R}{(1-a_R)a^{n_1}_R}\\ &= -\mu_R \frac{a_L(1-a^{n_1}_R)}{(1-a_R)}\\ &= -\mu_R \frac{(a_L-1)}{(1-a_R)}\end{aligned}$$

so that

$$\mu_L(1-a_R) = \mu_R(1-a_L) \tag{83}$$

Similarly, for any $n_1 > 0$, the intersection of each pair of BCB curves for the second family of orbits of complexity level one, corresponding to the symbolic sequence RL^{n_1}, given by Eqs. (16) and (17) leads to $\lambda_{n_1,1} = a_R a^{n_1}_L = 1$ and $\mu_R(1-a_L) = \mu_L(1-a_R)$ that means the intersection point belongs to the set (S) also in this case. Hence, the proposition is proved for the cycles of the first level of complexity.

Next we have to demonstrate that the same results hold also for complexity levels larger than one. As an example, let us consider the family of cycles of the second complexity level with symbolic sequences $LR^{n_1}(LR^{n_1+1})^{n_2}$. As these cycles represent the cycles of the *first* complexity level for the map (48) we can apply Eq. (81) for this map and obtain

$$A_L A^{n_2}_R = 1 \tag{84}$$

where the coefficients A_L and A_R are defined by Eq. (49). Inserting (49) into Eq. (84) we obtain immediately

$$(a_L a^{n_1}_R)(a_L a^{n_1+1}_R)^{n_2} = 1 \tag{85}$$

As the eigenvalue of a cycle with the symbolic sequence $LR^{n_1}(LR^{n_1+1})^{n_2}$ is given by $(a_L a^{n_1}_R)(a_L a^{n_1+1}_R)^{n_2}$, we conclude that Eq. (85) represents the condition that this cycle changes its stability. Obviously, the same reasoning can be applied for any other family of cycles of complexity level two and then iteratively for each next complexity level.

The proof that the intersection point of the BCB curves for any complexity level belongs to the set (S) is similar. For the same family $LR^{n_1}(LR^{n_1+1})^{n_2}$ of complexity level two we have

$$\tilde{M}_L(1 - A_R) = \tilde{M}_R(1 - A_L) \tag{86}$$

because we can consider again this family as a family of complexity level one for the map (48). From Eq. (86) we obtain, by using the definition of $\tilde{M}_R$ given by Eq. (49),

$$\tilde{M}_L(1 - A_R) = (A_L\mu_R + \tilde{M}_L)(1 - A_L)$$

and therefore

$$\tilde{M}_L(A_L - A_R) = A_L\mu_R(1 - A_L)$$

Then, by using $A_R = A_La_R$, we have

$$\tilde{M}_L(1 - a_R) = \mu_R(1 - A_L)$$

and eventually by substituting the definitions of $\tilde{M}_L$ and A_L we obtain

$$\left(\mu_La_R^{n_1} + \mu_R\frac{1 - a_R^{n_1}}{1 - a_R}\right)(1 - a_R) = \mu_R(1 - a_La_R^{n_1})$$

$$\mu_La_R^{n_1}(1 - a_R) + \mu_R(1 - a_R^{n_1}) = \mu_R(1 - a_La_R^{n_1})$$

$$\mu_La_R^{n_1}(1 - a_R) = \mu_R(a_R^{n_1} - a_La_R^{n_1})$$

$$\mu_L(1 - a_R) = \mu_R(1 - a_L)$$

that means the definition of the set (S). Iteratively the proof can be continued for each next complexity level. ∎

Note also that above the set (S) the *equations* for the BCB curves related to the existence of cycles are valid also but in this case the *inequalities* must be inverted. To demonstrate that let us consider the family of orbits of complexity level one corresponding to the symbolic sequences LR^n. In fact, the basic condition given by Eq. (9) is valid in any case (below and above the set (S)). However, the sign of the denominator $(1 - a_R^{n_1}a_L)$ in Eq. (9) changes at the boundary between the regular and the chaotic domains, as proven by Proposition 2. As long as we are in the regular domain the denominator is positive and the inequalities given in (11) and (13) hold. By contrast, in the chaotic domain the denominator $(1 - a_R^{n_1}a_L)$ is negative and therefore the inequalities (11) and (13) must be inverted. As an example, let us consider the periodicity region associated with the period-3 cycle LRR shown in Fig. 7. For parameters inside the stability region (shown red) the curve ξ^r_{LRR} represents the upper boundary of the region and the curve ξ^l_{LRR} the lower boundary. For parameters in the region

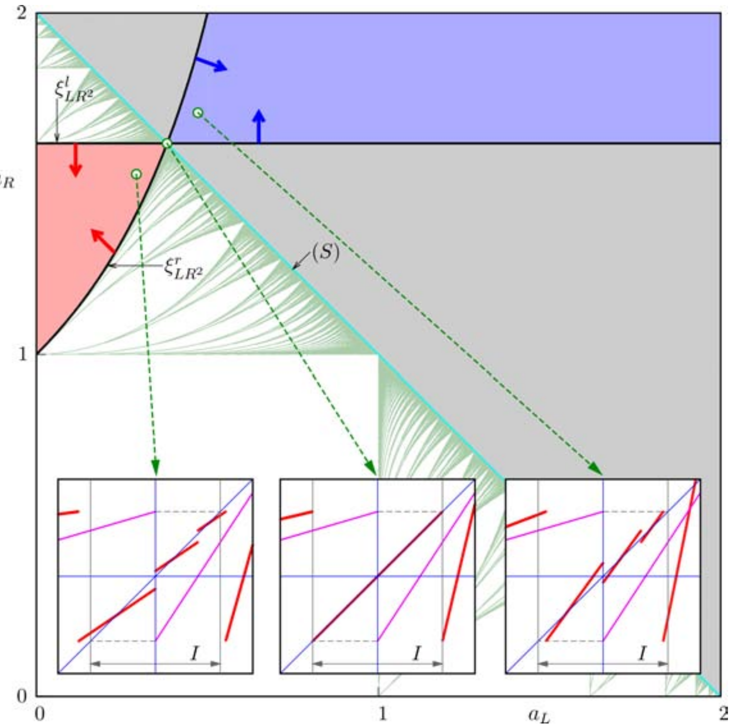


Fig. 7. Regions in the parameter plane (a_L, a_R) where the period-3 orbit LRR is stable (red), respectively unstable (blue). The insets show the system function and its third iterate at the marked points in the parameter space. Parameter setting: $\mu_L = 1$, $\mu_R = -1$.

(shown blue) the boundaries are exchanged. The insets show the graph of the map f in the absorbing interval I and the graph of the map f^3, whose fixed points define the stable 3-cycle. When the parameters belong to the set (S) then the three pieces of the map f^3 inside I coincide with the diagonal, covering the complete interval I so that all the points in I are periodic with period three.

The same reasoning holds also for all other orbits of any complexity level.

3.3. *Dynamics on the locus* (S)

Next the question arises, what is the dynamic behavior when a parameter point belongs to the set (S)? The property that we have seen in the example in Fig. 7 holds for any periodicity region. In fact, we prove the following

Proposition 3. *When the parameters belong to the set* (S) *at the intersection of a pair of BCB curves for a cycle with a given symbolic sequence, then each point of the absorbing interval* $I = [\mu_R, \mu_L]$ *is periodic and belongs to a cycle with the same symbolic sequence and hence with the same period and rotation number.*

Proof. Let us consider a point of the parameter space belonging to the set (S) which is the intersection point of a pair of BCB curves belonging to the first complexity level, associated with the symbolic sequence LR^{n_1}. Then considering the function $f_R^{n_1} \circ f_L$ we can show that

$$f_R^{n_1} \circ f_L(x) = x, \quad \forall\, n_1 > 0,\ \forall\, x \in (\mu_R, 0) \subset I \tag{87}$$

That means, the function $f_R^{n_1} \circ f_L$ for any n_1 is the identity map in $(\mu_R, 0)$ This implies that in the phase space, *inside the absorbing interval* $I = [\mu_R, \mu_L]$ *all the points are periodic with the same symbolic sequence*, because all functions $f_R^m \circ f_L \circ f_R^{n_1-m}$ are also identity maps in their corresponding subintervals of I.

To prove the identity (87) let us consider the expression given in (8). Then for any $x \in (\mu_R, 0)$ we have:

$$f_R^{n_1} \circ f_L(x) = (a_R^{n_1} a_L)x + \mu_L a_R^{n_1} + \mu_R \frac{1 - a_R^{n_1}}{1 - a_R} \tag{88}$$

Then from (77) and (81) we get

$$f_R^{n_1} \circ f_L(x) = (a_R^{n_1} a_L)x + \mu_L a_R^{n_1} - \mu_L a_R^{n_1} = x \tag{89}$$

Similarly, when we consider a point on (S) at the intersection of BCB curves for the second family of cycles of complexity level one (exchanging L and R) we have the same behavior due to the identity

$$f_L^{n_1} \circ f_R(x) = x, \quad \forall\, n_1 > 0,\ \forall\, x \in (0, \mu_L) \tag{90}$$

For the cycles associated with families of complexity two we can argue in the same way as before. For the corresponding composite map we will get the identity similar to Eq. (87). Hence, for this map each point of the absorbing interval is periodic and belongs to a cycle with the same symbolic sequence. Hereby this symbolic sequence belongs to complexity level one. As each point of an orbit of the composite map corresponds by definition to a sequence of points of the original map, all points of the absorbing interval are periodic for the original map too. In the same manner, the reasoning can be continued for all complexity levels larger than two. So the proof is complete. ■

If the point on the locus (S) is not related with a periodicity region, then the trajectories in I are quasiperiodic, dense in I and associated with an irrational rotation number. Hence, if the parameters belong to the set (S) then the map in the invariant interval I is topologically conjugated with a rotation of the circle: all the points are periodic if the rotation number is rational and quasiperiodic otherwise.

4. Summary

In this work, we recalled the technique introduced 50 years ago by Leonov for the calculation of border-collision bifurcation curves in piecewise-linear discontinuous 1D maps. Applied to the nested period adding bifurcation structure, this technique makes extensive use of the self-similarity of this bifurcation structure and allows to obtain analytical expressions for bifurcations involving periodic orbits with arbitrary large period and (at least in principle) complexity. Furthermore, we improved the technique proposed in the original work by Leonov and demonstrated that the shift of coordinates which was necessary in his work can in fact be avoided.

Additionally, we demonstrated that Leonov's approach can be used not only for the calculation of BCB curves. Due to its recursive nature, it is perfectly suited for proofs by induction of certain properties of the infinite nested period adding structure.

References

Avrutin, V. & Schanz, M. [2006] "On multi-parametric bifurcations in a scalar piecewise-linear map," *Nonlinearity* **19**, 531–552.

Avrutin, V., Schanz, M. & Banerjee, S. [2006] "Multi-parametric bifurcations in a piecewise-linear discontinuous map," *Nonlinearity* **19**, 1875–1906.

Avrutin, V. & Schanz, M. [2008] "On the fully developed bandcount adding scenario," *Nonlinearity* **21**, 1077–1103.

Avrutin, V., Eckstein, B. & Schanz, M. [2008a] "The bandcount increment scenario. I: Basic structures," *Proc. Roy. Soc. A* **464**, 1867–1883.

Avrutin, V., Eckstein, B. & Schanz, M. [2008b] "The bandcount increment scenario. II: Interior structures," *Proc. Roy. Soc. A* **464**, 2247–2263.

Avrutin, V. & Schanz, M. [2009] "Self-similarity of the bandcount adding: Calculation by map replacement," *Proc. Int. Workshop on Nonlinear Maps and Their Applications* (*NOMA'09*), Urbino, Italy, pp. 62–66.

Avrutin, V., Eckstein, B. & Schanz, M. [2009] "The bandcount increment scenario. III: Deformed structures," *Proc. Roy. Soc. A* **465**, 41–57.

Banerjee, S. & Grebogi, C. [1999] "Border collision bifurcation in two-dimensional piecewise smooth maps," *Phys. Rev. E* **59**, 4052–4061.

Banerjee, S., Karthik, M. S., Yuan, G. & Yorke, J. A. [2000a] "Bifurcations in one-dimensional piecewise smooth maps — Theory and applications in switching circuits," *IEEE Trans. Circ. Sys.* **47**, 389–394.

Banerjee, S., Ranjan, P. & Grebogi, C. [2000b] "Bifurcations in two-dimensional piecewise smooth maps — Theory and applications in switching circuits," *IEEE Trans. Circ. Syst.-I* **47**, 633–643.

di Bernardo, M., Feigin, M. I., Hogan, S. J. & Homer, M. E. [1999] "Local analysis of C-bifurcations in n-dimensional piecewise smooth dynamical systems," *Chaos Solit. Fract.* **10**, 1881–1908.

Fournier-Prunaret, D., Mira, C. & Gardini, L. [1997] "Some contact bifurcations in two-dimensional examples," *Grazer Mathematishe Berichte*.

Keener, J. P. [1980] "Chaotic behavior in piecewise continuous difference equations," *Trans. Amer. Math. Soc.* **261**, 589–604.

Leonov, N. N. [1959] "Map of the line onto itself," *Radiofisika* **2**, 942–956 (in Russian).

Leonov, N. N. [1962] "Discontinuous map of the straight line," *Dokl. Akad. Nauk SSSR* **143**, 1038–1041.

Maistrenko, Y. L., Maistrenko, V. L. & Chua, L. O. [1993] "Cycles of chaotic intervals in a time-delayed Chua's circuit," *Int. J. Bifurcation and Chaos* **3**, 1557–1572.

Maistrenko, Y. L., Maistrenko, V. L., Vikul, S. I. & Chua, L. O. [1995] "Bifurcations of attracting cycles from time-delayed Chua's circuit," *Int. J. Bifurcation and Chaos* **5**, 653–671.

Maistrenko, Y. L., Maistrenko, V. L. & Vikul, S. I. [1998] "On period-adding sequences of attracting cycles in piecewise linear maps," *Chaos Solit. Fract.* **9**, 67–75.

Mira, C. [1978] "Sur les structure des bifurcations des diffeomorphisme du cercle," *C. R. Acad. Sc. Paris Series A* **287**, 883–886.

Mira, C. [1987] *Chaotic Dynamics* (World Scientific, Singapore).

Mira, C., Gardini, L., Barugola, A. & Cathala, J.-C. [1996] *Chaotic Dynamics in Two-Dimensional Noninvertible Maps*, World Scientific Series on Nonlinear Science, Vol. 20 (World Scientific, Singapore).

Nusse, H. E. & Yorke, J. A. [1992] "Border-collision bifurcations including 'period two to period three' for piecewise smooth maps," *Physica D* **57**, 39–57.

Nusse, H. E. & Yorke, J. A. [1995] "Border-collision bifurcations for piecewise smooth one dimensional maps," *Int. J. Bifurcation and Chaos* **5**, 189–207.

Zhusubaliyev, Zh. T., Mosekilde, E., Maity, S. M., Mohanan, S. & Banerjee, S. [2006] "Border collision route to quasiperiodicity: Numerical investigation and experimental confirmation," *Chaos* **16**, 023122.

Zhusubaliyev, Zh. T., Soukhoterin, E. & Mosekilde, E. [2007] "Quasiperiodicity and torus breakdown in power electonic dc/dc converter," *Math. Comput. Simul.* **73**, 364–377.