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Snap-back Repellers in Non-smooth Functions

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Abstract—In this work we consider the homoclinic bifurcations of expanding periodic points. After Marotto, when homoclinic orbits to expanding periodic points exist, the points are called snap-back-repellers. Several proofs of the existence of chaotic sets associated with such homoclinic orbits have been given in the last three decades. Here we propose a more general formulation of Marotto's theorem, relaxing the assumption of smoothness, considering a generic piecewise smooth function, continuous or discontinuous. An example with a two-dimensional smooth map is given and one with a two-dimensional piecewise linear discontinuous map.

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1. INTRODUCTION

In the last decades, the dynamics associated with homoclinic orbits have been studied and developed by many researchers. We recall that the first results date back to Poincaré, who can be considered the founder of the qualitative theory of dynamical systems and the first who discovered the existence of chaotic dynamics. Numerous other studies have been done in the second half of the last century, mainly on the homoclinic orbits associated with saddle cycles in the study of flows. We recall, in addition to Birkhoff [1], Andronov and Pontryagin [2], Moser [3], the works of Abraham and Marsden [4], Smale [5], Afraimovich [6], Gavrilov and Shilnikov [7, 8], Guckenheimer and Holmes [9], Palis and Takens [10], Wiggins [11, 12], Kuznetsov [13], de Melo and van Strien [14], Gonchenko et al. [15–19]. Besides the transverse homoclinic and heteroclinic points of saddle cycles, also homoclinic orbits and chaotic dynamics associated with saddle-foci cycles have been studied, in particular, by Shilnikov and his collaborators. We also recall [20, 21] and the surveys books [22, 23]. New kinds of homoclinic orbits in maps with knot points are also shown in [24].

The homoclinic bifurcations occurring in flows are generally studied by use of discrete maps associated with a so-called Poincaré section, and thus are related to invertible maps. It follows that no homoclinic orbit can be associated with unstable foci or unstable nodes in a two-dimensional invertible map, or in general with repelling cycles in $n-$dimensional invertible maps, $n \geqslant 2$. However, in noninvertible maps we may have homoclinic orbits also associated with repelling or expanding cycles. A repelling cycle p is here one for which the Jacobian matrix, assuming the map smooth in p, has all the eigenvalues higher than 1 in modulus. Let p be a repelling cycle of a noninvertible map, in which the map is locally invertible and thus for the local inverse p is attracting. Marotto was the first to prove in [25] that homoclinic orbits may occur also for such repelling points, and that chaos is associated to the existence of homoclinic orbits. Indeed, his first work included a minor technical mistake, and he himself gave a corrected version in [26], after the appearance of several papers which, trying to correct the mistake, were providing less general

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proofs (as in Li and Chen [27]). The problem in the proof was related to the fact that a repelling fixed point is not necessarily expanding, where p is expanding only if

$$||f(x) - f(y)|| > s||x - y|| \tag{1.1}$$

for some $s > 1$ and for all the points $x \neq y$ in a suitable neighborhood of p, $B(p)$. However, as proved by Hirsh and Smale in [28], *a repelling fixed point is also locally expanding in a suitable norm* (although not necessarily in the usual Euclidean norm). Thus, for a differentiable noninvertible map $f : R^n \to R^n$, Marotto gave the following

Definition. *A fixed point p of f is a snap-back repeller if all the eigenvalues of $J_f(p)$ exceed 1 in magnitude, and there exists a point x_0 in a repelling neighborhood of p, and an integer m such that $x_m = p$ and $det(J_f(x_j)) \neq 0$ for $1 \leqslant j \leqslant m$, where $x_j = f^j(x_0)$.*

Clearly, a "repelling neighborhood" in the definition is one in which the map is locally invertible and the local inverse, say f_1^{-1}, is attracting in p, and the point x_0 in the definition is a homoclinic point. An example of full homoclinic orbit $\mathcal{O}(p)$ consists in taking the preimages of x_0 with the local inverse in p, and its forward iterates:

$$\mathcal{O}(p): \quad p \longleftarrow f_1^{-n}(x_0) \ldots f_1^{-1}(x_0),\ x_0,\ x_1,\ \ldots\ x_m = p. \tag{1.2}$$

It is also worth noting that by definition, this homoclinic orbit is nondegenerate, as it comes immediately from the following

Definition. *An homoclinic orbit $\mathcal{O}(p)$ of a map f is called nondegenerate if in all the points x_j of the orbit $J_f(x_j)$ is defined and $det(J_f(x_j)) \neq 0$.*

Then for a smooth map Marotto proves, in the sense of Li and Yorke [29], the following

Theorem 1 (Marotto [26]). *If f has a snap-back repeller then f is chaotic.*

It is plain that the same definitions apply for a $k-$cycle with $k > 1$ considering it as a fixed point of the map f^k.

It may be noticed that nothing is stated about the bifurcation which leads a repelling fixed point from no homoclinic orbit for it to a *snap-back repeller*, SBR for short. In [30], together with a different proof of the chaotic dynamics associated with a SBR in the smooth case, it was noticed that when the fixed point p is a repelling focus of a two-dimensional noninvertible map with an annular chaotic area $\mathcal{A}$ around p, the so-called SBR-bifurcation (leading to the first appearance of a homoclinic orbit of p) is associated with a *critical homoclinic orbit*. This bifurcation occurs when another rank-one preimage of p, different from p, merges with the external boundary of the chaotic area $\mathcal{A}$. That this first homoclinic orbit is critical is due to the fact that the boundary of the chaotic area is given by the images of a proper segment of the critical curves of the map, following the notation used in Mira et al. [31]. Moreover, as it was noticed in [30], the first SBR-bifurcation due to a critical homoclinic orbit is followed by an explosion of different homoclinic orbits of the same point p, and generally a cascade of degenerate homoclinic orbits occurs, followed by explosions of nondegenerate ones. The existence of SBR is often associated with the existence of a strange repeller (see for example in [32]), while its relation with repelling foci is mainly associated with attracting chaotic areas. And it is clear that also repelling nodes may be involved in some SBR bifurcations inside attracting chaotic areas.

In this work we shall generalize Theorem 1 by Marotto for a nondegenerate homoclinic orbit $\mathcal{O}(p)$ of a repelling fixed point p of a piecewise smooth noninvertible map f of R^n, continuous or piecewise continuous, proving the following

Theorem 2. *Let f be a piecewise smooth noninvertible map. Let p be a repelling cycle of f, in which the map is locally invertible, and all the eigenvalues of $J_f(p)$ exceed 1 in modulus. Let $\mathcal{O}(p)$ be a nondegenerate homoclinic orbit of p. Then in any neighborhood of $\mathcal{O}(p)$ there exists an invariant Cantor set Λ on which f is chaotic.*

Moreover, we further generalize the result to some *degenerate homoclinic orbits* $\mathcal{O}(p)$ such that in all the points x_j the map is locally invertible. This degeneracy may occur because there are points

x_j in the homoclinic orbit in which the Jacobian $J_f(x_j)$ is not defined or because it is defined but we have $\det(J_f(x_j)) = 0$. Anyhow the following result holds

Theorem 3. *Let f be a piecewise smooth noninvertible map. Let p be a repelling cycle of f, in which the map is locally invertible, and all the eigenvalues of $J_f(p)$ exceed 1 in modulus. Let $\mathcal{O}(p)$ be an homoclinic orbit of p and f locally invertible in each point of $\mathcal{O}(p)$. Then in any neighborhood of $\mathcal{O}(p)$ there exists an invariant Cantor like set Λ on which f is chaotic.*

The plan of the work is as follows. In Section 2 we shall prove Theorem 2, giving in Section 3 two examples of snap-back repellers via a smooth 2-dimensional map proposed by Gonchenko et al. [15, 16] and frequently used in the examples in [31]. Section 3 also includes an example with a 2-dimensional piecewise linear discontinuous map. The proof of Theorem 3 is given in Section 4.

2. PROOF OF THE HOMOCLINC THEOREM

Let us first recall some lemmas whose proof is nowadays well known (see [28, 33, 34]).

Lemma 1. *Let σ be the shift map acting on the space Σ_2 of one sided infinite sequences of two symbols $\{0, 1\}$, then σ is chaotic*[1]*. Any map which is topologically conjugated with the shift map is also chaotic.*

Lemma 2. *Let (X, d) be a metric space and $F : X \to X$ a map. If there exist two compact subsets $V_0 \subset X$ and $V_1 \subset X$ with $V_0 \cap V_1 = \varnothing$ such that $F(V_0) \supset V_0 \cup V_1$ and $F(V_1) \supset V_0 \cup V_1$ then:*

(i) the set $V_0 \cup V_1$ includes a closed invariant Cantor like set Λ, that is a set of closed compact sets A_α which are in one-to-one correspondence with the elements α of Σ_2 and $F(A_\alpha) = A_{\sigma(\alpha)}$.

(ii) A set A_α is a single point iff $\delta(A_\alpha) = 0$ where δ denotes the diameter of the set.

(iii) If $X \supseteq U \supset V_0 \cup V_1$ exists such that $F_0^{-1}(U) = V_0$, $F_1^{-1}(U) = V_1$ and $F(V_0) = F(V_1) = U$, where the inverses $F_{0,1}^{-1}$ are contractions in U, then $\delta(A_\alpha) = 0$ for any symbol sequence α of Σ_2.

(iv) If all the sets A_α are single points then Λ is a Cantor set on which F is topologically conjugated with the shift map.

Lemma 3. *Let (X, d) be a metric space and $f : X \to X$ a map. Let $F = f^n$ for some positive integer n. If F is chaotic on some invariant set $\Lambda \subset X$ then also f is chaotic on $\cup_{k=1}^{n} f^k(\Lambda)$.*

For convenience let us recall the standard proof of point (i) in Lemma 2 when a set $U \supset V_0 \cup V_1$ exists such that $F_0^{-1}(U) = V_0$ and $F_1^{-1}(U) = V_1$. Then defining $F^{-1} = F_0^{-1} \cup F_1^{-1}$ we have

$$
\begin{aligned}
&F^{-1}(U) = V_0 \cup V_1 \\
&F^{-2}(U) = F^{-1}(V_0) \cup F^{-1}(V_1) = V_{00} \cup V_{01} \cup V_{10} \cup V_{11} \subset F^{-1}(U) \\
&\qquad \ldots
\end{aligned}
$$

It is clear that $F^{-k}(U)$ includes 2^k disjoint sets and

$$\Lambda = \lim_{k \to \infty} F^{-k}(U) = \bigcap_{k \geqslant 0} F^{-k}(U). \tag{2.1}$$

(where $F^{-0} = F^0 = I$ is the identity function). Notice that up to now we have not used any assumption on the functions F_0^{-1} and F_1^{-1}, and the result stated in (i), which is purely geometrical, holds. Assuming that F_0^{-1} and F_1^{-1} are contraction mappings it follows that the elements of Λ are single points. However, as already commented in [34], a Cantor set of points Λ in the above process can be obtained also with less strong assumptions in these functions, for example it is enough to ask for the following stability property $\mathcal{S}$:

Stability property $\mathcal{S}$. *We say that a function F has the stability property in a set U if all the eigenvalues of $J_F(x)$ are less than 1 in modulus for any $x \in U$.*

---

[1] that is, σ has a positive topological entropy, is topologically transitive, and is chaotic in the sense of Li-Yorke [29] as well as in the sense of Devaney [33].

For our purpose it is enough to prove that the property of a Cantor set of points Λ is obtained assuming that the function F_0^{-1} satisfies the stability property $\mathcal{S}$ and F_1^{-1} is a contraction in U, so that point (*iii*) in the above Lemma 3 can be substituted by the following:

Lemma 4. *(iii′) If F is defined in a set $U \supset V_0 \cup V_1$ such that $F_0^{-1}(U) = V_0$, $F_1^{-1}(U) = V_1$ and $F(V_0) = F(V_1) = U$, where F_1^{-1} is a contraction in U and F_0^{-1} satisfies the stability property S, then $\delta(A_\alpha) = 0$ for any symbol sequence α of Σ_2.*

The proof of Lemma 4 comes reasoning by contradiction. Let us assume that for some set A_α we have the diameter $\delta(A_\alpha) > 0$, and let x and y be two points whose distance is $\delta(A_\alpha)$. Then considering the image $F(A_\alpha)$ we obtain a set whose diameter is wider, that is $\delta(F(A_\alpha)) = \delta(A_{\sigma(\alpha)}) > \delta(A_\alpha)$, because the function F is expanding in U and thus the inequality in (1.1) holds. It follows that in a finite number k of applications by F we obtain $\delta(F^k(A_\alpha)) > \delta(U)$ which contradicts the invariance of Λ in U.

Now, to prove Theorem 2 it is enough to show that when a nondegenerate homoclinic orbit $\mathcal{O}(p)$ exists, we can construct two compact subsets V_0 and V_1, $p \in V_1$, and that we can apply Lemma 2 with (*iii*) replaced by (*iii′*). Thus let us consider a repelling fixed point p of a noninvertible map f, denote by f_1^{-1} the local inverse in p and let $||.||$ be the suitable norm for which f_1^{-1} is a contraction in the local unstable set $W^u(p)$ (Hirsh and Smale [28] pp. 278–281).

Consider a suitable compact neighborhood $U_m(p) \subset W^u(p)$, and construct backward compact sets following the points of the homoclinic orbit such that $U_{m-1}(x_{m-1})$ satisfies $U_{m-1} \cap U_m = \varnothing$ and $f(U_{m-1}) = U_m$, $U_{m-2}(x_{m-2})$ satisfies $U_{m-2} \cap U_{m-1} = \varnothing$ and $f(U_{m-2}) = U_{m-1}, \ldots$ in each of which the function is locally smooth and we consider a unique local inverse. Let $j \leqslant 0$ be an integer (which necessarily exists) such that $U_j(x_j) \subset U_m(p)$. Let us define $H_0 = U_j(x_j)$ and let N_1 be the integer such that $f^{N_1}(H_0) = U_m(p)$. Now let us define $H_1 = f_1^{-N_2}(U_m(p))$ where the integer N_2 is chosen such that $H_0 \cap H_1 = \varnothing$. Let $N = N_1 N_2$ and define $V_0 = f_1^{-N_2}(H_0)$, $V_1 = f_1^{-N_1}(H_1)$, and $F = f^N$.

Then by construction we have $F(V_0) = U_m(p) \supset V_0 \cup V_1$ and $F(V_1) = U_m(p) \supset V_0 \cup V_1$. The local inverse $F_1^{-1} = f_1^{-N}$ is a contraction such that $F_1^{-1}(U_m(p)) = V_1$, and by properly choosing $U_m(p)$ and the integer j small enough, we can have a local inverse F_0^{-1} satisfying $F_0^{-1}(U_m(p)) = V_0$ by construction, that also satisfies the stability property $\mathcal{S}$.

Thus via the above Lemmas the proof of Theorem 2 is complete.

3. EXAMPLES

In this section we illustrate two examples, the first with a smooth two-dimensional map and the second with a discontinuous two-dimensional map.

3.1. A Smooth 2-Dimensional Example of SBR

Let us consider the following *noninvertible map* $f : R^2 \to R^2$

$$f : \begin{cases} x' = y \\ y' = -x^2 + cy + M \end{cases} \tag{3.1}$$

which was deduced in Gonchenko et al. [15, 16] as a truncated normal form for the first return maps defined near quadratic homoclinic tangencies in the cases of diffeomorphisms with saddle-foci fixed points whose unstable manifold is at least two-dimensional. Also it was used in many examples in [31] (in this or other forms, topologically conjugated to the present one).

The Jacobian of f is given by

$$J_f(x, y) = \begin{bmatrix} 0 & 1 \\ -2x & c \end{bmatrix} \tag{3.2}$$

and its determinant vanishes on a line which we call critical line LC_{-1} (following the notation used in [31]) of equation $x = 0$, and its image is the critical line $LC = f(LC_{-1})$ of equation $y = cx + M$, which separates the phase plane in two open regions denoted Z_0 and Z_2. A point $(x, y) \in Z_0$ has no rank-1 preimages while any point $(x, y) \in Z_2$ has two distinct rank-1 preimages given by:

$$f_{1,2}^{-1}(x', y') = (\pm\sqrt{cx' - y' + M}\ ,\ x'). \tag{3.3}$$

While a point $(x, y) \in LC$ has a unique rank-one preimage belonging to LC_{-1}. The further images of the curve LC, $LC_k = f^k(LC)$ for $k \geqslant 1$, are also called critical curves, of higher rank.

This map has two fixed points which appear by saddle-node bifurcation, say R the stable node and Q the saddle, given by

$$R = \left(\frac{(c-1) + \sqrt{(1-c)^2 + 4M}}{2}\ ,\ \frac{(c-1) + \sqrt{(1-c)^2 + 4M}}{2}\right)$$

$$Q = \left(\frac{(c-1) - \sqrt{(1-c)^2 + 4M}}{2}\ ,\ \frac{(c-1) - \sqrt{(1-c)^2 + 4M}}{2}\right).$$

For the parameters (c, M) in a suitable range the fixed point R is a repelling focus and an annular chaotic area $\mathcal{A}$ exists around it. An example is shown in Fig. 1a, for $c = 0.1$ and $M = 1.37$. The boundaries of the chaotic area are given by the images of an arc of the critical curve LC_{-1}, and thus are critical points of higher rank. In Fig. 1 we also illustrate the boundary of the basin of attraction which separates the points having bounded trajectories from those having divergent trajectories. The light gray points denote divergent trajectories, while the light blue points belong to the closure of the basin of attraction of the chaotic area $\mathcal{A}$. The unstable fixed point Q belongs to the basin boundary, as well as its stable set.

As shown in [30], the "hole" around R exists as long as the preimage of R distinct from itself, denoted $R_{-1} = f_2^{-1}(R)$, is external to the annular chaotic area (see Fig.1a), that is, as long as the repelling point R is not a SBR (and has no homoclinic orbits).

The first bifurcation leading to homoclinic orbits occurs at the parameters (c, M) for which we have the preimage R_{-1} on the boundary of the chaotic area $\mathcal{A}$, exactly on an arc of LC_3. Notice that this also means that an arc of the critical curve LC_4 crosses the fixed point R, but then (as it is fixed), also all the other critical arcs (the images) belonging to the critical curves LC_k for any $k > 4$ cross the fixed point R. This is a characteristic property of a SBR homoclinic bifurcation. Also, the chaotic area from annular becomes simply connected, and critical homoclinic orbits can be found (see Fig. 1b at $M = 1.432$). After this collision (or homoclinic bifurcation which leads R to be a SBR) the preimage R_{-1} is inside the chaotic area $\mathcal{A}$, and by taking further preimages of R_{-1} we can find infinitely many different nondegenerate homoclinic orbits of the type in (1.2).

We notice that increasing M the saddle fixed point Q becomes an unstable node, giving rise to a saddle 2-cycle $\{C_1, C_2\}$ on the frontier, bifurcated from Q. We show now that while the SBR bifurcation of the fixed point R is associated with an attracting chaotic area, the analogous bifurcation, i.e. the SBR bifurcation, of the second fixed point Q occurs inside a chaotic repeller.

The so-called final bifurcation of the chaotic attractor (see [30], also called external crisis of the chaotic attractor in [36]) occurs when the chaotic area $\mathcal{A}$ has a contact with its basin boundary (see Fig. 1c), after which the invariant set becomes a chaotic repeller.

This contact bifurcation is exactly the first tangent homoclinic bifurcation of the saddle 2-cycle $\{C_1, C_2\}$ belonging to the frontier of the basin. Notice how this also is associated with critical homoclinic orbits, due to the critical arcs which bound the chaotic area. After the contact we do not observe any attracting set, however we know that an invariant set exists and it is a chaotic repeller, to which the unstable node Q belongs. When the contact bifurcation occurs, as well as for some higher values of the parameter M, the repelling node Q is not a SBR, because it has no homoclinic orbit: points starting close to Q are repelled and will never come back. In order to detect when homoclinic orbits to Q also appear we can follow the images of the critical curves, taking the critical segment of LC_{-1} which was previously used for the boundary of the absorbing area. So we can see that approximately at $M = 1.9725$ an arc of the critical curve LC_3 crosses the fixed point Q, and thus also the critical arcs belonging to the images cross through Q. See Fig. 1d,

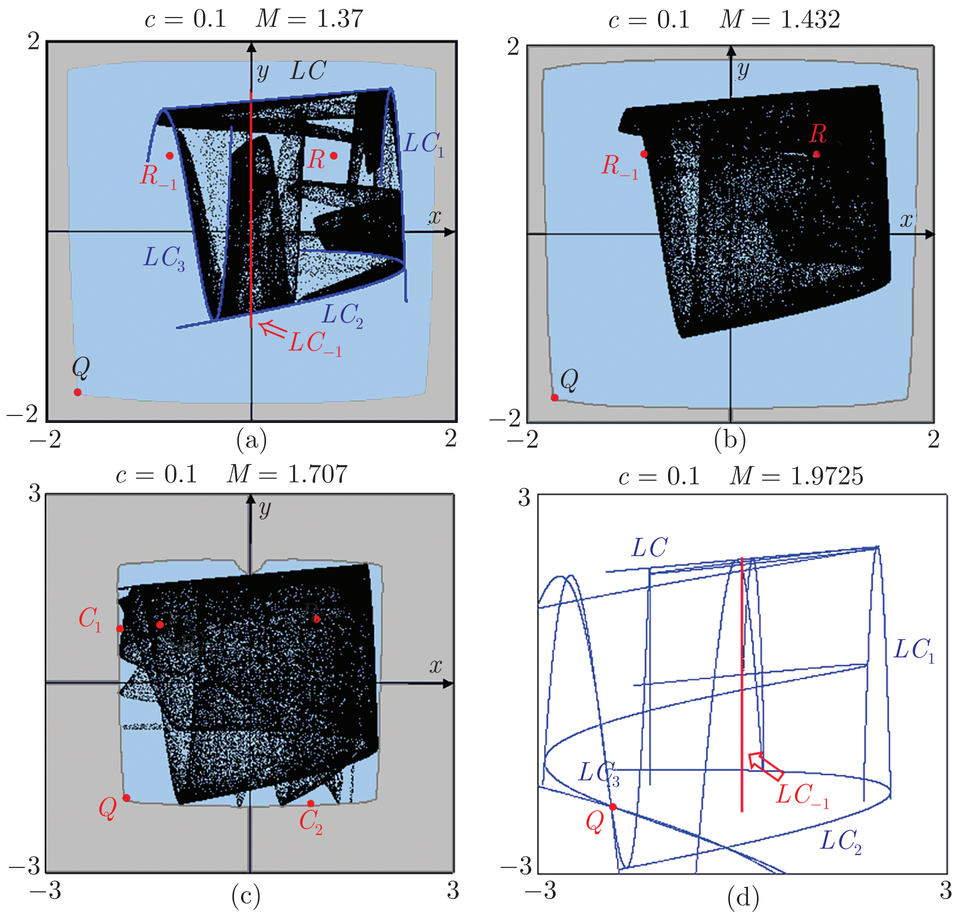

Fig. 1.

where besides the critical arc of LC_3 which is the first crossing though Q, some arcs of the critical curves LC_k for $k = 4, 5, 6$ are also shown through Q. Clearly this denotes the appearance of the first homoclinic orbits to Q. In fact, from $Q \in LC_3$ we can deduce that a point $q \in LC_{-1}$ exists such that $f^4(q) = Q$ and suitable preimages of q can be taken reaching some point x_0 in a neighborhood of Q (where Q is a stable node for f_2^{-1}). For higher values of M ($M > 1.9725$) the repelling node Q is a SBR, and nondegenerate homoclinic orbits of Q (as well as of R, of the 2-cycle $\{C_1, C_2\}$ and infinitely many other cycles) exist.

3.2. A Discontinuous 2-Dimensional Example of SBR

In this subsection we consider a piecewise linear two-dimensional map in canonic form proposed, in the continuous version, in several papers (see [37] and references therein), which is however here considered in a discontinuous form. The map f is given by two linear maps f_L and f_R which are defined in two half planes L and R:

$$f : (x, y) \mapsto \begin{cases} f_L(x, y),\ (x, y) \in L; \\ f_R(x, y),\ (x, y) \in R; \end{cases} \tag{3.4}$$

where

$$f_L : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \tau_L x + y + \mu_L \\ -\delta_L x \end{pmatrix},\ \ L = \{(x, y) : x \leqslant 0\}; \tag{3.5}$$

$$f_R : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \tau_R x + y + \mu_R \\ -\delta_R x \end{pmatrix},\ \ R = \{(x, y) : x > 0\}. \tag{3.6}$$

Here τ_L, τ_R are the traces and δ_L, δ_R are the determinants of the Jacobian matrix of the map f in the left and right half-planes, i.e., in L and R, respectively, $\mathbb{R}^2 = L \cup R$. The map is continuous when $\mu_L = \mu_R$, discontinuous otherwise. Here we consider an example with $\mu_L \neq \mu_R$. Following [31], the border line $x = 0$ is denoted LC_{-1}, and its backward and forward images by f_L and f_R are called *critical lines*.

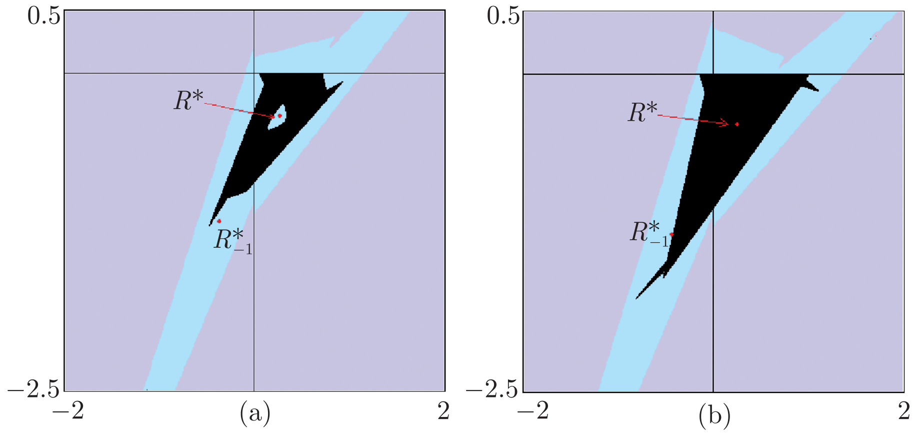

Fig. 2.

Let L^* and R^* denote the fixed points of f_L and f_R determined, respectively, by

$$\left(\frac{\mu_i}{1-\tau_i+\delta_i}, \frac{-\delta_i\mu_i}{1-\tau_i+\delta_i}\right), \ i = L, R.$$

L^* is the fixed point of the map f if $\mu_L/(1-\tau_L+\delta_L) \leqslant 0$, otherwise it is a so-called virtual fixed point which we denote by $\overline{L}^*$. Similarly, R^* is the fixed point of f if $\mu_R/(1-\tau_R+\delta_R) \geqslant 0$, otherwise it is a virtual fixed point denoted by $\overline{R}^*$. The stability of the fixed point R^* is defined by the eigenvalues $\lambda_{1,2(R)}$ of the Jacobian matrix of the map f_R, which are

$$\lambda_{1,2(R)} = \left(\tau_R \pm \sqrt{\tau_R^2 - 4\delta_R}\right)/2. \tag{3.7}$$

Let us consider here the following parameters constellation: $\tau_L = -1.4$, $\delta_L = -0.9$, $\mu_L = 0.9$, $\tau_R = -1.53$, $\delta_R = 1.3$, $\mu_R = 1$, at which we have that L^* is virtual while R^* is an unstable focus, around which an annular chaotic area exists (see Fig. 2a). The rank-1 preimage of R^* distinct from itself, that is, $R^*_{-1} = f_L^{-1}(R^*)$, given by:

$$f_L^{-1}(R^*) = \begin{pmatrix} \frac{\delta_R\mu_R}{\delta_L(1-\tau_R+\delta_R)} \\ \frac{\delta_L\mu_R-\tau_L\delta_R\mu_R}{\delta_L(1-\tau_R+\delta_R)} - \mu_L \end{pmatrix}$$

is external to the chaotic area (see Fig. 2a, where gray points are associated with divergent trajectories). Increasing the parameter δ_R the hole around R^* decreases and the transition to SBR occurs when the preimage R^*_{-1} has a contact with the boundary of the chaotic area (see Fig.2b), approximately at $\delta_R = 1.6$. After which the chaotic area becomes simply connected, and there exist infinitely many nondegenerate homoclinic orbits of R^*.

4. ON DEGENERATE HOMOCLINIC ORBITS OF EXPANDING POINTS

The results shown in Section 2 apply to homoclinic orbits which are nondegenerate. What occurs when a homoclinic orbit is degenerate has not been investigated up to now, and it is plain that there are several different occurrence to have a degenerate homoclinic orbit, here we take into account also degenerate homoclinic but keeping the local invertibility of the map.

There are two main points in the proof of Theorem 2: The first is that the map is locally invertible in a neighborhood of each point of the homoclinic orbit, the second is that the constructed local inverse F_0^{-1} satisfies the stability property $\mathcal{S}$. Now we shall keep only the first point.

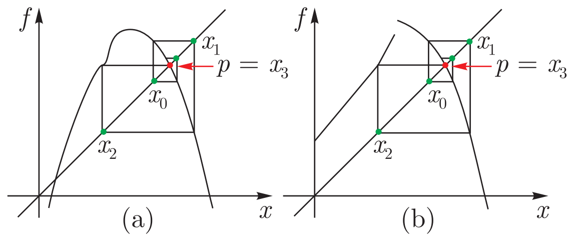

Fig. 3.

We can relax the assumption of nondegenericity assuming that in some point x_k of the homoclinic orbit the map is locally invertible, but we can have points in which $det(J_f(x_k)) = 0$ or points in which the Jacobian is not defined. As one-dimensional examples consider the cases shown in Fig. 3. In Fig. 3a a homoclinic orbit is clearly shown, and it is degenerate as in the point x_2 we have $f'(x_2) = 0$. The *good property* is that the local inverse is well defined, but the *bad property* is that the local inverse has not a bounded derivative in a neighborhood of x_2 or is not defined in x_2 (as in the example shown in Fig. 3b). However the local inverse exists and this is enough to prove, with the lemmas given in Section 2, that we can construct a Cantor like set on which f is invariant and chaotic, although the bad property prevents us to prove that the sets in the Cantor like construction are single points. So we prove now Theorem 3.

The proof clearly follows the same steps of the proof of Theorem 2, however the two functions there constructed have different properties. The local inverse $F_1^{-1} = f_1^{-N}$ is still a contraction such that $F_1^{-1}(U_m(p)) = V_1$, and we can have a local inverse F_0^{-1} satisfying $F_0^{-1}(U_m(p)) = V_0$ by construction, but F_0^{-1} may not satisfy the stability property $\mathcal{S}$. Thus we can apply only point (i) of Lemma 2, whose proof was recalled above, leading to the Cantor like set given in (2.1), which ends the proof.

Clearly the theorem works also for the case shown in Fig. 3b, where the homoclinic orbit is degenerate because in the point x_2 the function is not differentiable. However we remark that in this case the result of Theorem 2 applies, that is, the Cantor like set constructed in the proof is really a Cantor set of points. In fact, in this case we have that the local inverses have a bounded Jacobian in all the points of the considered neighborhood, except for the points of the homoclinic orbit, but this is enough to state that the a local inverse F_0^{-1} satisfying $F_0^{-1}(U_m(p)) = V_0$ by construction, also satisfies the stability property $\mathcal{S}$ in $U_m(p)\backslash p$, and we can conclude that Λ is a Cantor set of points.

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