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BIFURCATIONS AND MULTISTABILITY IN A CLASS OF TWO-DIMENSIONAL ENDOMORPHISMS

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1. INTRODUCTION

The sequence of bifurcations governing the transition to chaotic dynamics and the topological structure of chaotic invariants sets can be observed today in one-dimensional (1-d for short) maps, that is, in endomorphisms with a nonunique inverse. Then by now classic (although not well divulgated) results concerning the full-family of quadratic 1-d maps [1–9] has been extended, in its essential features, to generic 1-d endomorphisms by relating the structure of the invariant sets to the sequence of the *critical points* (or trajectory of a critical point) of the map [10–15]. In recent years such a kind of analysis has been generalized to two-dimensional (2-d, for brevity) endomorphisms, by exploiting the role of the 2-d analogue of the critical points, which are the *critical curves* of the map [10, 16–22].

We remark that 2-d maps with a nonunique inverse are frequently used in the mathematical modelling of many systems in engineering, physics, biology and economics [23–32]. In particular, a class of two-dimensional maps described (among others) in [33] is at the origin of the present study. That is, the class of maps described by

$$(x, z) \to (z, h(x)), \tag{1}$$

where $h(x)$ is a continuous function, piecewise continuously differentiable. We shall call a P-map one of the form given in (1), that is, such that $P(x, z) = (z, h(x))$, whose square is a map with separated variables, $P^2(x, z) = (h(x), h(z))$. Besides the work cited above, P-maps have been observed in other applicative models [34, 35].

It is clear that the dynamics occurring in P-maps are strictly related to those of the 1-d map $x' = h(x)$, and this relation, as will be described in the following sections, deserves perhaps some surprise. We shall see how rich the network creating cycles of P is, and that the coexistence of attracting cycles is the generic occurrence. In Section 2 we deduce the properties of P-maps, describing the characteristics of the local bifurcations which may occur, and proving the existence of infinitely many trapping sets belonging to "absorbing rectangles". We recall that a set $\mathcal{B}$ is *trapping* (or no-escaping, or mapped into itself) if for any initial condition $(x, z) \in \mathcal{B}$, the forward trajectory belongs to $\mathcal{B}$, that is, $\mathcal{P}^n(x, z) \in \mathcal{B}\ \forall n > 0$ or, in concise form, $\mathcal{P}(\mathcal{B}) \subseteq \mathcal{B}$, $\mathcal{B}$ is said to be *absorbing* if a neighbourhood U of $\mathcal{B}$ exists such that any initial condition in U has the ω-limit set in $\mathcal{B}$. We note also that we shall use the term invariant for mapping into itself exactly, that is, a set $\mathcal{A}$ is said to be *invariant* for $\mathcal{P}$

(or under $\mathcal{P}$) iff $\mathcal{P}(\mathcal{A}) = \mathcal{A}$. Clearly, nontrivial dynamics of P-maps occur when $h(x)$ is a noninvertible function, and in such cases the global bifurcations can also be described and related to the global bifurcations of the 1-d map $h(x)$. In the 2-d plane (x, z) we have critical curves, which are lines "issuing" from the critical points of $h(x)$, involved in the global bifurcations (homoclinic bifurcations of repelling cycles of P).

The properties of a P-map are then illustrated by an example. The model, chosen among those described in [33], is the Poincaré return map of a periodically impulsed oscillator, and is based on a 1-d dimensional function $h(x)$ which has two local extrema. As the dynamics of the 1-d maps are the fundamental substrate of the dynamics of P, those of the chosen example are presented in Section 3. This gives us the opportunity to apply some recent techniques, using critical points, to show that the dynamics are always bounded and that infinitely many full-boxes of bifurcations occur.

The correspondent dynamics of P (as a result of the properties presented in Section 2) are shown in Section 4. Some conclusions are drawn in Section 5.

2. PROPERTIES OF 2-d ENDOMORPHISMS P

In this section we shall point out some properties of the 2-d map P given by (1), that is

$$P(x, z) = (z, h(x)), \tag{2}$$

where $h(x)$ is assumed to be a generic continuous, piecewise continuously differentiable function. Moreover, we shall assume that $h(x)$ depends continuously on a real parameter μ, although we write $h(x)$ for short, instead of $h(x; \mu)$, or of $h_\mu(x)$.

The square of P is a map with separated variables

$$P^2(x, z) = (h(x), h(z)) \tag{3}$$

and each variable has the same 1-d dynamics, governed by the 1-d map

$$x' = h(x), \qquad x \in \mathbb{R}. \tag{4}$$

In the propositions which follow, when considering the eigenvalue of a cycle of $h(x)$, it is implicitly assumed, even if not explicitly stated, that $h(x)$ is differentiable at all points of the cycle. We recall also that the term "k-cycle" of a map is synonymous with "cycle of least period k", and in the statement "cycle of period k" it is understood that k is the least period. Fixed points correspond to $k = 1$. A k-cycle of $h(x)$ of periodic point a_1 is the set $\{a_1, a_2, \ldots, a_k\}$, where $a_{i+1} = h(a_i)$, for $i = 1, \ldots, (k - 1)$ and $a_1 = h(a_k)$. The a_i are called periodic points of the k-cycle. The set of periodic points is sometimes equivalently written, apart from the order of the points, as $\{h^i(a_1), i = 1, 2, \ldots, k\}$. In this last expression, any one of the periodic points can be used instead of a_1. Each periodic point is a fixed point of the map h^k ($h^k(a_i) = a_i$, $i = 1, 2, \ldots, k$). The eigenvalue of a k-cycle of h is given by $\lambda = \prod_{i=1}^{k} h'(a_i)$, where h' denotes the derivative of h.

Similar definitions hold for a k-cycle of the 2-d map P. A k-cycle of P with periodic point (a_1, b_1) is made up of the periodic points $\{P^i(a_1, b_1), i = 1, 2, \ldots, k\}$. Denoting by $J(x, z)$ the Jacobian matrix of P at (x, z), the eigenvalues of the k-cycles of P are the two eigenvalues of the matrix product of k Jacobian matrices: $J_k = J(a_k, b_k) \cdots J(a_1, b_1)$. The particular structure of P implies a particular structure of J_k. Indeed, J_k is diagonal if k is even, while its diagonal is made up of zeros if k is odd.

The fixed points of P belong to the intersection of the graphs of the two curves $z = x$ and $z = h(x)$, that is $x = h(x)$ and $z = h(z)$; it follows that fixed points of P are related to fixed points of h.

The following properties, are simple consequences of the particular structure of P and of the matrix J_k.

Property 1. Let a be a fixed point of h ($a = h(a)$) with eigenvalue $\lambda_a = h'(a)$, then (a, a) is a fixed point of P with eigenvalues $\lambda_1 = -\sqrt{\lambda_a}$ and $\lambda_2 = \sqrt{\lambda_a}$. If $\lambda_a > 0$ then the fixed point (a, a) is a star-node; if $\lambda_a < 0$ then (a, a) is a focus. If b is another fixed point of h with eigenvalues $\lambda_b = h'(b)$, then P possesses also the 2-cycle $\{(a, b), (b, a)\}$ with real eigenvalues $\lambda_1 = \lambda_a$ and $\lambda_2 = \lambda_b$. A 2-cycle of P exists iff h possesses two fixed points.

Clearly, if h has k fixed points, then P has k fixed points on the line $x = z$ and $k!/2!(k-2)!$ distinct 2-cycles. The periodic points of each 2-cycle are symmetric with respect to the line $x = z$.

Property 2. Let a be a fixed point of $h(x)$, then a point of the line $\{x = a\}$ is mapped by P into a point of the line $\{z = a\}$, and vice-versa. Therefore, the set $\{(x, s) : x = a\} \cup \{(x, z) : z = a\}$ is trapping for P (each set $\{(x, z) : x = a\}$ and $\{(x, z) : z = a\}$ is trapping for P^2). A point belonging to the bisectrix $\{(x, z) : z = x\}$ is mapped by P into a point belonging to the set $\{(x, z) : z = h(x)\}$, and vice versa. Therefore, the set $\{(x, z) : z = x\} \cup \{(x, z) : z = h(x)\}$ is trapping for P (each set $\{(x, z) : z = x\}$ and $\{(x, z) : z = h(x)\}$ is trapping for P^2).

Property 3. From $P^{2k}(x, z) = (h^k(x), h^k(z))$ we have

$$P^{2k}(x, z) = (x, z) \qquad \text{iff } x = h^k(x) \text{ and } z = h^k(z). \tag{5}$$

Therefore, cycles of P of even period $2k$ are associated with cycles of h of period k (odd or even).

From $P^{2m+1}(x, z) = (h^m(z), h^{m+1}(x))$ we have

$$P^{2m+1}(x, z) = (x, z) \qquad \text{iff } x = h^m(z) \text{ and } z = h^{m+1}(x) \tag{6}$$

that is, $x = h^{2m+1}(x)$ and $z = h^{2m+1}(z)$. Therefore, cycles of P of odd period $k = 2m + 1$ are associated with cycles of h of the same odd period k.

Property 4. The eigenvalues of a k-cycle of P of even period k are real with eigendirections parallel to the coordinate axes. Thus, a k-cycle of P of even period is either a node or a saddle, while a k-cycle of P of odd period is either a node or a focus.

Property 5. Let $\{a_i, i = 1, \ldots, k\}$ be a k-cycle of h of even period $k = 2m$, with eigenvalue $\lambda_a = \prod_{i=1}^{k} h'(a_i)$. Then P has $m = k/2$ distinct cycles of period $2k$, given by

$$\begin{array}{c} \{P^i(a_1, a_1), i = 1, \ldots, 2k\} \\ \{P^i(a_2, a_1), i = 1, \ldots, 2k\} \\ \vdots \\ \{P^i(a_m, a_1), i = 1, \ldots, 2k\}, \end{array} \tag{7}$$

each of the $2k$-cycles having the eigenvalues $\lambda_1 = \lambda_2 = \lambda_a$.

For the index i in (7) we may also write $i = 0, \ldots, 2k - 1$, as, from (5), $P^{2k}(a_j, a_1) = (a_j, a_1)$, but we have chosen to adopt the notation in (7) so as to make appear at a glance the period of the cycle.

We note that the cycles in (7) can also be written, respectively, as follows

$$\{P^i(a_{2m}, a_1),\ i = 1, \ldots, 2k\}$$
$$\{P^i(a_{2m-1}, a_1),\ i = 1, \ldots, 2k\}$$
$$\vdots$$
$$\{P^i(a_{m+1}, a_1),\ i = 1, \ldots, 2k\}$$

as $P^2(a_{2m}, a_1) = P(a_1, a_1), P^{2\cdot 2}(a_{2m-1}, a_1) = P(a_2, a_1), \ldots, P^{2m}(a_{m+1}, a_1) = P(a_m, a_1)$.

Property 6. Let $\{a_i, i = 1, \ldots, k\}$ be a k-cycle of h of odd period $k = 2m + 1$, $m \geq 1$, with eigenvalue $\lambda_a = \prod_{i=1}^{k} h'(a_i)$. Then P has $m = (k - 1)/2$ distinct cycles of even period $2k$, given by (7), each of which has the eigenvalues $\lambda_1 = \lambda_2 = \lambda_a$, and one cycle of odd period k given by

$$\{P^i(a_{m+1}, a_1),\ i = 1, \ldots, k\} \tag{8}$$

with eigenvalues $\lambda_1 = -\sqrt{\lambda_a}$ and $\lambda_2 = \sqrt{\lambda_a}$.

A note analogous to the previous one (referring to property 5) holds also for property 6. That is, the last m points of the k-cycle $\{a_i, i = 1, \ldots, k\}$ of h, $k = 2m + 1$, are associated with the same cycles of period $2k$ obtained above, in (7), while, by equation (6), the cycle of odd period k given in (8) can also be written as $\{(h^m(a_i), a_i), i = 1, \ldots, k\}$; however, in this last formulation the periodic points do not appear in the order in which they occur under iterated application of P.

We call cycles of *homogeneous type* the cycles of P related to the periodic points of a single cycle of h, as described in the properties 5 and 6.

The following property 7 characterizes cycles of P related to the periodic points of two distinct cycles of h, which we shall call of *mixed type*. That also two distinct cycles of h, of periods k and q say, are associated with cycles of P derives from the following observation. If a point a satisfies $a = h^k(a)$ and a point b satisfies $b = h^q(b)$, then $a = h^m(a)$ and $b = h^m(b)$, m being the least common multiple between k and q; thus, equation (5) applies, with $k = m$, and (a, b) turns out to be a periodic point of P of period $2m$.

Property 7. Let $\{a_i, i = 1, \ldots, k\}$ be a cycle of h of period k, $k \geq 1$ odd or even, with eigenvalue λ_a, and $\{b_i, i = 1, \ldots, q\}$ be a cycle of h of period q, $q \geq 1$ odd or even, with eigenvalue λ_b. Let m be the least common multiple between k and q, and n_1 and n_2 the two natural numbers such that $m = k \cdot n_1 = q \cdot n_2$. Then the 2-d map P has n_c, $n_c = (k \cdot q)/m$, distinct cycles of period $2m$, of mixed type, given by

$$\begin{aligned} &\{P^i(a_1, b_1),\ i = 1, \ldots, 2m\} \\ &\{P^i(a_2, b_1),\ i = 1, \ldots, 2m\} \\ &\qquad\vdots \\ &\{P^i(a_{n_c}, b_1),\ i = 1, \ldots, 2m\} \end{aligned} \tag{9}$$

each of which has the eigenvalues $\lambda_1 = \lambda_a^{n_1}$, $\lambda_2 = \lambda_b^{n_2}$.

As remarks to Property 7, let us discuss in detail some particular cases of this proposition.

Remark 1. Besides the hypotheses of proposition 7, assume $q = 1$, that is b is a fixed point of h. Then $m = k$ and $n_c = 1$, i.e. a single cycle of P of mixed type exists, of period $2k$, given by $\{P^i(a_1, b), i = 1, \ldots, 2k\}$, with eigenvalues $\lambda_1 = \lambda_a$ and $\lambda_2 = \lambda_b^k$. If also $k = 1$ (a is a fixed point of h) we obtain the single cycle of period 2 (of mixed type) associated with two fixed points of h, as already stated in property 1.

Remark 2. If the least common multiple is $m = k \cdot q$, then $n_c = 1$, i.e. a single cycle of P of mixed type is related to the given cycle of h. This cycle, of period $2kq$, is given by $\{P^i(a_1, b_1), i = 1, \ldots, 2kq\}$, with eigenvalues $\lambda_1 = \lambda_a^q$ and $\lambda_2 = \lambda_b^k$.

Remark 3. If $q = k$ (i.e. the two cycles have the same period), then $m = k$ and $n_c = k$, that is, k cycles of P of mixed type exist, of period $2k$, given by $\{P^i(a_1, b_1), i = 1, \ldots, 2k\}, \ldots, \{P^i(a_k, b_1), i = 1, \ldots, 2k\}$ with eigenvalues $\lambda_1 = \lambda_a$ and $\lambda_2 = \lambda_b$.

Remark 4. If we consider that h has also a cycle of period $2k$, say $\{d_i, i = 1, \ldots, 2k\}$, then it will be interesting to see how many cycles of P of mixed type are related to this $2k$-periodic cycle and the q-periodic cycle $\{b_i, i = 1, \ldots, q\}$, that is, the cycles given by

$$\{P^i(d_j, b_1), i = 1, \ldots, 2m'\}, \qquad j = 1, \ldots, n_c'.$$

If the natural number n_1 given in property 7 is odd, then $m' = 2m$ and $n_c' = n_c$, that is, the number of cycles is the same, and the period is doubled. If n_1 is even, then $m' = m$ and $n_c' = 2n_c$, that is, the number of cycles is doubled, and the period is the same.

From property 3 it follows that all the cycles of the 2-d map P are created in correspondence to some bifurcation occurring in the cycles of the 1-d map h, and from properties 5–7 it follows that local bifurcations of the cycles of P occur when local bifurcations in the cycles of h occur (due to the particular structure of the eigenvalues of the cycles of P). This correspondence is what we shall describe in the propositions of the following subsection. As we shall see, the bifurcations of cycles of P are often of particular type, due to the presence of two eigenvalues equal to 1 in absolute value. When this occurs we say that the cycle undergoes a *degenerate bifurcation*. We can have three kinds of degenerate bifurcations, depending on the sign of λ_1, λ_2. We say that a degenerate bifurcation is of fold-type if $\lambda_1 = \lambda_2 = 1$; of flip-type if $\lambda_1 = \lambda_2 = -1$; of saddle-type if $\lambda_1 = 1$ and $\lambda_2 = -1$. It is also clear that we use the term *standard bifurcation* for a cycle of P when only one of the two eigenvalues becomes equal to 1 in absolute value. We distinguish between standard bifurcation of fold-type (if $\lambda_1 = 1$), and of flip-type (if $\lambda_1 = -1$). Moreover, as regards to the Neimark–Hopf bifurcation of a cycle of P, we shall see that only a resonant case can occur.

2.1. *Cycles of P due to fold and flip bifurcations of h*

From properties 5 and 6 we can deduce that *multistability* is a characteristic property of the 2-d map P. In fact, whenever h possesses an attracting k-cycle of period $k > 2$, then P possesses more than one attracting cycle. From the same properties 5–6 it can be seen that to fold and flip bifurcations in the map h there correspond degenerate bifurcations in the map P, whose effects are described below.

Let us assume that at the value μ_k a fold bifurcation occurs in the map h, with generation of a couple of k-cycles, $k \geq 1$; that is, for $\mu = \mu_k + \varepsilon$ two k-cycles of h exist (which do not exist for $\mu = \mu_k - \varepsilon$), one attracting, say $\{a_i, i = 1, \ldots, k\}$, with eigenvalue $0 < \lambda_a < 1$, and the other repelling, say $\{b_i, i = 1, \ldots, k\}$, with eigenvalue $\lambda_b > 1$. At the fold bifurcation value μ_k the two k-cycles coincide, with common eigenvalue $\lambda_a = 1$. The case $k = 1$ reduces to a fold bifurcation of h generating a couple of fixed points, and is substantially described by property 1. Therefore, let us assume $k > 1$ and discuss separately the two cases k even and k odd.

Cycles of P generated by fold bifurcation of a k-cycle of h(x), k even, $k = 2m$, $m \geq 1$.

$\mu = \mu_k$. From property 5 we deduce that at $\mu = \mu_k$ there is the appearance in P of the m cycles (of homogeneous type and of period $2k$) given by (7), which did not exist before. All the new homogeneous cycles of P undergo (i.e. are created by) a degenerate bifurcation of fold-type, as both the eigenvalues of the homogeneous cycles are equal to 1.

From property 7 it follows that at $\mu = \mu_k$ there is also the appearance of the cycles of mixed type associated with the new k-cycle $\{a_i, i = 1, \ldots, k\}$ of h and all the pre-existing cycles of h. Let us say that n cycles of h (say $\gamma_j, j = 1, \ldots, n$, of period n_j and eigenvalue λ_j) already exist at $\mu = \mu_k$, distinct from the k-cycle born by fold bifurcation. Then all the new cycles of P of mixed type have one eigenvalue equal to 1, so that they all undergo a standard bifurcation, assuming that λ_j is different from 1 in absolute value. In what follows, we shall comment on the effects due to cycles γ_j with eigenvalues $|\lambda_j| > 1$ (which is the generic case), and the comments of different situations are obvious.

$\mu = \mu_k + \varepsilon$. For $\mu = \mu_k + \varepsilon$ (ε suitably small) the following $2k$ cycles of P of period $2k$ exist, according to property 5 (for cycles of homogeneous type) and to property 7 (for cycles of mixed type):

(i1) m cycles of homogeneous type, of period $2k$, attracting nodes

$$\{P^i(a_1, a_1), i = 1, \ldots, 2k\}, \{P^i(a_2, a_1), i = 1, \ldots, 2k\}, \ldots, \{P^i(a_m, a_1), i = 1, \ldots, 2k\},$$

each of which has the eigenvalues $\lambda_1 = \lambda_2 = \lambda_a$ ($\lambda_a < 1$);

(i2) m cycles of homogeneous type, of period $2k$, repelling nodes

$$\{P^i(b_1, b_1), i = 1, \ldots, 2k\}, \{P^i(b_2, b_1), i = 1, \ldots, 2k\}, \ldots, \{P^i(b_m, b_1), i = 1, \ldots, 2k\},$$

each of which has the eigenvalues $\lambda_1 = \lambda_2 = \lambda_b$ ($\lambda_b > 1$);

(i3) k cycles of mixed type, of period $2k$, saddles

$$\{P^i(a_1, b_1), i = 1, \ldots, 2k\}, \{P^i(a_2, b_1), i = 1, \ldots, 2k\}, \ldots, \{P^i(a_k, b_1), i = 1, \ldots, 2k\},$$

each of which has the eigenvalues $\lambda_1 = \lambda_a$ (<1) and $\lambda_2 = \lambda_b$ (>1).

Moreover, if h possesses other cycles, to the above ones we have to add the following:

(i4) all the cycles of mixed type associated with the k-cycle $\{a_i, i = 1, \ldots, k\}$ and the n pre-existing cycles γ_j of h. They have one eigenvalue belonging to (0, 1) ($\lambda_1 = \lambda_a^{n_1}$, n_1 suitable, depending on j). Thus, these are all saddles if the cycles γ_j of h are repelling;

(i5) all the cycles of mixed type associated with the k-cycle $\{b_i, i = 1, \ldots, k\}$ and the n pre-existing cycles γ_j of h. They have one eigenvalue greater than 1 ($\lambda_1 = \lambda_b^{n_1}$, n_1 suitable, depending on j). Thus these are all repelling nodes if the cycles γ_j of h are repelling.

Remark 5. Let us reassume the effect on P of a fold bifurcation in h occurring at $\mu = \mu_k$. Each of the m homogeneous cycles of P of period $2k$ appearing at the bifurcation value ($\mu = \mu_k$) gives rise, after bifurcation ($\mu = \mu_k + \varepsilon$), to four cycles of P: one cycle of homogeneous type, of period $2k$, attracting node, one cycle of homogeneous type, of period $2k$, repelling node, and two cycles of mixed type, of period $2k$, saddles. These cycles, described in (i1), (i2) and (i3), are the "effect" of the degenerate bifurcation of fold-type. The other cycles of mixed type in (i4) and (i5) are generally the "effect" of standard bifurcations of fold-type. That is, each cycle of mixed type born at $\mu = \mu_k$ gives rise, after the bifurcation, to two cycles of the same period, generally one saddle and one repelling mode.

Cycles of P due to a fold bifurcation of a k-cycle of h(x), k odd, $k = 2m + 1$, $m \geq 1$.

$\mu = \mu_k$. For the homogeneous cycles of P we now apply property 6 to deduce that at $\mu = \mu_k$ there is the appearance of the m cycles, of homogeneous type, of period $2k$, given by (7), with eigenvalues $\lambda_1 = \lambda_2 = 1$ (i.e. which undergo a degenerate bifurcation of fold-type), plus the cycle of odd period k, of homogeneous type, given by (8), with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$ (which undergoes a degenerate bifurcation of saddle-type). From property 7 it follows that there is also the appearance of the cycles of mixed type associated with the new k-cycle $\{a_i, i = 1, \ldots, k\}$ of h and with all the pre-existing cycles of h (γ_j, $j = 1, \ldots, n$, of period n_j and eigenvalue λ_j); these cycles of P have one eigenvalue $\lambda_1 = 1$.

$\mu = \mu_k + \varepsilon$. By property 6 (for cycles of homogeneous type) and property 7 (for cycles of mixed type), for $\mu = \mu_k + \varepsilon$ (ε small enough), P admits the following cycles:

(j1) m cycles of homogeneous type, of period $2k$, attracting nodes

$$\{P^i(a_1, a_1), i = 1, \ldots, 2k\}, \{P^i(a_2, a_1), i = 1, \ldots, 2k\}, \ldots, \{P^i(a_m, a_1), i = 1, \ldots, 2k\},$$

each of which has the eigenvalues $\lambda_1 = \lambda_2 = \lambda_a$ ($\lambda_a < 1$);

(j1.1) one cycle of odd period k, of homogeneous type, attracting node

$$\{P^i(a_{m+1}, a_1), i = 1, \ldots, k\}$$

with eigenvalues $\lambda_1 = -\sqrt{\lambda_a}$ and $\lambda_2 = \sqrt{\lambda_a}$ ($\lambda_a < 1$);

(j2) m cycles of homogeneous type, of period $2k$, repelling nodes

$$\{P^i(b_1, b_1), i = 1, \ldots, 2k\}, \{P^i(b_2, b_1), i = 1, \ldots, 2k\}, \ldots, \{P^i(b_m, b_1), i = 1, \ldots, 2k\},$$

each of which has the eigenvalues $\lambda_1 = \lambda_2 = \lambda_b$ ($\lambda_b > 1$);

(j2.1) one cycle of odd period k, of homogeneous type, repelling node

$$\{P^i(b_{m+1}, b_1), i = 1, \ldots, k\}$$

with eigenvalues $\lambda_1 = -\sqrt{\lambda_b}$ and $\lambda_2 = \sqrt{\lambda_b}$ ($\lambda_b > 1$);

(i3), (i4) and (i5) hold with $k = 2m + 1$.

Remark 6. Each of the m cycles of even period $2k$ undergoes a degenerate bifurcation of fold-type whose effect is the same as the one described in the previous case; that is, after bifurcation, for $\mu = \mu_k + \varepsilon$, it gives rise to four cycles of P, an attracting node, a repelling node and two saddles, all of period $2k$. A different effect has the degenerate bifurcation of the cycle of odd period k, which appears at $\mu = \mu_k$ with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$. After bifurcation, this cycle gives rise to one k-cycle attracting node (j1.1), one k-cycle repelling node (j2.1) (which are due to the eigenvalue $\lambda_2 = 1$), and one cycle saddle of double period $2k$ (one of those listed in (i3)) (which is due to the eigenvalue $\lambda_1 = -1$).

Now let us assume that increasing μ, the eigenvalue of the attracting k-cycle of h $\{a_i, i = 1, \ldots, k\}$ becomes negative and that a flip bifurcation of this cycle occurs at $\mu_{k\cdot 2}$. That is, at $\mu = \mu_{k\cdot 2}$ we have $\lambda_a = -1$, while for $\mu = \mu_{k\cdot 2} + \varepsilon$ the k-cycle $\{a_i, i = 1, \ldots, k\}$ is repelling with eigenvalue $\lambda_a < -1$ and an attracting $2 \cdot k$-cycle of h exists, say $\{d_i, i = 1, \ldots, 2k\}$, with eigenvalue λ_d, $0 < \lambda_d < 1$. Let us comment separately on the corresponding bifurcations occurring in the 2-d map P, for k even and k odd.

Cycles of P due to a flip bifurcation of a k-cycle of $h(x)$, k even, $k = 2m$, $m \geq 1$.

$\mu = \mu_{k\cdot 2}$. At $\mu = \mu_{k\cdot 2}$ the number and type of cycles of P is that given in (i1)–(i5), with $\lambda_a = -1$. Thus, any cycle listed in (i1) undergoes a degenerate bifurcation of flip-type; any cycle listed in (i3) undergoes a standard bifurcation of flip-type; and any cycle listed in (i4) undergoes a bifurcation, generally standard, which may be of flip-type or of fold-type, depending on the value of the integer $n_1(\gamma_j)$ (if $n_1(\gamma_j)$ is odd then one eigenvalue is $\lambda_1 = -1$, while if $n_1(\gamma_j)$ is even one eigenvalue is $\lambda_1 = 1$).

$\mu = \mu_{k\cdot 2} + \varepsilon$. For $\mu = \mu_{k\cdot 2} + \varepsilon$, applying properties 5 and 7, we get the following cycles:

(f0) all those given in (i1)–(i5), where saddles and attracting nodes become repelling nodes; plus the new ones:

(f1) k cycles of period $2.2k$, homogeneous, attracting nodes

$$\{P^i(d_1, d_1), i = 1, \ldots, 2.2k\}\ \{P^i(d_2, d_1), i = 1, \ldots, 2.2k\}, \ldots, \{P^i(d_k, d_1), i = 1, \ldots, 2.2k\},$$

each of which has the eigenvalues $\lambda_1 = \lambda_2 = \lambda_d$ ($\lambda_d < 1$);

(f2) k cycles of period $2.2k$, of mixed type, saddles

$$\{P^i(a_1, d_1), i = 1, \ldots, 2.2k\}\ \{P^i(a_2, d_1), i = 1, \ldots, 2.2k\}, \ldots, \{P^i(a_k, d_1), i = 1, \ldots, 2.2k\},$$

each of which has the eigenvalues $\lambda_1 = (\lambda_a)^2 > 1$ and $\lambda_2 = \lambda_d(\lambda_d < 1)$;

(f3) k cycles of period $2.2k$, of mixed type, saddles

$$\{P^i(b_1, d_1), i = 1, \ldots, 2.2k\}\ \{P^i(b_2, d_1), i = 1, \ldots, 2.2k\}, \ldots, \{P^i(b_k, d_1), i = 1, \ldots, 2.2k\},$$

each of which has the eigenvalues $\lambda_1 = (\lambda_b)^2 > 1$ and $\lambda_2 = \lambda_d$ ($\lambda_d < 1$);

(f4) the effect of the bifurcations occurring in the cycles described in (i4) is the generation of all the new cycles of P, of mixed type, associated with the $2k$-cycle $\{d_i, i = 1, \ldots, 2k\}$ and the pre-existing cycles γ_j of h, to which the remark 4 given above applies: if $n_1(\gamma_j)$ is odd then the effect of one eigenvalue $\lambda_1 = -1$ is the creation of cycles of double period, while if $n_1(\gamma_j)$ is even then the effect of one eigenvalue $\lambda_1 = 1$ is the creation of couples of cycles of the same period. If the cycles γ_j of h are repelling, then all these new cycles are saddles or repelling nodes.

Remark 7. The effects of the degenerate bifurcation of flip-type ($\lambda_1 = \lambda_2 = -1$), occurring at $\mu = \mu_{k\cdot 2}$ to each of the m cycles of P of period $2k$ given in (i1), gives rise, after bifurcation, for $\mu = \mu_{k\cdot 2} + \varepsilon$, to four cycles of P of double period. That is, each attracting node becomes a repelling node giving rise to two cycles of double period, attracting nodes, and to two cycles of double period, saddles (listed in (f1) and (f2) above). Each of the k cycles saddles of period $2k$ given in (i3) undergoes a standard bifurcation of flip-type ($\lambda_1 = -1$), and for $\mu = \mu_{k\cdot 2} + \varepsilon$ the saddle becomes a repelling node giving rise to one cycle of double period, saddle (listed in (f3) above), of mixed type.

It is clear that a cascade of flip bifurcations starting from a k-cycle of h, k even, will give rise to a cascade of bifurcations of the kind described in (f1)–(f4), with exponential increase in the number of cycles of P.

Cycles of P due to a flip bifurcation of a k-cycle of h(x), k odd, $k + 2m + 1$, $m \geq 1$.

$\mu = \mu_{k\cdot 2}$. At $\mu = \mu_{k\cdot 2}$ the number and type of cycles of P is that given in (j1)–(j2.1), (i3)–(i5), with $\lambda_a = -1$. Thus, any cycle listed in (j1) undergoes a degenerate bifurcation of flip-type; any cycle listed in (i3) undergoes a standard bifurcation of flip-type; and any cycle listed in (i4) undergoes a bifurcation, generally standard, which may be of flip-type or of fold-type, depending on the value of the integer $n_1(y_j)$. New comments deal only with the cycle of odd period given in (j1.1). That cycle, born as attracting node, becomes an attracting focus when the eigenvalue λ_a crosses the value 0 and becomes negative. Thus at $\mu = \mu_{k\cdot 2}$ this cycle undergoes a Neimark–Hopf bifurcation of resonant type, because its eigenvalues are $\lambda_1 = -i$ and $\lambda_2 = i$ (pure imaginary eigenvalues).

$\mu = \mu_{k\cdot 2} + \varepsilon$. For $\mu = \mu_{k\cdot 2} + \varepsilon$, we get the following cycles of P:

- all those existing at $\mu = \mu_{k\cdot 2}$, where saddles and attracting nodes become repelling nodes, and the attracting focus in (j1.1) becomes a repelling focus, plus the new ones;
- the cycles described in (f1)–(f4) with $k = 2m + 1$.

Remark 8. At $\mu = \mu_{k\cdot 2}$ each of the m cycles of P of even period $2k$ given in (j1) undergoes a bifurcation of degenerate or flip-type ($\lambda_1 = \lambda_2 = -1$), and after bifurcation, for $\mu = \mu_{k\cdot 2} + \varepsilon$, it becomes a repelling node giving rise to two cycles of double period, attracting nodes, and to two cycles of double period, saddles, giving $2m$ of the $(2m + 1)$ cycles listed in (f1) and $2m$ of the $(2m + 1)$ cycles listed in (f2). The remaining cycles (of period $2.2k$), one in (f1) and one in (f2), are related to the bifurcation of the cycle of P of odd period. The cycle of odd period k in (j1.1), from attracting node becomes an attracting focus, and at $\mu = \mu_{k\cdot 2}$ undergoes a Neimark–Hopf bifurcation of resonant type, with $\lambda_1 = -i$ and $\lambda_2 = i$. After bifurcation, this k-cycle becomes a repelling focus giving rise to two cycles of quadruple period $2.2k$, one homogeneous cycle attracting node, and one cycle of mixed type saddle.

We note that a cascade of flip bifurcations starting from a k-cycle of h, with k odd, corresponds to a particular case for P. In fact, the first flip bifurcation involves, for the cycle of P of odd period k, a Neimark–Hopf bifurcation of resonant type (with eigenvalues $\lambda_1 = -i$ and $\lambda_2 = i$). This resonant Hopf bifurcation produces cycles of even period (quadruple), so that the subsequent flip bifurcations of the cycles flip-generated, both for the map h and for the map P, correspond to flip bifurcations of cycles of even period.

In order to give an idea of the extremely high number of bifurcations occurring in the map P we can focus our attention for example to the case in which the 1-d map $h(x)$ is the standard logistic map $h(x) = \mu x(1 - x)$, and consider the value of μ, say μ_3, at which a fold bifurcation of the map $h^3(x)$ occurs, creating two cycles of period $k = 3$. At $\mu = \mu_3 + \varepsilon$ the 2-d map P has the following new cycles (which do not exist for $\mu = \mu_3 - \varepsilon$):

- one homogeneous cycle of period 6 attracting node (j1);
- one homogeneous cycle of period 3 attracting node (j1.1);
- one homogeneous cycle of period 6 repelling node (j2);
- one homogeneous cycle of period 3 repelling node (j2.1);
- three cycles of mixed type of period 6 saddles (i3);
- from (i4): infinitely many cycles saddles of mixed type, obtained by applying property 7 to the attracting 3-cycles of h and to any of the cycles of h which already exist at $\mu = \mu_3 - \varepsilon$ (we know that these are infinitely many, of any period, but clearly numerable, and all repelling);
- from (i5): infinitely many cycles repelling nodes of mixed type, obtained by applying property 7 to the repelling 3-cycles of h and to any of the cycles of h which already exist at $\mu = \mu_3 - \varepsilon$.

We have considered above the case k odd with $k \geq 3$, so that the case $k = 1$ is excluded. Let us briefly comment here on this simple case. Consider a fixed point of h, say $a = h(a)$, which flip bifurcates through the eigenvalue $\lambda = -1$, giving rise to an attracting 2-cycle $\{d_1, d_2\}$. It is easy to see the cycles of P related only to these cycles, a and $\{d_1, d_2\}$, of h. The fixed point (a, a) of P, attracting focus, undergoes a Neimark–Hopf bifurcation of resonant type, with $\lambda_1 = -i$ and $\lambda_2 = i$. After the bifurcation, this fixed point becomes a repelling focus, giving rise to two cycles of period 4: one homogeneous cycle of period 4, attracting node, $\{P^i(d_1, d_1), i = 1, \ldots, 4\}$ and one cycle of mixed type, of period 4, saddle $\{P^i(a, d_1), i = 1, \ldots, 4\}$. If, besides the fixed point a, other cycles of h are present, then (from property 7) other cycles of P exist, of mixed type, which also bifurcate and give rise to other new cycles of P.

2.2. *Trapping sets of p and critical curves*

We have already met in property 2 examples of trapping sets of P. Besides the coordinate axes, the set $\{(x, z): z = h(x)\} \cup \{(x, z): x = z\}$ is a trapping for P. This is just one of the infinitely many trapping sets of P that can be obtained by use of the graph of powers of $h(x)$. In fact, let us consider a point r of the plane belonging to the set of equations of $z = h^k(x)$, say $r = (x, h^k(x))$; then $P(x, h^k((x)) = (h^k(x), h(x))$, that is, r is mapped into a point belonging to the set of equations $x = h^{k-1}(z)$. The reverse also holds, because $P(h^k(x), h(x)) = (h(x), h^{k+1}(x))$ belongs again to the set of equations $z = h^k(x)$. Thus, each set $\{(x, z): z = h^k(x)\}$ and $\{(x, z): x = h^{k-1}(z)\}$, for any $k \geq 1$, is a trapping for P^2, and their union is trapping for P. We have so proved the following property.

Property 8. Each set $\{(x, z): z = h^k(x)\} \cup \{(x, z): x = h^{k-1}(z)\}$, for $k \geq 1$, is trapping for P.

From the structure of the periodic points of cycles of P of homogeneous type, described in properties 5 and 6, it follows that homogeneous cycles of P belong to trapping sets of the kind described above. Let us call *basic trapping sets* those involving powers h^i with $i \leq k$ being k the period of a cycle $\{a_i, i = 1, \ldots, k\}$ of h ($k = 2m$ or $k = 2m + 1$). The homogeneous cycles of P related to this cycle of h are given in (7) if $k = 2m$ and in (7)–(8) if $k = 2m + 1$. In any case, a cycle of P of even period ($2k$) is one listed in (7). Let us consider a periodic point of P of one of the cycles listed in (7). As $P^{2k}(a_j, a_1) = (a_j, a_1)$ we can consider $(a_j, a_1), j \in \{1, \ldots, m\}$. Then, being $a_j = h^{j-1}(a_1)$ we have $(a_j, a_1) = (h^{j-1}(a_1), a_1)$, which implies that the homogeneous cycle of P with periodic point (a_j, a_1) belongs to the set $\{x = h^{j-1}(z)\}$, and thus to the basic trapping set $\{(x, z) : z = h^j(x)\} \cup \{(x, z) : x = h^{j-1}(z)\}$. Writing $a_1 = h^{k-j+1}(a_j)$ we have $(a_j, a_1) = (a_j, h^{k-j+1}(a_j))$, which implies that the homogeneous cycle of P with periodic point (a_j, a_1) belongs to the set $\{z = h^{k-j+1}(x)\}$, and thus to the trapping set $\{(x, z) : z = h^{k-j+1}(x)\} \cup \{(x, z) : x = h^{k-j}(z)\}$.

While for a homogeneous cycle of P of odd period we have, using the same notation as above, $j = m + 1$ and $k = 2m + 1$, so that $(a_{m+1}, a_1) = (h^m(a_1), a_1)) = (a_{m+1}, h^{m+1}(a_{m+1}))$, which implies that the homogeneous cycle of P of odd period with periodic point (a_{m+1}, a_1) belongs to a single basic trapping set: $\{(x, z) : z = h^{m+1}(x)\} \cup \{(x, z) : x = h^m(z)\}$. We have so proved the following property.

Property 9. Each homogeneous cycle of even period listed in (7) belongs to two distinct basic trapping sets. The periodic point $(a_j, a_1), j \in \{1, \ldots, m\}$ belongs to $\{(x, z) : z = h^j(x)\} \cup \{(x, z) : x = h^{j-1}(z)\}$ and to $\{(x, z) : z = h^{k-j+1}(x)\} \cup \{(x, z) : x = h^{k-j}(z)\}$. While a homogeneous cycle of P of odd period, given in (8), belongs to the single basic trapping set $\{(x, z) : z = h^{m+1}(x)\} \cup \{(x, z) : x = h^m(z)\}$.

The 2-d map P is clearly an endomorphism with a nonunique inverse iff the 1-d map h is an endomorphism with a nonunique inverse. In such a case, the critical curves of P are related to the critical points of h (we follow the index notation and the terminology used in [10, 11]). Let s_{-1} be a critical point of rank-0 of h (i.e. a point where $h(x)$ attains a local minimum or a local maximum). Let $s_0 = h(s_{-1})$ be the related critical point of rank-1 and, in general, let $s_i = h^{i+1}(s_{-1})$ be the related critical points of rank-($i + 1$), $i \geq 1$. Then, the straight line of equation $x = s_{-1}$ is a critical line of rank-0 of P, say LC_{-1}. The corresponding critical curves LC_i of P are straight lines parallel to the coordinate axes, "issuing" from the critical points of h: LC_0 is the line $z = s_0$, LC_1 the line $x = s_0$, LC_2 the line $z = s_1$, LC_3 the line $x = s_1$, and so on.

In particular, if I is an absorbing interval of h (trapping or invariant) bounded by the critical points s_0 and s_1, then the segments resulting from the intersection of the critical lines $z = s_0$, $x = s_0$, $z = s_1$ and $x = s_1$, give the boundary of an absorbing square $I \times I$ in the phase plane (x, z), trapping or invariant for P. Moreover, as we shall recall in Section 3, inside a wide absorbing interval, invariant for h, generally there exists a set of *cyclical absorbing intervals* (trapping or invariant for h) of order k, $k > 1$, of the form $I = \bigcup_{i=1}^{k} I_i$, where $I_{i+1} = h(I_i)$, $h(I_k) \subseteq I_1$, that is, $h^k(I_i) \subseteq I_i$, for $i = 1, \ldots, k$, and the boundary of each interval I_i consists of critical points of h. The existence of such cyclical absorbing intervals of order k, for the 1-d map h, implies the existence of cyclical absorbing rectangles for the 2-d map, bounded by segments of critical lines. The number of distinct sequences of these rectangles, and their order, can be determined by reasoning as in the properties 5 and 6 above. That is, all the Cartesian products $I_i \times I_j$, for $i, j = 1, \ldots, k$, give rectangles in the phase plane (x, z) belonging to cyclical sets of the 2-d map P, obtained as follows.

Property 10. Let $I = \bigcup_{i=1}^{k} I_i$ be a set of cyclical absorbing intervals of h, of order k. If k is even, say $k = 2m$, then the 2-d map P admits m distinct sequences of cyclical absorbing rectangles of order $2k$, given by

$$\begin{aligned} &P^i(I_1 \times I_1), \qquad i = 1, \ldots, 2k \\ &P^i(I_2 \times I_1), \qquad i = 1, \ldots, 2k \\ &\qquad\vdots \\ &P^i(I_m \times I_1), \qquad i = 1, \ldots, 2k. \end{aligned} \tag{10}$$

If k is odd, $k = 2m + 1$, then P admits m distinct sequences of cyclical absorbing rectangles of order $2k$, given in (10), plus one sequence of cyclical rectangles of odd period k, given by

$$P^i(I_{m+1} \times I_1), \qquad i = 1, \ldots, k. \tag{11}$$

Applications of the properties of P-maps will be discussed in Section 4, where an example is shown, which refers to a 1-d map $h(x)$ of bimodal shape. As we shall be advantaged from the knowledge of the bifurcations occurring in the 1-d map h, in the next section we introduce the map $h(x)$ of applicative interest and perform the analysis of its dynamics.

3. SOME PROPERTIES OF THE 1-d MAP $h(x)$ OF BIMODAL SHAPE

In this section we consider the 1-d map defined in [33]

$$x' = h(x); \qquad h(x) = -Dx + Bg(Fx), \tag{12}$$

where

$$g(s) = \begin{cases} 4\mu s(1 - s) & \text{if } 0 < s < 1; (\mu > 0) \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

The parameters D, B and F satisfy the following constraints $0 < D < 1$, $-1 < B < 0$ and $F < 0$. Once the values of D, B and F are fixed, we consider the dynamics of the one-parameter family of maps $h(x)$ defined in (12)–(13) as a function of the parameter μ (strength of the impulse function in the applicative model). In all the figures reported in this paper, the numerical examples are obtained with the values $D = 0.62386015$, $B = -0.790837347$, $F = -0.85446789$, while the value of μ is reported in each figure caption. The qualitative shape of h is shown in Fig.1.

We note first that both for the 1-d map h in (12) and the related 2-d map P, the dynamics are always bounded. In fact, $h(x)$ is linear for $Fx \geq 1$ (i.e. $x \leq 1/F$) and for $Fx \leq 0$ (i.e. $x \geq 0$), with slope equal to $-D$. Being $-1 < -D < 0$, no trajectory of the 1-d map $h(x)$ can be divergent, and being $P^2(x, z) = (h(x), h(z))$, no trajectory of the 2-d map P can be divergent.

The origin is the unique fixed point of h, globally attracting, for $0 < \mu < \mu_1 = (1 + D)/4BF$ (in the case considered here $\mu_1 \simeq 0.6$), and in this range h is an invertible map. For $\mu > \mu_1$ one more fixed point of h exists, given by $x^* = (1/F)(1 - (1 + D)/4\mu BF)$, at first locally attracting. Moreover, for $\mu > \mu_1$ the map h possesses two local extrema, and it becomes an endomorphism of type $Z_1 - Z_3 - Z_1'$. A local maximum occurs at the fixed point O, which is thus a critical point of rank-k, for any $k \geq 0$. A local minimum occurs at the critical point of

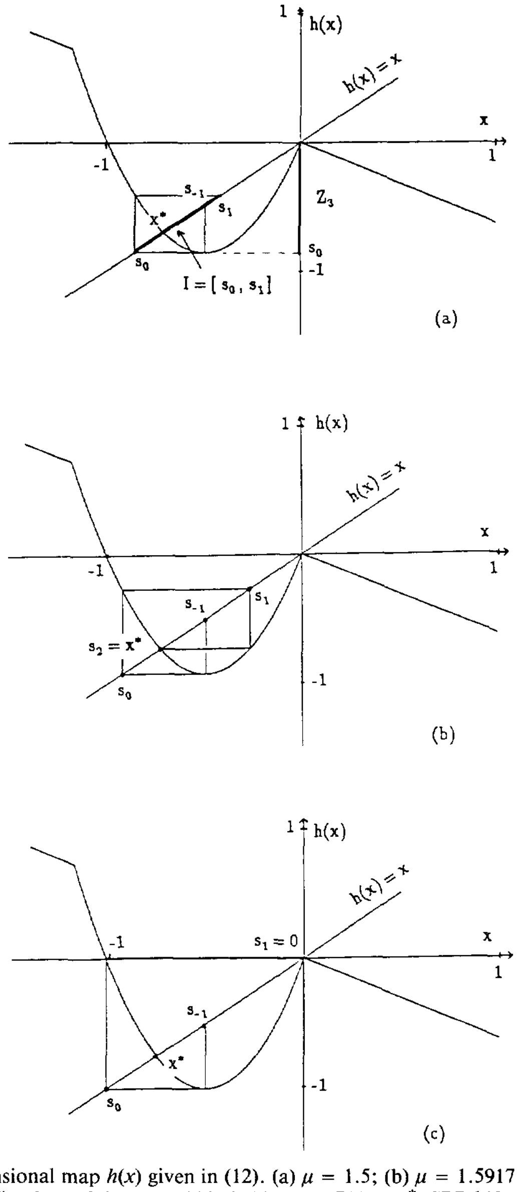


Fig. 1. One-dimensional map $h(x)$ given in (12). (a) $\mu = 1.5$; (b) $\mu = 1.5917 \simeq \mu^*_{1.2}$, SBR bifurcation of x^*, closure of the first box of the second kind; (c) $\mu = 1.711 \simeq \mu^*_1$, SBR bifurcation of the origin, closure of the first box of the first kind.

rank-0 given by $s_{-1} = \frac{1}{2}(1 - D/4\mu BF)$. Its images $f^i(s_{-1}) = s_{i-1}$, $i \geq 1$, are the critical points of rank-i. The interval Z_3, locus of points for which three distinct rank-1 preimages exist, is bounded by the critical points of rank-1: $Z_3 =]s_0, 0[$ (see Fig. 1a). On increasing μ beyond μ_1, the fixed point x^* crosses the point s_{-1} and its eigenvalue becomes negative. The flip bifurcation of x^* occurs at $\mu = \mu_{1.2} = (3 + D)/4BF$ ($\mu_{1.2} \simeq 1.34$). For $\mu > \mu_{1.2}$ the critical points s_0 and s_1 give the boundary of an absorbing interval, $I = [s_0, s_1]$, which becomes invariant when the critical point s_1 crosses s_{-1} (as in Fig. 1a). It is clear that as long as the interval $I = [s_0, s_1]$ belongs to the interval $\bar{Z}_3$ (where overbar denotes the closure), the map h is homeomorphic to the quadratic Myberg's map $f(x) = x^2 - c$, or, equivalently, to the logistic map $f(x) = vx(1 - x)$. Thus, the same dynamics occur in I, and I being globally absorbing, except for the origin and its preimages, we can state that the dynamics of h are completely known in this range. A complex sequence of bifurcations of type "box-within-a-box" occurs, as described by Mira in 1975 [3] and we refer to [11] for its complete description. Let us recall here only some elementary features, to be used later. The first box of the first kind starts at μ_1, when the attracting fixed point x^* appears, bifurcating from the other fixed point O. The end of this box occurs at μ_1^*, when the critical point s_1 merges into O, and is characterized by the first homoclinic bifurcation of the fixed point O (in our example $\mu_1^* \simeq 1.71$, see Fig. 1c). The first box of the second kind starts at the flip bifurcation of x^*, at $\mu_{1\cdot2}$, and ends at $\mu_{1\cdot2}^*$, when the critical point s_2 merges into x^*, marking the first homoclinic bifurcation of the fixed point x^* (in our example $\mu_{1\cdot2}^* \simeq 1.59$, see Fig. 1b). Inside these boxes, a complex structure of sequences of boxes of the first kind and of the second kind occur, all having a self-similar structure. In general, a box of the first kind $\Omega_k = [\mu_k, \mu_k^*]$ opens at the fold bifurcation which gives rise to a couple of k-cycles of h, say $\{a_i,\ i = 1, \ldots, k\}$, attracting, with eigenvalue $0 < \lambda_a < 1$, and $\{b_i,\ i = 1, \ldots, k\}$, repelling, with eigenvalue $\lambda_b > 1$. Then the attracting cycle undergoes a flip bifurcation ($\lambda_a = -1$) at $\mu_{k\cdot2}$, and this value opens the corresponding box of the second kind $\Omega_{k\cdot2} = [\mu_{k\cdot2}, \mu_{k\cdot2}^*] \subset \Omega_k$. The closure of a box occurs when a critical point s_i merges for the first time into the repelling cycle which started the box, marking a homoclinic bifurcation of this cycle. That is at $\mu = \mu_{k\cdot2}^*$ the first homoclinic bifurcation of the cycle $\{a_i,\ i = 1, \ldots, k\}$ (made up of critical points) occurs, while a homoclinic bifurcation of the cycle $\{b_i,\ i = 1, \ldots, k\}$ (made up of critical points) occurs at $\mu = \mu_k^*$. For a description of the relation between the "box-within-a-box" bifurcation structure and the homoclinic bifurcations of cycles we refer to [36]. Let us call (as in [36], following Marotto's notation [37]) the first homoclinic bifurcation of a cycle as its snap-back-repeller bifurcation, SBR bifurcation for short. We recall that the SBR bifurcation of a cycle of the quadratic map, cycle different from the origin, is followed by other homoclinic bifurcations, or homoclinic explosion (causing the sudden appearance of infinitely many new homoclinic orbits of the same cycle). Such values are always characterized by the merging of a critical point of h into the same repelling cycle.

It is difficult to observe numerically the complex sequence of bifurcations, because after the first Myrberg's cascade [1] (or Feigenbaum sequence [7]) of flip bifurcations, started from the flip bifurcation of x^*, and accumulating at $\mu = \mu_{1s}$, we enter in the chaotic regime. We know that at any value of μ beyond μ_{1s}, $\mu \in]\mu_{1s}, \mu_1^*[$, the attracting set of h is either a cycle (with periodic points in I) or a chaotic set (a critical chaotic set, or some cyclical chaotic intervals, belonging to I). When the second case occurs we say that the dynamics of h are chaotic in the strict sense, or that strict chaos occurs. However, also in the first case (i.e. when an attracting cycle of h exists), the computed trajectory seems often aperiodic (apart from values of μ in rare boxes, or windows). This is due to the fractal structure of the basin of attraction of the cycle, whose boundary consists in an invariant fuzzy set on which h is chaotic. In such cases we say that nonstrict chaos occurs (the computed trajectories appear as chaotic, although an attracting cycle exists). Strict chaos occurs only at bifuraction values. Among such bifurcation values (which include, for example, μ_{1s} and other similar values μ_{ks}, as well as values of nonclassical bifurcations [11]), there are also the ends of boxes of the first kind and of the second kind.

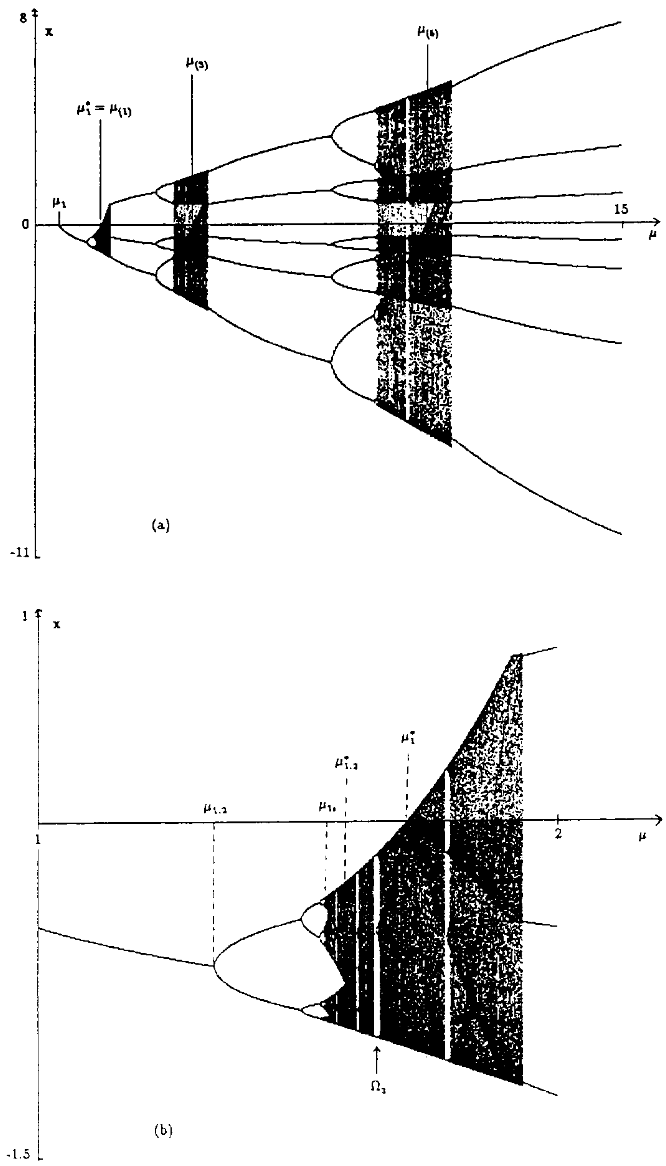
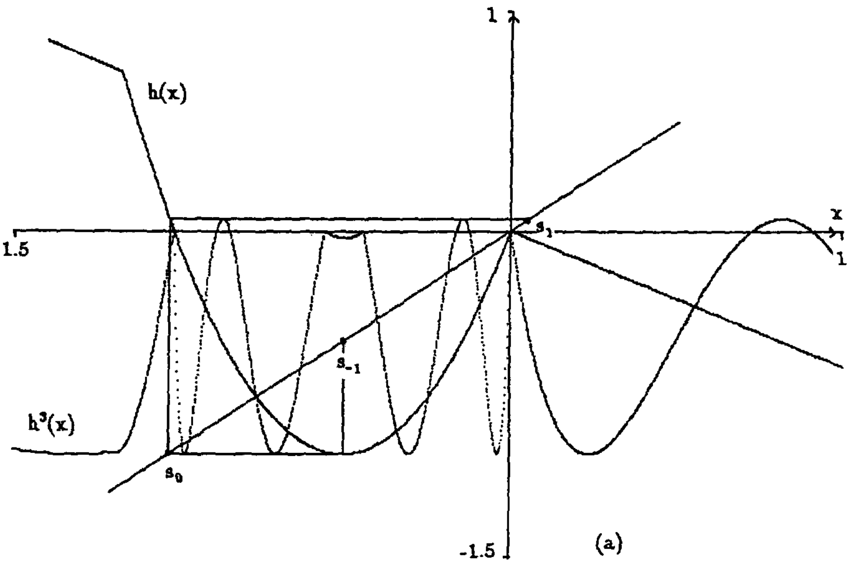


Fig. 2. (a) Bifurcation diagram of $h(x)$ for $0 < \mu < 15$; (b) enlargement of a portion of (a), for $1 < \mu < 2$. $\mu_1 \simeq 0.6$, $\mu_{1.2} \simeq 1.34$, $\mu^*_{1.2} \simeq 1.5917$, $\mu^*_1 \simeq 1.711$, $\mu_{(3)} \simeq 4.033$, $\mu_{(5)} \simeq 10$.

At $\mu = \mu^*_{k\cdot 2}$ and $\mu = \mu^*_k$ the dynamics of h are chaotic in cyclical intervals of order k, bounded by critical points, invariant for h^k. A rough idea of the bifurcations comes from the bifurcation diagram of $h(x)$ as a function of μ, reported in Fig. 2b, which shows an enlargement of Fig. 2a. We note that in such bifurcation diagrams it is difficult to distinguish between values of μ of strict chaos or of nonstrict chaos.



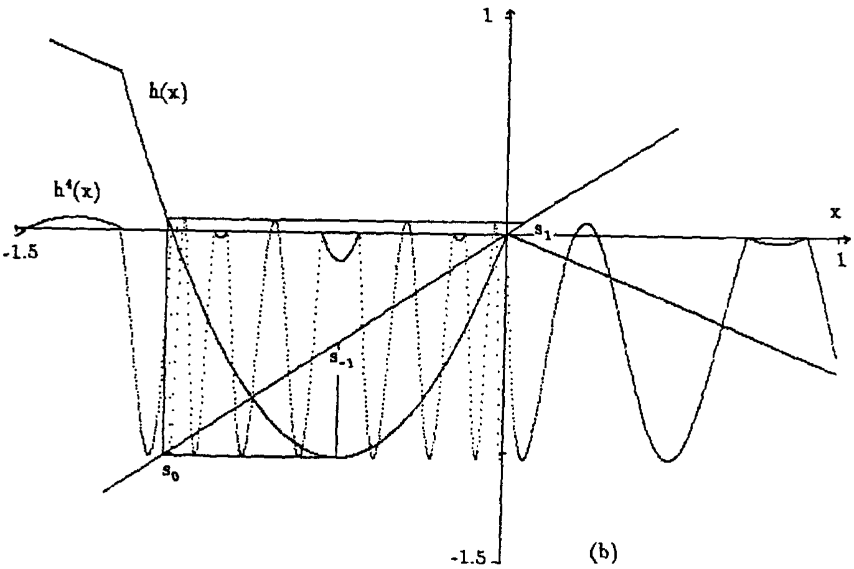


Fig. 3. Graph of $h^3(x)$ in (a) and of $h^4(x)$ in (b), at $\mu = 1.73 > \mu_1^*$.

As stated above, in the range $\mu_{,1} < \mu < \mu_1^*$ we have $I = [s_0, s_1] \subset \bar{Z}_3 = [s_0, 0]$, and $I = \bar{Z}_3$ at $\mu = \mu_1^*$. This SBR bifurcation of the origin determines also the end of the range in which the cubic shape of h plays no role. As O is also a critical point of h, and the merging of a critical point s_i into a different critical point (here the origin O), causes an important qualitative change. For $\mu > \mu_1^*$ the critical point O belongs to the absorbing interval $I = [s_0, s_1]$, as well

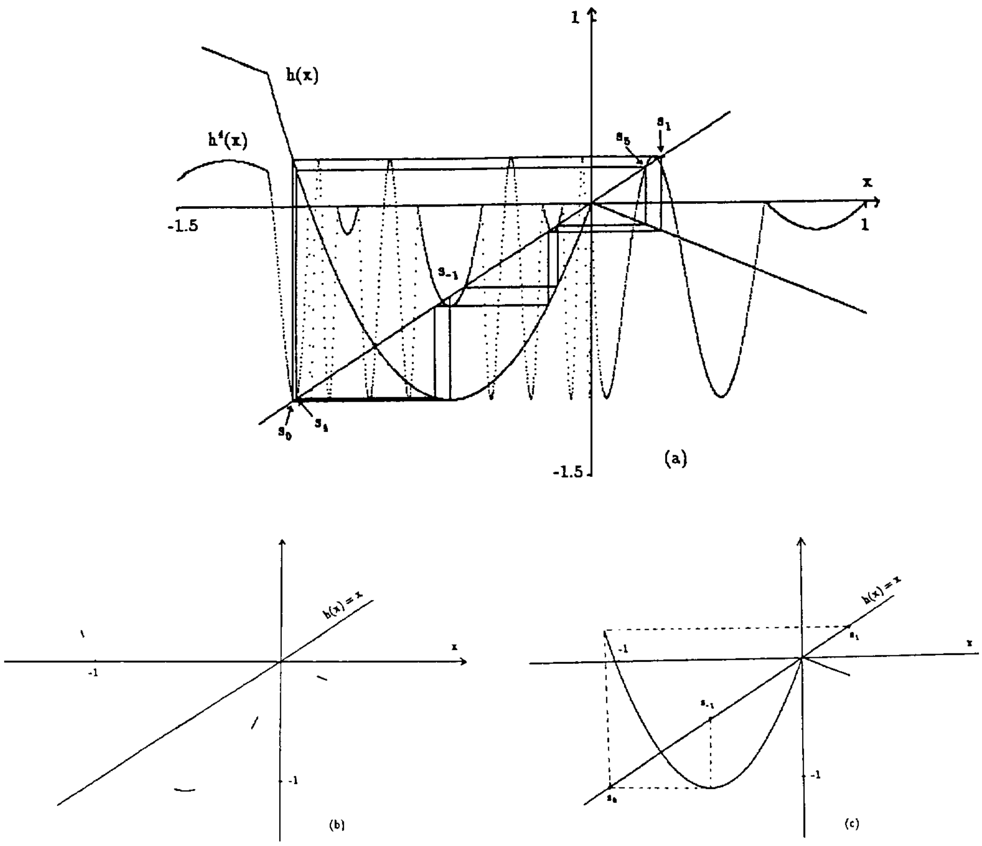


Fig. 4. $\mu = \mu_4^* = 1.793$. (a) The critical points $\{s_4, s_5, s_6, s_7\}$ give the repelling 4-cycle born at the beginning of the box, at $\mu = \mu_4 = 1.7837$. (b) A trajectory of $h(x)$ at $\mu = \mu_4^* = 1.793$, when the dynamics are chaotic on 4-cyclic intervals. (c) A trajectory of $h(x)$ after the SBR bifurcation, at $\mu = 1.794 > \mu_4^*$, exhibiting aperiodic behaviour in the whole absorbing interval $I = [s_0, s_1]$.

as infintely many preimages of O, of any rank. This causes the appearance of new foldings in the graph of $h^k(x)$ for $k \geq 3$, that is, new maxima, where $h^k(x)$ assumes the value 0, and new local minima. See the graph of $h^3(x)$ in Fig. 3a and that of $h^4(x)$ in Fig. 3b for μ beyond μ_1^*. As μ increases, these foldings in the graph of h^k approach the bisectrix causing a fold bifurcation of h^k, high values of k occurring first. Each of these fold bifurcations opens a box of first kind analogous to the box $\Omega_k = [\mu_k, \mu_k^*]$ of the previous regime. For example, at $\mu_4 \simeq 1.7837$ a fold bifurcation of h^4 starts a box of the first kind which closes at $\mu_4^* \simeq 1.793$, when the critical point s_4 falls into the repelling 4-cycle born at the beginning of the box, as shown in Fig. 4a. Note that even if h has a cubic shape when μ belongs to a box, we can determine cyclical absorbing intervals bounded by critical points s_i inside which h^k is homeomorphic to the logistic map. The cubic shape of h becomes relevant after the closure of a box, even if we are in a regime of nonstrict chaos, the computed trajectories seem aperiodic in the wide absorbing interval $I = [s_0, s_1]$. For example, the trajectory shown in Fig. 4c illustrates the "explosion"

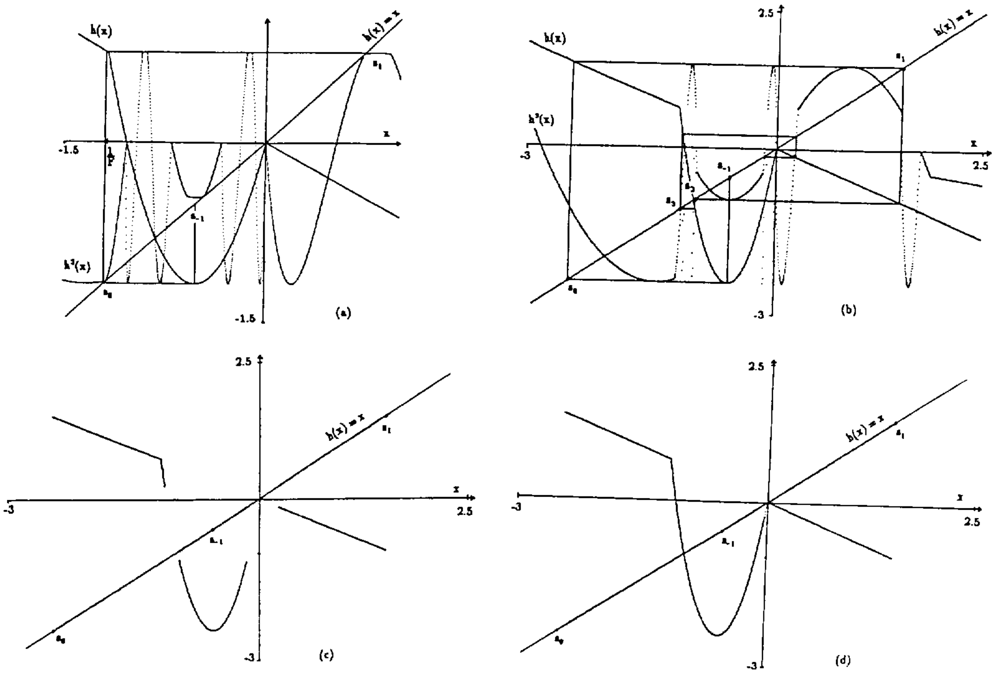


Fig. 5. (a) μ = 1.935. Fold bifurcation of $h^3(x)$; s_0 is just below $1/F$. (b) μ = 3.53, SBR bifurcation of the repelling 3-cycle born at μ = 1.935, now made up of critical points, $\{s_3, s_4, s_5\}$. (c) A trajectory of $h(x)$ at μ = 3.53, the dynamics are chaotic on 3-cyclic intervals. (d) A trajectory of $h(x)$ after the bifurcation, at μ = 3.54, with aperiodic behaviour in the whole absorbing interval $I = [s_0, s_1]$.

which follows the closure of the box of the 4-cycle of Fig. 4b, where h has 4-cyclic chaotic intervals. The explosion is due to the fact that outside the cyclical absorbing intervals, infinitely many repelling cycles exist, forming a repelling invariant set on which h is chaotic, which gives the boundary, fuzzy (or chaotic), of the basin of attraction of the absorbing intervals. The homoclinic bifurcation occurring at the closure of a box, corresponds to a contact between the cyclical chaotic intervals and the boundary of their basin of attraction, also called "crisis" by Grebogi *et al.* [38–40]. After the contact, a trajectory escapes the "old" cyclical absorbing intervals and is confined to the wider absorbing interval $I = [s_0, s_1]$.

A fold bifurcation of h^3 in what we may call "second regime", opens a box of the first kind at $\mu \simeq 1.935$. This fold bifurcation is particular because it occurs soon after the crossing of the critical point s_0 through the point $1/F$ (after which the critical point s_0 will enter the linear part of the shape of $h(x)$). This crossing causes a flat minimum in h^3 (see Fig. 5a), which, in its turn, causes a long interval of μ-values with an attracting 3-cycle, as can be seen in Fig. 2a. The flip bifurcation of the 3-cycle occurs at $\mu \simeq 3.067$ (beginning of the corresponding box of the second kind), giving an attracting 6-cycle, which flip bifurcate at $\mu \simeq 3.51$, and so on. The end of this box of a 3-cycle occurs at $\mu \simeq 3.53$ (see Fig. 5b), when the critical point s_3 falls into the repelling 3-cycle born at the beginning of the box. After this homoclinic bifurcation we see an explosion of the chaotic trajectory in the wider absorbing interval $I = [s_0, s_1]$ (see Fig. 5c, d).

As we have said before, the cubic shape of h determines the structure of the bifurcations for $\mu > \mu_1^*$ because the existence of two critical points of rank-1, s_0 and O characterizes the foldings in the shape of powers of $h(x)$. Let us summarize the essential features:

- at $\mu = \mu_{(1)}$ $(=\mu_1^* \simeq 1.71)$ we have $s_1 = h^2(s_{-1}) = 0$, after which new foldings appear in h^3, and in the graph of $h^k(x)$ for $k \geq 3$; $s_0 = 1/F$ occurs before the fold bifurcation of h^3, generating a wide box of the 3-cycle;
- at $\mu = \mu_{(3)}$ $(\mu_{(3)} \simeq 4.033)$ we have $s_3 = h^4(s_{-1}) = 0$, after which new foldings appear in h^5, and in the graph of $h^k(x)$ for $k \geq 5$; $s_2 = 1/F$ occurs before the fold bifurcation of h^5, generating a wide box of the 5-cycle;
- at $\mu = \mu_{(5)}$ $(\mu_{(5)} \simeq 10)$ we have $s_5 = h^6(s_{-1}) = 0$, after which new foldings appear in h^7, and in the graph of $h^k(x)$ for $k \geq 7$; $s_4 = 1/F$ occurs before the fold bifurcation of h^7, generating a wide box of the 7-cycle; and so on.

The regimes $[\mu_{(1)}, \mu_{(3)}]$, $[\mu_{(3)}, \mu_{(5)}]$, ..., are infinitely many because as μ increases all the critical points of odd index merge eventully into the origin, that is, $s_i = h^{i+1}(s_{-1}) = 0$ occurs for all the odd values of i. We note that while in the quadratic map the SBR bifurcation of the origin is the last one (beyond which the attraction set is at infinity), in the map with two critical points of rank-0 also the SBR bifurcation of the origin (which is its first homoclinic bifurcation) is followed by other homoclinic bifurcations. In fact, the values which determine the above regimes, $\mu = \mu_{(i)}$ for i odd, are values of homoclinic bifurcations of the origin, characterized by the merging of the critical point s_i into the origin itself.

The dynamics of h are always bounded in the globally absorbing interval $I = [s_0, s_1]$; in it we can detect k-cyclical absorbing intervals (when μ belongs to the box related to some k-cycle), where h^k is homeomorphic to a quadratic map.

4. SOME PROPERTIES OF THE 2-d MAP P RELATED TO (12)

Let us turn now to the dynamics of the 2-d map P

$$(x, z) \rightarrow (z, h(x)), \text{ where } h(x) \text{ is defined in (12) with (13)} \tag{14}$$

so that we can refer to the properties of the 1-d map discussed in the previous section, and apply the results of Section 2.

The origin is the unique fixed point of P, globally attracting, for $0 < \mu < \mu_1 = (1 + D)/4BF$, and in this range P is an invertible map. For $\mu > \mu_1$ one more fixed point of P exists, S^*, given by $z^* = x^* = (1/F)(1 - (1 + D)/4\mu BF)$ (at first an attracting node), and a 2-cycle saddle given by $\{(x^*, 0), (0, x^*)\}$. For $\mu > \mu_1$ the map P becomes an endomorphism of type $Z_1 - Z_3 - Z_1'$. The region Z_3 is the rank-1 image of the strip bounded by the two critical lines of rank-0, called LC_{-1}, of equation $x = s_{-1}$ and $x = 0$. Thus, the region Z_3 is the strip bounded by the two critical lines of rank-1, called LC_0, of equation $z = s_0$ and $z = 0$. To the absorbing interval $I = [s_0, s_1]$ of h corresponds, for the 2-d map P, the absorbing square $I \times I$, bounded by the critical lines $z = s_0, x = s_0, z = s_1, x = s_1$. The fixed point S^* (k-cycle for $k = 1$, odd) becomes an attracting focus and then a repelling focus via a resonant Neimark–Hopf bifurcation at $\mu = \mu_{1.2}$, from which an attracting 4-cycle node $\{(x_1, x_1), (x_1, x_2), (x_2, x_2), (x_2, x_1)\}$ (where $x_1 - x_2$ is the 2-cycle of h) and a 4-cycle saddle $\{(x_1, x^*), (x^*, x_2), (x_2, x^*), (x^*, x_1)\}$ originate. At the same time the 2-cycle saddle becomes a repelling node via flip bifurcation, giving rise to a 4-cycle saddle $\{(x_1, 0), (0, x_2), (x_2, 0), (0, x_1)\}$. In this way, by use of the properties 5, 6 and 7 and of the observations related to fold and flip bifurcations of h, we can

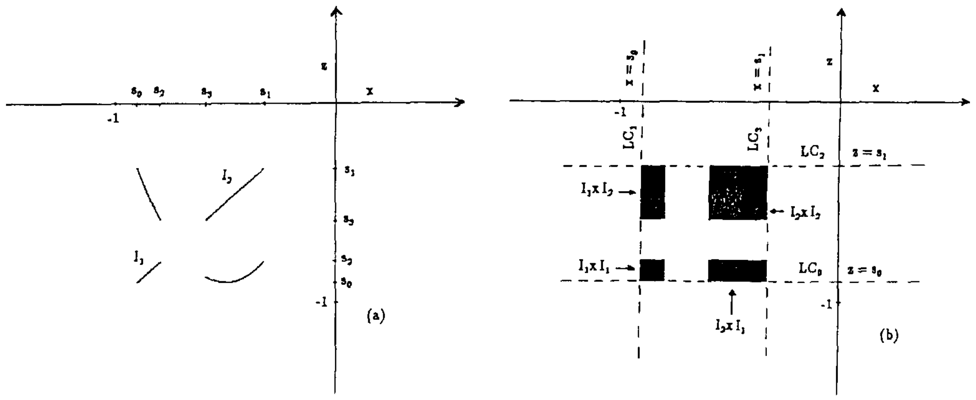


Fig. 6. Trajectories of P defined in (14) at $\mu = 1.56$ (h has the 2-cyclic absorbing intervals $I_1 - I_2$, $I_1 = [s_0, s_2]$, $I_2 = [s_3, s_1]$). (a) A trajectory of P on the trapping set $\{(x, z) : z = h(x)\} \cup \{(x, z) : x = z\}$. (b) An aperiodic computed trajectory in the 4-cyclic rectangles $I_i \times I_j$, $i, j = 1, 2$.

analyse the complex structure of the bifurcations of P. In particular, when the 2-cycle $x_1 - x_2$ of h flip bifurcate into an attracting 4-cycle, then the regime of multistability starts for the 2-d map P (this flip bifurcation gives rise to two distinct attracting 8-cycles of P). For any value of μ in the first regime, $\mu_1 < \mu < \mu_1^*$, we know that either an attracting k-cycle ($k > 2$) of h exists (and thus, either $k/2$, if k is even, or $(k + 1)/2$, if k is odd, attracting cycles of P exist), or strict chaos occurs in some critical set or in some cyclical intervals (and thus P is chaotic in critical sets or in cyclical rectangles). At $\mu = \mu_1^*$, P is chaotic (strict chaos) in the whole square $I \times I$. The same comments hold for $\mu > \mu_1^*$ as we have seen that also in the subsequent regimes a complex structure of boxes exists, related to cycles of any period, $k \geq 3$, $k \geq 5$, $k \geq 7, \ldots$.

Figure 6a shows an example of a trajectory on the trapping set $\{(x, z) : z = h(x)\} \cup \{(x, z) : x = z\}$, for $\mu_{1.2} < \mu < \mu_{1.2}^*$. An initial condition outside this trapping set gives the trajectory of Fig. 6b, that is, P shows chaotic dynamics (nonstrict chaos) in the 4-cyclic absorbing rectangles $I_i \times I_j$, $i, j = 1, 2$ (which correspond to the 2-cyclic absorbing intervals $I_1 - I_2$ of h, $I_1 = [s_0, s_2]$, $I_2 = [s_3, s_1]$). Trajectories of P for $\mu_3 < \mu < \mu_3^*$, μ in the first regime, in the box of a 3-cycle of h, just before the homoclinic bifurcation which closes the box (when $h(x)$ has 3-cyclic absorbing intervals $I_1 - I_2 - I_3$, $I_1 = [s_0, s_3]$, $I_2 = [s_4, s_1]$, $I_3 = [s_5, s_2]$) are shown in Fig. 7.

Figure 7a shows a set of 6-cyclical absorbing rectangles $P^i(I_1 \times I_1)$, $i = 1, \ldots, 6$; Fig. 7b shows a set of 3-cyclical absorbing rectangles $P^i(I_2 \times I_1)$, $i = 1, 2, 3$ (property 10 of Section 2); Fig. 7c shows the attractor in the trapping set $\{(x, z) : z = h^4(x)\} \cup \{(x, z) : x = h^3(z)\}$, while Fig. 7d shows the attractor in the trapping set $\{(x, z) : z = h^5(x)\} \cup \{(x, z) : x = h^4(z)\}$.

Figure 8 shows some trajectories of P for μ just beyond μ_3^*; in Fig. 8a we observe the explosion of the trajectory in the whole absorbing $I \times I$, $I = [s_0, s_1]$; in Fig. 8b the trajectory belongs to the trapping set $\{(x, z) : z = h^4(x)\} \cup \{(x, z) : x = h^3(z)\}$ and in Fig. 8c the trajectory belongs to the trapping set $\{(x, z) : z = h^5(x)\} \cup \{(x, z) : x = h^4(z)\}$.

Figure 9 shows trajectories of P corresponding to the trajectories of $h(x)$ in Fig. 4 (μ at the end of a box of a 4-cycle). The two distinct 8-cyclic absorbing rectangles of P are shown in Fig. 9a and Fig. 9b, while Fig. 9c shows the explosion of a chaotic trajectory in the absorbing square $I \times I$, soon after the end of the box.

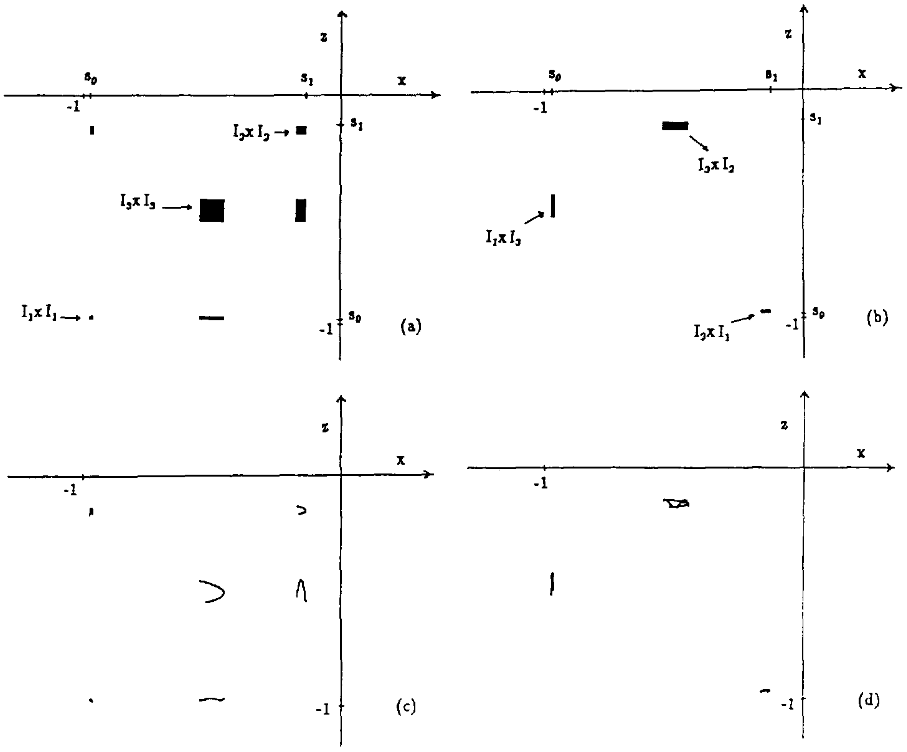


Fig. 7. Trajectories of P defined in (14) at $\mu = 1.657$ ($<\mu_3^*$), for which h has 3-cyclic absorbing intervals $I_1 - I_2 - I_3$. (a) A trajectory in the 6-cyclic absorbing rectangles $P^i(I_1 \times I_1)$, $i = 1, \ldots, 6$. (b) A trajectory in the 3-cyclic absorbing rectangles $P^i(I_1 \times I_1)$, $i = 1, 2, 3$. (c) A trajectory in the trapping set $\{(x, z) : z = h^4(x)\} \cup \{(x, z) : x = h^3(z)\}$. (d) A trajectory in the trapping set $\{(x, z) : z = h^5(x)\} \cup \{(x, z) : x = h^4(z)\}$.

5. CONCLUSIONS

We have taken into consideration a family of 2-d maps P of the form $(x, z) \to (z, h(x))$, where $h(x)$ is a 1-d endomorphism defined by a continuous map, piecewise continuously differentiable. A variety of interesting dynamical behaviours in this family are pointed out, including multistability, i.e. coexistence of distinct attracting sets (cycles or cyclical chaotic rectangles); degenerate bifurcations of cycles, and the existence of infinitely many trapping sets in the form of graphs of functions (on which invariant sets can be defined). Moreover, the cycles of a P-map can be classified as a function of the cycles of the 1-d map $x \to h(x)$. An applicative example in which the 1-d map $h(x)$ is an endomorphism with three inverses, has been discussed in detail, by exploiting the relation between the bifurcation structure of the invariant sets and the structure of the critical points of $h(x)$. Particular emphasis has been put on homoclinic bifurcations because of their role in the transition from regular to chaotic dynamics, both in the 1-d dynamics of $h(x)$ and in the 2-d dynamics of P. The bifurcations occurring in the 2-d map P, described by the properties of Section 2, have been shown by the chosen applicative example.

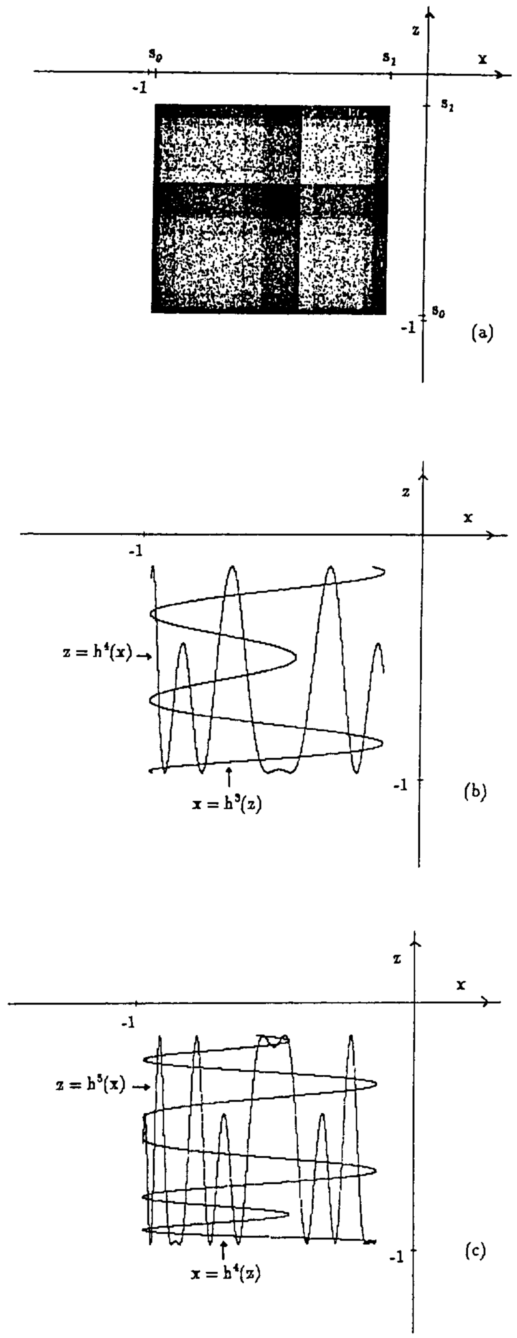


Fig. 8. Trajectories of P defined in (14) at $\mu = 1.658$ ($>\mu_3^*$). (a) An aperiodic trajectory in the absorbing square $I \times I$, $I = [s_0, s_1]$. (b) A trajectory in the trapping set $\{(x, z) : z = h^4(x)\} \cup \{(x, z) : x = h^3(z)\}$. (c) A trajectory in the trapping set $\{(x, z) : z = h^5(x)\} \cup \{(x, z) : x = h^4(z)\}$.

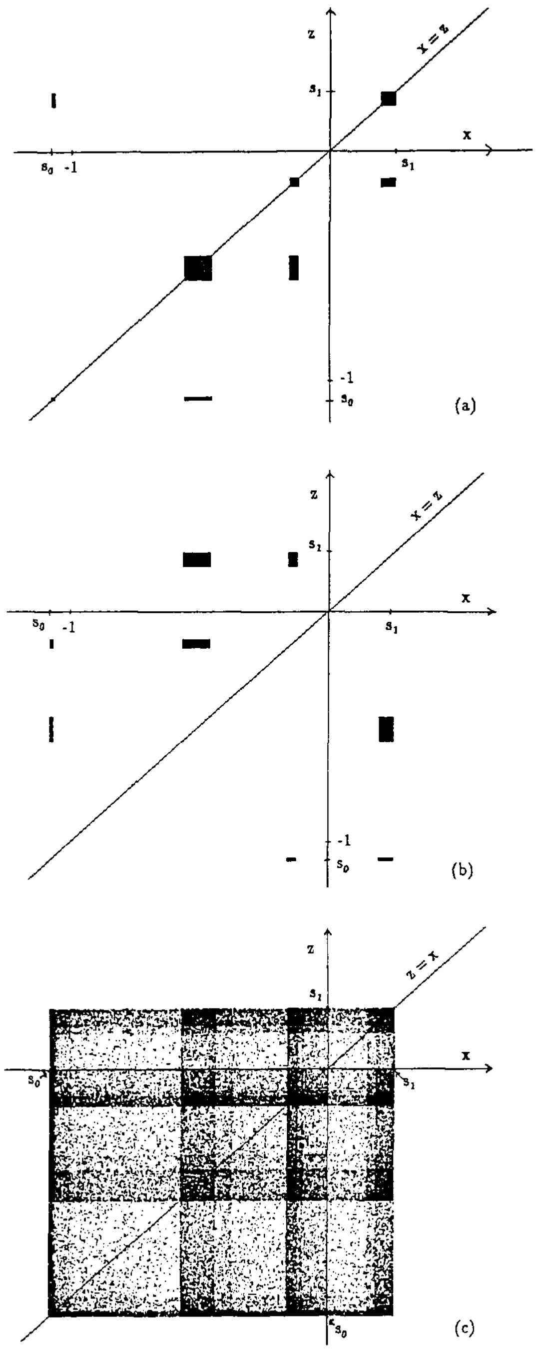


Fig. 9. Trajectories of P at $\mu = 1.793$, at which h has 4-cyclic absorbing intervals $I_1 - I_2 - I_3 - I_4$. (a) 8-Cyclic absorbing rectangles $P^i(I_1 \times I_1)$, $i = 1, \ldots, 8$. (b) 8-Cyclic absorbing rectangles $P^i(I_2 \times I_1)$, $i = 1, \ldots, 8$. (c) $\mu = 1.794$, aperiodic trajectory in the absorbing square $I \times I$, $I = [s_0, s_1]$.

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REFERENCES

1. MYRBERG P. J., Iteration der reellen Polynome zweiten Grades I, II, III, *Ann. Acad. Sci. Fenn.* **A256**, 1-10 (1958); **A268**, 1-10 (1959); **A336**, 1-10 (1963).
2. LI T. Y. & YORKE J., Period three implies chaos, *Am. Math. Mon.* **82**, 985-992 (1975).
3. MIRA C., Accumulations de biforcations et structure boites emboitées dans les récurrences et transformations ponctuelles, in *Proc. of the VII International Conference on Nonlinear Oscillations.* (ICNO), Berlin, Sept. 1975, pp. 81-93. Akademic Verlag, Berlin (1977).
4. GUMOWSKI I. & MIRA C., Sur les récurrences, ou trasformations ponctuelles, du premier ordre, avec inverse non-unique, *C. r. Acad. Sci. Paris* **A280**, 905-908 (1975).
5. GUMOWSKI I. & MIRA C., Accumulations de biforcations dans une récurrence, *C. r. Acad. Sci. Paris* **A281**, 45-48 (1975).
6. GUMOWSKI I. & MIRA C., Box-within-a-box bifurcation structure and the phenomenon of chaos, *Informatica* **76** (1976).
7. FEIGENBAUM M. J., Quantitative universality for a class of nonlinear transformations, *J. Stat. Phys.* **19**, 25-52 (1978).
8. FEIGENBAUM M. J., Universal behavior in nonlinear systems, *Los Alamos Sci.* **1**, 4-27 (1980).
9. GUCKENHEIMER J., On the bifurcations of maps of the interval, *Inv. Math.* **39**, 165 (1977).
10. GUMOWSKI I. & MIRA C., *Dynamique Chaotique.* Cepadues, Toulouse (1980).
11. MIRA C., *Chaotic Dynamics.* World Scientific, Singapore (1987).
12. COLLET P. & ECKMANN J. P., *Iterated Maps on the Interval as Dynamical Systems.* Birkhauser, Boston (1980).
13. DEVANEY R. L., *An Introduction to Chaotic Dynamical Systems.* Addison-Wesley, Reading (1989).
14. FOURNIER-PRUNARET D., Structure de biforcations d'un endomorphisms défini par un polynome du troisième degré, *C. r. Acad. Sci. Paris* **294**(1), 455-458 (1982).
15. FOURNIER-PRUNARET D., The bifurcation structure of a family of degree one circle endomorphisms, *Int. J. Bifurcation Chaos* **1**(4), 823-838 (1991).
16. MIRA C., Sur quelques propriétés de la frontiére de stabilité d'un point double d'une récurrence et sur un cas de biforcation de cette frontiére, *C. r. Acad. Sci. Paris* **A262**, 951-954 (1966).
17. MIRA C. & ROUBELLAT F., Cas ou le domaine de stabilité d'un ensemble limite attractif d'une récurrence du deuxiéme ordre n'est pas simplement connexe, *C. r. Acad. Sci. Paris* **A268**, 1657-1660 (1969).
18. GARDINI L., ABRAHAM R. H., FOURNIER-PRUNARET D. & RECORD D., A double logistic map, *Int. J. Bifurcation Chaos* **4**(1), 145-176 (1994).
19. MIRA C., Complex dynamics in two-dimensional endomorphisms, *Nonlinear Analysis* **4**(6), 1167-1187 (1980).
20. BARUGOLA A., CATHALA J. C. & MIRA C., Annular chaotic areas, *Nonlinear Analysis* **10**(11), 1223-1236 (1986).
21. MIRA C. & NARAYANINSAMY T., On two behaviors of two-dimensional endomorphisms: role of the critical curves, *Int. J. Bifurcation Chaos* **3**(1), 187-194 (1993).
22. MIRA C., FOURNIER-PRUNARET D., GARDINI L., KAWAKAMI H. & CATHALA J. C., Basin bifurcations of two-dimensional non invertible maps: fractalization of basins, *Int. J. Bifurcation Chaos* **4**(2), 343-381 (1994).
23. CATHALA J. C., Sur la dynamique complexe et la détermination d'une zone absorbante pour un systéme a données échantillonnées décrit par une récurrence du second ordre, *R.A.I.R.O. Auto. Syst. Analysis Contr.* **16**(2), 175-193 (1982).
24. MORRIS H. C., RYAN E. E. & DODD R. K., Snap-back repellers and chaos in a two-dimensional discrete population model with delayed recruitment, *Nonlinear Analysis* **7**(6), 571-621 (1983).
25. HOGG T. & HUBERMAN B. A., Generic behavior of coupled oscillators, *Phys. Rev.* **A29**(1), 275-281 (1984).
26. VAN BISKIRK R. & JEFFRIES C., Observation of chaotic dynamics of coupled nonlinear oscillator, *Phys. Rev.* **A31**(5), 3332-3357 (1985).
27. LORENZ H. W., *Nonlinear Dynamical Economics and Chaotic Motion.* Springer, New York (1989).
28. MIRA C., *Systèmes asservis non linéaires.* Hermès, Paris (1990).
29. KOLYADA S. & SHARKOVSKY A. N., On topological dynamics of triangular maps of the plane, in *Proc. ECIT-89*, pp. 177-183. World-Scientific, Singapore (1991).
30. GARDINI L., Some global bifurcations of two-dimensional endomorphisms by use of critical lines, *Nonlinear Analysis* **18**(4), 361-399 (1992).
31. SHARKOVSKY A. N., YU L. MAISTRENKO & ROMANENKO E. YU., *Difference Equations and their Applications.* Kluwer Academic Publishers, Dordrecht (1993).

32. MAISTRENKO V. L., MAISTRENKO YU. L. & SUSHKO I. M., Noninvertible two-dimensional maps arising in radiophysics, *Int. J. Bifurcation Chaos* **4**(2), 383–400 (1994).
33. LUPINI R., LENCI S., & GARDINI L., Poincaré maps of impulsed oscillators and two-dimensional dynamics (to appear).
34. CHAPEAU-BLONDEAU F., Analysis of neutral networks with chaotic dynamics, *Chaos, Solit. Fract.* **3**(2), 133–139 (1993).
35. NAIMZADA A., Universita' Bocconi, Milano (private communication).
36. GARDINI L., Homoclinic bifurcations in n-dimensional endomorphisms, due to expanding periodic points, *Nonlinear Analysis* **23**(8), 1039–1089 (1994).
37. MAROTTO J. R., Snap-back repellers imply chaos in $\mathbb{R}^n$, *J. math. Analysis Applic.* **63**, 199–223 (1978).
38. GREBOGI C., OTT E. & YORKE J. A., Crises, sudden changes in chaotic attractors and transient chaos, *Physica* **D7**, 181–200 (1983).
39. McDONALD S. W., GREBOGI C., OTT E. & YORKE J. A., Structure and crises of fractal basin boundaries, *Phys. Lett.* **A107**(2), 51–54 (1985).
40. McDONALD S. W., GRENOGI C., OTT E., & YORKE J. A., Fractal basin boundaries, *Physica* **D17**, 125–153 (1985).