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About a route to fractalization of basins generated by noninvertible plane maps

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Abstract

The paper deals with the solutions generated by a two-dimensional noninvertible map defined by a cubic polynomial. The map is of the $(Z_1 - Z_3 - Z_1)$ type, i.e. the plane is divided into three unbounded open areas: one Z_3 generating three real rank-one preimages, bordered by two regions Z_1 generating only one real rank-one preimage. This short study concerns the bifurcations leading a basin in a non fractal situation into a fractal one (multiply connected basin, or non connected basin). In this route to fractalization the first step is the existence of a chaotic attractor which after destabilization gives rise to a strange repeller, limit set for the the basin elements, when it is no longer simply connected. More specially one of such routes is considered, when the strange repeller is said to have a structure with "high density".

1 Introduction

This paper is devoted to one of the routes leading to the fractalization of a basin generated by a two-dimensional noninvertible map. Publications Mira *et al.* [1994, 1996] have shown that such maps may lead to a fractal basin the elements of which have a *strange repeller* as fractal limit set. Generally this set results from the destabilization of a chaotic attractor (said "*chaotic area*" for noninvertible maps) before the fractalization occurrence. Then *the fractal properties of the basin depend strongly on the chaotic attractor structure*, at the origin of this situation. This because the resulting strange repeller constitutes the "skeleton", the framework of the fractal basin. The following example of a $(Z_1 - Z_3 - Z_1)$ two-dimensional family of maps T, defined by two polynomials: a linear one, and a cubic one, in the form:

$$\begin{aligned} x' &= y \\ y' &= ax + bx^2 + cx^3 + dy \end{aligned} \qquad (1)$$

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illustrates a route to a fractal basin from a chaotic attractor with a "*high degree*" of chaos, leading to a "*high degree*" of fractalization (these two terms will be roughly defined below).

The map (1) depends on the parameters a, b, c, d. It is reminded that two-dimensional maps are identified by a symbolism based on the configuration of regions Z_k of the plane, each point of Z_k having k distinct rank-one real preimages. For the map (1), $(Z_1 - Z_3 - Z_1)$ means that the plane is divided into three unbounded open areas: one Z_3 generating three real rank-one preimages, bordered by two regions Z_1 generating only one real rank-one preimage. Here the boundaries of the regions Z_k, $k = 1, 3$, are made up of two parallel straight lines L, L', branches of the rank-one critical curve LC, locus of points such that two determinations of the inverse map T^{-1} are merging on the set $LC_{-1} = L_{-1} \cup L'_{-1}$, made up of two vertical parallel straight lines. The set LC_{-1} is given by equaling to zero the Jacobian determinant of T. *It is worth noting that noninvertible polynomial maps are incompletely identified by their degree.* Indeed two-dimensional quadratic maps may lead to regions Z_k, for which the highest integer k is either 2, or 4. For two-dimensional cubic maps the highest integer k may be either 3, or 5, or 7, or 9. The map complexity depends on the highest value of k.

It is known that basins generated by two-dimensional noninvertible maps may be either simply connected, or multiply connected, or non connected, depending on the situation of their boundary with respect to the critical set LC (Mira *et al.* [1994, 1996]). This dependance occurs in different ways. It is reminded that a basin is multiply connected when it is pierced by infinitely many holes, called *lakes* in the above references. When it is nonconnected it is generally made up of infinitely many elements (called *islands*) without any connection. For language convenience by "*high degree*" of fractalization (resp. "*high degree*" of chaos) we mean that lakes, or islands, (resp. the orbits of the chaotic attractor) cover tightly a whole region of the (x, y) plane. These rough definitions are sufficient for the paper purpose. They will come into an evident view from the basin numerical representation in the (x, y) plane (cf. for example Figures 5, 6).

The basic "mechanism" of the basin with "high degree" of fractalization can be summarized as follows for the map family (1). A suitable parameter choice gives rise to two attractors: a stable fixed point O with a non connected basin $D(O)$, and a chaotic attractor (d) (chaotic area) with a multiply connected basin $D(d)$, located in a bounded region of the (x, y) plane. The closure $\overline{D} = \overline{D}(O) \cup \overline{D}(d)$ of the union of the two basins is the basin of bounded orbits. A parameter variation destroys (d) leading to a simply connected basin $D \equiv D(O)$, and the formation of a *strange repeller* (denoted SR). SR is an unstable fractal set constituted by unstable cycles with increasing period, unstable set of saddles cycles, their limit set when the period tends toward infinity, and the increasing rank preimages (arborescent sequence) of all these points. As a repulsive set, the strange repeller belongs to a basin boundary $\partial D(O)$ (Mira *et al.* [1996]a), and is located inside the domain bounded by the "external" boundary $\partial_e D(O)$, which is also the boundary of the domain of unbounded orbits. When $D(O)$ is simply connected SR gives rise to *chaotic transients* toward the fixed point O. When $D(O)$ is non connected or multiply connected, SR gives rise to a *fractal basin*. In the first case the SR points are not limit points of lakes, or islands. An initial point in the region containing SR generates an orbit which has an erratic behavior during more or less iterations, before a regular convergence toward the stable fixed point O. In the second case the SR points are limit points of lakes (for multiply connected basins), or islands (for nonconnected basins). Then SR constitutes the "skeleton", i.e. a *nucleus* of the corresponding fractal basin. As in Mira *et al.* [1994, 1996], for language convenience, in relation with basin structure, this paper uses geographic analogies: *sea* for the domain of divergent orbits, *continent* for domain of bounded orbits, *lakes*, *islands*, respectively as elements of multiply connected and nonconnected basins.

Section 2 is devoted to some reminders about general properties of two-dimensional noninvertible maps. Section 3 describes the bifurcation giving rise to the strange repeller. Section 4 concerns the bifurcations leading to the fractal structure of the multiply connected basin and the nonconnected one. The conclusion deals with the problems induced by numerical simulations for studying the generation of basins.

2 Some reminders

We consider a two-dimensional noninvertible map T, not specially (1). As in any neighborhood of a point of the critical set LC there are points for which at least two distinct inverses are defined, LC_{-1} is a set of points for which the Jacobian determinant of T vanishes. The set LC satisfies the relations $T(LC_{-1}) = LC$, and $T^{-1}(LC) \supseteq LC_{-1}$. $LC_k = T^k(LC), = 1, 2, 3, ...,$ constitute the rank-$(k+1)$ critical set of T.

A closed and invariant set Ω is called an attracting set if some neighborhood U of Ω exists such that $T(U) \subset U$, and $T^n(X) \to \Omega$ as $n \to \infty$, $\forall X \in U$. An attracting set Ω may contain one, or several attractors coexisting with sets of repulsive points (strange repellers) giving rise to either *chaotic transients* towards these attractors, or *fuzzy boundaries* of their basin (Mira [1987], (Mira *et al.* [1996]). As an attracting set a *chaotic area* (d) is an invariant *absorbing area* (cf. Mira *et al.* [1996] p.188) bounded by arcs of critical curves LC_n ($n = 0, 1, 2, ..., p$, $LC_0 \equiv LC$, p being a finite or an infinite integer), inside which a numerical simulation shows a stable chaotic behavior. "Chaotic" may be considered either in a "*non-strict sense*", or in a "*strict sense*". "Chaotic in a *non-strict sense*" means that the observed dynamics presents no regularity from a numerical simulation (always implying finite precision and finite number of iterations). "Chaotic in *strict sense*" means that it is possible to prove that the set (d) is a true strange attractor.

The open set $B = \bigcup_{n\geq 0} T^{-n}(U)$ is the total basin of Ω, i.e. B is the open set of points X whose forward trajectories (set of increasing rank images of X) converge towards Ω. B, as its boundary ∂B, is invariant under backward iteration T^{-1} of T, but not necessarily invariant by T :

$$T^{-1}(B) = B, \quad T(B) \subseteq B, \qquad T^{-1}(\partial B) = \partial B, \quad T(\partial B) \subseteq \partial B$$

The strict inclusion holds iff B contains points of a Z_0 region, which is not the case for the map (1). If Ω is a connected attractor (particular example: Ω is a fixed point), the *immediate basin* B_0 of Ω, is defined as the widest connected component of B containing Ω.

We remark that $T^{-1}(\partial B) = \partial B$ implies that ∂B must contain the set of preimages of any of its cycles, i.e. must contain the stable set W^s of any cycle of T belonging to ∂B. It is worth noting that, for unstable node and focus cycles, the stable set W^s is made up of the set of increasing rank preimages of cycle points (such a set does not exist in the case of an invertible map). For a saddle cycle, W^s is made up of the local stable set W^s_l, associated with the determination of the inverse map which let invariant this cycle, and its increasing rank preimages.

When Ω is the widest attracting set of a map T, its basin D (called *continent* in Mira *et al.* [1994, 1996]) is the open set D containing Ω such that its closure $\overline{D}$ is the locus of points of the plane having bounded trajectories. Its complementary set, denoted by D' (i.e. $\overline{D} \cup D' = \mathbf{R}^2$) when it is non void, is the basin (called *sea* in Mira *et al.* [1996a]) of an attracting set at infinite distance (on the Poincaré's equator), i.e. the locus of points of the plane having divergent trajectories. In such a case the two basins have a common boundary (the separating set). A map T may also possess no attracting set at finite distance (when only repellers, chaotic or not, exist at finite distance). In such a case, the locus of points of the plane having bounded trajectories belongs to the boundary $\partial D'$ of D' (and $\mathbf{R}^2 = \overline{D'}$). Then $\partial D'$ may be a strange repeller giving rise to a chaotic transient toward the Poincaré's equator. Here these bounded trajectories have no "physical" sense, because they suppose an infinite accuracy on the data, and the absence of disturbance on the system which is modelled by the map.

Conditions of existence of a non connected basin, and a multiply connected basin are given in Mira *et al.* [1994, 1996]. They depend on the geometric situation of basins boundary with respect to the critical set LC. This occurs in several ways, leading to many global bifurcations characterizing the transitions: simply connected basin $\leftrightarrow$ nonconnected one, simply connected basin $\leftrightarrow$ multiply connected one, multiply connected basin $\leftrightarrow$ non connected one. Such transitions are generally related to homoclinic and heteroclinic bifurcations.

3 Generation of a strange repeller by the map T

The map T in (1) has three fixed points $O = (0, 0)$, P and Q:

$$\begin{aligned} x_P &= [-b - \sqrt{\Delta}]/(2c), \qquad x_Q = [-b + \sqrt{\Delta}]/(2c), \qquad \Delta = b^2 - 4c(a + d - 1) \\ y_P &= x_P, \qquad y_Q = x_Q \end{aligned}$$

Their multipliers (eigenvalues) are denoted by S_1 and S_2. We consider the parameter set $a = -0.75$, $c = -0.5$, $d = 0.25$, $1.84 \leq b < 2.3$.

For $b = 1.84$ the fixed points O and P are two stable foci. The fixed point Q is an unstable node ($S_1 < -1$, $S_2 > 1$) associated with a period two saddle C_2^j, $j = 1, 2$. The closure of the basin $D(\Omega)$ of the attracting set $\Omega = O \cup P$ is the domain of bounded orbits. The boundary $\partial D(\Omega)$ of this basin is an invariant closed curve made up of the stable manifolds $W^s(C_4^j)$, $W^s(C_4'^j)$ of two period 4 saddles C_4^j, $C_4'^j$, $j = 1, 2, 3, 4$, ending at two period 4 unstable nodes N_4^j, $N_4'^j$ (Figure 1). The basin $D(O)$ of O is non connected, $D_0(O)$ being the immediate basin (the part containing O). The basin $D(P)$ of P is multiply connected. The common boundary $\partial D(O) \cap \partial D(P)$ is made up of the stable manifold $W^s(C_2^j)$, $j = 1, 2$, of the period two saddle C_2^1, C_2^2. The limit set of the basins $D(O)$, $D(P)$ is $\partial D(\Omega)$ made up of the stable manifolds $W^s(C_4^j)$, $W^s(C_4'^j)$. The basins $D(O)$, $D(P)$ and their limit set have not a fractal structure.

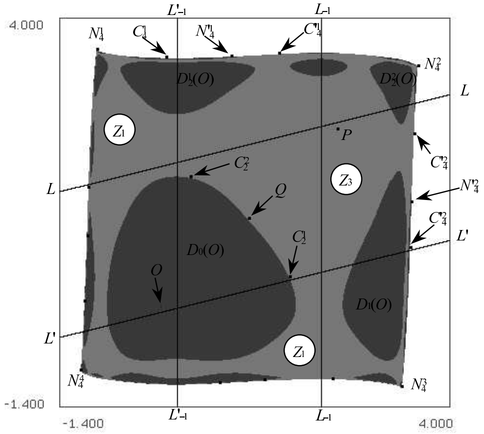

Figure 1: $a = -0.75$, $c = -0.5$, $d = 0.25$, $b = 1.84$. *The dark grey region is the non connected basin $D(O)$ of the stable fixed point O. The grey region is the multiply connected basin $D(P)$ of the stable fixed point P. The white part is the domain of diverging orbits.*

When b increases from $b = 1.84$, the fixed point P undergoes a Neïmark bifurcation turning into an unstable focus surrounded by a stable invariant closed curve,which grows to be a chaotic area (d) (cf. Figure 2 for $b = 2$).

The closure of the basin $D(\Omega)$ of the attracting set $\Omega = O \cup (d)$ is the domain of

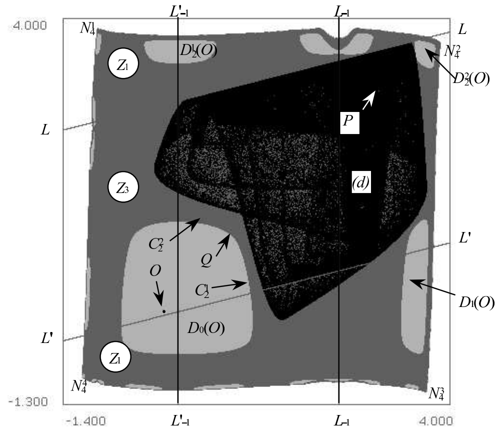

Figure 2: $a = -0.75$, $c = -0.5$, $d = 0.25$, $b = 2$. *The grey region is the non connected basin $D(O)$ of the stable fixed point O. The dark grey region is the multiply connected basin $D(d)$ of the chaotic area (d). The white part is the domain of diverging orbits.*

bounded orbits. With b increasing from $b = 1.84$, the period 4 saddles C_4^j, $C_4'^j$, give rise to a classical period doubling bifurcation followed by the creation of unstable period $4k2^i$ cycles, $i = 0, 1, 2, ...$, $k = 1, 3, ...$, and after by unstable cycles with a period different from $4k2^i$. All these cycles are located on the basin boundary $\partial D(\Omega)$ (cf. Figure 3 with cycles until the period 13).

Considering the one-dimensional map T_r reduced to the repulsive invariant closed curve $\partial D(\Omega)$, this situation means that the dynamics is chaotic on this closed curve. Then the set of the corresponding unstable cycles, and their increasing rank preimages, have a fractal structure on $\partial D(\Omega)$.

Consider now the parameter set $2 \leq b < 2.3$. The fixed point O remains a stable focus. The points P and Q are respectively an unstable focus and an unstable node ($S_1 < -1$, $S_2 > 1$). The period two saddle C_2^j, $j = 1, 2$, continues to exist in the domain bounded by $\partial D(\Omega)$. For $b = 2$ the map T has two attractors: the fixed point O and a chaotic area (d) (Figure 2) bounded by arcs of rank-k critical curves LC_k, $k = 0, 1, 2, 3, 4$, $LC_0 \equiv LC$, $LC_k = T^k(LC)$. A large segment of the critical line L belongs to the (d) boundary. The corresponding basins are denoted $D(O)$ and $D(d)$, with

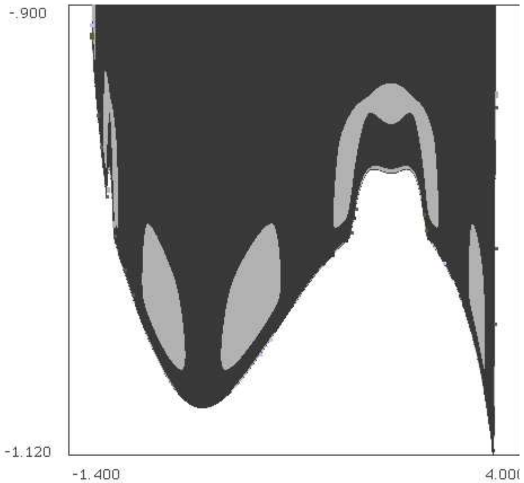

Figure 3: *Enlargement of the figure 2 lower part.*

$D(\Omega) = D(O) \cup D(d)$. The basin $D(O)$ is non connected, $D_0(O)$ being the immediate basin (the part containing O). $D(d)$ is multiply connected. The common boundary $\partial D(O) \cap \partial D(d)$ is made up of the stable manifold $W^s(C_2^j)$, $j = 1, 2$, of the period two saddle C_2^1, C_2^2. The limit set of the basins $D(O)$, $D(d)$ is $\partial D(\Omega)$ where the dynamics is chaotic. The basins $D(O)$, $D(d)$ have not a fractal structure, but their limit set is fractal. In this sense we shall say that they have a "*limit fractal structure*".

For $b = b_{f1} \simeq 2.004993$ a contact bifurcation between the chaotic attractor (d) and the stable manifold $W^s(C_2^j)$ occurs. It results a homoclinic bifurcation by tangency of $W^s(C_2^j)$ with the branch of the unstable manifold $W^u(C_2^j)$ tending toward $(\overline{d})$. Then (d), destroyed for $b = b_{f1} + \varepsilon$ as small as ε may be, gives rise to a strange repeller SR. Figure 4 ($b = 2.007$) shows the basin $D(O)$ of the fixed point O. The "granular" region inside $D(O)$ reproduces the ancient basin $D(d)$. It corresponds to the strange repeller SR. The boundary $\partial D(O)$ of the basin $D(O)$ is $\partial D(O) = \partial_e D(O) \cup SR$.

The bifurcation $b = b_{f1}$ changes the non connected basin $D(O)$ into a simply connected one with the formation of a strange repeller from the chaotic area (d).

4 Basin fractalization

For $b = 2.022$, a bay H_0 has been generated via a tangential contact of $\partial_e D(O)$ with the critical line L (Mira *et al., 1994, 1996*) near a local minimum of the $\partial_e D(O)$ ordinate,

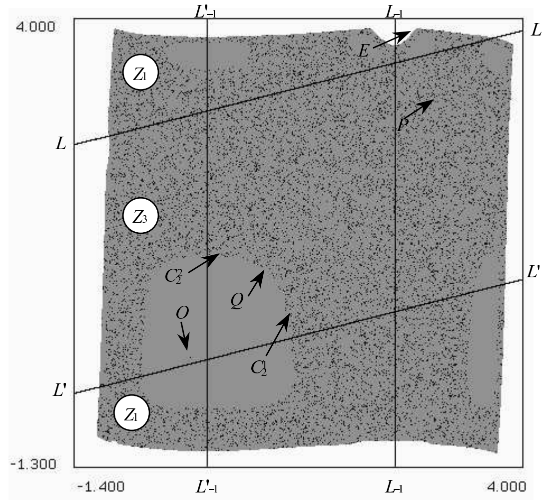

Figure 4: $a = -0.75$, $c = -0.5$, $d = 0.25$, $b = 2.007$. *The grey region is the basin $D(O)$ of the stable fixed point O. This basin is now simply connected. Its boundary contains a strange repeller SR, located in the region with black points, and born from the destabilisation of the chaotic area (d). The white part is the domain of diverging orbits.*

when $b = b_{f2} \simeq 2.02113$. So the basin $D(O)$ is now multiply connected (cf. Fig 5 $b = 2.022$, where the lakes are white colored as the sea). The rank-one lake is $H_1 = T^{-1}(H_0)$, made up of the union of two inverses of T. The third inverse of ∂H_0 gives an arc of $\partial_e D(O)$. The lake H_0 belongs to Z_3, and the rank-$n+1$ lakes $T^{-n}(H_1) = H_{n+1}^{i_1,\dots,i_n}$, $n = 1, 2, ...$, form an arborescent sequence when $n \to \infty$. These lakes have SR and $\partial_e D(O)$ as limit set. Note that the point E, local maximum of the $\partial_e D(O)$ ordinate, belongs to Z_1. Below it will play a role in the bifurcation changing the multiply connected basin into a non connected one. Figures 5 ($b = 2.022$) and 6 ($b = 2.05$) show the lakes covering tightly the largest part of the domain bounded by $\partial_e D(O)$, the one inside the SR region represented in the "granulated" region of Figure 4. Such a situation is due to the high density of the iterated points of the chaotic attractor (d) just before the bifurcation $b = b_{f1}$. As said above one has a "high degree" of lakes fractalization.

The bifurcation $b = b_{f2}$ changes the simply connected basin $D(O)$ into a multiply connected one.

For $b = b_{f3} \simeq 2.07$, the point E of $\partial_e D$ belongs to the critical segment L. A contact bifurcation between L and the boundary $\partial_e D$ occurs near the point E (Figure 7). This situation is an existence limit for the bay H_0. When $b > b_{f3}$ the point E belongs to Z_3

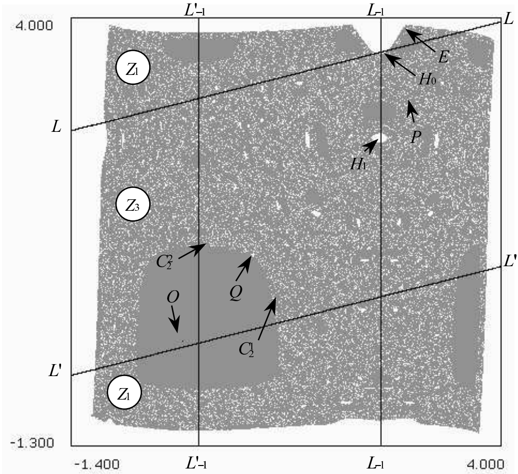

Figure 5: $a = -0.75$, $c = -0.5$, $d = 0.25$, $b = 2.022$. *The grey region is the basin $D(O)$ of the stable fixed point O. This basin is now multiply connected, with creation of a bay H_0 and lakes $T^{-n}(H_0)$, $n = 1, 2, 3, ...$, the limit set of which is the strange repeller appearing in Figure 4. The white part is the domain of diverging orbits.*

(Figs. 8, 9, respectively for $b = 2.09$, $b = 2.2$). Then the sea "penetrates" the lakes,and $D(O)$ becomes non connected.

The bifurcation $b = b_{f3}$ changes the multiply connected basin $D(O)$ into a non connected one.

5 Conclusion

Numerical simulations are used for obtaining the paper results, in particular for the basins drawing, and to discuss their bifurcations. Such a method is based on finite precision of calculus. It can give only a "*macroscopic" view* of the map behavior, and thus implies a critical analysis of results ([Mira C., 2000]). For a "*microscopic" point of view*, we note that a Newhouse's theorem states that in any neighborhood of a C^r-smooth ($r \geq 2$) dynamical system, there exist regions of the space of dynamical sys-

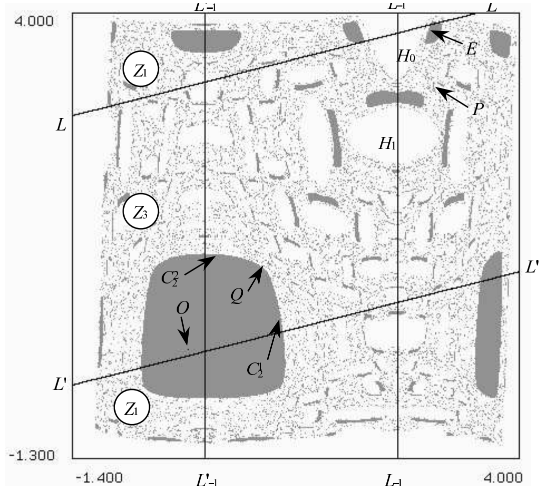

Figure 6: $a = -0.75$, $c = -0.5$, $d = 0.25$, $b = 2.05$. *The grey region is the basin $D(O)$ of the stable fixed point O. The multiply connected basin is such that the surface of lakes is increasing. The white part is the domain of diverging orbits.*

tems (or a parameter space) for which systems with homoclinic tangencies (then with structurally unstable, or nonrough homoclinic orbits) are dense. Domains having this property are called *Newhouse regions*. This result was completed by V.S. Gonchenko, D.V. Turaev & L.P. Shilnikov ([Gonchenko *et al.* 1993]) who asserts that systems with infinitely many homoclinic orbits of any order of tangency, and with infinitely many arbitrarily degenerate periodic orbits, are dense in the Newhouse regions of the space of dynamical systems. This fact has an important consequence: *systems belonging to a Newhouse region are such that a complete study of their dynamics and bifurcations is impossible* ([Shilnikov, 1997]). Indeed in many smooth cases, due to the finite time of a simulation, what appears numerically as a chaotic (strange) attractor may contain a "large" hyperbolic subset in presence of a finite or an infinite number of stable periodic solutions. Generally such stable solutions have large periods, and narrow "oscillating" tangled basins, which are impossible to exhibit numerically due to the finite time of observation, and unavoidable numerical errors. So it is only possible to consider some of the characteristic properties of the system, their interest depending on the problem nature. Such complex behaviors occur for p-dimensional flows, $p > 2$, and thus for $p \geq 2$ invertible and noninvertible maps.

From a "*macroscopic point*" *of view* (the one considered in this paper) the union of the numerous, and even infinitely many stable solutions, which are stable cycles for a map, forms an attracting set. By definition a numerical simulation is made from a limited

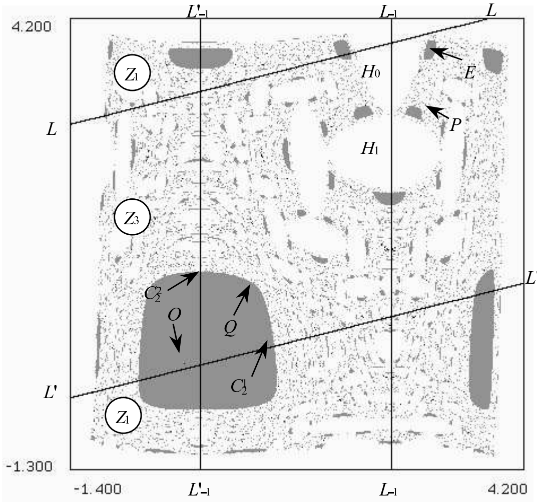

Figure 7: $a = -0.75$, $c = -0.5$, $d = 0.25$, $b = 2.07$. *For* $b = b_{f3} \simeq 2.07$, *the point* E *of* $\partial_e D$ *belongs to the critical segment* L. *A contact bifurcation between* L *and the boundary* $\partial_e D$ *occurs near the point* E. *This situation is an existence limit for the bay* H_0. *The grey region is the basin* $D(O)$ *of the stable fixed point* O. *The white part is the domain of diverging orbits.*

number of iterations. Consider the case of a noninvertible map which *numerically* shows a chaotic attractor, after elimination of a transient, made up of a sufficiently large set of "first" iterations. Then either the numerical simulation "reproduces" points of a chaotic area, related to a "strict" strange attractor in the mathematical sense, or represents a very long transient toward an attracting set including stable cycles of large period in the above conditions. The first case for example is that of some piecewise smooth maps (i.e. with isolated points of nonsmoothness), not permitting stable cycles (i.e. the Jacobian determinant cannot be sufficiently small). In the second case, supposing iterations without error (which numerically has no sense), the transient would be the one toward a stable cycle having a period larger than the number of iterations, this transient occurring inside a very narrow basin, tangled with similar basins of the other stable cycles of large period. In presence of unavoidable numerical errors, the iterate points cannot remain inside the same narrow basin. They sweep across the narrow tangled basins of cycles of the attracting set. Then they reproduce a chaotic area bounded by segments of critical curves, ([Mira C., 2000]). This means that the chaotic area (as bounded by critical segments) is that observed numerically, but is not a true strange attractor from a "microscopic" point of view. So in this smooth case the numerical simulation may

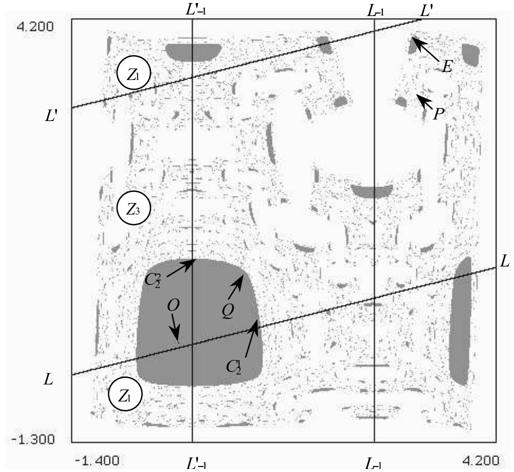

Figure 8: $a = -0.75$, $c = -0.5$, $d = 0.25$, $b = 2.09$. *The point* E *belongs to* Z_3 *Then the domain of divergent trajectories (white part) "penetrates" the lakes, and* $D(O)$ *(grey colored) becomes non connected.* $D(O)$ *is made up of an immediate basin containing* O, *and of infinitely many non connected parts (islands) the limit set of which is the strange repeller appearing in Figure 4.*

be a transient toward an attracting set located inside the chaotic area, with successive jumps in different very narrow basins due to numerical errors. Just after the bifurcation destroying the chaotic area these very small basins persist, but numerically they cannot appear in presence of "dominant" basins. Then the resulting strange repeller may coexist with such "microscopic" basins. In the nonsmooth case the chaotic area is a true strange attractor when stable cycles cannot exist.

The macroscopic point of view concerns all the mathematical models which have a "physical sense", i.e. which imply a finite precision. In this framework the related notion of chaotic area (cf. [Mira *et al.*, 1996]) constitutes an important characteristic of the system dynamics, even if in the smooth case it is impossible to discriminate numerically a situation of a strange attractor in the mathematical sense, from that of an attracting set made up of stable cycles with very large period. Indeed in some way this notion permits to "free ourselves" from the impossiblity of a complete dynamics study, noted by L. P. Shilnikov, by adoption of a "physical" point of view in the case of noninvertible maps. The "macroscopic" study of the paper describes one of the possible route leading to a basin fractalization. It corresponds to a contact of the basin boundary with the map critical set. Inside the closure of the simply connected basin obtained before the bifurcation, this supposes the existence of a strange repeller, coming from the destabilization of a chaotic area. The publication [Mira *et al.*, 1996] has shown that two-

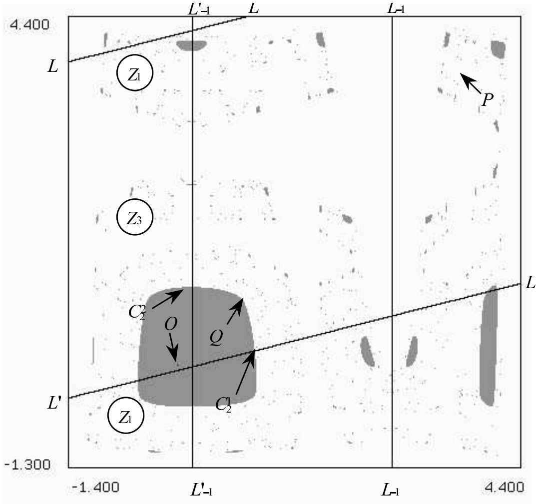

Figure 9: $a = -0.75$, $c = -0.5$, $d = 0.25$, $b = 2.2$. *The same as for Figure 3, but the domain of divergent trajectories (white part) has increased with respect to Figure 8.*

dimensional noninvertible maps can generate such attractors (and so the corresponding strange repellers) with a lot of different fractal structures. So, in the general case, the fractal structures of basins can present a large variety of situations. Due to the creation of new singularities, this variety is increased when the functions defining the map have a denominator which can vanish on some set of points (cf. [Bischi *et al.*, 1999] and [Bischi *et al.*, 2003]).

References

[Bischi *et al.*, 1999] Bischi G. I., Gardini L., & Mira C.. "Plane maps with denominators. I Some generic properties". *Int. J. Bifurcation and Chaos*, *9*(1)(1999), 119-153.

[Bischi *et al.*, 2003] Bischi G. I., Gardini L., & Mira C.. "Plane maps with denominators II. Noninvertible maps with simple focal points". *Int. J. Bifurcation and Chaos*, *13*(8)(2003), 2253-2277.

[Gonchenko *et al.* 1993] Gonchenko V.S., Turaev D.V. & Shilnikov L.P. [1993] "On models with a non-rough homoclinic Poincaré curve," *Physica D 62* (1993), 1-14.

[Mira *et al.*, 1994] Mira, C., Fournier-Prunaret, D., Gardini, L., Kawakami, H. and Cathala, J.C., "Basin bifurcations of two-dimensional noninvertible maps: fractalization of basins" *International Journal of Bifurcation and Chaos*, vol.4, No.2, (1994), 343-381.

[Mira *et al.*, 1996] Mira C., Gardini L., Barugola A. & Cathala J.C. [1996] *Chaotic dynamics in two-dimensional noninvertible maps* (World Scientific Series on Nonlinear Sciences, Series A, vol. 20, 630 pages).

[Mira C., 2000] Mira C. "Chaos and fractal properties induced by noninvertibility of models in the form of maps".*Chaos Solitons & Fractals*, No 11 (2000), 251-262.

[Shilnikov, 1997] L. Shilnikov. "Mathematical problem of nonlinear dynamics: a tutorial". *Int. J. Bifurcation and Chaos*, *7*(9), (1997), 1953-2001.

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