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FROM THE BOX-WITHIN-A-BOX BIFURCATION ORGANIZATION TO THE JULIA SET. PART I: REVISITED PROPERTIES OF THE SETS GENERATED BY A QUADRATIC COMPLEX MAP WITH A REAL PARAMETER

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Properties of the different configurations of Julia sets J, generated by the complex map T_Z: $z' = z^2 - c$, are revisited when c is a real parameter, $-1/4 < c < 2$. This is done from a detailed knowledge of the fractal bifurcation organization "*box-within-a-box*", related to the real Myrberg's map $T : x' = x^2 - \lambda$, first described in 1975. Part I of this paper constitutes a first step, leading to Part II dealing with an embedding of T_Z into the two-dimensional noninvertible map $\overline{T} : x' = x^2 + y - c$; $y' = \gamma y + 4x^2 y$, $\gamma \geq 0$. For $\gamma = 0$, $\overline{T}$ is semiconjugate to T_Z in the invariant half-plane ($y \leq 0$). With a given value of c, and with γ decreasing, the identification of the global bifurcations sequence when $\gamma \to 0$, permits to explain a route toward the Julia sets. With respect to other papers published on the basic Julia and Fatou sets, Part I consists in the identification of J singularities (the unstable cycles and their limit sets) with their localization on J. This identification is made from the symbolism associated with the "box-within-a-box" organization, symbolism associated with the unstable cycles of J for a given c-value. In this framework, Part I gives the structural properties of the Julia set of T_Z, which are useful to understand some bifurcation sequences in the more general case considered in Part II. Different types of Julia sets are identified.

Keywords: Noninvertible map; Julia set; fractal set; stability; basin; global bifurcation.

1. Introduction

This paper is the first part of a double publication devoted to a study of a common basis: the fractal bifurcation organization called "*box-within-a-box*" (translation of "*boîtes emboîtées*" in French). In the simplest case this configuration is generated by the one-dimensional quadratic map. This paper involves the real Myrberg's noninvertible map $x' = x^2 - \lambda$, inside the interval $-1/4 \leq \lambda \leq 2$. The first description of the box-within-a-box organization was given in [Gumowski & Mira, 1975] and [Mira, 1975] before the introduction of the word "fractal".

Quoting these publications Guckenheimer [1980] called it "*embedded boxes*". Further to these publications, the books by Gumowski and Mira [1980a, 1980b], Mira [1987], Mira *et al.* [1996] have furnished more elaborated presentations of this topic.

The identification of the fractal "*box-within-a-box*" organization of the one-dimensional quadratic map was made on the following bases: the Myrberg's results [1963], and a nonclassical bifurcation resulting from the merging of two singularities of different nature, an unstable periodic point with a rank-r image of the minimum of $x' = x^2 - \lambda, r = 2, 3, \dots$ [Mira, 1975].

The Myrberg's results can be summarized as follows.

- All the bifurcations values occur into the interval $-1/4 \le \lambda \le 2$.
- The number N_k of all possible cycles having the same period k, and the number $N_\lambda(k)$ of bifurcation values giving rise to these cycles, increases very rapidly with k.
- The cycles having the same period k differ from each other by the type of cyclic transfer (permutation) of one of their points by k successive iterations by T. These permutations were defined by Myrberg using a binary code constituted by a sequence of $(k-2)$ signs $[+, -]$. More or less explicitly the Myrberg's papers give an extension of this notion to the case $k \to \infty$, and to general orbits (iterated sequences).
- For $\lambda < \lambda_{(1)s} = 1,401155189, \dots$, the number of singularities (cycles) is finite. For $\lambda \ge \lambda_{(1)s}$, the number of singularities is infinite, and the situation is chaotic (stable, or unstable chaos). The parameter $\lambda_{(1)s}$ is a limit point of bifurcation values of period doubling of cycles of period 2^i, $i = 0, 1, 2, \dots$, (*Myrberg cascade* is called "*spectrum*" by Myrberg [1963], and called *Feigenbaum cascade* after Feigenbaum [1978]).
- The following cascades of bifurcations: "stable period $k2^i$ cycle $\to$ unstable period $k2^i$ cycle $+$ stable period $k2^{i+1}$ cycle", $i = 0, 1, 2, 3, \dots$, $k = 1, 3, 4, \dots$, occurs when λ increases. When $i \to \infty$, the bifurcations have a limit point $\lambda^j_{(k)s}$, $\lambda_{(1)s} < \lambda^j_{(k)s} < 2$, j characterizing the permutation of the period k cycle.
- It is possible to classify all the cycles of binary codes via an ordering law (*Myrberg's ordering law*).
- A binary code can be associated with the λ-value resulting from accumulation of bifurcations such that $i \to \infty$, or $k \to \infty$. This rotation sequence satisfies the ordering law.

All these fundamental results have passed over in silence, as they are unknown to the wide public, and not cited in contemporary papers dealing with this subject (subject quite popular since 1978). Most parts of these results are now very often attributed to authors who rediscovered them after, using other forms of quadratic map such as the logistic map, or maps of the unit interval.

When c is real, this first part of the double publication shows how the knowledge of this bifurcation organization permits a better understanding of some "microscopic" properties of the Julia set J, created by the complex map T_Z, $z' = z^2 - c$, where c is a real parameter. Its two-dimensional real form is:

$$T_Z : \begin{cases} x' = x^2 - y^2 - c \\ y' = 2xy \end{cases} \tag{1}$$

By introducing a second parameter γ, from a situation where the Julia set does not exist, the second Part will explain some bifurcation routes, leading to different configurations of Julia sets J generated by T_Z, when $\gamma \to 0$. This is made by an "indirect" embedding of T_Z into a two-dimensional family of noninvertible maps $\overline{T}$:

$$\overline{T} : \begin{cases} x' = x^2 + y - c \\ y' = \gamma y + 4x^2 y \end{cases} \tag{2}$$

with $-1/4 \le c \le 2$, $\gamma \ge 0$. This embedding is not a "direct" one because its link with T_Z is not obtained by equating directly the parameter γ to zero. Indeed the maps family is characterized by the fact that $\overline{T}_{\gamma=0}$ is semiconjugate to T_Z in the invariant half-plane $\{(x, y) : y \le 0\}$ (cf. [Agliari *et al.*, 2003, 2004]), i.e. $\overline{T}_{\gamma=0} \circ h = h \circ T_Z$, where $h(x, y) = (x, -y^2)$. In this half-plane $\overline{T}_{\gamma=0}$ is equivalent (i.e. semiconjugate) to the two-dimensional map T_Z. Then the properties of the different Julia set configurations, obtained for fixed values of parameter c, are also revealed from a bifurcation study when γ decreases from 1 to 0. For $\gamma = 0$ the basin boundary in $y \le 0$ is a fractal set nowhere smooth, except for particular values of c at which J is a circle ($c = 0$), or a segment ($c = 2$).

Remind that if T is a map, $X' = TX$, a *period k cycle* is a set of k consecutive points satisfying the relations $T^k X = X$, $T^r X \ne X$, $0 < r < k$. In the case of a general complex map $z' = \varphi(z)$ (not specially a polynomial one) a *Julia set* includes the points of all unstable cycles of any period $k = 1, 2, \ldots$, their limit sets, and their increasing rank preimages (some properties of this set are recalled below). The cycle *multipliers* (eigenvalues) of the two-dimensional quadratic map T_Z, are real and equal, $S_1 = S_2 = S$. The paper shows that each of these cycles located on the x-axis is exactly identified by the symbolism $(k; j)$, or a more elaborated one called "*embedded representation*" (cf. Sec. 2.2), where j characterizes the cycle points permutation by k iterations (cf. Sec. 2.1). As for the cycles with $y \neq 0$, they have the same characterization, because progressively they belong to $y = 0$ when c increases until $c = 2$.

The Julia set J of T_Z is a perfect set, closure (derived set, or set of the limit points, E') of the set E of all the unstable periodic points (cf. [Julia, 1918; Fatou, 1919, 1920], see also [Blanchard, 1984], the books of selected papers edited by Devaney and Keen [1988], Beardon [1991] and Devaney [1994]). The source of the fundamental results concerning J properties are the works of Julia and Fatou. Often the papers published after, quoting these authors, do not mention their exact contribution, which makes fuzzy the contributions after those authors. In this paper, the corresponding page numbers of the original French publications are given, when necessary. Regarding this point, in order to facilitate the reading of the Julia and Fatou papers, it is important to indicate the following two basic original symbolisms. Set J is called E' by Julia and F (with a rounded type) by Fatou. The map is written $z' = \varphi(z)$ in Julia, $z' = R(z)$ in Fatou.

In relation with the published papers, it is worth to note that sometimes the same word has different meanings according to the mathematical "schools", which is a source of misunderstanding and mix-up. So in the classical nonlinear mechanic field, and also for R. Thom (see his book *Stabilité Structurelle et Morphogénèse* [1972]), a *basin of attraction* is related to an asymptotically stable stationary state i.e. an attractor, and not to a *semi-stable* state (or "*neutral*" as denoted by other authors).

This text also uses the term "*chaotic attractor*" or "*chaotic intervals*" (behavior on the x-axis) as attracting sets, and the term "*basin*" in any case of chaotic attractors. This vocabulary requires an explanation. Indeed in the case of a cycle, "*basin of attraction*" is classically used when it is a topological attractor (with eigenvalue $|S| < 1$) and it is not used when it is neutral ($|S| = 1$) even if it attracts almost all the points of a domain. Generally "chaotic intervals", or a "critical chaotic set" are not topological attractors, even if they attract almost all the points of a domain, thus the term "basin of attraction" (as for the neutral cycle) cannot be strictly used.

We remind that an attractor (or topological attractor) A of a map T is defined as an invariant set for which there exists a neighborhood U such that $\lim T^n(U) = A$, which is not the case when $|S| = 1$. If this attractor has a "domain of influence" of positive measure it is called "*attractor in Milnor sense*". As for a topological attractor, for an attractor in the Milnor sense we shall use the term "*basin*" (which for a topological attractor means basin of attraction). Similarly when we have a chaotic attractor, it is generally an "attractor in Milnor sense". It is the case for the map restricted to the x-axis, when the boundary of a "*cyclical chaotic interval*" includes a repelling cycle, so it cannot be an attractor, but only an "attractor in Milnor sense". For the two-dimensional map the basin does not exist, the Julia set (a *dendrite*) becoming the boundary of the domain of diverging orbits.

The notion of *critical point*, which plays a fundamental role in the study of a complex map $z' = \varphi(z)$, also must be clarified. Originally in the papers of Julia, Fatou, and the other authors of the French school of iteration (end of the 19th century, and beginning of the 20th one), this notion is related to the inverse map. A *critical point of the inverse of the map* (as explicitly written in these papers) is the image C of a point such that $d\varphi/dz = 0$ (for example see [Julia, 1918, p. 51]). From the second half of the 20th century, in the most part of the papers published in English a critical point is presented as a point satisfying $d\varphi/dz = 0$, without any reference to the inverse map. Following the voluminous literature existing on noninvertible maps, this paper uses the definition of the French school of iteration (which is the Julia–Fatou one). So in this paper a *critical point*, or a *rank-one critical point*, *is a point for which at least two coincident rank-one preimages exist*. The forward images of a critical point are also called critical points, clearly of higher rank.

The Julia set J has also other properties resulting from the previous ones given above.

(i) J is completely invariant (i.e. forward and backward invariant) so that it includes all the increasing rank images and all the increasing rank preimages of any of its points.

(ii) J is also given by the closure of the set of all the preimages of any of its points. Thus, *a fortiori*, all the increasing rank preimages of E are everywhere dense on J.

(iii) For a *polynomial map*, in the (x, y) plane J constitutes the boundary of the basin of the point at infinity ($z = \infty$), i.e. it bounds the domain of divergent orbits [Fatou, 1920, p. 85]. Indeed, making the variable change $z = 1/Z$, the point at infinity is now $Z = 0$, with a multiplier (eigenvalue) $S = 0$, point also called superstable or superattracting.

(iv) In the case of the map T_Z with $-1/4 < c < 2$ *generally* J is also the basin boundary of an attracting set on the x-axis. Here "generally" is related to the fact that in this interval of c-values particular bifurcations values are excluded, giving situations in which the basin does not exist. Here two cases, developed in Sec. 3, are possible. The first one is related to a *basin*, in the Julia–Fatou sense, toward a point, or a cycle (also called *neutral*) located on J, its multiplier being $|S| = 1$. In the second case (*dendrite*) J is the boundary of the basin of the point at infinity, but does not separate another basin. We recall that the set of c-values giving rise to a dendrite is a set of positive Lebesgue measure.

It is worth noting that the last paragraph (p. 73) of Chapter 4 in [Fatou, 1920] underlines the interest of finding, in the general case, the necessary and sufficient condition for a "*continuous*" variation of J, when the parameters vary, this independently of the local behavior of the attractor. In the particular case of the quadratic maps family T_Z, with c real, the box-within-a-box organization gives the solution of the Fatou problem. Indeed the qualitative properties of J change when a $(k; j)$ (unstable) cycle first with $y \neq 0$ belongs to $y = 0$, after crossing a c bifurcation value for which the cycle multiplier (eigenvalue) is $|S| = 1$. For each of the $(k; j)$ cycles, this paper shows that the knowledge of the box-within-a-box bifurcations organization permits to define a c-open interval associated with an attracting $(k; j)$ cycle, where J has such a continuous behavior. Such an open interval is bounded by c-values such that $|S(k; j)| = 1$. This gives a first step to discern between the J properties. Inside each interval the multiplier $S(k; j) = 0$ separates two different local behaviors near the $(k; j)$ attracting cycle: $S > 0$ with a regular convergence of orbits, $S < 0$ with an "alternate" convergence.

For the clarity of this paper, we also have to define two qualifiers, specifically used in this paper, which are related to the properties of the Julia set J. The first one is the J *structure*, which is only related to the identification of the localization of the $(k; j)$ unstable cycles in the plane, i.e. of well-defined subsets of the Julia set J. The propositions in Sec. 4 provide such information. At this step the J outline is not yet considered. The second qualifier is the J *shape* directly related to its outline. So a same J structure can correspond to different shapes, which can be identified from a numerical simulation. For example, in the interval $-1/4 < c < 3/4$ (related to the attracting fixed point), J has the same structure, but with a continuous evolution of four shapes described in Sec. 5.3. This shape evolution depends on the distance of the period 2 cycle ($y \neq 0$, which attains $y = 0$ for $c \geq 3/4$) from the x-axis.

This paper, as the numerous others published since 1965, does not pretend to give new fundamental results with respect to the Julia and Fatou contribution, which defined the basic situations without a computer help. The purpose of the paper is only to show how the bifurcations symbolism related to the box-within-a-box organization (described in Sec. 2) permits to obtain a first ordering of the Julia sets generated by (1), when c is real, $-1/4 \leq c \leq 2$. Incidentally, we note that the section of the Mandelbrot set by the real axis, obtained numerically and shown in many relatively recent papers, is well identified by the box-within-a-box organization.

After this introduction, Sec. 2 is devoted to a rundown of the box-within-a-box bifurcation organization of the Myrberg's map $x' = x^2 - c$. Section 3 is devoted to the Julia sets generated by T_Z. Some general Julia and Fatou results are reminded, with a particular view concerning the polynomial map T_Z. The propositions about the structure identification of the Julia set are given in Sec. 4, from the bifurcation values of the Myrberg's map, and the intervals they define, inside which the J evolution is continuous. From this information Sec. 5 describes five well defined types of Julia sets, and their shape evolution inside intervals where the J evolution is continuous.

2. Box-Within-a-Box Bifurcations Organization of Unimodal Real Maps. Rundown

2.1. *Some basic properties*

A unimodal map is a one-dimensional noninvertible map, $x' = f(x, \lambda)$ (λ is a real parameter), defined by a function f with only one extremum. Here x is assumed to be real, and that the x-axis is made up of two open intervals: Z_2, each point of which has two distinct rank-one preimages, and Z_0 each point of which has no real preimage. Such a map is said of $Z_0 - Z_2$ type. In particular, a quadratic map belongs to this type for correctly chosen parameter values. The fractal "*box-within-a-box*" (translation of "*boîtes emboîtées*" in French) bifurcations structure, or "*embedded boxes*" according to Guckenheimer [1980], was first identified in the case of unimodal maps with negative Schwarzian derivative [Gumowski & Mira, 1975] and [Mira, 1975]. A more complete presentation is given in the books Gumowski and Mira [1980], Mira [1987], Mira *et al.* [1996]. The basic fractal bifurcation organization is generated by the simplest case of unimodal maps, given by the quadratic map:

$$T : x' = x^2 - \lambda \tag{3}$$

Here x is a real variable, and for this map, called *Myrberg's map* [Myrberg, 1963], the real parameter c of the map $\overline{T}$ (2) is written λ. The inverse map T^{-1} is defined by $x = \pm\sqrt{x' + \lambda}$. So the x-axis is made up of the intervals Z_2 ($x' > -\lambda$), Z_0 ($x' < -\lambda$). The rank-one image $C = T(C_{-1})$ of the ordinate minimum C_{-1} ($x = 0$) of the map function is the *rank-one critical point* (in the Julia–Fatou sense), $x(C) = -\lambda$. It has two merging rank-one preimages at $T^{-1}(C) = C_{-1}$, C separating Z_0 and Z_2. A rank-r critical point C_{r-1} is obtained after r iterations of C_{-1} (or equivalently $r - 1$ iterations of C, considered as the rank-one critical point $C_0 \equiv C$). The set of increasing rank critical points is denoted by E_c and its limit set by E'_c, (derived set of E_c). The map T is characterized by the following properties.

(a) The parameter interval $\Omega_1 = [\lambda_{(1)_0}, \lambda_1^*]$, $\lambda_{(1)_0} = -1/4$, $\lambda_1^* = 2$, called *overall box* contains all the bifurcations values of (3). In the interval $-1/4 < \lambda < 2$ the map possesses a unique attractor, which in the simple cases is an asymptotically stable (or attracting) fixed point, or an attracting period k cycle, or a chaotic attractor. The value $\lambda_{(1)_0} = -1/4$ corresponds to a *fold bifurcation* giving rise to two fixed points q_i, $i = 1, 2$, with multiplier (or eigenvalue) $S = 2x(q_i)$: q_1 always unstable ($S > 1$), and q_2 ($S < 1$, attracting when $-1 \le S < 1$). In the interval $\lambda < \lambda_{(1)_0} = -1/4$ no real fixed point exists. The value $\lambda = \lambda_1^* = 2$ is a basic nonclassical bifurcation related to the merging of the unstable fixed point q_1 with the rank-two critical point $C_1 = T(C) = T^2(C_{-1}) = q_1$. For this parameter value $x(C_1) = x(q_1) = 2$, $x(C) = x(q_1^{-1}) = -2$, $T^{-1}(q_1) = q_1 \cup q_1^{-1}$. When $0 < \lambda < \lambda_1^*$ the invariant segment $[q_1^{-1}, q_1]$ is the closure of the basin of the absorbing segment $\overline{CC_1}$, containing the unique attractor.

For the parameter value $0 < \lambda < \lambda_1^* = 2$ the segment $\overline{CC_1}$ is *absorbing*, an absorbing segment (d') being bounded by two critical points, such that the increasing rank images of any point of its neighborhood $U(d')$, from a finite number of iterations, enter into (d') and cannot get away after entering.

For the parameter value $\lambda = \lambda_1^*$ the segment $\overline{CC_1}$ is chaotic and merges with $[q_1^{-1}, q_1]$, $x(q_1^{-1}) = x(C) = -2$, $x(q_1) = x(C_1) = 2$. All the possible cycles have been created, and they belong to $\overline{CC_1}$. Then $\overline{CC_1}$ is *invariant but not absorbing*. From an initial condition x_0, $-2 < x_0 < 2$, the map generates a bounded orbit, belonging to the interval $[q_1^{-1}, q_1]$, which is very sensitive to very small changes of x_0. The repelling cycles constitute a real set E which is dense in the whole interval $[-2, 2]$ (as well as their preimages of any rank), that is, the derived set (set of limit points) $E' = [-2, 2]$ is perfect (see more details in [Julia, 1918; Gumowski & Mira, 1980; Mira, 1987]).

When $\lambda > 3/4$ the fixed point q_2 is always repelling with $S(q_2) < -1$, and a period 2 cycle appeared from q_2. This cycle, made up of two points q_{2i}, $i = 1, 2$, has the multiplier $S(q_{2i}) = 4 - 4\lambda$, attracting ($|S(q_2)| < 1$) if $3/4 < \lambda < 5/4$. The value $\lambda = \lambda_{b1} = 3/4$ is a *flip bifurcation*. Increasing values of λ generate a sequence of flip bifurcations $\lambda = \lambda_{bm}$ for period 2^m cycles, $m = 1, 2, \ldots$, with an accumulation value $\lim_{m\to\infty} \lambda_{bm} = \lambda_{1s} \simeq 1.401155189$. At this particular bifurcation value $\lambda = \lambda_{1s}$, the corresponding attractor is an invariant set with Cantor like structure called *critical attractor* A_{cr} (see, among others, [Guckenheimer & Holmes, 1983]). When $\lambda < \lambda_{1s}$ the number of repelling (or unstable) cycles is finite, each cycle has a period 2^m which has been created after crossing through the value λ_{bm}. When $\lambda = \lambda_{1s} + \varepsilon$, $\varepsilon > 0$, $\varepsilon \to 0$, infinitely many repelling period 2^i cycles ($i = 0, 1, 2, \ldots$) exist, the ones created by the above sequence of flip bifurcations. The parameter interval $\omega_1 \equiv [\lambda_{(1)_0}; \lambda_{1s}]$ is called the Myrberg *spectrum*, denomination used in this text. It corresponds to the sequence (cascade) of period doubling bifurcations from the fixed point q_2 $(i = 0)$.

(b) The number N_k of all possible cycles having the same period k, and the number $N_\lambda(k)$ of bifurcation values leading to these cycles, increase very quickly with k (cf. pp. 93–97 of [Mira, 1987] for the relations giving N_k and $N_\lambda(k)$). Cycles with the same period k differ from each other by the *permutation* (cyclic transfer) of their points by successive iterations of T. Each k-cycle can be identified by the symbolism $(k; j)$, j being an index characterizing this permutation. Afterward j will be simply called "permutation" in place of "permutation of the cycle points via k iterations". Let $(k; j)$ be one of such cycles. It can be generated from two basic bifurcations: either a fold one, or a flip one. The *fold bifurcation* generates two *basic cycles* at $\lambda = \lambda^j_{(k)_0}$: $(k; j)_{S>1}$ and $(k; j)_{S<1}$, $k \neq 2$. With increasing values of λ, a cascade of *flip bifurcations* is created from the cycle $(k; j)_{S<1}$, giving rise to a sequence of $(k2^i; j_i)_{S<1}$ cycles with accumulation, when $i \to \infty$, at a value λ^j_{ks}, $\lambda_{1s} < \lambda^j_{ks} < 2$. Here j_i is the permutation related to the related period 2^i cycle, generated in the interval ω_1. Myrberg also calls "*spectrum*" the parameter interval $\omega^j_k = [\lambda^j_{(k)_0}; \lambda^j_{ks}]$, $k = 1, 3, 4, \ldots$. The interval ω^j_k is made up of parameter intervals corresponding to attracting cycles of period $k2^i$, $i = 0, 1, 2, \ldots$. In ω^j_k the flip bifurcation of a $(k2^{m-1}; j_{m-1})$ cycle is denoted λ^j_{kbm}, $m = 1, 2, \ldots$. The cycle symbolisms $(k; j)$ and $(k2^i; j, p_i)$ are related to what is called a *nonembedded representation* in [Mira, 1987] and [Mira *et al.*, 1996]. This symbolism, which identifies precisely every cycle, is of wide interest and importance in the description of the complex dynamics of one-dimensional unimodal maps. The complex and fractal behaviors can be described also with other analytical tools, as for instance, the kneading theory or symbolic dynamics. Nevertheless such theories do not identify the cycles generated by the map, and so are not able to explain their origin in the complex bifurcations organization, as the parameter λ increases from $\lambda_{(1)_0} = -1/4$ to $\lambda^*_1 = 2$. For a given value $\lambda = \lambda_g$ of λ, the "box-within-a-box" bifurcation organization permits the identification of all the cycles born for $\lambda < \lambda_g$.

For $\lambda > \lambda^*_1 = 2$, $[q_1^{-1}, q_1] \subset \overline{CC_1}$, the only attractor is the point at infinity, and no other bifurcation takes place. The derived set E' (without the point at infinity) constitutes the nonwandering set $E' \subset [q_1^{-1}, q_1]$. The map T has generated all the possible cycles, which are real and repelling, and E' is an invariant *Cantor set* (and thus *totally disconnected*). This set, which constitutes the basin boundary of the fixed point at infinity, is everywhere disconnected (discontinuous in [Fatou, 1919, p. 260]).

The situation equivalent to the one at λ^*_1 (but now with an absorbing set inside $\overline{CC_1}$) is met for each $(k; j)$ cycle with multiplier $S > 1$ (thus generated by a fold bifurcation), for a value $\lambda = \lambda^{*j}_k$. In this case λ^{*j}_k is the least λ-value such that the critical points $C_k = T^k(C)$, $C_{k+1}, \ldots, C_{2k-1}$ merge into k points of the $(k; j)$ cycle with $S > 1$. Considering the map T^k, for k intervals bounded by critical points of well-defined rank, the value λ^{*j}_k reproduces qualitatively the situation of T when $\lambda = \lambda^*_1$. So similarly to the case $\lambda = \lambda^*_1$, when $\lambda = \lambda^{*j}_k$ the map T gives rise to k nonconnected intervals constituting a *k-cyclic chaotic segment* denoted CH^j_k which attracts almost all (i.e. except for a set of zero Lebesgue measure) the points of $]q_1^{-1}, q_1[\backslash CH^j_k$. CH^j_k is made up of the k cyclic chaotic segments $\overline{CC_k}, \overline{C_1C_{k+1}}, \ldots, \overline{C_{k-1}C_{2k-1}}$.

(c) The permutation (cyclic transfer) of one of the points of a $(k; j)$ cycle, via k successive iterations by T, can be defined either in a binary form (*Myrberg's rotation sequence*), or a decimal one (*decimal rotation sequence*) [Mira, 1987]. Each rotation sequence is associated with a well-defined index $j = 1, 2, \ldots, N_\lambda(k)$. These rotation sequences are ordered according to the Myrberg's ordering law [Myrberg, 1963; Mira, 1987], and the index j gives not only the place of any cycle in this ordering, but also the birth order of the bifurcations, when λ increases from $\lambda_{(1)_0} = -1/4$. Note that a necessary and sufficient condition for a permutation of k integers to be one of a cycle generated by a unimodal map is given in pp. 136–138 of [Mira, 1987].

2.2. *Description of the bifurcations organization*

The bifurcations organization described here in the case of (3), concerns the whole family of unimodal maps (i.e. $Z_0 - Z_2$ ones) with negative

Schwarzian derivative, which are topologically conjugated with (3) in some correctly chosen parameter range. Globally the organization is characterized by the existence of a parameter interval $\Omega_1 = [\lambda_{(1)_0}, \lambda_1^*]$ (*overall box*), inside which all the possible bifurcations occur. This overall box contains intervals reproducing the Ω_1 properties in a configuration of "Russian dolls" type. Out of Ω_1 no bifurcation occurs. Ω_1 is generated from the two *basic* period $k = 1$ cycles, i.e. the fixed points q_1 and q_2. Taking into account the Myrberg spectrum ω_1 related to the fixed point q_2 ($S < 1$), the box Ω_1 is defined by:

$$\Omega_1 = [\lambda_{(1)_0}, \lambda_1^*] = \omega_1 \cup \Delta_1 \quad \Delta_1 =]\lambda_{1s}, \lambda_1^*]$$

The description of the box-within-a-box organization implies a specific symbolism. So considering the cycle $(2^i; p_i)$ generated inside the spectrum ω_1, the symbol "2^i" is not used for cycles of even period born from a fold bifurcation, or a flip bifurcation related to a *basic cycle* appearing out of ω_1. So with such a symbolism the period 2^2 is different from the period 4, $2^3 \neq 8$, and $2^3 \neq 4.2^1$, 4.2^1 being the period of the cycle born from the flip bifurcation of the period 4 cycle which appears from a fold bifurcation $\lambda^1_{(4)_0} \in \Delta_1$. Cycles different from $(2^i; p_i)$ can appear only for $\lambda \in \Delta_1$. The interval $\lambda < \lambda_{(1)_0} = -1/4$ corresponds to the absence of fixed points (except the point at infinity), or cycles, and every orbit is divergent. For $\lambda > \lambda_1^* = 2$ all the possible period k cycles have been created. They are repelling, and the map has the properties indicated in Sec. 2.1.

Two basic cycles $(k; j)$ ($k = 3, 4, \ldots$), issued from the same fold bifurcation, one with $S > 1$, the other with $S < 1$, generate a parameter interval, provisionally denoted $\hat{\Omega}_k$, having the same behavior as Ω_1, $\hat{\Omega}_k \subset \Delta_1$. The box $\hat{\Omega}_k$ is denoted Ω_k^j, if k is a prime number, or if it is not contained in another interval $\hat{\Omega}_{k'}$, k being a multiple of k'. Then Ω_k^j is called *rank-one box*, or *box* $(k; j)$ (*embedded representation*) with:

$$\Omega_k^j = [\lambda^j_{(k)_0}, \lambda_k^{*j}], \quad \Omega_k^j = \omega_k^j \cup \Delta_k^j \subset \Delta_1,$$

$$\Delta_k^j =]\lambda^j_{ks}, \lambda_k^{*j}]$$

The index k is the *basic period* (or rank-one basic period) of all the cycles generated in the box Ω_k^j, and j the *basic permutation* (or rank-one basic permutation) of the points of these cycles. The interval ω_k^j is the *spectrum* $(k; j)$, and includes the Myrberg's cascade (or period doubling cascade of flip bifurcations) generated from the basic $(k; j)$-cycle of the box with $S \leq -1$. Considering T^k the box Ω_k^j reproduces all the bifurcations contained in the box Ω_1, in the same order (self similarity property), for a set of cycles (of the map T) having periods multiple of k (but not all the possible cycles with these periods). Let $\Omega_{k_1}^{j_1}$ be one of such boxes. Inside $\Omega_{k_1}^{j_1}$ it is possible to define *rank-two* boxes $\Omega_{k_1.k_2}^{j_1,j_2} = [\lambda^{j_1,j_2}_{(k_1.k_2)_0}, \lambda^{*j_1,j_2}_{k_1.k_2}] \subset \Delta_{k_1}^{j_1}$, related to two $(k_1.k_2; j_1, j_2)$ basic cycles, which for $(T^{k_1})^{k_2}$ reproduce in the same order all the bifurcations of the box $\Omega_{k_1}^{j_1}$, and so those of Ω_1:

$$\begin{aligned}\Omega_{k_1.k_2}^{j_1,j_2} &= [\lambda^{j_1,j_2}_{(k_1.k_2)_0}, \lambda^{*j_1,j_2}_{k_1.k_2}] \\ &= \omega^{j_1,j_2}_{k_1.k_2} \cup \Delta^{j_1,j_2}_{k_1.k_2} \subset \Delta^{j_1}_{k_1}, \\ \Delta^{j_1,j_2}_{k_1.k_2} &=]\lambda^{j_1.j_2}_{(k_1.k_2)s}, \lambda^{*j_1,j_2}_{k_1.k_2}]\end{aligned}$$

All the cycles generated inside $\Omega_{k_1.k_2}^{j_1,j_2}$ have a *rank-one basic period* k_1, a *rank-one basic permutation* j_1, a *rank-two basic period* k_1k_2 and a *rank-two basic permutation* (j_1, j_2). Similarly, from a couple of basic cycles $(k_1, \ldots, k_a; j_1, \ldots, j_a)$, one with $S < 1$, the other with $S > 1$, *rank-a boxes* embedded into a rank-$(a-1)$ box are defined:

$$\begin{aligned}\Omega^{j_1,\ldots,j_a}_{k_1,\ldots,k_a} &= [\lambda^{j_1,\ldots,j_a}_{(k_1,\ldots,k_a)_0}, \lambda^{*j_1,\ldots,j_a}_{k_1,\ldots,k_a}] \\ &= \omega^{j_1,\ldots,j_a}_{k_1,\ldots,k_a} \cup \Delta^{j_1,\ldots,j_a}_{k_1,\ldots,k_a} \subset \Delta^{j_1,\ldots,j_{a-1}}_{k_1,\ldots,k_{a-1}}\end{aligned}$$

with cycles having *rank-p basic periods*, $p = 1, \ldots, a$, and *rank-p basic permutations*. Moreover $\Omega^{j_1,\ldots,j_a}_{k_1,\ldots,k_a} \subset \Omega^{j_1,\ldots,j_{a-1}}_{k_1,\ldots,k_{a-1}}$, $a = 1, 2, \ldots$. The boundary parameter $\lambda^{*j_1,\ldots,j_a}_{k_1,\ldots,k_a}$ of each of these boxes ($a = 1, 2, \ldots$) corresponds to the merging of well-defined critical points with a repelling basic cycle having the multiplier $S > 1$. Boxes $\Omega^{j_1}_{k_1}, \ldots, \Omega^{j_1\ldots,j_a}_{k_1,\ldots,k_a} \ldots$ are called *boxes of first kind*. The representation of these boxes is given in Fig. 1(a), with the enlargement in Fig. 1(b).

Boxes of second kind can be defined as follows. Consider another type of bifurcation parameter λ^*, now defined from a repelling cycle with $S < -1$, born from a flip bifurcation. The first and largest box is $\Omega_{2^1} \equiv [\lambda_{b1}, \lambda^*_{2^1}] \subset \Omega_1$, $\lambda^*_{2^1}$ ($k = 1$) corresponding to $C_2 \equiv q_2$ ($S < -1$), $\lambda^*_{2^1} \simeq 1.543689013$. Similarly boxes $\Omega_{2^m} \equiv [\lambda_{bm}, \lambda^*_{2^m}]$, $\lambda^*_{2^m}$ ($k = 1$), corresponding to critical points merging with the period 2^{m-1} cycle ($S < -1$) from rank 2^{m+1}, can be defined with

$$\Omega_{2^m} \subset \Omega_{2^{m-1}} \subset \cdots \subset \Omega_{2^1} \subset \Omega_1$$

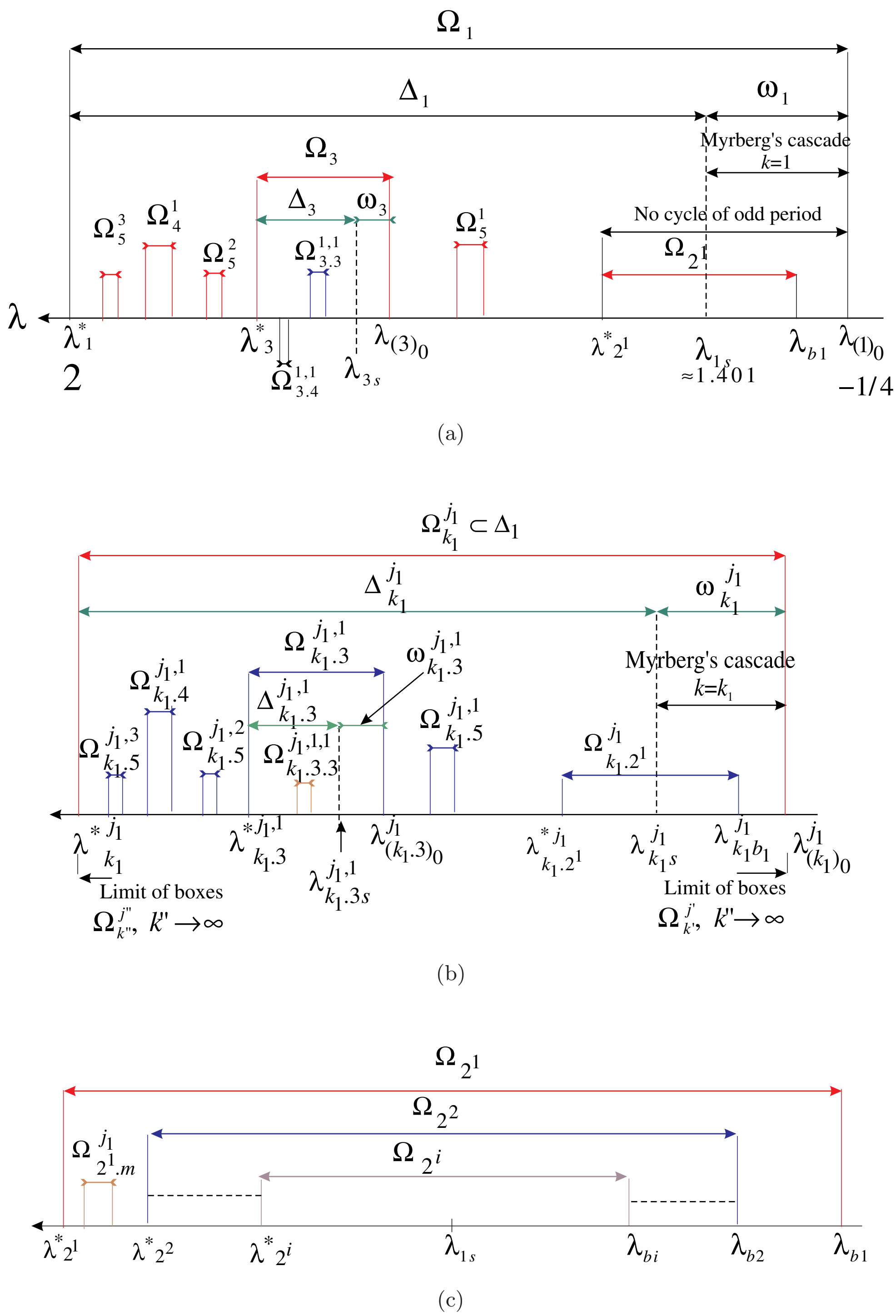


Fig. 1. Fractal "box-within-a-box" (or embedded boxes) bifurcations structure. (a) General view. (b) Enlargement of the box of first kind $\Omega_{k_1}^{j_1}$. (c) Box of second kind Ω_{2^1}.

Then each interval $[\lambda^*_{2^{m+1}}, \lambda^*_{2^m}] \subset \Omega_{2^m}$ contains boxes, self-similar with Ω_1, denoted

$$\Omega^{p_m, j_1, j_2, \ldots, j_a}_{2^m k_1, k_2, \ldots, k_a} \subset \Delta^{p_m, j_1, j_2, \ldots, j_{a-1}}_{2^m k_1, k_2, \ldots, k_{a-1}}, \quad a = 1, 2, \ldots$$

Each of such boxes is defined from the basic cycle $(2^m k_1, k_2, \ldots, k_a; p_m, j_1, j_2, \ldots, j_a)$ with the rank-one basic period 2^m, and the rank-one basic permutation p_m of the period 2^m cycle generated inside ω_1. For $\lambda = \lambda^*_{2^m}$ the map T gives rise to m nonconnected intervals constituting a m-cyclic chaotic segment denoted $CH^{p_m}_{2^m}$. With the box-within-a-box symbolism, note that a period $2^m k_1$

is not a $k_1 2^m$ cycle because they are not generated in the same box. The box-within-a-box symbolism implies a cycle identification related with a well-defined box.

Considering now a box $\Omega_{k_1}^{j_1}$, bifurcations $\lambda = \lambda_{k_1 2^m}^{*j_1,p_m}$, $m = 1,2,\ldots, k_1 = 1,3,4,\ldots$, can be defined in an equivalent way. They are characterized by the fact that the critical points, from the rank $k_1 2^{m+1}$, merge into the unstable period $k_1 2^{m-1}$ cycle ($S < -1$). Considering the flip bifurcation $\lambda = \lambda_{k_1 bm}^{j_1}$, generating the attracting cycle $(k_1 2^m; j_1, p_m)$, the interval $\lambda_{k_1 bm}^{j_1} \leq \lambda \leq \lambda_{k_1 2^m}^{*j_1}$ defines a box of second kind, denoted $\Omega_{k_1 2^m}^{j_1} \subset \Omega_{k_1}^{j_1}$. When $m \to \infty$, the two boundaries of $\Omega_{k_1 2^m}^{j_1}$ tend toward $\lambda_{k_1 s}^{j_1}$, with $\lambda_{k_1 2^m}^{*j_1} > \lambda_{k_1 s}^{j_1}$. Here k_1 is the rank-one basic period of all the $\Omega_{k_1 2^m}^{j_1}$ cycles, j_1 the corresponding rank-one basic permutation, $k_1 2^m$ the rank-two basic period, (j_1, p_m) the rank-two basic permutation, p_m being the permutation related to the period 2^m cycle of the ω_1 spectrum. For $\lambda = \lambda_{k_1 2^m}^{*j_1}$ the map T gives rise to km nonconnected intervals constituting a km-cyclic chaotic segment denoted $CH_{k_1 2^m}^{j_1,p_m}$.

Other boxes of the second kind of embedded versions can be defined, for example:

$$\Omega_{k_1,k_2,\ldots,k_a 2^m}^{j_1,j_2,\ldots,j_a,p_m} = [\lambda_{k_1,\ldots,k_a bm}^{j_1,\ldots,j_a}, \lambda_{k_1,k_2,\ldots,k_a 2^m}^{*j_1,j_2,\ldots,j_a}] \subset \Omega_{k_1,k_2,\ldots,k_a}^{j_1,j_2,\ldots,j_a}$$

from the cycle $(k_1, k_2, \ldots, k_a 2^{m-1}; j_1, j_2, \ldots, j_a, p_m)$ with $S = -1$, and

$$\Omega_{k_1,k_2,\ldots,k_{r-1}.2^m.k_{r+1},\ldots,k_a}^{j_1,j_2,\ldots,j_{r-1},p_m,j_{r+1},\ldots,j_a} = [\lambda_{k_1,k_2,\ldots,k_{r-1},k_r bm,k_{r+1},\ldots,k_a}^{j_1,j_2,\ldots,j_{r-1},p_m,j_{r+1},\ldots,j_a}, \lambda_{k_1,k_2,\ldots,k_{r-1},2^m,k_{r+1},\ldots,k_a}^{*j_1,j_2,\ldots,j_{r-1},p_m,j_{r+1},\ldots,j_a}] \subset \Omega_{k_1,k_2,\ldots,k_a}^{j_1,j_2,\ldots,j_a}$$

from the cycle $(k_1.k_2, \ldots, k_{r-1}.2^{m-1}.k_{r+1}, \ldots, k_a; j_1, j_2, \ldots, j_{r-1}, p_m, j_{r+1}, \ldots, j_a)$ with $S = -1$. More complex boxes, with several periods 2^q in the k_i sequences characterizing a cycle period, can be defined.

Figure 1 represents the fractal "box-within-a-box" (or embedded boxes) bifurcations structure, with self-similarity properties. The organization of the set Ω_1 is similar to that of its parts (the above defined boxes), even if these parts are infinitesimal.

2.3. *Properties*

Consider the map (3) and increasing values of the parameter λ. In this case, the multiplier S of a cycle $(k; j)_{S>1}$ increases, and the multiplier S of a cycle $(k; j)_{S<-1}$ decreases. So these cycles become more and more repelling, and they cannot disappear by bifurcation. The following properties result from the fractal box-within-a-box organization:

(a) Let $[k, j]$ (nonembedded representation), $k = 1,3,4,\ldots$, be the given basic cycle of the box Ω_k^j with $S < 1$. For $\lambda \geq \lambda_{ks}^j$ the spectrum ω_k^j has generated an invariant set with Cantor like structure $Cs[k, j]$, made up of all the repelling $(k2^i, j_i)$-cycles, $i = 0,1,2,\ldots$, with multiplier $S < -1$, and their limit set, born from the flip bifurcations of ω_k^j.

(b) Let $(k_1; j_1)$ be the basic cycle ($S < 1$) of the rank-one box $\Omega_{k_1}^{j_1}$. For $\lambda \geq \lambda_{k_1}^{*j_1}$ the box $\Omega_{k_1}^{j_1}$ has generated an invariant set with a Cantor like structure $Cs[k_1, j_1]$. This fractal set is made up of infinitely many Cantor like sets, $Cs[k_1 k_2; j_1, j_2], \ldots, Cs[k_1 k_2, \ldots, k_a; j_1, j_2, \ldots, j_a]$ (embedded representation)..., $a = 1,2,\ldots,\infty$, generated from the infinitely many boxes embedded into $\Omega_{k_1}^{j_1}$.

(c) For $\lambda \geq \lambda_{k_1}^{*j_1}$ the map T (thus not only the box $\Omega_{k_1}^{j_1}$ as in (b)) has generated infinitely many invariant sets with Cantor like structure related to the infinitely many boxes created for $\lambda < \lambda_{k_1}^{*j_1}$.

(d) For $\lambda < \lambda_{(k_1)s}^{j_1}$ the map has generated infinitely many invariant sets with Cantor like structure related to the infinitely many boxes created for $\lambda < \lambda_{(k_1)_0}^{j_1}$.

(e) For $\lambda \geq \lambda_1^*$, T has generated all the possible cycles (which are repelling), created from the infinitely many boxes embedded into the over all box Ω_1, and all belong to an invariant set with Cantor like structure included in the interval $[q_1^{-1}, q_1]$.

(f) For any $\lambda \geq \lambda_{1s}$ the map T has generated a Cantor like invariant set on which the restriction of T is chaotic, which includes infinitely many repelling cycles defined from the properties (a)–(d).

Properties (a)–(d) describe a "microscopic" view of the generation of Cantor like structures, while (e)–(f) give the global result of such a generation. When the map variable is complex, $z' = z^2 - c$, the properties (a)–(d) have the interest of detecting what are the subsets of the Julia set J which become real when the real parameter c increases, and the J

subsets, located out of the real axis, which are just about to become real.

For any $\lambda \geq \lambda_{1s}$ denote Λ_λ^* the fractal invariant set belonging to $[q_1^{-1}, q_1]$ which includes all the unstable cycles created for values of the parameter lower than λ (whose bifurcation organization is well defined and represented in Fig. 1) together with all their preimages and limit points. When $\lambda_{1s} \leq \lambda < \lambda_1^*$ from any initial point $x_0 \in [q_1^{-1}, q_1] \backslash \Lambda_\lambda^*$, after a number $N(x_0)$ of iterations (the number depending on the initial point) the trajectory enters an ε-neighborhood of the unique attracting set existing in $\overline{CC_1}$. Then when the point x_0 is sufficiently close to Λ_λ^*, $N(x_0)$ may be quite high and the orbit (discrete trajectory) possesses a *chaotic transient*. When $\lambda > \lambda_1^*$, and $x_0 \in]q_1^{-1}, q_1[\backslash \Lambda_\lambda^*$, $N(x_0)$ denotes the number of iterations occurring in the interval $]q_1^{-1}, q_1[$, after which the point is mapped outside $]q_1^{-1}, q_1[$ and the orbit diverges tending toward infinity. When the point x_0 is close to Λ_λ^* then $N(x_0)$ may be quite high and the trajectory possesses a chaotic transient.

As remarked above, on the x-axis, the repelling cycles, their increasing rank preimages, and their limit points, have a fractal organization when $\lambda \geq \lambda_{1s}$. For each point of the parameter λ-axis, $\lambda \geq \lambda_{1s}$, the fractal structure of the map singularities is completely identified from the box-within-a-box bifurcation organization. Consider $\lambda \in \omega_k^j$, with λ sufficiently near $\lambda_{(k)0}^j$ so that the map has an attracting cycle $(k; j)$. For the map T^k this cycle gives k attracting fixed points P_i, $i = 1, \ldots, k$, each of which with an immediate basin $d_0(P_i)$, and a total nonconnected basin $d(P_i) = \bigcup_{n>0} (T^{-k})^n d_0(P_i)$. The total basins $d(P_i)$ have a fractal structure. The set Λ_λ^* constitutes a strange repeller, which belongs to the boundary of $\bigcup_{n=1}^k d(P_i)$.

2.4. *Limit sets of boxes and resulting properties*

Let $\lambda > \lambda_{1s}$. Consider the critical set E_c (i.e. the orbit of the critical point) $E_c = \{T^n(C), n \geq 0\}$, its derived set E'_c, the set E of repelling cycles ($|S| > 1$), and its derived set E'. The set E' contains sets of accumulation points of increasing *classes* in the Pulkin' sense [1950]. So E belongs to the class 1, a *limit point of class p* being accumulation point of points of class $q < p$ (also see [Mira, 1987, pp. 99–100]). The situation $p \to \infty$ is characteristic of a fractal set.

Adapting the Fatou's results [1919] to the case of a real variable, the following properties can be deduced [Mira, 1987, pp. 156–160]:

(i) When T has an attracting cycle ($|S| < 1$), a point of the critical set E_c or its derived E'_c does not belong to the set E of repelling cycles ($|S| > 1$), or to its derived set E'.

(ii) When $E \cup E'$ contains points of $E_c \cup E'_c$, then some bifurcation occurs, giving either a neutral cycle with $|S| = 1$, or some chaotic attracting set, say for $\lambda = \hat{\lambda}$. In this last case T generates either a critical attractor A_{cr} or k-cyclic chaotic segments ($k \geq 1$) in the interval $\overline{CC_1}$ (for $k = 1$ the chaotic interval is bounded by the critical points C and C_1). For example, λ_k^{*j} (or any closure of a box of first kind), $\lambda_{k2^i}^{*j}$ (or any closure of a box of second kind) and λ_{ks}^j (Myrberg's limit point of flip bifurcations sequence) are particular $\hat{\lambda}$ values. When $\lambda = \lambda_k^{*j}$, k points of E_c and their increasing rank images merge into k points of E. When $\lambda = \lambda_{k2^i}^{*j}$, $k2^i$ points of E_c and their increasing rank images merge into $k2^i$ points of E. When $\lambda = \lambda_{ks}^j$ the whole set E'_c coincides with the critical attractor A_{cr} and belongs to E'.

From these considerations, a first set of properties related to the different types of limit sets of sequences of rank-one boxes Ω_r^h ($r = 3, 4, \ldots$) can be given (more details are given in [Mira, 1987, pp. 156–160, 166–174]).

(a) Consider a rank-one box of first kind $\Omega_k^j = [\lambda_{(k)_0}^j, \lambda_k^{*j}]$, $k = 3, 4, \ldots$, and its boundaries. For $\lambda < \lambda_{(k)_0}^j$ the parameter value $\lambda_{(k)_0}^j$ (at which the set E'_c consists in the $(k; j)$ neutral cycle) is a limit point of rank-one boxes of first kind $\Omega_{k'}^{j'}$ with $k' \to \infty$. For $\lambda > \lambda_k^{*j}$ the value λ_k^{*j} is a limit point of rank-one boxes $\Omega_{k''}^{j''}$ with $k'' \to \infty$. For $\lambda < \lambda_k^{*j}$, λ_k^{*j} is a limit of a subset of rank-a boxes, $a > 1$, embedded into Ω_k^j. The value λ_k^{*j} is such that E_c includes the repelling $(k; j)_{S>1}$ cycle (i.e. C is either periodic or preperiodic), the set E_c is without accumulation points.

(b) Inside each Ω_k^j box the bifurcation values λ_{ks}^j is a limit point for $\lambda > \lambda_{ks}^j$ of $\lambda_{k2^i}^{*j}$ values when $i \to \infty$, and for $\lambda < \lambda_{ks}^j$ of the flip bifurcations generated in the interval ω_k^j. The value λ_{ks}^j is such that the whole set E'_c coincides with the critical attractor A_{cr} (i.e. the invariant set with

Cantor like structure $\Lambda^*_\lambda \subset [q_1^{-1}, q_1]$). Moreover the critical point C belongs to the set E'_c and the set E'_c belongs to E'.

(c) Parameter values of type $\hat{\lambda}$, denoted $\tilde{\lambda}$, exist as limit of boxes Ω_r^h, without belonging to a box boundary. For example $\tilde{\lambda} \simeq 1.89291098791$ for which $q_2 \equiv C_3$ (and similar values exist for each $k \geq 3$ at which $q_2 \equiv C_k$, [Mira, 1982, 1987]). Then $\overline{CC_1}$ is an absorbing chaotic segment, giving rise to a nonclassical invariant measure (cf. [Couot & Mira, 1983; Mira, 1987, pp. 156–160, 166–174], see also [Thunberg, 2001] and references therein). At such particular bifurcation values (in which the attracting set of the map is a chaotic interval, or cyclical chaotic intervals) the set E_c includes a repelling cycle (i.e. C is either periodic or preperiodic, the set E_c is without accumulation points).

(d) Due to the self-similarity property, (a)–(c) also recur for embedded rank-a boxes, $a > 1$, with adaptations related to their rank, for example, now $\overline{CC_1}$ contains some cyclic chaotic segment giving rise to a nonclassical invariant measure.

We note that for $\lambda = \lambda_k^{*j}$ the cyclic chaotic segment CH_k^j, made up of the k segments $\overline{CC_k}$, $\overline{C_1C_{k+1}}, \ldots, \overline{C_{k-1}C_{2k-1}}$, contains all the cycles created inside the Ω_k^j box, and their limit sets. Its complementary part $\overline{CC_1} \backslash CH_k^j$ inside $\overline{CC_1}$ contains all the repelling cycles created for $\lambda < \lambda^j_{(k)0}$. A value $\tilde{\lambda}$, limit of boxes Ω_r^h, is such that $\overline{CC_1}$ contains all the cycles created for $\lambda < \tilde{\lambda}$, except the point q_1 (period one cycle).

We end this section summarizing briefly the properties of this family of maps, as λ varies in the interval $-1/4 \leq \lambda \leq 2$.

For any value of λ almost all the points x of the interval $]q_1^{-1}, q_1[$ (i.e. apart from at most a set of points of zero Lebesgue measure) have the same asymptotic behavior, which sometimes is called metric attractor A_λ, due to this property, and independently on its nature. This metric attractor A_λ can only be one of the following three typologies ([Blockh & Lyubich, 1991], see also [Sharkovsky *et al.*, 1997]):

(1) a k-cycle (of any period $k \geq 1$, either stable ($|S| < 1$), or neutral ($|S| = 1$));
(2) a critical attractor (A_{cr}) with Cantor like structure, of zero Lebesgue measure;
(3) k-cyclic chaotic intervals, $k \geq 1$.

In the case (1) the generic omega limit set $\omega(x)$ is equal to the omega limit set of the critical point C, and the trajectory of C tends to the k-cycle, stable or neutral $A_\lambda, \omega(C) = A_\lambda$. In the case in which $S = 0$ we have $E_c = A_\lambda$ and $E'_c = \varnothing$ while when $|S| \leq 1$ and $S \neq 0$ we have $A_\lambda = E'_c$ and $E_c \cap E'_c = \varnothing$, so that $E'_c \cap J \neq \varnothing$ when $|S| = 1$.

In the case (2) the generic omega limit set $\omega(x)$ is equal to $\omega(C) = E'_c$ (that is $A_{cr} = E'_c$) and $C \in E'_c$ (so that $E_c \subset E'_c$).

In the case (3) the critical point C is either periodic or preperiodic, merging into a repelling cycle ($|S| > 1$), which is called a critical periodic orbit. Thus E_c consists in a finite set of points, which are not limit point of critical points, however the critical periodic orbit belongs to the chaotic intervals A_λ, so that $E_c \cap E \neq \varnothing$ and $E_c \cap A_\lambda \neq \varnothing$.

Let us define as λ_p the set of parameter values in the box Ω_1 ($[-1/4; 2]$) at which the typology (1) occurs, λ_{cr} and λ_{ch} respectively the set of parameter values in the same interval $[-1/4, 2]$ at which the typology (2) and (3) respectively occurs. Then it is important to notice that the set λ_p consists of infinitely many nontrivial intervals having a fractal structure in the interval $[-1/4, 2]$ and dense in it (i.e. $\overline{\lambda}_p = [-1/4, 2]$). These intervals are the Myrberg spectra without their boundary λ^j_{ks}. The set λ_{cr} is a completely disconnected set of zero Lebesgue measure while the set λ_{ch} is a completely disconnected set of positive Lebesgue measure (for the proofs we refer to [Thunberg, 2001] and references therein). Thus the set in which we have chaotic attracting sets, above denoted with $\hat{\lambda}$, is given by their union, that is, $\hat{\lambda} = \lambda_{cr} \cup \lambda_{ch}$ and is a set of positive Lebesgue measure.

As recalled in the previous sections, when the parameter λ varies in the interval $-1/4 \leq \lambda \leq 2$ sequences of "boxes" occur, with the related bifurcations. Each box of the first kind is opened by a fold bifurcation giving rise to a pair of cycles, such a box of first kind closes when the cycle with $S > 1$ becomes critical for the first time (i.e. the first time that a critical point merges in it, at its first homoclinic bifurcation). Inside each box of first kind the cycle with $S < 1$ starts an infinite sequence of flip bifurcations, each of which opens a box of second class which closes when it becomes critical for the first time (i.e. at its first homoclinic bifurcation). Such sequences of boxes have a fractal structure due to the self-similar property. All the boundaries of boxes of first or second class are bifurcation values. At all the opening values the map is of typology

(1), while all the closure values are global (homoclinic) bifurcations (belonging to the set λ_{ch}), and the map is of typology (3). Inside each box of first kind there exists a limit value of boxes of second kind at which the map is of typology (2) (for example, those previously denoted as λ_{ks}^{j} belonging to the set λ_{cr}). Particular bifurcation values of λ are those which are limit points of other bifurcation values (for example, boundaries of boxes of first class), such bifurcation values belong to the set λ_{ch} and the map is of typology (3). In particular, when the critical point C is periodic or preperiodic (to an unstable cycle) the map is of typology (3).

Remark. The results related to the above item (c) (i.e. to typology 3) are generally attributed to [Misiurewicz, 1981], the parameter values λ_1^*, $\lambda_{2^1}^*$,λ_k^{*j}, $\hat{\lambda}$, $\tilde{\lambda}$ and their embedded forms being called "*Misiurewicz points*" by Blanchard [1984], or other authors. Nevertheless these values were identified before, from the years 1975, without using the same language (cf. [Gumowski & Mira, 1975, 1980a, 1980b; Mira, 1975, 1976, 1978, 1979, 1982, 1987; Couot & Mira, 1983]). This identification permitted the ordering of the Myrberg spectra in the framework of the fractal "*box-within-a-box*" bifurcations organization.

2.5. *General occurrence of the embedded boxes organization*

As already remarked in Sec. 2.1 the embedded boxes organization generated by the Myrberg's map T (3) also occurs for other types of unimodal maps. Particularly in the case of the general form of quadratic map $y' = ay^2 + 2by + c$, a linear change of variable $y = \alpha x + \beta$ leads to (3) with $\lambda = b^2 - ac - b$, $a\alpha = 1$, $a\beta = -b$. Moreover, particular classes of bimodal maps (maps with two extrema, i.e. $Z_1 - Z_3 - Z_1$ maps) create such bifurcations organization related to each of the two possible attractors (cf. [Gumowski & Mira, 1980, pp. 401–418]). For multimodal maps locally this organization may also exist.

The fractal embedded boxes organization described in Secs. 2.1–2.3 shows that if the Myrberg's map has a cycle with a period different from 2^i, $i = 0, 1, 2, \ldots$, that is for $\lambda > \lambda_{1s}$, then T has already generated infinitely many repelling cycles which belong to a strange repeller (as stated in Sec. 2.3). This property may occur also in multimodal maps. It gives a test permitting to state the presence of a strange repeller, and also the existence of any homoclinic trajectory of a repelling cycle permits to state this existence (and an homoclinic explosion of a repelling cycle occurs whenever a critical point is merging with a repelling cycle, which corresponds to the existence of chaotic dynamics on intervals). In the case of a two-dimensional map T, such a dynamic behavior may occur on a one-dimensional manifold, in which case we can say that it contains a strange repeller, generated by the one-dimensional map resulting from the restriction of T to this manifold.

3. Julia Set Properties from the Box-Within-a-Box Ones

3.1. *Some basic general Julia Fatou results*

The introduction has already recalled some basic properties (cf. (i) to (iii)) of the Julia set J generated by a complex map $z' = \varphi(z)$ (not necessarily a quadratic polynomial). In particular J is a perfect set including the set E of all the unstable cycles of any period $k = 1, 2, 3, \ldots$, their derived set (or set of limit points) E' (Julia notation), $J \equiv E'$. This section gives more properties denoted below (P1)–(P7). In this framework it is reminded (see (iii) in Sec. 1) that the point at infinity is an attracting fixed point with multiplier $S = 0$ (being also a critical point), when $\varphi(z)$ is a polynomial.

- (P1) The basin of an attracting fixed point (or a cycle) is either simply connected (as it is always the case for the point $z = \infty$ when the map is polynomial), or nonconnected with infinite order (i.e. made up of infinitely many nonconnected components). The basin of $z = \infty$ is bounded by the Julia set J. Generally J is nowhere differentiable [Julia, 1918; Fatou, 1920].
- (P2) If more than two attracting fixed points, or cycles, exist, at most one of these attractors can have a simply connected basin, the other basin being made up of infinitely many distinct domains [Fatou, 1920, p. 79].
- (P3) When a fixed point z^* is such that its multiplier is $|S| = 1$, it always belongs to the Julia set J. The point z^* is a limit point for the increasing rank images of the critical point related to the branch of the inverse map $\varphi^{-1}(z)$ related to this point, i.e. $z^* \in E'_c$ (cf. Sec. 2.1). The convergence toward z^* is

called "*singular*". Then the Julia set J has a numerable set of points where the tangent can be defined [Julia, 1918, pp. 52–53, 222–243; Fatou, 1919, p. 163, Chaps. II and IV].

(P4) The basin of an attracting fixed point (or a cycle) always contains a critical point [Julia, 1918, p. 129]).

(P5) Let α be an attracting fixed point of the map $z' = \varphi(z)$, and D_0 its immediate basin, supposed to be simply connected. If the boundary ∂D_0 of D_0 does not include a point image of a critical point of the inverse map φ^{-1}, or limit of increasing rank preimages of a critical point, ∂D_0 has no tangent at any of its points, except when ∂D_0 is a circle, or a straight line, or an arc of circle, or a segment of straight line [Fatou, 1920, p. 240].

(P6) The structure of J is self-similar (now called fractal) (cf. [Julia, 1918, p. 49], and [Fatou, 1920]).

(P7) J is either a simple closed Jordan curve, or made up of infinitely many closed Jordan curves (C^u) and their limit points [Julia, 1918, p. 52]. In this last case J contains double points everywhere dense on J. Each point of a (C^u) curve is a limit point of curves external to the one considered, their dimensions tending toward zero.

3.2. *General properties of the quadratic map T_Z, c being real*

Let us recall some other particular features of J generated by the quadratic polynomial map T_Z in the interval $c \geq -1/4$. They were proved by Julia [1918] and Fatou [1919, 1920], and differently presented by Blanchard [1984], Devaney [1986]. The parameter λ of the real map $x' = x^2 - \lambda$ in Sec. 2 is now denoted c, and the cycles multipliers of the two-dimensional map T_Z are real and $S_1 = S_2 = S$.

(a) Except the cases $c = 0$, $c = 2$, the Julia set J is a fractal set (cf. Sec. 3.1, P3).

(b) For $c = 0$ J is the circle with radius 1 (i.e. the circle $|z| = 1$ in the complex representation), on which the map T_Z is topologically conjugated with the map of the circle into itself $f(\theta) = 2\theta$, $\theta \in [0, 2\pi]$ (cf. [Julia, 1918, p. 103; Fatou, 1920, p. 226]).

For $c = 2$ J is the interval $[-2, 2]$ [Julia, 1918, p. 52, 186].

(c) For $c > 2$, the critical point C belongs to the domain of divergent trajectories, then J is the complementary set of this domain, and is everywhere disconnected [Fatou, 1920, p. 84]. J is a Cantor set (on which the map is topologically conjugated with the shift map [Devaney, 1986]).

(d) For any $c > -1/4$ the restriction of T_Z to J is a chaotic map [Devaney, 1986]).

(e) For $-1/4 \leq c < 3/4$ J is made up of a simple (i.e. without multiple points) Jordan closed curve (cf. (P7) and [Julia, 1918, p. 52, 188–213]), fractal for $c \neq 0$. The shape of J in the interval $-1/4 \leq c < 0$ is sometimes called *petal-like*. For $3/4 \leq c < c_{1s} \simeq 1.401155189$, J is a closed continuous curve, which may have a parametric representation as $x = f(t)$, $y = g(t)$, having multiple points everywhere dense on itself. J is made up of infinitely many curves, each one being a simple closed Jordan curve (cf. (P7) and [Julia, 1918, p. 52, 220–222]).

(f) The last paragraph (p. 73) of Chapter 4 in [Fatou, 1920] notices that, when J (denoted F by Fatou) contains points of $E_c \cup E'_c$, the corresponding parameter of the map can be related to what is a bifurcation (even if this word is not used). Indeed Fatou says that examples show this situation, which is in the parameter space a boundary separating two regions where J varies continuously (for polynomials, see also the contribution of Douady in the book edited by Devaney [1994]). Fatou also notes that, in the general case, it would be interesting to find the necessary and sufficient condition for a continuous variation of J when the parameters vary. When c is a real parameter, the knowledge of the box-within-a-box organization permits to define the boundaries separating c intervals where J varies continuously, as this will appear below.

Among the situations $E'_c \subset J$ consider the particular case for which a fixed point, or cycle (limit of increasing rank critical points), $z^* \in J$ has a multiplier $|S| = 1$ (Secs. 3.4 and 3.5), thus with only a *basin* toward this cycle of J, attractor in the Milnor sense on the x-axis. This case is a bifurcation one, as indicated in Sec. 2. Except this case, when J contains points of $E_c \cup E'_c$ the corresponding situations are given by the values $c = \hat{c}$ (cf. Sec. 2.4, λ becoming now the parameter c). For example, such values are c_k^{*j}, or $\tilde{c}$, for which a rank-r critical point (belonging to E_c) C_{r-1}, $C_0 \equiv C$, merges with a point of a repelling cycle, and

c^j_{ks} (limit of period doubling bifurcations), and their embedded forms in rank-a boxes, $a > 1$. For such c-values J is not the basin boundary of an attracting set on the x-axis different from the point at infinity, and its shape nowadays is called a *dendrite*, see for example [Devaney, 1986] for $c = c^*_{2^1} \simeq 1.543689013$ ($C_2 \equiv q_2$). In such cases J is made up of a "base", the segment $[q_1^{-1}, q_1]$ of the x-axis, and an "arborescent" subset of J for $y \neq 0$.

(g) In the special case $c = 0$ the Julia set J of T_Z is a circle, with the fixed point q_2 (multiplier $S = 0$) as center. For $-1/4 \leq c < 0$, J has a shape presenting infinitely many bumps (see below Fig. 10), called above "petal like". When $c > 0$ first (i.e. before another shape change) the J shape appears as made up of infinitely many spikes (see below Fig. 11), the continuous evolution of the J shape occurring through the circle case.

3.3. *Specific properties of T_Z related with the box-within-a-box ones*

We turn now to more specific properties of the two-dimensional map T_Z (1), in the interval $-1/4 \leq c \leq 2$. From $T_Z(x, y) = T_Z(-x, -y)$ the symmetry property of the map, with respect to the origin, results. Thus the preimages of any point different from the origin are symmetric with respect to $(0, 0)$, as all the backward invariant sets, in particular J. This two-dimensional map T_Z has only one rank-one critical point, $C = T_Z(0, 0) = (-c, 0)$ belonging to the x-axis. This axis is an invariant set (the restriction to $y = 0$ is the Myrberg's map), thus all the critical points of any rank, images of C, belong to the x-axis. The same is not true for the preimages. Indeed it is easy to see that all the points of the plane, different from $(0, 0)$, possess two distinct rank-one preimages. *Only the points of the x-axis $(x, 0)$ with $x > x(C) = -c$ have two distinct, symmetric, rank-one preimages belonging still to the x-axis (they are those related to the Myrberg's map). All the other points of the plane have two distinct rank-one preimages not belonging to the x-axis* (and symmetric with respect to the origin). In particular, it is the case of the points $(x, 0)$ with $x < x(C)$, which have two distinct rank-one preimages belonging to the y-axis. For example, with $-1/4 \leq c \leq 2$ consider the two rank-one preimages of the segment $[q_1^{-1}, -c]$ on the x-axis, where q_1^{-1} is the preimage of the repelling fixed point q_1, different from this point. Such preimages give the two symmetric segments of the y-axis, $0 \leq y \leq \sqrt{-(q_1^{-1} + c)}$ and $-\sqrt{-(q_1^{-1} + c)} \leq y \leq 0$. Notice that the width of the segment $[q_1^{-1}, -c]$ (and that of its rank-one preimages) decreases as c increases, and tends to zero, which occurs for $c = 2$ with $q_1^{-1} = -c = -2$.

Due to the fact that all the critical points belong to the x-axis, it follows that *the attractor of the map T_Z at finite distance (related to the orbit of the critical point C) can only belong to the x-axis*, and so it is the attractor of the Myrberg's map. Divergent orbits always exist, thus the Julia set J bounds the basin of divergent trajectories, and may be also the boundary of the attractor on the x-axis. *For the map T_Z all the possible cycles always exist in the plane, at any value of c. Depending on the parameter value, some of them may be on the x-axis, and all the other outside* (necessarily repelling, thus belonging to J).

For example, for $-1/4 \leq c < 3/4$ only the two fixed points of T_Z belong to the x-axis, all the other k-cycles, $k > 1$, (which are repelling) have their ordinate $y \neq 0$, and belong to the Julia set J (here made up of a simple Jordan closed curve). Clearly, as all these cycles of period $k > 1$ have ordinates $y \neq 0$, the same property also occurs for all their preimages of any rank. While for the two fixed points on the x-axis, only a subset of their increasing rank preimages also exists with $y \neq 0$ from a certain rank.

As said in Sec. 1 two qualifiers, related to the Julia set J properties, can be used. The first one is the J *structure*, which identifies the set of $(k; j)$ unstable cycles belonging to $y = 0$, and the one belonging to $y \neq 0$, this without any relation with the J outline. The second qualifier is qualitative, and concerns the J *shape* directly related to its outline. This last qualifier is essentially related to the numerical simulation of J, but qualitatively depends on the ordinate ($y \neq 0$) of the first cycle which will attain the x-axis from a c-increase.

The structure and the shape of J change as the parameter c increases, starting from the value $c = -1/4$ [case of Fig. 7(a)]. As c increases, the positions of the repelling cycles with $y \neq 0$ (and thus the Julia set J) changes continuously as long as no bifurcation occurs on the x-axis, which involves the dynamics of the Myrberg's map $x' = x^2 - \lambda$. Thus *every bifurcation occurring in the Myrberg's map, also implies a bifurcation in the structure of J*. Generally the bifurcations of the Myrberg's map involve the appearance of cycles on the x-axis, or better: the transition of cycles already existing in the plane (outside the x-axis, on the set J), to the x-axis. Thus J can have continuous changes only in the interval of values corresponding to the existence of an attracting cycle on the x-axis, where no bifurcation occurs. Stated in other words, *when c increases, all the bifurcations of the Julia set J are associated with bifurcations of the Myrberg's map, and often correspond to transitions of cycles from $y \neq 0$ to the x-axis, from which they can never escape*. Now consider the properties of the Myrberg's map mentioned in Sec. 2.4, by emphasizing the related properties of the Julia set J.

For any value of c, $-1/4 \leq c \leq 2$ the structure of J is related to the structure of the unique metric attractor A_c existing on the x-axis for the Myrberg's map:

(P′1) When A_c is a stable k-cycle (of any period $k \geq 1$, $|S| < 1$), then J changes continuously in the interval of c for which $-1 < S < 1$ (as described in Secs. 5.3 and 5.4). The values $S = +1$ and $S = -1$ are bifurcation values for J, described in Secs. 5.1 and 5.2.

(P′2) When A_c is a critical set A_{cr} (with Cantor like structure of zero Lebesgue measure on the x-axis), or when A_c consists of k-cyclic chaotic intervals ($y = 0$), $k \geq 1$, then J is at a bifurcation value and it is a *dendrite*.

In the case (P′1) when the cycle is stable, ($|S| < 1$), then the trajectory of the critical point C is either periodic (superstable case), or tends to the stable k-cycle, and no point of $E_c \cup E'_c$ belongs to J. In this case J separates two basins: the basin of A_c, and the basin of divergent trajectories (the point at infinity being also an attractor for T_Z). When the cycle is neutral ($|S| = 1$) then J is at a bifurcation. One has $A_c = E'_c$ which is the neutral k-cycle, and $E_c \cap E'_c = \varnothing$ but now the periodic orbit E'_c belongs to J ($E'_c \subset J$). In this neutral case J separates the basin of the point at infinity (domain of divergence), and a *basin* toward A_c, which is a set of positive measure also for T_Z, in the two-dimensional phase plane.

The set of parameter values of the interval $[-1/4; 2]$, where a continuous variation of J occurs, is related to intervals inside the Myrberg's spectra, which are contained in all the boxes of first and second kind, in a self-similar way (Sec. 2.4). Each one of such intervals is bounded by two consecutive bifurcations of a given spectrum. So the continuous variations of J occur in infinitely many nontrivial intervals having a fractal structure in $[-1/4; 2]$ and dense in it.

As already noticed, in the case (P′2) the dendrite structure of J is related to two different situations, in each of which the invariant set A_c of the Myrberg's map has chaotic dynamics, and J is the boundary of the basin of divergent trajectories but not the boundary of an attractor on the x-axis (the invariant set A_c in fact belongs to J itself). The two different situations, related to the two different kinds of chaotic sets on the x-axis, have different properties in terms of limit sets of the critical point of the map T_Z. When c belongs to $c_{cr} \subset \hat{c}$, then A_c is a critical set A_{cr} (cf. Sec. 2.5 for the definition of A_{cr} and c_{cr}) with a Cantor like structure of zero Lebesgue measure. Then $A_{cr} = E'_c$, $C \in E'_c$ so that $E_c \subset E'_c \subset J$. While when c belongs to $c_{ch} \subset \hat{c}$ (cf. Sec. 2.4), then A_c consists of k-cyclic chaotic intervals, the critical point C is either periodic or preperiodic, merging into a repelling cycle ($|S| > 1$), which is called a critical repelling periodic orbit. Thus $E_c \cap J \neq \emptyset$. The set of parameter values $\hat{c} = c_{cr} \cup c_{ch}$, for which the case (P′2) of the dendrite structure of J occurs, is a completely disconnected set of positive Lebesgue measure.

When the case of (P′1) occurs, the so-called *filled Julia set* (or filled-in Julia set) $\mathcal{F}(J)$ is the set of all points (x, y) that have a bounded orbit (i.e. nondiverging trajectories). It is given by the closure of the basin of the stable (or neutral) k-cycle on the x-axis. The frontier of $\mathcal{F}(J)$ is J. Clearly in the cases in (P′2) the filled Julia set $\mathcal{F}(J)$ reduces to the dendrite J.

The properties of J, issued from the knowledge of the box-within-a-box organization, comes from the fact that the subset of E (repelling cycles of the map T_Z) belonging to the x-axis for $-1/4 \leq c < 2$, their preimages and limit sets, can be well identified by the symbolism described in Sec. 2, and such sets are involved in any bifurcation of J. We recall that all such bifurcations are of codimension two for the map T_Z because we have always $S_1 = S_2$ (a pair of cycles from the region $y \neq 0$ reaches the x-axis when $S = +1$, a cycle from the region $y \neq 0$ merges with a cycle on the x-axis when $S = -1$). The bifurcation values of the parameter c, defined by the box-within-a-box organization, permit to bound intervals where J changes continuously. So five different types of structure of the Julia set J can be identified, for c-values in the interval $-1/4 \leq c < 2$.

Three types are related to the c bifurcation values, and two to intervals where J has a continuous change, i.e. intervals corresponding to the existence of an attracting cycle on the x-axis, where no bifurcation occurs.

The *first type* occurs at each fold bifurcation ($S = +1$) on the x-axis. For $c = c_{(1)_0} = -1/4$, it will be considered of *class A*. For $c = c^j_{(k)_0}$, or more generally $c^{j_1,\dots,j_a}_{(k_1,\dots,k_a)_0}$ fold bifurcations giving rise to a pair of cycles of period $k_1, k_2, \dots, k_a$, it will be considered said of *class B*.

The *second type* occurs at each flip bifurcation on the x-axis ($S = -1$) belonging to a Myrberg spectrum. For $c = c_{bm} \in \omega_1$, it will be considered of *class A*. For $c = c^j_{k_{bm}} \in \omega^j_k$, or more generally $c^{j_1,\dots,j_a}_{(k_1,\dots,k_a)_{bm}} \in \omega^{j_1,\dots,j_a}_{k_1,\dots,k_a}$, $m = 1, 2, 3, \dots$, it will be considered of *class B*.

The *third type* occurs when J changes continuously in intervals $c_{(1)_0} < c < c_{b1}$, $c_{(1)_0} = -1/4$, $c_{b1} = 3/4$. It will be considered of *class A*. For $c^j_{(k)_0} < c < c^j_{k_{b1}}$, $k \ge 3$, or more generally $c^{j_1,\dots,j_a}_{(k_1,\dots,k_a)_0} < c < c^{j_1,\dots,j_a}_{(k_1,\dots,k_a)_{b1}}$, at which T_Z has an attracting cycle on the x-axis with multipliers $-1 < S < 1$, it will be considered of *class B*. Its immediate basin boundary is made up of k ($k = 1, 2, \dots$), or $k_1, \dots, k_a$, simple (i.e. without multiple points) Jordan closed curves. For $k > 2$ the points of these curves are accumulation of other such curves.

The *fourth type* occurs when J changes continuously in intervals $c_{bm} < c < c_{b(m+1)}$, $m = 1, 2, \dots$. Then it will be considered of *class A*. For $c^j_{k_{bm}} < c < c^j_{k_{b(m+1)}}$, $k \ge 3$, or more generally $c^{j_1,\dots,j_a}_{(k_1,\dots,k_a)_{bm}} < c < c^{j_1,\dots,j_a}_{(k_1,\dots,k_a)_{b(m+1)}}$, at which T_Z has an attracting cycle on the x-axis with multipliers $-1 < S < 1$, it will be considered of *class B*. For $k = 1$ ($c_{bm} < c < c_{b(m+1)}$) J is made up of infinitely many closed Jordan curves (C^u) and their limit points. In this last case J contains double points everywhere dense on J. Each point of a (C^u) curve is a limit point of curves external to the one considered, their dimensions tending toward zero [Julia, 1918, p. 52].

The *fifth type*, corresponds to the dendrite structure of J, at each value of c belonging to the set $\hat{c} = c_{cr} \cup c_{ch}$ (which includes values such as c^{*j}_k, c^j_{ks}, $\tilde{c}$ and their embedded forms in rank-a boxes, $a > 1$), except for the value $c = 2$.

So the "*class A*" indicates that the considered parameters belong to the first Myrberg spectrum. With the "*class B*" they belong to embedded spectra ω^j_k, or $\omega^{j_1,\dots,j_a}_{k_1,\dots,k_a}$. A given type associated with one of the two classes (except the fifth type) is related to a well defined structure (in Sec. 1 sense) of the Julia set, as it will be shown in Sec. 4.

3.4. *Consequences*

It is worth to note another remarkable property of the so-called filled Julia set $\text{Ⅎ}(J)$ in the cases defined by (P′1), and of the Julia set J in the cases defined by (P′2) (see above) for $c \in [-1/4, 2]$. In the case (P′1) $\text{Ⅎ}(J)$ is given by the closure of the set of all the preimages of the segment $[q_1^{-1}, q_1]$ on the x-axis. In the case (P′2) J is also given by the closure of the set of all the preimages of the segment $[q_1^{-1}, q_1]$ on the x-axis, that is:

$$\text{Ⅎ}(J) = C_l \left\{ \bigcup_{n \ge 0} T_Z^{-n}([q_1^{-1}, q_1]) \right\} \quad \text{in the cases (P}'1) \tag{4}$$

$$J = C_l \left\{ \bigcup_{n \ge 0} T_Z^{-n}([q_1^{-1}, q_1]) \right\} \quad \text{in the cases (P}'2) \tag{5}$$

where C_l denotes the closure of the set. Indeed considering the segment $[q_1^{-1}, q_1]$ of the x-axis, and the arborescent set of its increasing rank preimages, the property stated above is clearly true when $c = 2$ as in this case for any $n \ge 0$ we have $T_Z^{-n}([q_1^{-1}, q_1]) = [q_1^{-1}, q_1] = J$. When $c \in [-1/4, 2[$ the subset $[q_1^{-1}, -c[$ of the segment $[q_1^{-1}, q_1]$ is the one from which all the preimages have $y \ne 0$. So the two rank one preimages of the segment $[q_1^{-1}, q_1]$ on the x-axis are: the segment itself and the segment on the y-axis with $-\sqrt{-(q_1^{-1} + c)} \le y \le \sqrt{-(q_1^{-1} + c)}$, intersecting the other at the origin. Then all the increasing rank preimages consist of arcs issuing from (or crossing) the further preimages of the origin (on the x-axis and on the y-axis occurring whenever a preimage of some rank of the origin is a point belonging to $[q_1^{-1}, -c]$). The rank-one preimages of this first segment on the y-axis consist of two arcs issuing from (or transversally crossing) the two points of the x-axis belonging to $T^{-1}(x = 0)$ (which are the same points of $T_Z^{-1}(0, 0)$) and symmetric with respect to the x-axis. And so on. Considering $\bigcup_{n \ge 0} T_Z^{-n}([q_1^{-1}, q_1])$ we get infinitely many arcs which belong to the filled Julia set, or to J, because all such points do not have divergent trajectories.

Thus considering the closure of this set we get the whole filled Julia set, or the whole Julia set J. In fact, it is enough to consider the closure of the preimages of the point q_1 to get the whole frontier J, and thus the equality is obviously true in (5) when J is a dendrite. Otherwise in the cases (P′1), the set J (clearly contained in $C_l(\bigcup_{n\geq 0} T_Z^{-n}([q_1^{-1}, q_1])))$ is the frontier of a basin, and is necessarily on the boundary of the set, which thus consists in the filled Julia set $\mp(J)$.

We remark that the width of the first segment on the y-axis belonging to the rank-one preimage of $[q_1^{-1}, q_1]$ is $2\sqrt{-(q_1^{-1}+c)}$ and tends to zero as c increases towards 2. Thus the structure of J is more and more "contracted" on the x-axis, as c increases. Moreover, at each value of c, say $c = \bar{c}$, all the repelling cycles of the Myrberg's map (cycles of T_Z with $y = \emptyset$ belonging to J) belong to the subset $J \cap [q_1^{-1}, q_1]$, with the subset of their increasing rank preimages on the x-axis and their limit points. While the part of the Julia set J with $y \neq \emptyset$ includes all the other repelling cycles of T_Z (as all exist in the plane at any value of c) and still outside the x-axis. It is clear that such cycles and their preimages (all with $y \neq \emptyset$) are only limit points of the preimages of the interval $[q_1^{-1}, q_1]$. All such repelling cycles belonging to J (but not to the x-axis) will enter the x-axis at higher values of the parameter c, at the other bifurcations occurring for $c > \bar{c}$ in the Myrberg's map.

4. Propositions on the Julia Set Structure

4.1. *Generalities*

It is recalled that the notion of *structure* is only related to the identification of the position of $(k; j)$ unstable cycles and their limit sets in the plane, i.e. to the geometrical situation of well defined subsets of the Julia set J, this without any relation with the J outline (or shape). So a same structure of J can correspond to different shapes, which can be identified from a numerical simulation. Until now only a coarse view of the plane situation of the unstable cycles has been given for a non "bifurcated" $c = c_g$ parameter value: the ones located on the x-axis, generated for all the bifurcations of the interval $c_{(1)_0} \leq c < c_g$, the ones having $y \neq 0$, which are associated with the bifurcations of the interval $c_g < c < c_1^*$. The purpose of this section is to refine the identification of J subsets in $y \neq 0$.

Another description of the filled Julia set $\mp(J)$ in the cases (P′1) is obtained by considering the immediate basin, denoted $d_0(A_c) \subset (y = 0)$ of the Myrberg's real map $x' = x^2 - c$. The boundary $\partial d_0(A_c)$ of $d_0(A_c)$ belongs to J, then clearly

$$\mp(J) = C_l\left(\bigcup_{n\geq 0} T_Z^{-n}(d_0(A_c))\right) \tag{6}$$

and J is the boundary of $\mp(J)$.

Using the box-within-a-box symbolism, the above properties can be presented as follows. Consider the restriction of T_Z to the x-axis, that is the Myrberg's map T ($x' = x^2 - c$), and for $c^j_{(k_1)_0} < c < c^j_{kb1}$ the stable basic cycle $(k_1; j_1)$ of the box $\Omega^{j_1}_{k_1}$. The corresponding k_1 stable fixed points of the map T^{k_1} have as immediate basins the open segments $d^n_0(k_1; j_1) \subset (y = 0)$, $n = 1, 2, \ldots, k_1$, bounded by the associated $(k_1; j_1)$ unstable fixed points of T^{k_1}, with $S > 1$, and some well defined of their preimages until the rank k (Sec. 2.3). The boundaries of the other parts of the total basins (on $y = 0$) are made up of all the repelling cycles, created on the x-axis for $c < c^{j_1}_{(k_1)0}$ (the lower boundary of the box $\Omega^{j_1}_{k_1}$), their derived set, the increasing rank preimages (on $y = 0$) of all these points. Inside each of the immediate basins of the k_1 stable fixed points of T^{k_1}, and on their boundary, the dynamics reproduces the behavior inside the basin (and on its boundary) of the stable fixed point q_2 with $-1 < S < 1$, obtained when $c_{(1)0} = -1/4 < c < c_{1b1} = 3/4$. An equivalent property occurs in the intervals $c_{bm} < c \leq c_{b(m+1)}$ (belonging to the spectrum ω_1), for a period 2^m cycle, and $c^{j_1}_{k_1bm} < c \leq c^j_{k_1b(m+1)}$ (interval belonging to $\omega^{j_1}_{k_1}$) a period $k_1 2^m$ cycle.

When we consider the points of the two-dimensional phase plane of T_Z, with $-1/4 < c < 3/4$, the Julia set J is made up of all the unstable cycles of any period (belonging to the plane with $y \neq 0$, and entering the x-axis when the parameter c belongs to Ω_1), their limit points and their increasing rank preimages. The set J is the basin boundary of the fixed point q_2 (first cycle of the Myrberg spectrum ω_1, with multiplier $|S| < 1$). The basin is an open simply connected domain.

We note that the permutation of the abscissae of a cycle $(k; j)$ with $y \neq 0$, whatever be $c < c^j_{(k)0}$, is also that of the Myrberg's map. Indeed when c increases each cycle attains the x-axis, hence permitting to identify the cycle from the Sec. 2 data. The only difference is for a cycle $(k2^m; j, p_m)$ resulting from a flip bifurcation, for which each of the $k2^{m-1}$ pairs of its points have the same abscissa. This is due to the flip bifurcation on the x-axis, coming from the merging of a pair of cycles from the region with $y \neq 0$ with a stable cycle of period $k/2$ on the x-axis. From this property, even in this case the permutation of the cycle abscissae permits to identify the period $k2^m$ cycle.

Due to the properties of self-similarity, for $c^j_{(k)0} < c < c^j_{kb1}$ the immediate basin $D_0(k; j)$ of the stable $(k; j)$ cycle has a boundary $\partial D_0(k; j)$, which is a subset of the Julia set J. The set $\partial D_0(k; j)$ limits k domains, which are the immediate basins $D^n_0(k; j)$, $n = 1, 2, \ldots, k$, of the k stable fixed points of T^k_Z. Each one, with its boundary $\partial D^n_0(k; j)$, reproduces locally the dynamics obtained for $-1/4 < c < 3/4$, $D^n_0(k; j) \cap (y = 0) = d^n_0(k; j)$, and for $\partial d^n_0(k; j)$, the boundary of $d^n_0(k; j)$, $\partial D^n_0(k; j) \cap (y = 0) = \partial d^n_0(k; j)$. For $c^{j_1}_{(k_1)0} \leq c < c^{j_1}_{k_1 b1}$ the repelling cycles on the x-axis are those created for $c < c^{j_1}_{k_1 b1}$, property which results from the box-within-a-box organization.

Definition 4.1

(a) The Julia set J, or one of its subset, is said to have the Julia–Fatou configuration (A1) when it is a simple closed Jordan curve.

(b) The Julia set J, or one of its subset, is said to have the configuration (A2) in the following conditions. (i) It is a continuous closed curve, but having double points everywhere dense on itself. (ii) It is the union of infinitely many curves (C^u) with their limit points, each (C^u) being a simple closed Jordan curve. (iii) Each point of a (C^u) curve is a limit point of curves external to the one considered, their dimensions tending toward zero.

These two configurations are described in p. 52 of [Julia, 1918] and proved in pp. 158–175, (also see of [Fatou, 1920, p. 91]). A configuration (A1) different from the one obtained for $c = -1/4$ [see below Fig. 7(a)] is represented in Fig. 2 at $c = 0.72$. A configuration (A2) is represented in Fig. 3, at $c = 1.22$.

4.2. *First interval of a Myrberg spectrum*

From the Julia–Fatou results, and the above considerations, for the intervals $c_{(1)0} = -1/4 < c < c_{b1} = 3/4$, and $c^j_{(k)0} < c < c^j_{kb1}$, the J structure (in the sense defined in Sec. 1) is now well identified by the following propositions on the Julia set structure:

Proposition 1a. *Let c be the parameter value of the interval $c_{(1)0} = -1/4 < c < c_{b1} = 3/4$.*

(i) *The Julia set J is the basin boundary of the stable fixed point q_2.*

(ii) *J contains all the unstable cycles, and their limit sets, generated inside the box Ω_1. $J \cap (y = 0) = q_1 \cup q_1^{-1}$, $T^{-1}(q_1) = q_1 \cup q_1^{-1}$, and so contains only one unstable cycle, the fixed point $q_1 (S > 1)$.*

(iii) *J has the Julia–Fatou configuration (A1), i.e. it is a simple closed Jordan curve.*

Proposition 1a′. *Let c be the fold parameter value $c_{(1)0} = -1/4$, Proposition 1a holds changing the stable fixed point q_2, into the neutral fixed point.*

Propositions 1a and 1a′ result from the above considerations. Figure 2 represents the Julia filled set for $c = 0.72$. This figure shows the positions of the cycles $(2^1; p_1)$ (α_1, α_2), $(2^2; p_2)$ $(\eta_1, \eta_2, \eta_3, \eta_4)$, the period 3 cycle $(\sigma_1, \sigma_2, \sigma_3)$ generated in the box Ω^1_3, and the period 6 cycle (six blue points $\sigma^{1,1}_{2^1.3}$) of the box $\Omega^{1,1}_{2^1.3} \subset \Omega_{2^1}$. The evolution of these cycles will be followed in the next figures.

The symbolism of the following proposition is defined in Sec. 2.2 dealing with boxes of first kind.

Proposition 1b. *Let c be the parameter of the interval $c^{j_1}_{(k_1)0} < c < c^{j_1}_{k_1 b1}$, generating the stable cycle $(k_1; j_1)$, $k_1 = 3, 4, \ldots$, (multiplier $-1 < S < 1$). Let $\tilde{J}^{j_1}_{k_1}$ be the J subset of all the unstable cycles with a rank-one basic period k_1, a rank-one permutation j_1, generated inside the interval $c^{j_1}_{(k_1)0} \leq c \leq c^{*j_1}_{k_1}$ (box $\Omega^{j_1}_{k_1}$). Let $D_0(k_1; j_1)$ be the immediate basin of the stable cycle $(k_1; j_1)$.*

(i) *All the cycles, and their limit sets, generated for $c < c^{j_1}_{k_1 b1}$, belong to the J subset $J \cap (y = 0)$.*

(ii) *$\tilde{J}^{j_1}_{k_1} \subset J$ belongs to the immediate basin boundary $\partial D_0(k_1; j_1)$ of the stable cycle $(k_1; j_1)$.*

(iii) *Except the unstable cycle $(k_1; j_1)$ (multiplier $S > 1$) all the $\tilde{J}^{j_1}_{k_1}$ cycles have an ordinate $y \neq 0$.*

(iv) *The unstable cycles entering the x-axis for $c > c^{*j_1}_{k_1}$ are such that $y \neq 0$. They are limit points of the increasing rank preimages of $\tilde{J}^{j_1}_{k_1}$. Out of*

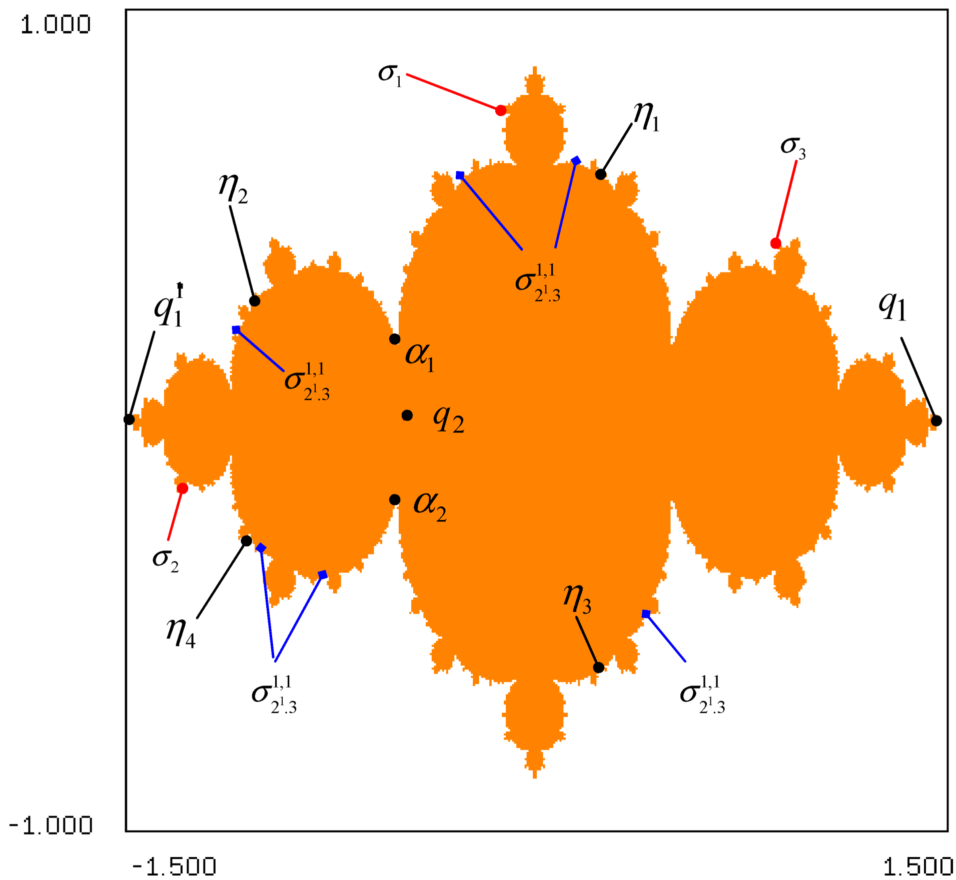


Fig. 2. The Julia filled set for $c = 0.72$, related to a Julia–Fatou configuration (A1). The Julia set J bounds the basin toward the stable fixed point q_2 ($-1 < S < 0$). The points α_1, α_2 are those of the period two cycle $(2^1; p_1)$. This figure also represents the period four cycle $(2^2; p_2)$ (points $\eta_1, \eta_2, \eta_3, \eta_4$), the period three cycle $(\sigma_1, \sigma_2, \sigma_3)$, entering the x-axis when c is in the box Ω_3^1, and the period six cycle (six blue points $\sigma^{1,1}_{2^1.3}$) when c is in the box $\Omega^{p_1,1}_{2^1.3} \subset \Omega_{2^1}$. The Julia set J presents infinitely many excrescences with a "base" having a decreasing length, tending toward zero for $c \to c_{b1} = 3/4$. The origin of such excrescences is due to the fact that when $c \to c_{b1}$ the two points ($y < 0$ and $y > 0$) of the unstable period 2^1 cycle $\alpha_1 \cup \alpha_2 \in J$ tend toward the stable fixed point q_2 on the x-axis. This creates locally, in the basin of q_2, a narrow vertical section bounded by α_1 and α_2, the increasing rank preimages of which are related to the fractal set of excrescences. When $c = c_{b1}$ the period two cycle merges with q_2, the section length becoming equal to zero, which leads to the basic Julia–Fatou configuration (A2).

the $D_0(k_1; j_1)$ closure, $\tilde{J}^{j_1}_{k_1}$ itself belongs to the limit set of these increasing rank preimages.

(v) *Each component $\partial D^n_0(k_1; j_1)$, $n = 1, 2, \ldots, k_1$, of $\partial D_0(k_1; j_1)$ has the Julia–Fatou configuration (A1), i.e. it is a simple closed Jordan curve. The unstable cycles generated for $c < c^{j_1}_{k_1 b1}$ (on the x-axis) belong to the limit set of the increasing rank preimages of $\partial D^n_0(k_1; j_1)$, which intersect $y = 0$ symmetrically.*

(vi) *The above properties recur for the first interval $c^{j_1,\ldots,j_a}_{(k_1,\ldots,k_a)0} < c < c^{j_1,\ldots,j_a}_{(k_1,\ldots,k_a)b1}$ of every spectrum $\omega^{j_1,\ldots,j_a}_{k_1,\ldots,k_a}$ of embedded first kind boxes $\Omega^{j_1,\ldots,j_a}_{k_1,\ldots,k_a}$.*

When $c^{j_1}_{(k_1)0} \le c < c^{j_1}_{k_1 b1}$, first we note that each boundary $\partial d^n_0(k_1; j_1) \equiv \partial D^n_0(k_1; j_1) \cap (y = 0)$ is made up of one of the k_1 points of the unstable $(k_1; j_1)$ cycle (with $S > 1$) and, among its rank-k_1 preimages, the nearest preimage of this cycle point. For $c = c^{*j_1}_{k_1}$ we note that each component of $CH^{j_1}_{k_1}$ (Sec. 2.2), k_1-cyclic chaotic segment on the x-axis, is bounded by the same points. The statement (i) results from the above considerations. The claims, (ii), (iii) are directly due to the k_1 periodicity, associated with the permutation j_1, of the immediate basin boundary $\partial D_0(k_1; j_1)$. Indeed according to the box-within-a-box cycles organization, except the cycles generated inside the box $\Omega^{j_1}_{k_1}$ ($c^{j_1}_{(k_1)0} \le c \le c^{*j_1}_{k_1}$) no other cycle multiple of k_1 with a basic period k_1, and a basic permutation j_1 (cf. Sec. 2.2) can exist on $\partial D_0(k_1; j_1)$. For $c = c^{*j_1}_{k_1}$

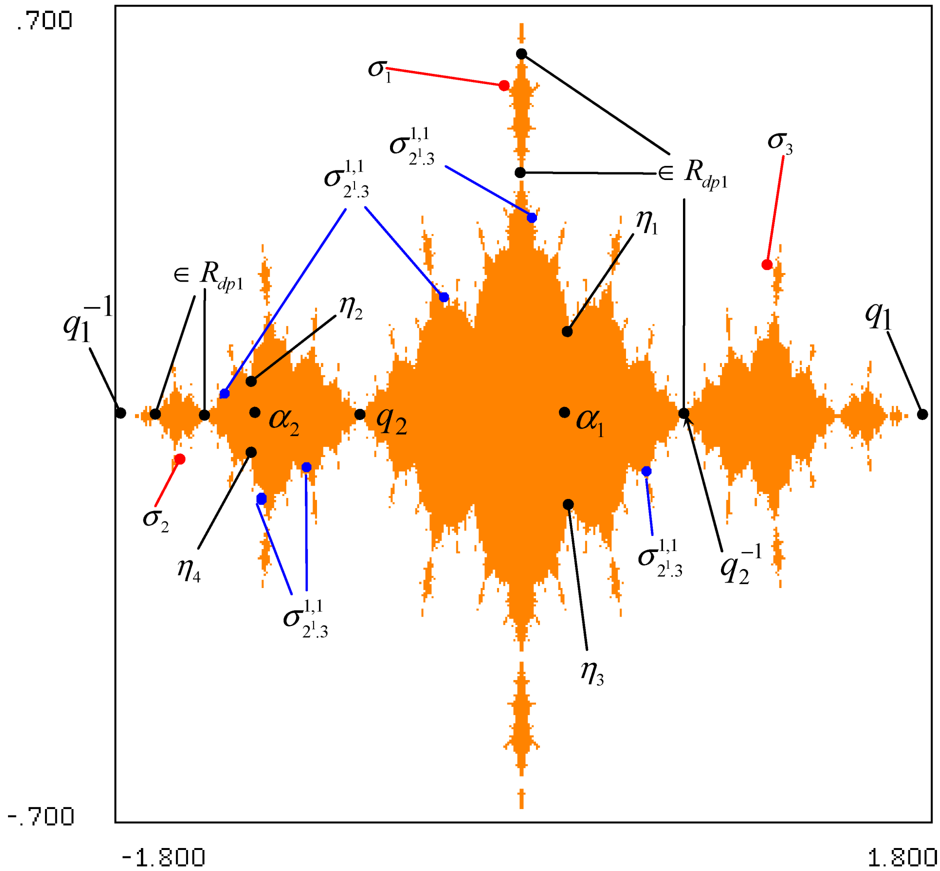


Fig. 3. Interval $c_{b1} < c < c_{b2}$, $c = 1.22$, basic Julia–Fatou configuration (A2). The filled Julia set is the basin toward the stable period 2^1 cycle $\alpha_1 \cup \alpha_2$. The set R_{dp1} is made up of the unstable fixed point q_2 and its increasing rank preimages. The colored points are those of the unstable cycles defined in Fig. 2. The period four cycle $(2^2; p_2)$ (points $\eta_1, \eta_2, \eta_3, \eta_4$), and the period six cycle (six blue points $\sigma^{1,1}_{2^1.3}$) belong to the immediate basin boundary of the stable period 2^1 cycle. The period three cycle $(\sigma_1, \sigma_2, \sigma_3)$ belongs to the remaining part of the basin boundary.

all the $CH^{j_1}_{k_1}$ unstable cycles $(k_1, \ldots, k_a; j_1, \ldots, j_a)$, $a = 1, 2, \ldots$, with a period multiple of k_1 comes from unstable cycles $(y \neq 0)$ which entered the x-axis for a c-value, $c^{j_1}_{(k_1)0} \leq c < c^{*j}_k$. Indeed when c increases in the interval $c^{j_1}_{k_1 b1} \leq c < c^{*j_1}_{k_1}$, the unstable cycles of $\tilde{J}^{j_1}_{k_1} \cap (y \neq 0)$ are those which progressively enter on the x-axis, more precisely in the intervals defined by the boundaries of the former $d^n_0(k; j)$ (now being not basin parts). The point (ii) also reflects the self similarity property between J for $-1/4 < c < 3/4$, which contains all the unstable cycles generated in Ω_1, and $\tilde{J}^{j_1}_{k_1}$ which for $c^{j_1}_{(k_1)0} < c < c^{j_1}_{k_1 b1}$ contains all the unstable cycles generated inside the box $\Omega^{j_1}_{k_1}$. About the other points of Proposition 1b, it is clear that the cycles generated for $c > c^{*j}_k$ are limit points $(y \neq 0)$ of the increasing rank preimages of $\tilde{J}^{j_1}_{k_1}$, and out of the $D^n_0(k_1; j_1)$ closure, each point of $\tilde{J}^{j_1}_{k_1}$ is a limit of a J subset made up of increasing rank preimages of $\tilde{J}^j_k$. As J is a simple closed Jordan curve for $c_{(1)0} = -1/4 < c < c_{b1} = 3/4$ (cf. [Julia, 1918; p. 52]), $\tilde{J}^{j_1}_{k_1}$ belongs to k_1 simple closed Jordan curves in the interval $c^j_{(k)0} \leq c < c^j_{kb1}$. Points (iii)–(vi) are justified by properties given in Secs. 3.3 and 3.4, by self similarity properties related to the immediate basin between the intervals $c_{(1)0} < c < c_{b1}$, $c^j_{(k)0} < c < c^j_{kb1}$, and their embedded forms.

Proposition 1b′. *Let c be the fold parameter value $c = c^{j_1}_{(k_1)0}$. Proposition 1b holds changing the stable cycle $(k_1; j_1)$ into the neutral cycle $(k_1; j_1)$, and adapting the boundaries of the parameter intervals.*

This Proposition results from the above considerations, when $c \to c^{j_1}_{(k_1)0}$ from decreasing c values.

4.3. *Interval bounded by the two first flip bifurcations of the basic Myrberg spectrum* ω_1

The symbolism of this section is defined in Sec. 2.2 dealing with boxes of second kind. The basic Myrberg spectrum is ω_1, defined by $c_{(1)0} = -1/4 < c < c_{1s} = 1.401, \ldots$. The subinterval of ω_1 here considered is $c_{b1} = 3/4 < c < c_{b2} = 5/4$, bounded by the two first consecutive flip bifurcations. In this interval the attractor is the stable cycle $(2^1; p_1)$, located on the x-axis. With the box-within-a-box symbolism, consider that 2^m denotes the period of a cycle born from the bifurcation $S = -1$ (so $2^2 \neq 4$, $2^3 \neq 8, \ldots$), and that a period denoted $2^m k$ is different from the period denoted $k2^m$. As shown by Julia and Fatou, the Julia set J has the Julia–Fatou configuration (A2), called here *basic configuration* (*A2*) (cf. Fig. 3), bounded on the x-axis by the unstable fixed point q_1 and its rank-one preimage q_1^{-1} different from q_1 $(T^{-1}(q_1) = q_1 \cup q_1^{-1})$.

Proposition 2(a1). *Let c be the parameter value inside the interval $c_{b1} = 3/4 < c < c_{b2} = 5/4$, inside the spectrum ω_1, interval bounded by the two first consecutive flip bifurcations and generating the stable cycle $(2^1; p_1)$. Let $\tilde{J}_{2^1}$ be the J subset of all the unstable cycles (i.e. with their limit sets), generated inside the interval $c_{b1} < c \leq c^*_{2^1}$ (box of second kind Ω_{2^1}), cycle which has a rank-one basic period 2^1 associated with the permutation p_1.*

- (i) *$J \cap (y = 0)$ contains $q_1 \cup q_1^{-1} \cup q_2$, and the subset of all the increasing rank preimages of q_2, located on $(y = 0)$.*
- (ii) *$\tilde{J}_{2^1} \subset J$ belongs to the immediate basin boundary $\partial D_0(2^1; p_1)$ of the stable period 2^1 cycle. The point q_2 is common to the two components $\partial D_0^n(2^1; p_1)$, $n = 1, 2$, of $\partial D_0(2^1; p_1)$.*
- (iii) *The J cycles different from the $\tilde{J}_{2^1}$ ones belong to $(y \neq 0)$, and enter the x-axis for c in the interval $c^*_{2^1} < c < c^*_1 = 2$.*
- (iv) *The Julia set J has the Julia–Fatou configuration (A2). The set J is connected but bounds nonconnected open domains. Among these domains 2^1 of them belong to the immediate basin $\partial D_0(2^1; p_1)$ of the stable period 2^1 cycle.*

Consider the immediate basin boundary $\partial D_0(2^1; p_1)$, of the stable period 2^1 cycle, and the basin boundary of each of the 2^1 stable fixed points generated by $T_Z^{2^1}$. The boundary $\partial D_0(2^m; p_m)$ is made up of two components $\partial D_0^n(2^1; p_1)$, $n = 1, 2$, with $\partial D_0^1(2^1; p_1) \cap \partial D_0^2(2^1; p_1) = q_2$. The statement (i) results from Sec. 3 considerations. The assertion (ii) is directly due to the 2^1 periodicity of $\partial D_0(2^1; p_1)$, associated with the permutation p_1. Indeed 2^1 and p_1 are respectively the rank-one basic period and the rank-one basic permutation for all the cycles generated in the interval $c_{b1} \leq c \leq c^*_{2^m}$ (box of second kind Ω_{2^1}). No other cycle with an even period 2^{1+r}, $r = 0, 1, 2, \ldots$, and a permutation p_{1+r}, exists in this interval. For $c = c^*_{2^1}$, $\tilde{J}_{2^1}$ belongs to the two-cyclic chaotic segment $CH^{p_1}_{2^1}$, having q_2 as common point. All its unstable cycles (generated inside Ω_{2^1}), now with $y = 0$, come from unstable cycles $(y \neq 0)$ which entered the x-axis for a c-value, $c_{b1} < c < c^*_{2^1}$. It is clear that the other unstable cycles of J with $y \neq 0$ are generated in the interval $c^*_{2^1} < c < c^*_1$ (point (iii) of Proposition 2a). The total basin of the stable period 2^1 cycle is nonconnected, but with a connected boundary J.

As shown by Julia and Fatou, J is made up of the union of infinitely many closed curves (C^u) and so has the Julia–Fatou configuration (A2) (cf. Sec. 4.1) limiting nonconnected open areas (cf. Fig. 3). The points set $R_{dp1} = \mathcal{C}_l(\bigcup_{r \geq 0} T^{-r}(q_2))$ corresponds to contacts between these curves, the points of which are dense on J. On the x-axis $J \cap (y = 0)$ is made up of q_2 and its increasing rank preimages, tending toward $q_1 \cup q_1^{-1}$, which belong to $R_{dp1} \cap (y = 0)$. The points of R_{dp1} belong to the connected basin boundary $\partial D(2^1; p_1)$ of the stable period 2 cycle located on $y = 0$. Each of these points *separates two bordering* (*adjacent*) *nonconnected parts of the total basin* $D(2^1; p_1)$.

For $y \neq 0$ the shape of $J \cap (y = 0)$ is reproduced on the fractal set of arcs given by $T_Z^{-r}([q_1^{-1}, q_1])$ for $r > 0$. It is clear that the other unstable cycles of J with $y \neq 0$ are those becoming stable in the interval $c^*_{2^1} < c < c^*_1$ (point (iii) of Proposition 2a). The total basin of the stable period 2^m cycle is nonconnected, but with a connected boundary J. Point (iv) is due to the properties of self-similarity of the embedded boxes.

Figure 3 is obtained from Fig. 2 after the merging of the two points α_1, α_2 of the unstable cycle $(2^1; p_1)$ into the stable fixed point q_2, the cycle $(2^1; p_1)$ becoming stable on the x-axis, and q_2 unstable. It results in a breaking of the of the simply connected basin Fig. 2 into pieces separated by the set of R_{dp1} points. Figure 3 illustrates the properties described in the Proposition. So the unstable period six cycle (six blue points $\sigma^{1,1}_{2^1.3}$) of the box $\Omega^{1,1}_{2^1.3} \subset \Omega_{2^1}$, as all the unstable cycles of Ω_{2^1} are located on the immediate basin boundary $\partial D_0(2^1; p_1)$ of the stable cycle $(2^1; p_1)$. The unstable period three basic cycle (three red points σ_i, $i = 1, 2, 3$), generated in the box Ω^1_3 out of Ω_{2^1}, belongs to the total basin boundary, but does not belong to $\partial D_0(2^1; p_1)$. The sequence of bordering (adjacent) nonconnected parts of the total basin intersecting the y-axis, is made up of decreasing open domains on both sides of $y = 0$, symmetrical with respect to $y = 0$, with ordinates $-\sqrt{-(q_1^{-1} + c)} < y < -\sqrt{-(q_1^{-1} + q_2)}$ and $\sqrt{-(q_1^{-1} + q_2)} < y < \sqrt{-(q_1^{-1} + c)}$. Their boundaries form two plaits, first rank preimages of boundaries of basin parts on both sides of the x-axis, with $-(q_1^{-1} + c) \le x \le q_2$. The increasing rank preimages of $\partial D_0(2^1; p_1)$ give rise to the J configuration (A2). Remark that these parts belong to J subsets constituting well defined levels of "strata" starting from the immediate basin.

4.4. *Interval bounded by the two first flip bifurcations of a Myrberg spectrum ω^j_k*

Proposition 2(b1). *Let c be the parameter value inside the interval $c^{j_1}_{k_1 b1} < c < c^{j_1}_{k_1 b2}$, of the spectrum $\omega^{j_1}_{k_1}$, $k_1 = 3, 4, \ldots$, generating the stable cycle $(k_1 2^1; j_1, p_1)$. Let $\tilde{J}^{j_1,p_1}_{k_1 2^1}$ be the J subset of all the unstable cycles, generated inside the interval $c^{j_1}_{k_1 b1} < c \le c^{*j_1}_{k_1 2^1}$ (box of second kind $\Omega^{j_1,p_1}_{k_1 2^1}$). Let $\hat{J}^{j_1}_{k_1}$ be the J subset of all the unstable cycles, generated inside the interval $c^{*j_1}_{k_1 2^1} < c \le c^{*j_1}_{k_1}$, located inside the box of first kind $\Omega^{j_1}_{k_1}$. Let $\partial D_0(k_1 2^1; j_1, p_1)$ be the immediate basin boundary of the stable point of the cycle $(k_1 2^1; j_1, p_1)$, made up of $k_1 2^1$ components $\partial D^n_0(k_1 2^1; j_1, p_1)$, $n = 1, 2, \ldots, k_1 2^1$.*

(i) *The subset $J \cap (y = 0)$ is made up of all the unstable cycles, their limit sets, born in the interval $c_{(1)0} \le c \le c^{j_1}_{k_1 b1}$.*

(ii) *$\tilde{J}^{j_1,p_1}_{k_1 2^1} \subset J$ belongs to the immediate basin boundary $\partial D_0(k_1 2^1; j_1, p_1)$. Each point of the unstable cycle $(k_1; j_1)_{S<-1}$ (located on the x-axis) is common to $\partial D^n_0(k_1 2^1; j_1, p_1)$ and $T^{k_1}_Z[\partial D^n_0(k_1 2^1; j_1, p_1)]$. This situation gives rise to a J subset $\tilde{J}^{j_1}_{k_1}$ made up of k_1 pairs $\overline{\partial D}^r_0$ of connected sets, $r = 1, \ldots, k_1$, $\tilde{J}^{j_1}_{k_1} = \bigcup_{r=1}^{k_1} \overline{\partial D}^r_0$.*

(iii) *Each pair $\overline{\partial D}^r_0$ is linked with a subset $\hat{J}(^{j_1}_{k_1})^r$ of $\hat{J}^{j_1}_{k_1}$. As $\overline{\partial D}^r_0$ the set $\hat{J}^{j_1}_{k_1} = \bigcup_{r=1}^{k_1} \hat{J}(^{j_1}_{k_1})^r$ and the set $\tilde{J}^{j_1}_{k_1} \cup \hat{J}^{j_1}_{k_1}$ are periodic of period k_1, associated with the permutation j_1. Each of the k_1 elements $\overline{\partial D}^r_0 \cup \hat{J}(^{j_1}_{k_1})^r$ of the set $\tilde{J}^{j_1}_{k_1} \cup \hat{J}^{j_1}_{k_1}$ has the basic Julia–Fatou configuration (A2) of Fig. 3, bounded on the x-axis by a point of the unstable cycle $(k_1; j_1)_{S>1}$ and its rank-k_1 preimage the nearest to this cycle point.*

(iv) *The unstable cycles generated for $c^{*j_1}_{k_1} < c \le c^*_1$ are such that $y \ne 0$, and are limit points of the increasing rank preimages of $\tilde{J}^{j_1}_{k_1} \cup \hat{J}^{j_1}_{k_1}$. The set $\tilde{J}^{j_1}_{k_1} \cup \hat{J}^{j_1}_{k_1}$ itself belongs to the limit set of these increasing rank preimages.*

(v) *The above properties recur for each interval $c^{j_1,\ldots,j_a}_{(k_1,\ldots,k_a)b1} < c < c^{j_1,\ldots,j_a}_{(k_1,\ldots,k_a)b2}$ of the spectrum $\omega^{j_1,\ldots,j_a}_{k_1,\ldots,k_a}$ of the embedded box of first kind $\Omega^{j_1,\ldots,j_a}_{k_1,\ldots,k_a}$.*

The property (i) results from considerations given in the previous section. As for (ii) it is directly due to the $k_1 2^1$ periodicity, with the permutation (j_1, p_m), of the immediate basin boundary $\partial D_0(k_1 2^1; j_1, p_1)$. Indeed according to the box-within-a-box cycles organization, except the cycles generated inside the box $\Omega^{j_1,p_1}_{k_1 2^1}$ ($c^{j_1}_{k_1 b1} < c \le c^{*j_1,p_1}_{k_1 2^1}$) no other cycle multiple of k_1 with a rank-two basic period $k_1 2^1$, and a rank-two basic permutation (j_1, p_1) (cf. Sec. 2.2) exists on the immediate basin boundary. For point (iii) it is clear that $\tilde{J}^{j_1}_{k_1} \cup \hat{J}^{j_1}_{k_1}$ is periodic with the period k_1 and the permutation j_1. Due to the self-similarity property, each of its k_1 elements reproduces the situation of points (ii)–(iv) of Proposition 2(a1). The situations of the present points (iv) and (v) are also due to the properties of self-similarity. The J subset $\tilde{J}^{j_1}_{k_1} \cup \hat{J}^{j_1}_{k_1}$ must belong to a subset of the limit points of the increasing rank preimages of $\tilde{J}^{j_1}_{k_1} \cup \hat{J}^{j_1}_{k_1}$, clearly out of the domain bounded by its external boundary, the other increasing rank preimages having as limit set the unstable cycles, and their limit sets, entering the x-axis for c in the interval $c^{*j_1}_{k_1} < c \le c^*_1$.

4.5. *Interval bounded by two consecutive flip bifurcations of the Myrberg spectrum ω_1*

The interval considered here is defined by $c_{bm} < c < c_{b(m+1)}$ ($\subset \omega_1$), $m = 2, 3, \ldots$, inside which the map gives rise to the stable cycle $(2^m; p_m)$. As a first step, starting from Fig. 3 let us compare its configuration with the ones in Figs. 4–6, obtained respectively for c-values of the interval with $m = 2, 3, 4$. In these figures, due to their informative and "central" illustrative role, the following basic cycles are represented:

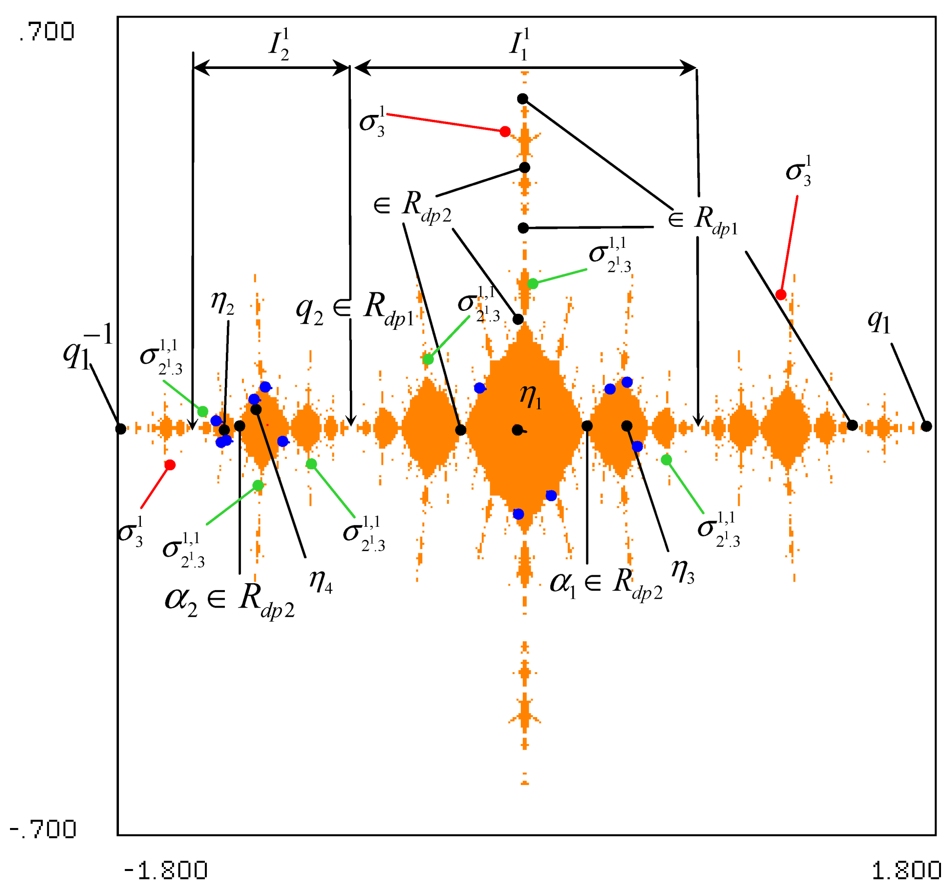
The $(2^m; p_m)$ cycles, $m = 0$ (fixed point q_2), $m = 1$ (points $\alpha_1 \cup \alpha_2$), $m = 2$ (η_i, $i = 1, \ldots, 4$), and for $m = 3, 4$, the corresponding stable cycle on the x-axis, are light blue colored.

The unstable period six cycle (six points $\sigma^{1,1}_{2^1.3}$) generated with $S < 1$ in the interval $c^*_{2^2} < c < c^*_{2^1}$ containing the box $\Omega^{1,1}_{2^1.3} \subset \Omega_{2^1}$, i.e. one of the two period three cycles of T^2_Z generated in the box Ω_{2^1}.

The unstable period 12 cycle (12 points) generated with $S < 1$ in the interval $c^*_{2^3} < c < c^*_{2^2}$ containing the box $\Omega^{1,1}_{2^2.3} \subset \Omega_{2^2}$, i.e. one of the two period three cycles of T^2_Z generated in the box Ω_{2^2}.

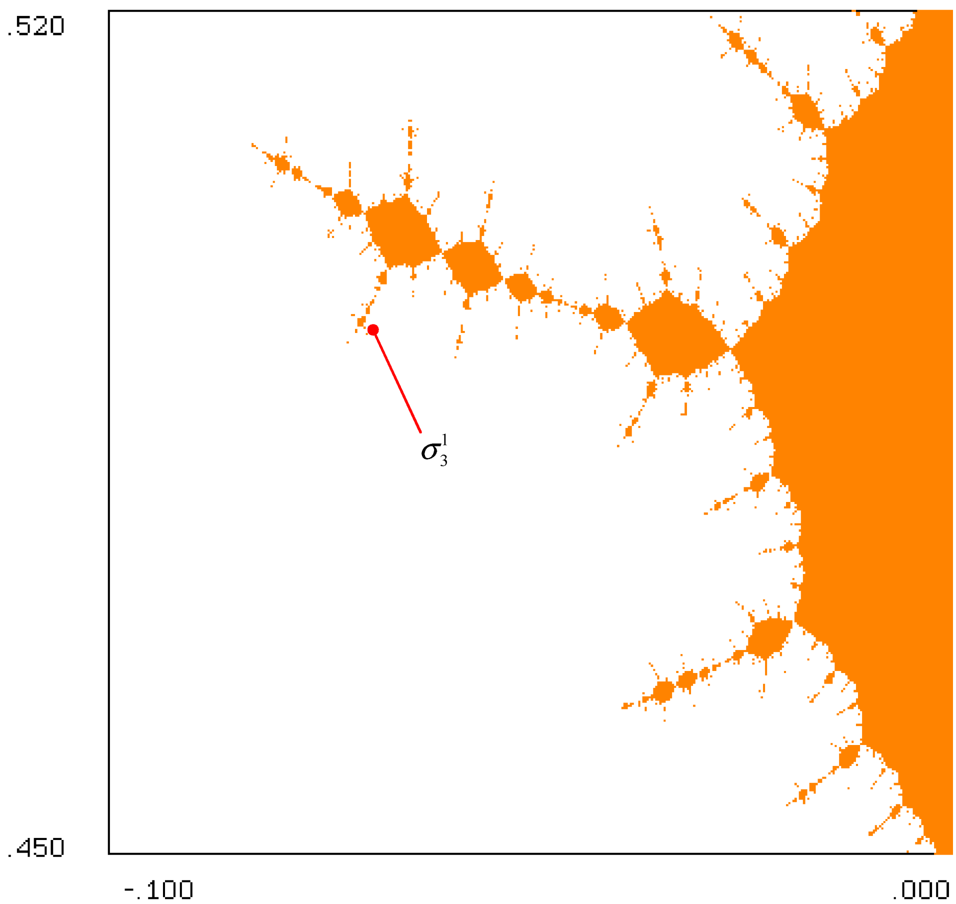
The unstable period 3 basic cycle (three points σ_i, $i = 1, 2, 3$) generated with $S < 1$ in the box Ω^1_3 out of Ω_{2^1}.

With the above mentioned cycles these figures illustrate an evident first property: the immediate basin boundary $\partial D_0(2^m; p_m)$ of the stable cycle $(2^m; p_m)$ (on the x-axis) being periodic of period 2^m, $\partial D_0(2^m; p_m)$ contains all the unstable cycles entering the x-axis for c in the interval $c_{bm} < c \leq c^*_{2^m}$ (box of second kind Ω_{2^m}), then with a rank-one



(a)

Fig. 4. Interval $c_{b2} < c < c_{b3}$, $c = 1.34$, the filled Julia set is a basin toward the stable period 2^2 cycle $(2^2; p_2)$ (points η_i, $i = 1, \ldots, 4$). (a) The Julia set J contains the set R_{dp1}, now accumulation of points of the set R_{dp2}, made up of the unstable period 2^1 cycle $\alpha_1 \cup \alpha_2$, and its increasing rank preimages. The blue points are those of one of the two period 12 basic cycles $(2^2.3; p_2, 1)$ of the box $\Omega^{p_2,1}_{2^23}$ contained in the interval $c^*_{2^3} < c < c^*_{2^2}$. They are located on the immediate basin boundary $\partial D_0(2^2; p_2)$ of the stable cycle $(2^2; p_2)$. The six green points $\sigma^{1,1}_{2^13}$, located in the rank-one J layer, are those of the period 6 basic cycle of the box $\Omega^{p_1,1}_{2^13}$ contained in the interval $c^*_{2^2} < c < c^*_{2^1}$. The three red points σ^1_3, located in the rank-two layer, are those of the cycle $(3; 1)$, generated inside the box Ω^1_3 located inside the interval $c^*_{2^1} < c < c^*_1 = 2$. (b) Enlargement of the rank-two layer containing a point of the cycle $(3; 1)$.



(b)

Fig. 4. (*Continued*)

basic period 2^m associated with the permutation p_m. Let $\tilde{J}_{2^m}$ be this subset of J.

A second property appears from the visible "central" basic configuration (A2) located on both sides of the y-axis, which reproduces Fig. 3 that was generated for $m = 1$, and which surrounds two points of the stable $(2^m; p_m)$ cycle. This configuration is repeated 2^{m-1} times.

Now a R_{dph} set is defined from the unstable cycles $(2^q; p_q) \in J \cap (y = 0)$, $q = 0, 1, \ldots, m-1$, born in the interval $c_{(1)0} = -1/4 \leq c \leq c_{bm}$. On the x-axis each point of a $(2^{h-1}; p_{h-1})$ cycle is an accumulation point of a subset of increasing rank preimages of a point of the cycle $(2^h; p_h)$, $0 < h < m$. Each of the unstable cycles $(2^{h-1}; p_{h-1})$, $0 \leq h \leq m$ (the cycle $(2^0; p_0)$ is the fixed point q_2), belonging to $y = 0$ and generated for $c < c_{bh}$, gives rise to a set R_{dph} of multiple points of J:

$$R_{dph} = \bigcup_{r \geq 0} T^{-r}[(2^h; p_h)]$$

Note that here the closure of $\bigcup_{r\geq 0} T^{-r}[(2^h; p_h)]$ is not considered. For $m = 1$ the points set belongs to $R_{dp1} = \bigcup_{r\geq 0} T^{-r}(q_2)$, defined in the previous section, and separating two bordering (adjacent) nonconnected parts of the total basin $D(2^1; p_1)$.

Two different kinds of R_{dph} sets can be distinguished:

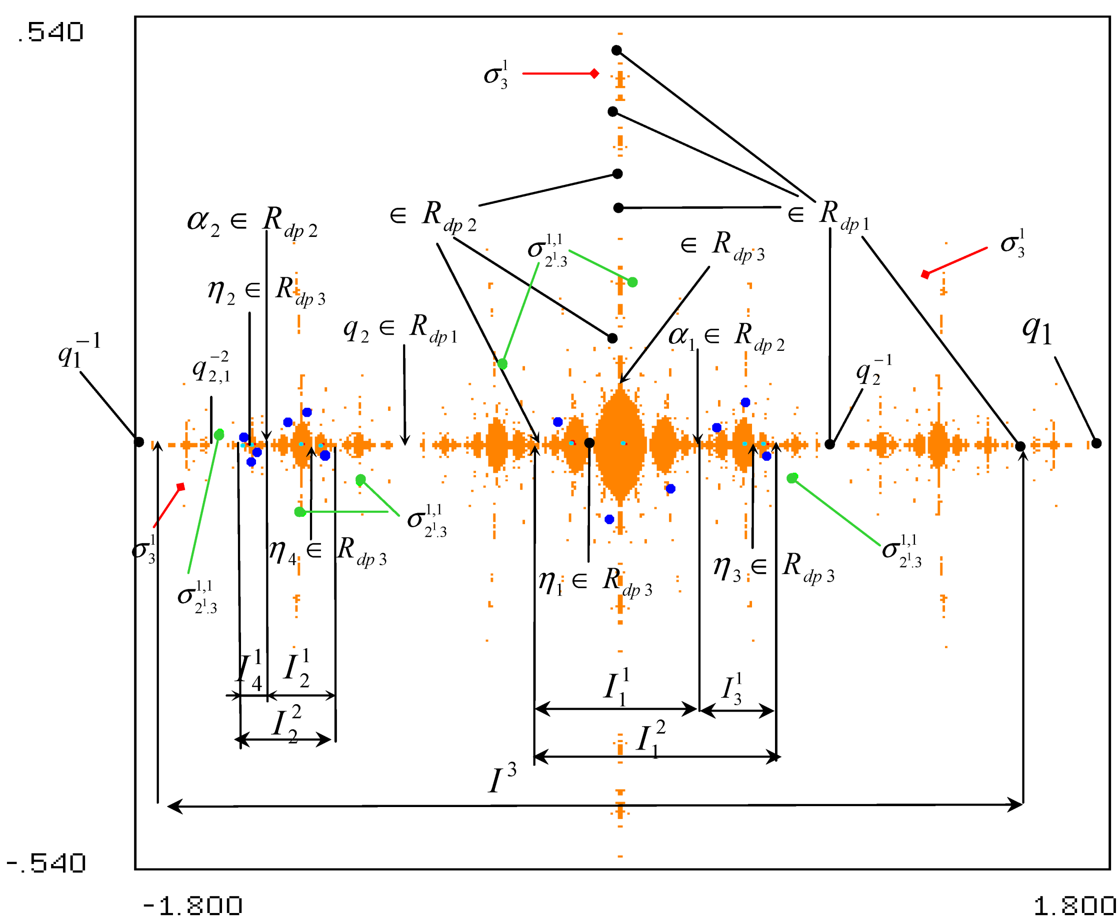
The first is related to points of R_{dpm} which separate two bordering (*adjacent*) *nonconnected parts of $D(2^m; p_m)$*. They are generated from the points of the unstable cycle $(2^{m-1}; p_{m-1})$ located on the x-axis, and their increasing rank preimages. Each component $\partial D_0^n(2^m; p_m)$ of the immediate basin boundary of one of the 2^m stable fixed points of $T_Z^{2^m}$, has a common point (a point of the cycle $(2^{m-1}; p_{m-1})$) with the immediate basin $T_Z^{2^{m-1}}[\partial D_0^n(2^m; p_m)]$ of another fixed point of $T_Z^{2^m}$. This gives rise to the 2^{m-1} pairs of connected sets $\overline{\partial D}_{0m}^r$ $r = 1, \ldots, 2^{m-1}$, via a common point of the unstable cycle $(2^{m-1}; p_{m-1})$. We shall say that the two immediate basin boundaries $\partial D_0^n(2^m; p_m)$ and $T_Z^{2^{m-1}}[\partial D_0^n(2^m; p_m)]$, have a *strong linkage*, or are *strongly linked*, through the unstable cycle $(2^{m-1}; p_{m-1})$. Similarly their increasing rank preimages generate sequences of plaits of the total basin which are strongly linked inside a J *subset*, *having the basic configuration* (*A2*) of Fig. 3.

The second kind of R_{dph} *sets is related to the points of* $R_{dp(m-1)} \in \partial D(2^m; p_m)$, $0 \le h < m$, *a subset of which limits a* J *subset denoted* $\hat{J}_{2^{m-1}}^{(A2)}$, *having the basic configuration* (*A2*) of Fig. 3. *It is the case of each of the* $2^{m-1}\hat{J}_{2^{m-1}}^{(A2)}$ *surrounding two points of the stable cycle* $(2^m; p_m)$. We shall say that each of the $R_{dp(m-1)}$ points introduces a *strong linkage between two adjoining* $\hat{J}_{2^{m-1}}^{(A2)}$, but a weak linkage between plaits of strongly linked parts of the basin $D(2^m; p_m)$.

The third kind of R_{dph} *sets is related to the points of* $R_{dph} \in \partial D(2^m; p_m)$, $0 \le h < m-1$, *limit points of a sequence of* J *subsets having the basic configuration* (*A2*) of Fig. 3. They produce *weak linkage between* $\hat{J}_{2^{m-1}}^{(A2)}$.

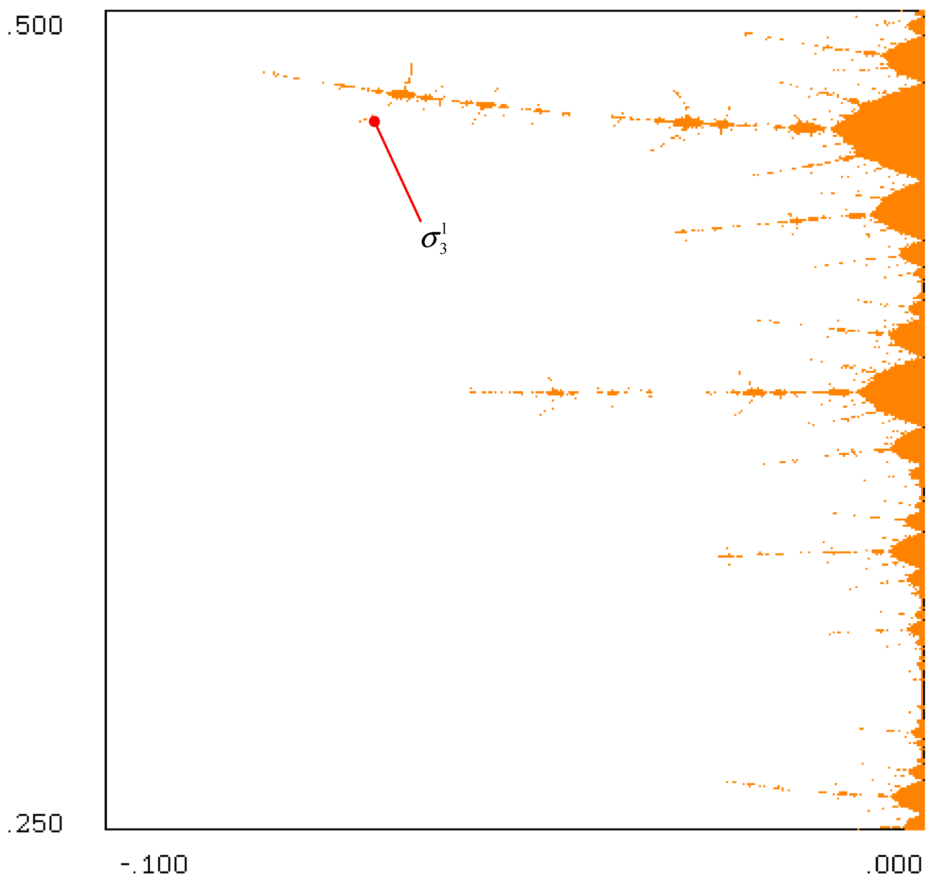
This last situation is due to the fact that when $c > c_{bm}$ the points of each unstable period 2^q cycle, $q < h < m-1$, belonging to the basin boundary $\partial D(2^m; p_m) = J$ are limit points for a subset of increasing rank preimages of the period 2^h cycles located on the x-axis.

It is worth noting that for $c > c_{bm}$ each unstable cycle $(2^g; p_g)$, $g > m$, $y \ne 0$, is symmetric with



(a)

Fig. 5. Interval $c_{b3} < c < c_{b4}$, $c = 1.385$. (a) The filled Julia set is a basin toward the stable period 8 cycle $(2^3; p_3)$. A new set R_{dp3} has been created. It is made up of the unstable period 4 cycle $(2^2; p_2)$ (points $\eta_1, \eta_2, \eta_3, \eta_4$) and its increasing rank preimages having the set R_{dp2} as limit set. Now the blue points of the basic cycle $(2^2.3; p_2, 1)$ have moved from the immediate basin boundary of the x-axis to the rank-one layer of the Julia set. The cycles $\sigma_{2^13}^{1,1}$, σ_3^1, are located on the rank two and three layers, respectively. (b) Enlargement of the rank-two layer containing a point of the cycle (3; 1).



(b)

Fig. 5. (*Continued*)

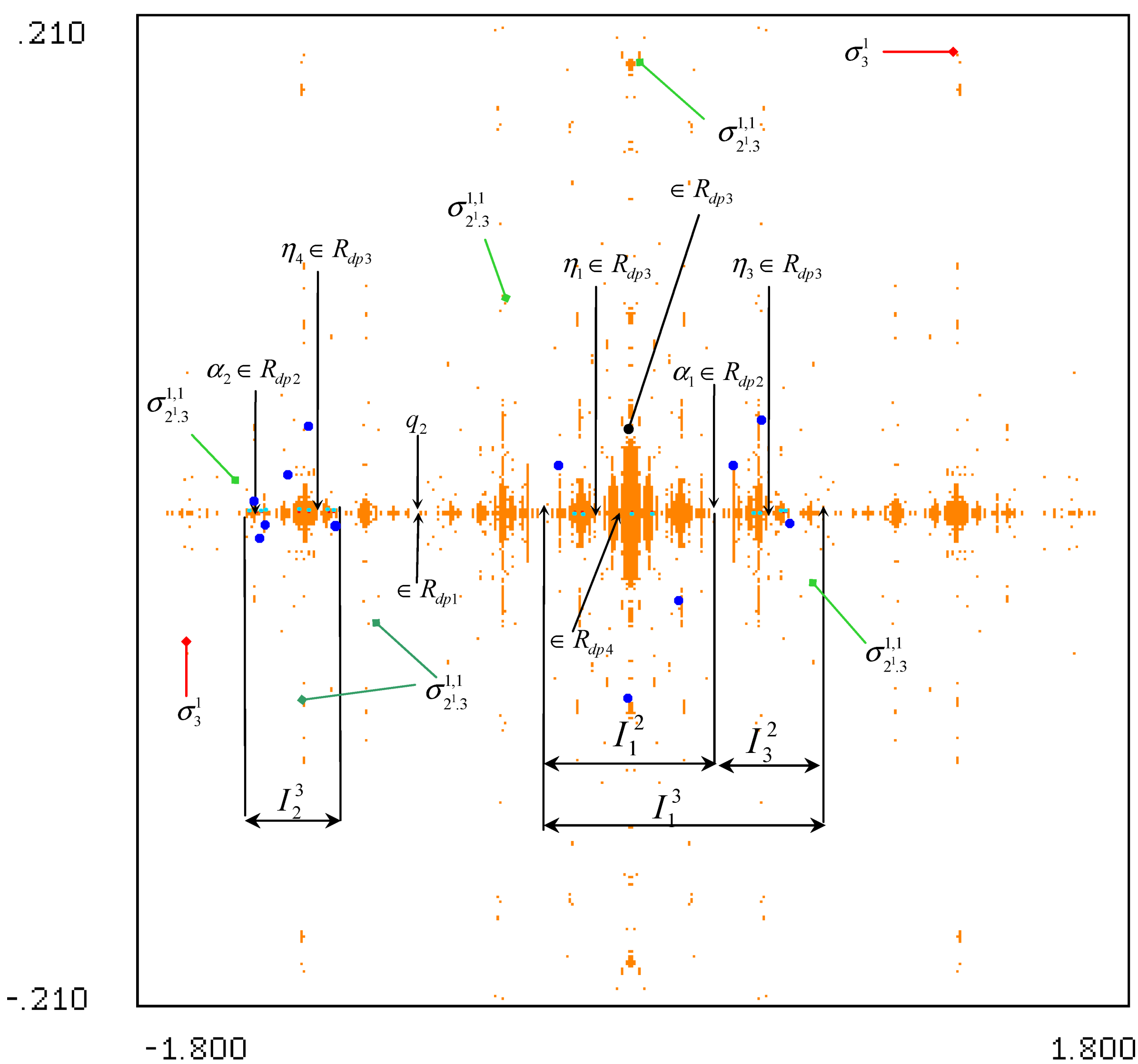
respect to the x-axis, i.e. it is made up of 2^{g-1} couples of points, the two points of each couple having the same abscissa. When $c = c_{bg}$ the 2^{g-1} couples merge into the stable $(2^{g-1}; p_{g-1})$ cycle of the x-axis, and for $c > c_g$ they turn into the $(2^g; p_g)$ cycle of the x-axis, stable for $c_{bg} < c < c_{b(g+1)}$. For $c > c_{b(g+1)}$, becoming unstable the $(2^g; p_g)$ cycle gives rise to the $R_{dp(g-1)}$ set.

Let us return to Fig. 4(a) ($m = 2$). The immediate basin boundary $\partial D_0(2^2; p_2)$ contains all the unstable cycles, and their limit sets, generated in the interval $c_{b2} \leq c \leq c^*_{2^2}$ (box Ω_{2^2}). Among them the unstable period twelve cycle (12 blue points) entering the x-axis (with $S < 1$) when c is in the box $\Omega^{1,1}_{2^2.3} \subset \Omega_{2^2}$, are represented. This figure also shows a "central" configuration (A2) on both sides of the y-axis, with an outline equivalent to Fig. 3, bounded by a subset of the R_{dp1} points (weak linkage), made up of the fixed point q_2 and a subset of its increasing rank preimages. These points limit a set, denoted $\hat{J}^{(A2)1}_{2^{m-1}}$ for which the (C^u) curves of the basic configuration (A2) have R_{dp2} points (strong linkage) as multiple points, and bound nonconnected basin parts, organized in plaits with decreasing size, when the distance increases from the immediate basin of the stable fixed points η_1, η_3 of $T_Z^{2^2}$. The same situation takes place for the set denoted $\hat{J}^{(A2)2}_{2^1}$, containing the immediate basin boundary of the fixed points η_2, η_4 of $T_Z^{2^2}$. These two (A2) basic configurations have a strong linkage, giving rise to the couple $\overleftrightarrow{K}_{2^{m-1}} = \overleftrightarrow{K}_{2^1} = \hat{J}^{(A2)1}_{2^1} \cup \hat{J}^{(A2)2}_{2^1}$, having the point $q_2 \in R_{dp1}$ in common. They intersect the x-axis at the intervals I^1_1 and I^1_2 [cf. Fig. 4(a)]. This shows that $\overleftrightarrow{K}_{2^1} \backslash \partial D_0(2^2; p_2)$ contains all the cycles, generated in the interval $c^*_{2^2} < c < c^*_{2^1}$ which belongs to the box of second kind. $\Omega_{2^{m-1}} \supset \Omega_{2^m}$. Among them the unstable period 6 cycle (six green colored points $\sigma^{1,1}_{2^1.3}$) entering the x-axis (with $S < 1$) when c is in the box $\Omega^{1,1}_{2^1.3} \subset \Omega_{2^1}$, is represented in Fig. 4(a).

So for $m = 2$ three layers containing well identified unstable cycles and their limits can be identified: $\partial D_0(2^2; p_2)$, $\overleftrightarrow{K}_{2^1} \backslash \partial D_0(2^2; p_2)$, and the remaining part of J which contains the cycles becoming stable in the interval $c^*_{2^1} < c \le c^*_1$ (see the period three cycle σ_3^1, red points in Fig. 4(a)).

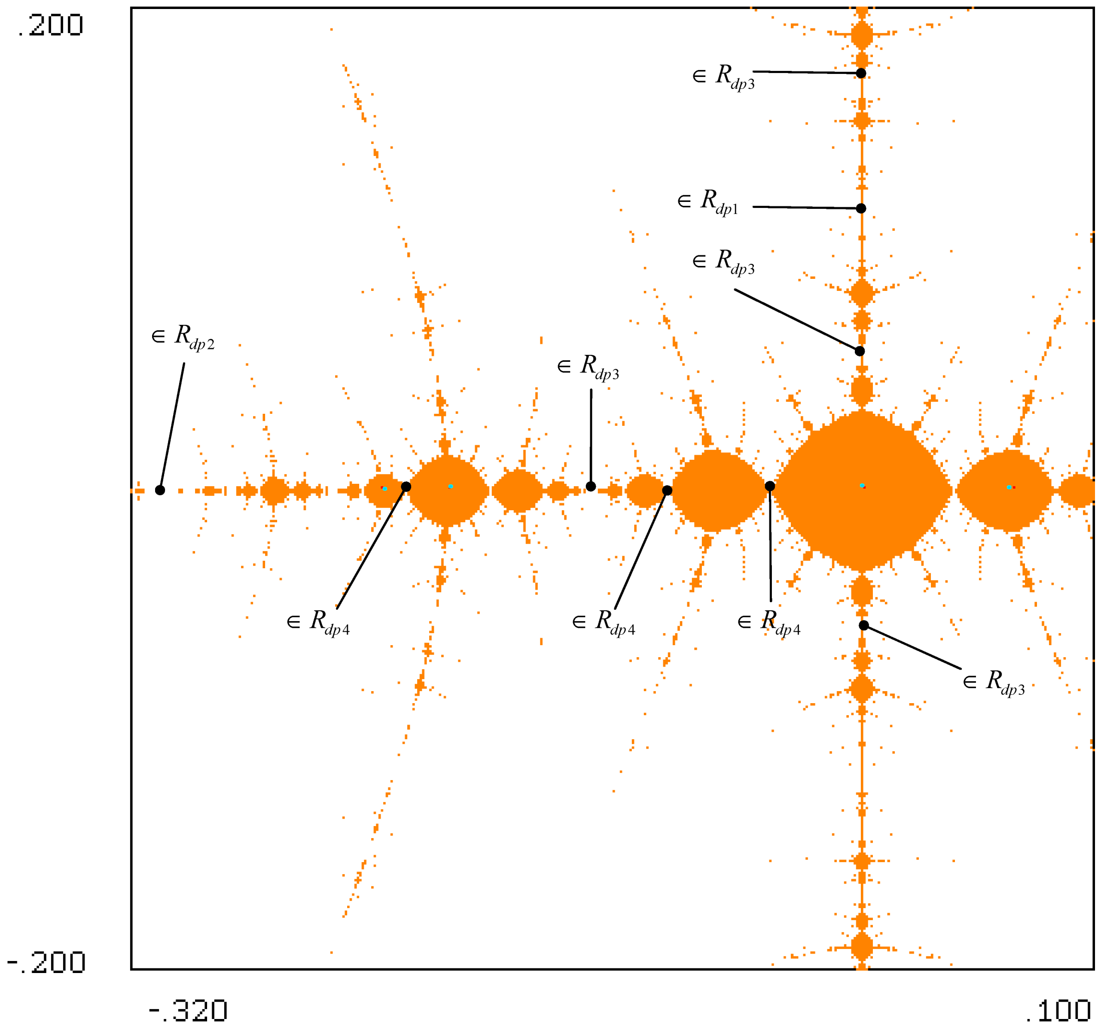
For $m = 3$, Fig. 5 shows 2^2 basic configurations (A2) $\hat{J}_{2^2}^{(A2),r}$, $r = 1, \ldots, 2^2$. Each $\hat{J}_{2^2}^{(A2),r}$ contains two components of the immediate basin boundary of the stable cycle $(2^3; p_3)$, and is adjacent to another $\hat{J}_{2^2}^{(A2),r'}$ through a point of the $(2^1; p_1)$ cycle. The set $\overleftrightarrow{K}_{2^2} = \bigcup_{r=1}^{2^2} \hat{J}_{2^2}^{(A2),r}$ is periodic with period 2^2. Two components $\hat{K^i}_{2^1}$ of $\overleftrightarrow{K}_{2^2}$ can be associated such as $\hat{K^i}_{2^1} = \hat{J}_{2^2}^{(A2),i} \cup \hat{J}_{2^2}^{(A2),t+i}$, $i = 1, 2^1 = 2^{m-2}$, $t = 2^1$, $\hat{J}_{2^2}^{(A2)t+i} = T_Z^{t+i}[\hat{J}_{2^2}^{(A2)i}]$, obtained from a linkage via a point of the cycle $(2^1; p_1)$ (points α_1, α_2 of the x-axis) belonging to the R_{dp2} set. $\overleftrightarrow{K}_{2^1} = \bigcup_{i=1}^{2^1} \hat{K}^i_{2^1}$ is a set of period 2^1, $\overleftrightarrow{K}_{2^1} \cap (y = 0) = I_1^2 \cup I_2^2$.

On the x-axis, the two components $\hat{K^i}_{2^1}$ of $\overleftrightarrow{K}_{2^1}$ are weakly linked through a subset S_{2^1} of increasing rank preimages of $\overleftrightarrow{K}_{2^1}$ having the fixed



(a)

Fig. 6. Interval $c_{b4} < c < c_{b5}$, $c = 1.3965$. (a) The filled Julia set is a basin toward the stable period 16 cycle $(2^4; p_4)$. A new set R_{dp4} has been created. It is made up of the unstable period 8 cycle $(2^3; p_3)$ of the x-axis, and its increasing rank preimages having the set R_{dp3} as limit set. Now the blue points of the basic cycle $(2^2.3; p_2, 1)$ have moved from the rank-one layer to the rank-two layer of the Julia set. The cycles $\sigma^{1,1}_{2^1 3}$, σ_3^1, are located on the rank three and four layers, respectively. (b) Enlargement of a part of the filled Julia set.



(b)

Fig. 6. (*Continued*)

point q_2 (i.e. a cycle $(2^{m-3}; p_{m-3})$ belonging to the R_{dp1} set) as limit point. The union of these components is the set $\overleftrightarrow{K}_{2^0}$ intersecting the x-axis on a segment I_3. We have: $\partial D_0(2^3; p_3) \subset \overleftrightarrow{K}_{2^2} \subset \overleftrightarrow{K}_{2^1} \subset \overleftrightarrow{K}_{2^0}$.

A first layer is the immediate basin boundary $\partial D_0(2^3; p_3)$, which contains all the points of the unstable cycles with a *basic period* 2^3, and their limit sets, generated inside the interval $c_{b3} < c \le c^*_{2^3}$ (box of second kind Ω_{2^3}), then with a rank-one basic period 2^3 associated with the permutation p_m.

The second layer is $\overleftrightarrow{K}_{2^2} \backslash \partial D_0(2^3; p_3)$ which contains all the points of the unstable cycles, and their limit sets, generated inside the interval $c^*_{2^3} < c \le c^*_{2^2}$ (inside the box of second kind $\Omega_{2^{m-1}}$).

The third layer is $\overleftrightarrow{K}_{2^1} / \overleftrightarrow{K}_{2^2}$. It contains all the points of the unstable cycles, and their limit sets, generated inside the interval $c^*_{2^2} < c \le c^*_{2^1}$ (inside the box of second kind Ω_{2^1}).

A fourth layer $\overleftrightarrow{K}_{2^0} / \overleftrightarrow{K}_{2^1}$ contains the cycles becoming stable in the interval $c^*_{2^1} < c \le c^*_1$.

Notations. Let c be the parameter value inside the interval $c_{bm} < c < c_{b(m+1)}$ $(\in \omega_1)$, $m = 2, 3, \ldots$, which gives rise to the stable cycle $(2^m; p_m)$.

$\hat{J}^{(A2),r}_{2^{m-1}}$, $r = 1, \ldots, 2^{m-1}$, is the J subset reproducing the Fig. 3 outline (basic (A2) configuration) but bounded by a subset of $R_{dp(m-1)}$ points, increasing rank preimages of the unstable cycle $(2^{m-2}; p_{m-2})$ $(y = 0)$. A subset of $\hat{J}^{(A2),r}_{2^{m-1}}$ contains two elements of the immediate basin boundary of the stable cycle $(2^m; p_m)$.

$\overleftrightarrow{K}_{2^{m-1}} = \bigcup_{r=1}^{2^{m-1}} \hat{J}^{(A2),r}_{2^{m-1}}$ is a set of period 2^{m-1}.

Two components of $\overleftrightarrow{K}_{2^{m-1}}$ are associated such as $\hat{K^i}_{2^{m-2}} = \hat{J}^{(A2),i}_{2^{m-1}} \cup \hat{J}^{(A2),t+i}_{2^{m-1}}$, $i = 1,\ldots,2^{m-2}$, $t = 2^{m-2}$, $\hat{J}^{(A2),t+i}_{2^{m-1}} = T_Z^{t+i}[\hat{J}^{(A2),i}_{2^{m-1}}]$, obtained from a linkage via a point of the cycle $(2^{m-2};p_{m-2})$ of the x-axis belonging to the $R_{dp(m-1)}$ set.

$\overleftrightarrow{K}_{2^{m-2}} = \bigcup_{i=1}^{2^{m-2}} \hat{K^i}_{2^{m-2}}$ contains the set of unstable cycles with the basic period 2^{m-2}.

Recursively 2^{m-s} rank-s subsets of the Julia set J, denoted $\hat{K^i}_{2^{m-s}}$ ($s \le m$, $m > 1$, $i = 1,\ldots,2^{m-s}$), are defined. Each one made up of the association of two neighboring sets $\hat{K^i}_{2^{m-(s-1)}}$ weakly linked from a point of the cycle $(2^{m-s};p_{m-s})$ of the x-axis belonging to $R_{dp(m-s+1)}$, and also by the subset of increasing rank preimages of these neighboring sets having the point of the cycle $(2^{m-s};p_{m-s})$ as accumulation point.

$\overleftrightarrow{K}_{2^{m-s}} = \bigcup_{i=1}^{2^{m-s}} \hat{K^i}_{2^{m-s}}$ contains the set of unstable cycles with the basic period 2^{m-s}.

$\partial D_0(2^m;p_m) \subset \overleftrightarrow{K}_{2^{m-1}} \subset \cdots \subset \overleftrightarrow{K}_{2^{m-s}}$

$L^s_{2^m}$ is the rank-s layer of J, defined by ($m \ge 3$, $s = 1,2,\ldots,m$):

$$L^s_{2^m} = \overleftrightarrow{K}_{2^{m-s}} \backslash \overleftrightarrow{K}_{2^{m-(s-1)}}$$

Then:

Each $\hat{K^i}_{2^{m-s}}$ bounds a subset of the basin of the stable $(2^m;p_m)$ cycle on the x-axis, which contains 2^s points of this cycle.

$L^s_{2^m}$ contains all the cycles, and their limits, created in the interval $c^*_{2^{m-(s-1)}} < c \le c^*_{2^{m-s}}$.

Figure 6 illustrates the situation for $m = 4$.

When $m = \infty$, $c = c_{1s} = \lim_{m\to\infty} c^*_{2^n} = \lim_{m\to\infty} c_{bm}$, the Julia set J is a dendrite. Infinitely many layers with associated families of cycles result as limit of the above situations when $m \to \infty$. The basins D_0 and D now do not exist, ∂D_0 degenerates into the Cantor set on the x-axis, made up of the limit of the stable cycle $(2^m;p_m)$ when $m \to \infty$. In this case the segment $[q_1^{-1};q_1]$ of the x-axis is the limit of infinitely many layers.

Proposition 2(a2). *Let c be the parameter value inside the interval $c_{bm} < c < c_{b(m+1)} (\subset \omega_1), m = 2,3,\ldots,$ which gives rise to the stable cycle $(2^m;p_m)$, and the J subsets $\hat{J}^{(A2)}_{2^{m-1}}$, $\overleftrightarrow{K}_{2^{m-s}}$ defined above.*

(i) *The cycles $(2^q;p_q) \in J \cap (y = 0), q = 0,1,2,\ldots,m-1$, are unstable and born in the interval $c_{(1)0} = -1/4 \le c \le c_{bm}$.*

(ii) *Each component $\partial D_0^n(2^m;p_m)$, $n = 1,2,\ldots,2^m$, of the immediate basin boundary $\partial D_0(2^m;p_m)$ of the stable period 2^m cycle $(2^m;p_m)$, has a common point with $T_Z^{2^{m-1}}[\partial D_0^n(2^m;p_m)]$. The boundary $\partial D_0(2^m;p_m)$ contains all the unstable cycles generated inside the interval $c_{bm} < c \le c^*_{2^m}$ (box of second kind Ω_{2^m}).*

(iii) *The J subset $\hat{J}^{(A2)}_{2^{m-1}}$ contains all the cycles with a basic period 2^{m-1}, generated in the interval $c^*_{2^m} < c < c^*_{2^{m-1}}$.*

(iv) *The J layer $L^s_{2^m}$ contains all the cycles created in the interval $c^*_{2^{m-(s-1)}} < c \le c^*_{2^{m-s}}$. The last layer $L^m_{2^m}$ contains all the cycles created in the interval $c^*_{2^1} < c \le c^*_1 = 2$.*

4.6. *Interval bounded by two consecutive flip bifurcations of the Myrberg spectrum ω_k^j*

Let c be the parameter value inside the interval $c^{j_1}_{k_1bm} < c < c^{j_1}_{k_1b(m+1)}$ (spectrum $\omega^{j_1}_{k_1}$) $m = 2,3,\ldots,k_1 = 3,4,\ldots,$ bounded by two consecutive flip bifurcations, generating the stable cycle $(k_12^m;j_1,p_m)$. Considering the map T^{k_1}, as in Sec. 4.5, $R^{j_1}_{k_1dph}$ sets are defined from the unstable $(k_12^h;j_1,p_h)$, $0 \le h \le m$, cycles of the x-axis

$$R^{j_1}_{k_1dph} = \bigcup_{r\ge 0} T^{-r}[(k_12^h;j_1,p_h)].$$

It is the same for the following sets:

$\hat{J}^{j_1(A2)r}_{k_12^{m-1}}$, $r = 1,\ldots,k_12^{m-1}$, is the J subset reproducing the Fig. 3 outline but bounded by a subset of $R^{j_1}_{k_1dp(m-1)}$ points, increasing rank preimages of the unstable cycle $(k_12^{m-2};j_1,p_{m-2})$ ($y = 0$). A subset of $\hat{J}^{j_1(A2)r}_{k_12^{m-1}}$ contains two components of the immediate basin boundary $\partial D_0(k_12^m;j_1,p_m)$.

$\overleftrightarrow{K}^{j_1}_{k_12^{m-1}} = \bigcup_{r=1}^{2^{m-1}} \hat{J}^{j_1(A2)r}_{k_12^{m-1}}$ contains unstable cycles with the basic period k_12^{m-1}.

Two components of $\overleftrightarrow{K}^{j_1}_{k_12^{m-1}}$ are associated such as $\widehat{K}^i_{k_12^{m-2}}{}^{j_1} = \hat{J}^{j_1(A2)i}_{k_12^{m-1}} \cup \hat{J}^{(A2)t+(i-1)}_{2^{m-1}}$, $i = 1,\ldots,k_12^{m-2}$, $t = 2^{m-2}$, $\hat{J}^{j_1(A2)t}_{k_12^{m-1}} = T_Z^t[\hat{J}^{(A2)1}_{2^{m-1}}]$ from a linkage by a point of the cycle $(k_12^{m-2};j_1,p_{m-2})$ of the x-axis belonging to the $R^{j_1}_{k_1dp(m-1)}$ set.

$\overleftrightarrow{K}^{j_1}_{k_1 2^{m-2}} = \bigcup_{i=1}^{k_1 2^{m-2}} \widehat{K^i}^{j_1}_{k_1 2^{m-2}}$ contains unstable cycles with the basic period $k_1 2^{m-2}$.

Recursively $k_1 2^{m-s}$ rank-s subsets of the Julia set J, denoted $\widehat{K^i}^{j_1}_{k_1 2^{m-s}}$ ($s \leq m$, $m > 1$, $i = 1, \ldots, k_1 2^{m-s}$), are defined. Each one is made up of the association of two neighboring sets $\widehat{K^i}^{j_1}_{k_1 2^{m-(s-1)}}$ weakly linked from a point of the cycle $(k_1 2^{m-s}; j_1, p_{m-s})$ of the x-axis belonging to $R^{j_1}_{k_1 dp(m-s+1)}$, and also by the subset of increasing rank preimages of these neighboring sets having the point of the cycle $(k_1 2^{m-s}; j_1, p_{m-s})$ as accumulation point.

$\overleftrightarrow{K}^{j_1}_{k_1 2^{m-s}} = \bigcup_{i=1}^{k_1 2^{m-s}} \widehat{K^i}^{j_1}_{k_1 2^{m-s}}$ contains unstable cycles with the basic period $k_1 2^{m-s}$.

$2\partial D_0(k_1 2^m; j_1, p_m) \subset \overleftrightarrow{K}^{j_1}_{k_1 2^{m-1}} \subset \cdots \subset \overleftrightarrow{K}^{j_1}_{k_1 2^{m-s}}$.

$L^{j_1,s}_{k_1 2^m}$ is the rank-s layer of J defined by ($m \geq 3$, $s = 1, 2, \ldots, m$)

$$L^{j_1,s}_{k_1 2^m} = \overleftrightarrow{K}^{j_1}_{k_1 2^{m-s}} \backslash \overleftrightarrow{K}^{j_1}_{k_1 2^{m-(s-1)}}.$$

Proposition 2(b2). *Let c be the parameter value inside the interval $c^{j_1}_{k_1 bm} < c < c^{j_1}_{k_1 b(m+1)}$, $m = 2, 3, \ldots, k_1 = 3, 4, \ldots$, bounded by two consecutive flip bifurcations, generating the stable cycle $(k_1 2^m; j_1, p_m)$. Let $\hat{J}^{j_1(A2)}_{k_1 2^{m-1}}$, $\overleftrightarrow{K}^{j_1}_{k_1 2^{m-s}}$ the J subsets defined above*

(i) *The subset $J \cap (y = 0)$ contains all the unstable cycles, and their limit sets, born in the interval $c_{(1)0} \leq c < c^{j_1}_{k_1 b(m+1)}$.*

(ii) *Each component $\partial D^n_0(k_1 2^m; j_1; p_m)$, $n = 1, 2, \ldots, k_1 2^m$, of the immediate basin boundary $\partial D_0(k_1 2^m; j_1; p_m)$ of the stable period $k_1 2^m$ cycle $(k_1 2^m; j_1, p_m)$, has a common point with $T_Z^{k_1 2^{m-1}}[\partial D^n_0(k_1 2^m; j_1, p_m)]$. $\partial D_0(k_1 2^m; j_1, p_m)$ contains all the unstable cycles generated inside the interval $c^{j_1}_{k_1 bm} < c \leq c^{j_1 *}_{k_1 2^m}$ (box of second kind $\Omega^{j_1}_{k_1 2^m}$).*

(iii) *The J subset $\hat{J}^{j_1(A2)}_{k_1 2^{m-1}}$ contains the closure of all the cycles generated in the interval $c^{j_1 *}_{k_1 2^m} < c < c^{j_1 *}_{k_1 2^{m-1}}$.*

(iv) *The J layer $L^{j_1,s}_{k_1 2^m}$ contains all the cycles created in the interval $c^{j_1 *}_{k_1 2^{m-(s-1)}} < c \leq c^{j_1 *}_{k_1 2^{m-s}}$. The last layer $L^{j_1,m}_{k_1 2^m}$ contains the closure of all the cycles created in the interval $c^{j_1 *}_{k_1 2^1} < c \leq c^*_1 = 2$.*

Proposition 2b′. *Let c be the flip parameter values $c_{k_1 bm}$, $m = 1, 2, \ldots$, of the spectrum ω_1, $c^{j_1}_{k_1 bm}$, $k_1 = 3, 4, \ldots$, of the spectrum $\omega^{j_1}_{k_1}$, generating the neutral cycle $(k_1 2^{m-1}; j_1, p_{m-1})$ ($S = -1$). Propositions 2(b1) and 2(b2) hold changing the stable cycle $(k_1 2^m; j_1, p_m)$ into the neutral cycle, and the immediate basin into the immediate convergence, and adapting the boundaries of the parameter intervals.*

Proposition 2b′ is deduced from Propositions 2(b1) and 2(b2), when $c \to c_{bm}$, $c \to c^{j_1}_{k_1 bm}$ with decreasing values.

5. Properties of the Different Types of Julia Set

On the basis of intervals defined by bifurcation values of the Myrberg's map, Sec. 3.3 has defined five different types of Julia set. Except the dendrites case these types can be differentiated between a *class A*, when c belongs to the spectrum ω_1, and a *class B*, when c belongs to an embedded spectrum ω^j_k. The propositions in Sec. 4 have provided the plane situation of well defined subsets of the Julia set J for four of these types. This section completes the Julia set properties of each type, and describes the different J outlines generated inside a same type.

5.1. *First type of Julia sets. Multiplier $S = +1$*

This type is generated for parameter values $c = c_{(1)0}$ (*class A*), $c = c^j_{(k)_0}$ (*class B*), or more generally the embedded forms $c = c^{j_1,\ldots,j_a}_{(k_1,\ldots,k_a)_0}$, which are the first boundary of Myrberg spectra. Consider that the J structure (i.e. location of the unstable cycles in the plane) depends strongly on the class, even if for class B the outline of J subsets reproduces the J one for class A.

The *class A* case, $c = c_{(1)0}$, fold bifurcation of the spectrum ω_1, is the simplest one. This parameter value gives a situation in the (x, y) plane described in of [Julia, 1918, pp. 231–237], of [Fatou, 1920, pp. 91–92, pp. 240–242], and J has the Julia–Fatou configuration (A1) i.e. it is a simple closed Jordan curve. The Julia set J contains all the unstable cycles generated inside the box Ω_1, and $J \cap (y = 0)$ contains the neutral fixed point $q_1 \equiv q_2$

($S = 1$), "left" limit of the increasing rank critical points, then belonging to E'_c, and its rank-one preimage q_1^{-1}. According to (P3) in Sec. 3.1 J cannot be the basin boundary of an attracting set on the x-axis. It does not satisfy the Fatou theorem [1920, p. 240] recalled in (P5) in Sec. 3.1. The Julia set J limits only the basin of the point at infinity (domain of divergence), and a *basin* toward $q_1 \equiv q_2$ adjoining this point. At $q_1 \equiv q_2$ ($c_{(1)0} = -1/4$) J presents a cusp point with a horizontal tangent. Moreover, for a numerable set of points, increasing rank preimages of $q_1 \equiv q_2$, the tangent to J can be defined [Fatou, 1920]. Nevertheless J is nowhere differentiable, because at a cusp point a function is not differentiable. The Julia set J is a Jordan curve without double points. Fatou [1920, p. 242] identified the J outline as equivalent to that of a von Koch curve, i.e. it is fractal. This case is shown in Fig. 7(a), where the J outline has a fractal petal-like aspect, and is the boundary of the brown region (the filled Julia set $\digamma(J)$). The Julia set J has the properties given in Proposition 1a′.

The *class B* of this first type, defined by the opening $c = c^j_{(k)_0}$, $k > 2$ of any spectrum $\omega^j_k \subset \Omega^j_k$, has the properties given in Proposition 1b′. As mentioned in Sec. 3.3, the class A petal-like shape of J is fractally reproduced at $c = c^j_{(k)_0}$.

The fractal structure, observed on the x-axis with the Myrberg's map, is reproduced considering all the preimages of any rank of the segment $[q_1^{-1}, q_1]$, $T_Z^{-n}([q_1^{-1}, q_1])$ for $n > 0$, which however is more and more "contracted" on the x-axis, as c increases. An example is shown in Figs. 7(b)–7(d) (at the beginning $c = c^1_{(3)_0} = 7/4$ of the box of first kind associated with the three-cycle). A period three petal-like outline is fractally reproduced on both sides of the x-axis, and along "rays" ending at points of the immediate basin [Fig. 7(c)]. Clearly a similar behavior occurs for any $c = c^{j_1,\ldots,j_a}_{(k_1,\ldots,k_a)_0}$ of the rank-a box $\Omega^{j_1,\ldots,j_a}_{k_1,\ldots,k_a}$.

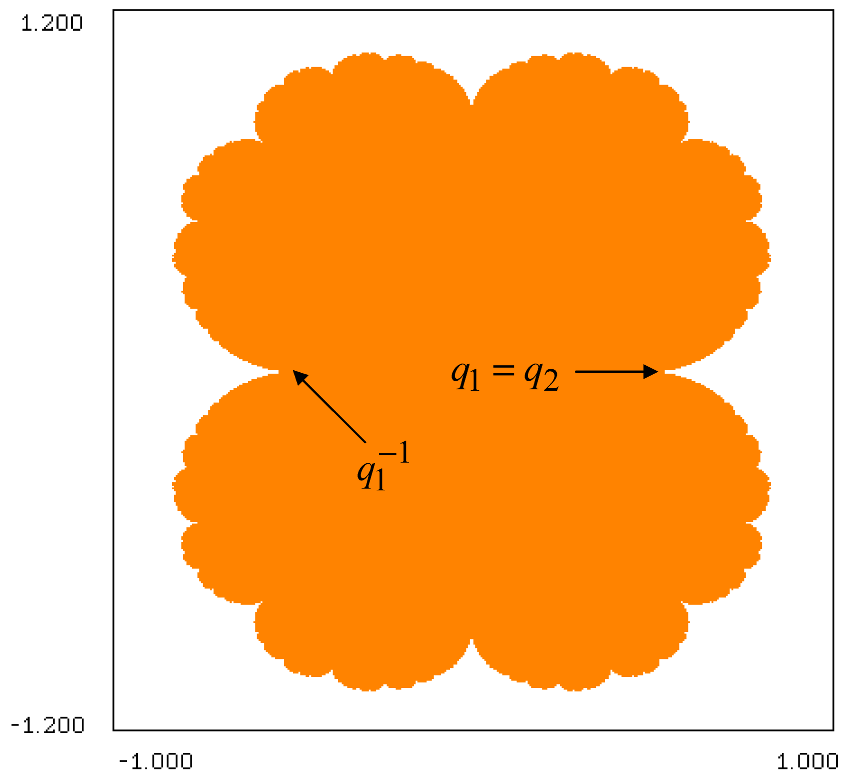
5.2. *Second type of Julia sets. Multiplier $S = -1$*

The simplest case is the first flip bifurcation of the Myrberg spectrum ω_1 (class A situation) i.e. $c = c_{b1} = 3/4$, with $S = -1$ for the fixed point q_2 which is neutral and belongs to J. According to (P3) given in Sec. 3.1, J cannot be the basin boundary of an attracting set on the x-axis (on both sides the point q_2 is the limit of increasing rank critical points, i.e. $q_2 \in E'_c$). The Julia set J limits only the basin of the point at infinity (domain of divergence), and a *basin toward* q_2 adjoining this point. This means that the convergence toward q_2 is *singular* in the Julia sense. At q_2 J has a vertical tangent, and J has a numerable set of points, increasing rank preimages of q_2, where the tangent can be defined. Elsewhere J has no tangent. Figures 8(a) and 8(b) show this situation where the brown region is the basin toward q_2 (i.e. the filled Julia set $\digamma(J)$), the white one (which is touching q_2) is the basin of the point at infinity. The vertical tangent at q_2 is such that locally two "arcs" of J, belonging to two lobes of the basin toward q_2, are on the same side of this tangent [Fig. 8(b)], giving a hollow for an arc and a bump for the other. For $c = c_{b1} = 3/4$, J is connected and has the Julia–Fatou basic configuration (A2). Remark that for the Myrberg's map $x' = x^2 - c$ (reduction of T_Z to the x-axis) q_2 is stable and not neutral.

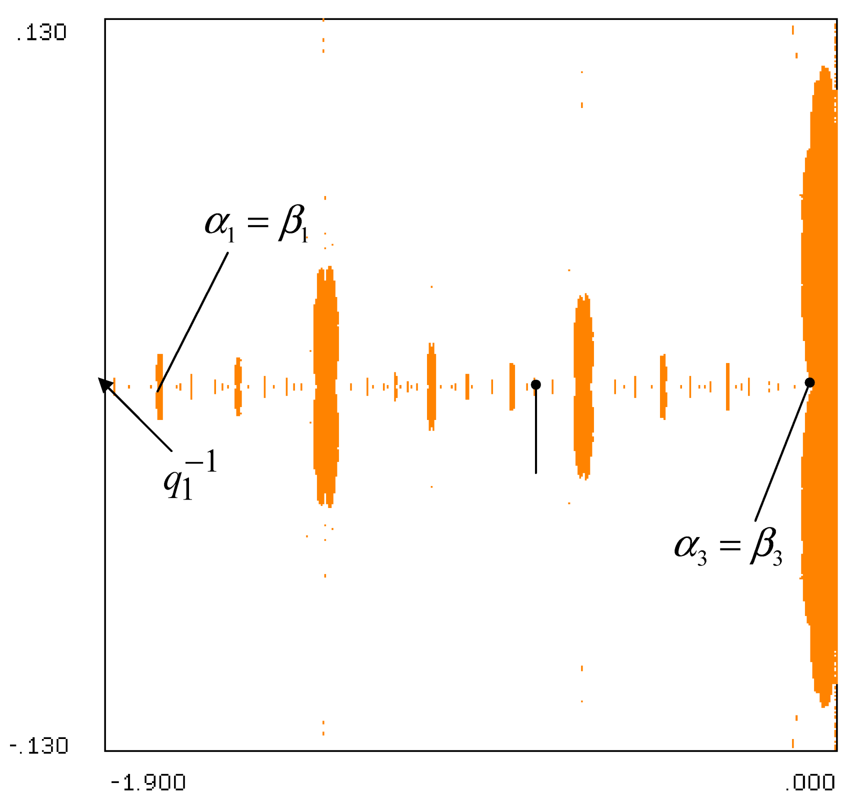
Each $c = c_{bm}$ values, $m = 2, 3, \ldots$, leads to properties of Proposition 2a′. Each of the 2^{m-1} components of the boundary of the immediate basin associated with the neutral period 2^{m-1} cycle, is connected with the Julia–Fatou basic configuration (A2).

Figure 9 shows the structure of J and filled Julia set $\digamma(J)$ at the second flip bifurcation, occurring for $c = c_{b2} = 5/4$, $J \cap (y = 0) = [\mathcal{C}_l(\bigcup_{r\geq 0} T^{-r}(E_1^2))] \cap (y = 0)$, E_1^2 being made up of q_1, q_2 and the period 2 cycle (α_1, α_2) with $S = -1$, born on the x-axis from q_2 for $c = c_{b1} = 3/4$, $\mathcal{C}_l$ indicating the closure of the set. The same behavior occurs for flip values $c = c_{bm}$ of the ω_1 spectrum, introducing E_1^m the finite set of the m repelling cycles of the x-axis of period 2^p, $0 \leq p < m$, the cycle of period 2^0 being the fixed point q_2.

When $c = c^j_{kbm}$ (class B), $J \cap (y = 0)$ contains infinitely many Cantor like sets Cs (cf. Sec. 2.3) born for $c < c^j_{(k)0}$ and the unstable cycles with their increasing rank preimages located on $(y = 0)$, born for $c^j_{(k)0} \leq c < c^j_{kbm}$. So the set $J \cap (y = 0)$ is a well defined fractal set. The basin intersects the x-axis including $k2^{m-1}$segments invariant by $T_Z^{k2^{m-1}}$, and their increasing rank preimages located on the segment $[q_1^{-1}, q_1]$. All the points of J with $y \neq \emptyset$ consist of points of all the other repelling cycles existing in the plane and still outside the x-axis (which will enter the x-axis at higher values of parameter c), and their limit points.

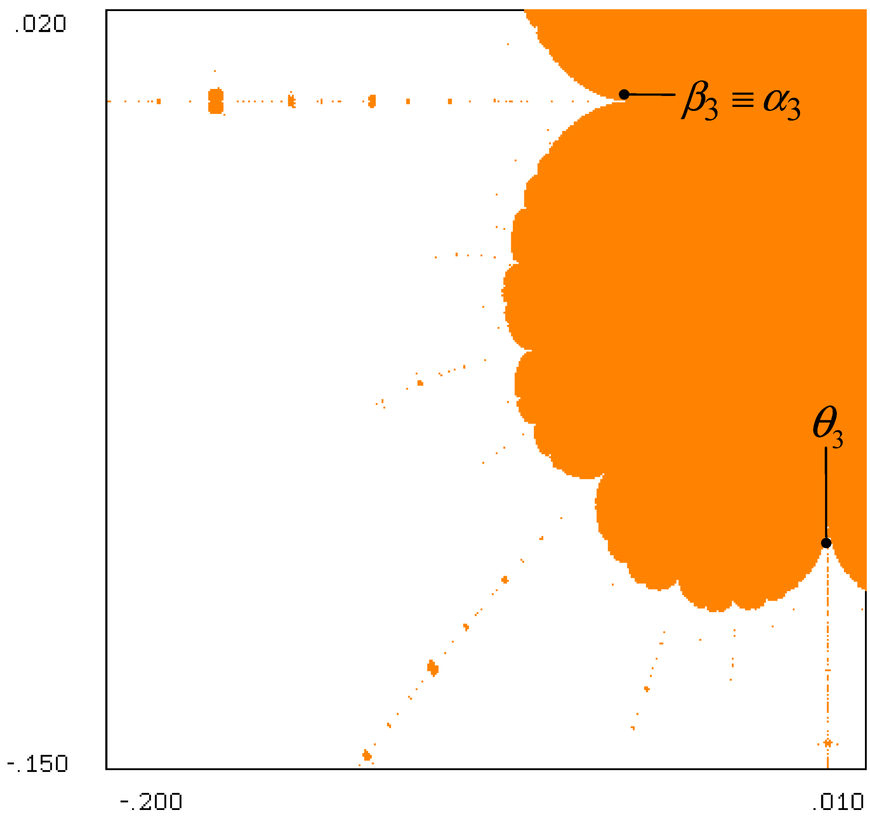


(a)

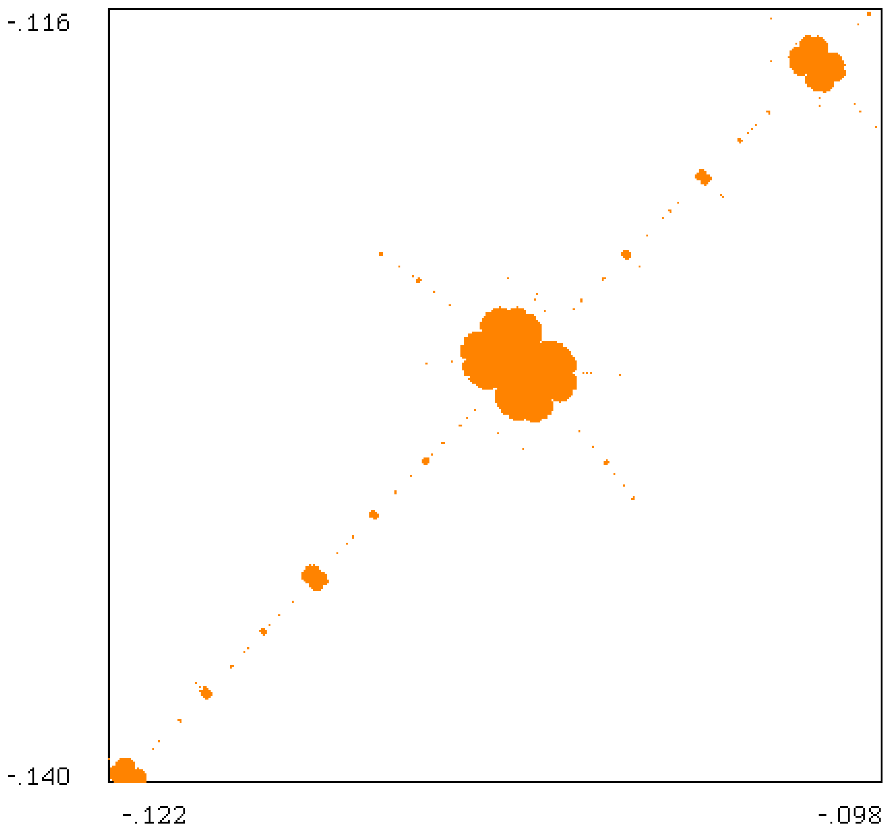


(b)

Fig. 7. (a) Filled Julia set (of petal-like type) for the fold bifurcation $c = c_{(1)_0} = -1/4$. The filled Julia set is a basin toward the neutral fixed point $q_2 \equiv q_1 \in J$ $(S = 1)$. The Julia set J has a numerable set of points where the tangent can be defined, but these points are cusps, so J remains nowhere differentiable. (b) Partial view of the filled Julia set (symmetric with respect to $x = 0$) for the fold bifurcation $c = c^1_{(3)_0} = 7/4$. The two period three cycles $(\alpha_1, \alpha_2, \alpha_3)$ with $S \leq 1$, $(\beta_1, \beta_2, \beta_3)$ with $S \geq 1$ merge at this parameter value and $S = 1$. (c) and (d) represent two enlargements.



(c)



(d)

Fig. 7. (*Continued*)

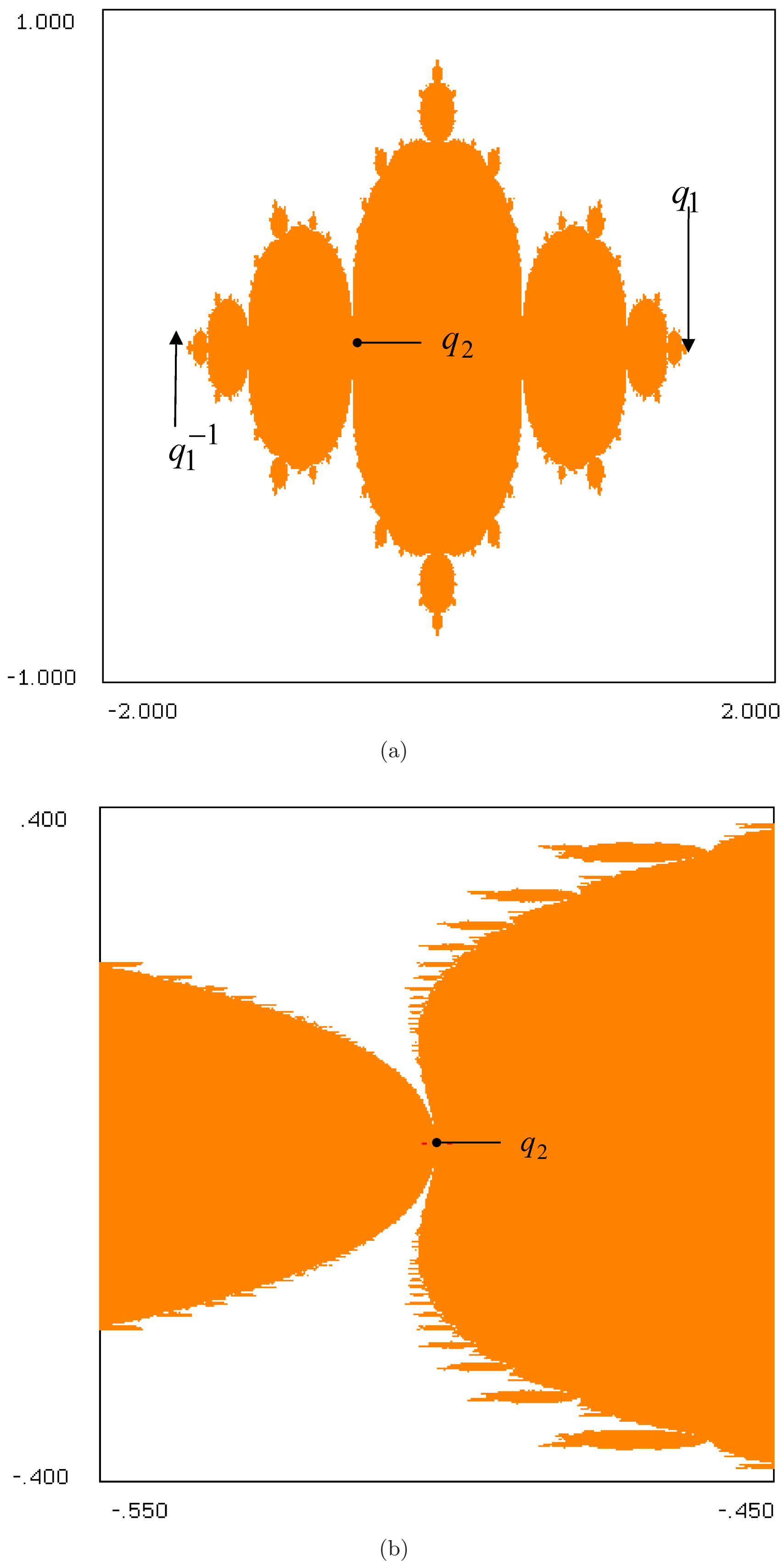


Fig. 8. (a) Filled Julia set for the first flip bifurcation $c_{b1} = 3/4$. The filled Julia set is a basin toward the neutral fixed point q_2 $(-1 < S < 0)$, which results from the merging of the points α_1 and α_2 of the period two cycle on the x-axis (cf. Fig. 2 caption). It results in the basic Julia–Fatou configuration (A2). (b) Enlargement in the neighborhood of q_2.

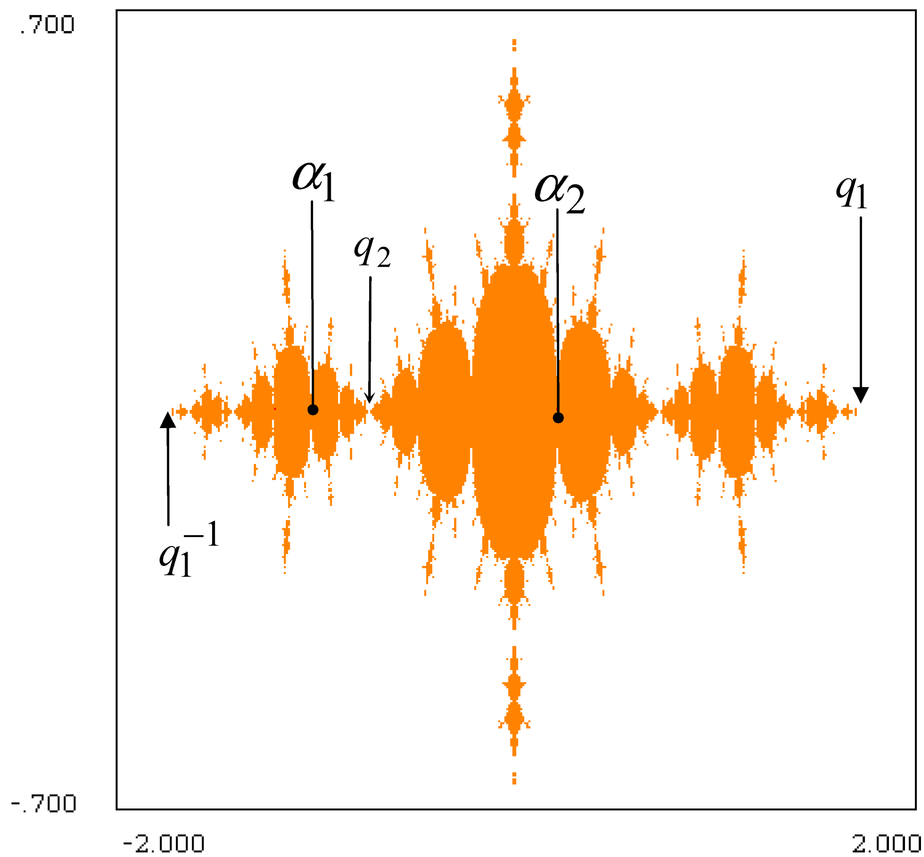


Fig. 9. Filled Julia set for the flip bifurcation $c = c_{b2} = 5/4$. The period 2 cycle (α_1, α_2), which belongs to the Julia set J, is neutral $(S = -1)$ and the filled Julia set is a basin toward (α_1, α_2). The Julia set J has a numerable set of points where the tangent can be defined, but J remains nowhere differentiable.

The flip values $c = c^{j_1,\ldots,j_a}_{(k_1,\ldots,k_a)_{bn}}$ of a rank-a box repeat the same behavior.

5.3. *Third type of Julia set*

A continuous variation of J occurs inside well defined intervals, beginning with the fold bifurcation of a Myrberg spectrum. The simplest case is *class A* for the interval $c_{(1)0} < c < c_{b1}$, $c_{(1)0} = -1/4$, $c_{b1} = 3/4$, belonging to ω_1. The corresponding structure of the Julia set is given by Proposition 1a. Types of *class B* are related to intervals $c^j_{(k)_0} < c < c^j_{kb1}$ $(k > 2)$ belonging to ω^j_k, or more generally for an interval $c^{j_1,\ldots,j_a}_{(k_1,\ldots,k_a)_0} < c < c^{j_1,\ldots,j_a}_{(k_1,\ldots,k_a)_{b1}}$ belonging to $\omega^{j_1,\ldots,j_a}_{k_1,\ldots,k_a}$, with equivalent intervals for boxes of second kind. The corresponding structure of the Julia set is given in Proposition 1b.

Julia set type of class A (cf. Proposition 1a)

In the interval, $c_{(1)0} < c < c_{b1}$, $c_{(1)0} = -1/4$, $c_{b1} = 3/4$, the stable (attracting) cycle is the fixed point q_2 $(k = 1)$. The Julia set J has a fractal outline (except for $c = 0$) with a continuous modification of its shape when c increases, passing from a petal-like outline to two other forms, the last ones tending to Fig. 8(a) when $c \to c_{b1}$.

In the interval $c_{(1)0} < c < 0$ the shape has a bumpy fractal aspect (*petal-like*). This aspect results from a continuous modification of the case $c = c_{(1)0} = -1/4$, but now contrary to $c = c_{(1)0}$ the set J has nowhere a tangent. The unstable fixed point q_1 $(y = 0)$, located at a cusp point for the fold bifurcation value $c = c_{(1)0}$, has moved to the right, and q_2 is attracting in the brown region of Fig. 10, which now represents the basin of this point. The only points of J, located on the x-axis, are q_1 and q_1^{-1}, $T^{-1}(q_1) = q_1 \cup q_1^{-1}$, while $T_Z^{-1}(q_1^{-1})$ includes the points $(x = 0,\ y = \pm\sqrt{-(q_1^{-1} + c)})$. The segment $-\sqrt{-(q_1^{-1} + c)} < y < \sqrt{-(q_1^{-1} + c)}$ on the y-axis, given by $T_Z^{-1}(]q_1^{-1}, -c[)$, belongs to the basin of q_2. When $c \to 0$, the J bumps progressively become less and less pronounced up to attain the circle $|z| = 1$ at $c = 0$, for which the q_2 multiplier is $S = 0$.

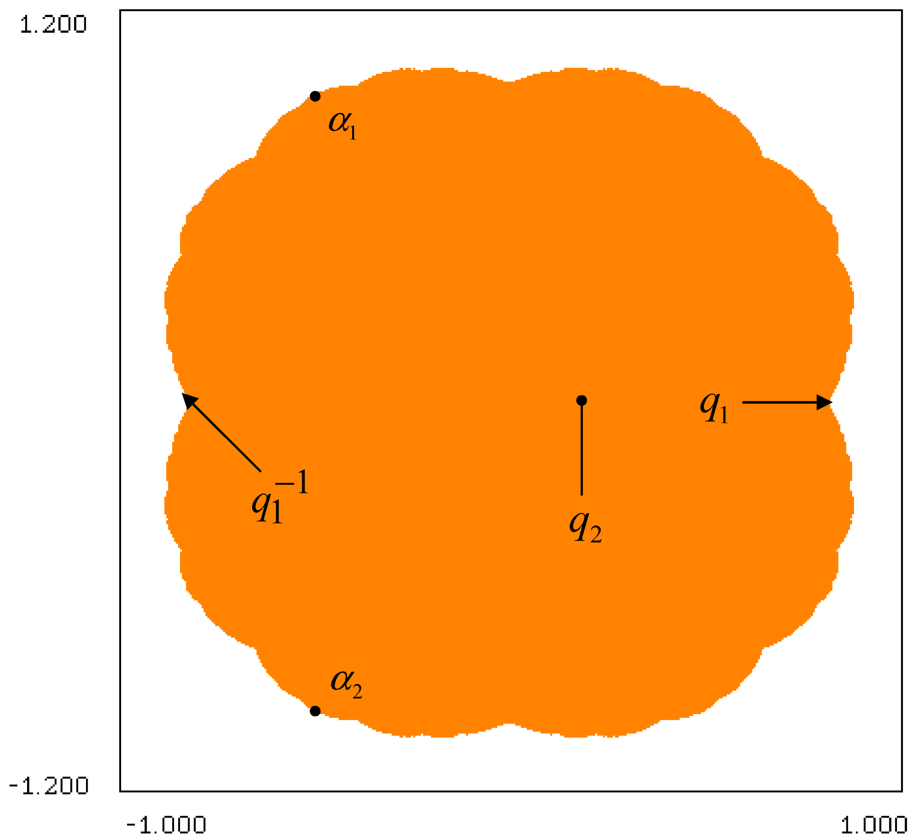


Fig. 10. Filled Julia set (of petal-like type) for $c = -0.15$. The filled Julia set is the basin of the stable fixed point q_2 $(0 < S < 1)$. The points (α_1, α_2) are those of the period 2 cycle.

In the interval $0 < c \leq c_{b1} = 3/4$. Figure 11 ($c = 0.25$) and Fig. 12 ($c = 0.5$) show the continuous evolution of J (boundary of the brown region, basin of the fixed point q_2) from the circle $|z| = 1$ ($c = 0$) to the flip bifurcation shown in Fig. 8(a) ($c = c_{b1} = 3/4$). The value $c = 0$ is a boundary separating two different J outlines: the petal-like one from an outline presenting infinitely many "spikes" in a fractal configuration (Fig. 11). When c increases, then progressively the simply connected basin of q_2 (bounded by the simple Jordan closed curve J) presents infinitely many excrescences (Figs. 12 and 2) with a "base" having a decreasing length, which tends toward zero when $c \to c_{b1}$. The origin of such excrescences is easily explained, considering that when $c \to c_{b1}$ the two points ($y < 0$ and $y > 0$) of the unstable period 2^1 cycle $\alpha_1 \cup \alpha_2 \in J$ tend toward the stable fixed point q_2 on the x-axis. Indeed inside the basin of q_2, this situation near q_2 creates a narrow vertical section (Fig. 2), bounded by α_1 and α_2, the increasing rank preimages of which are related to the fractal set of excrescences.

When $c = c_{b1}$ the period two cycle merges with $q_2 \equiv \alpha_1 \equiv \alpha_2$, the section length becoming equal to zero [Fig. 8(a)]. This results in J the rank-one infinite set $R_{dp1} \subset J$ of double points (q_2 and its increasing rank preimages) when $c \geq c_{b1}$ (cf. Sec. 4).

Julia set type of class B (Proposition 1b)

The simplest form corresponds to intervals $c^j_{(k)_0} < c < c^j_{kb1}$ ($k > 2$) belonging to ω^j_k. T_Z has a $(k; j)$ attracting cycle on the x-axis with multipliers $-1 < S < 1$. The immediate basin boundary of the stable $(k; j)$ cycle is made up of k ($k = 3, 4, \ldots$) simple (i.e. without multiple points) Jordan closed curves (with the Julia–Fatou configuration (A1)). Inside each interval the multiplier $S(k; j) = 0$ separates two different local behaviors near the $(k; j)$ cycle: $S > 0$ with a regular convergence of orbits, $S < 0$ with an "alternate" convergence.

Now, contrarily to $c = 0$, the value $c = c^j_k(S = 0)$, giving the $(k; j)$ cycle multiplier $S = 0$ (separating orbits with a regular local convergence, and an "alternate" one) is no longer a boundary separating the petal-like J outline, from the one presenting infinitely many "spikes". Indeed in this

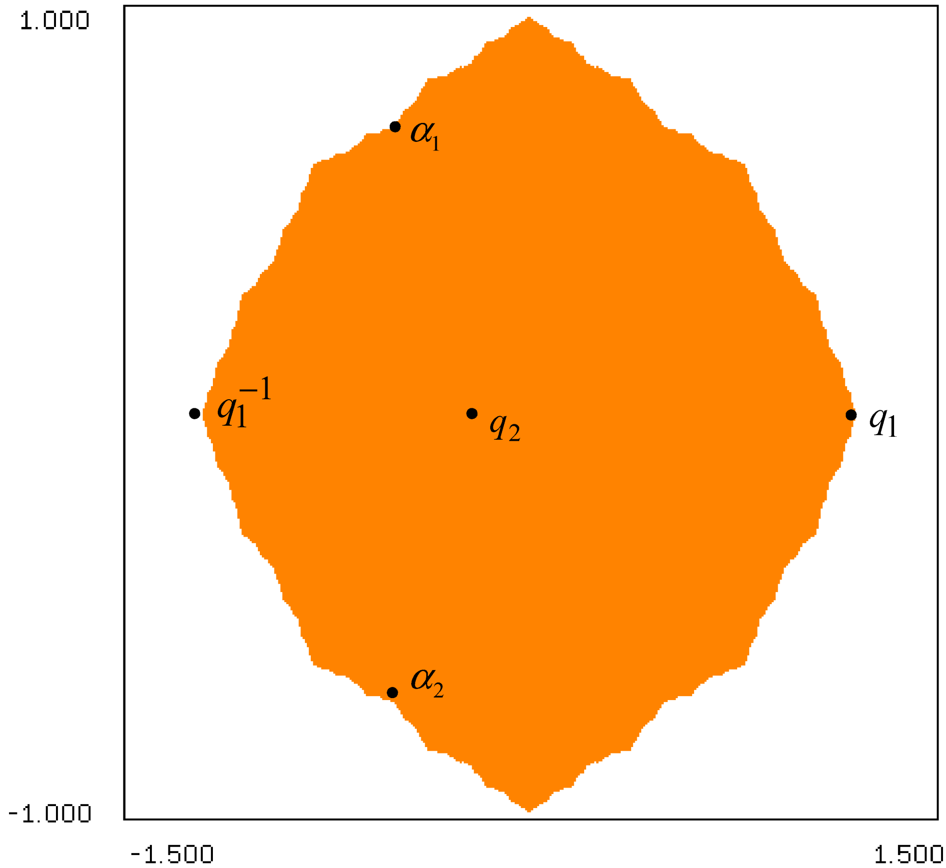


Fig. 11. Filled Julia set for $c = 0.25$. The filled Julia set is a basin toward the stable fixed point q_2 $(-1 < S < 0)$. The points (α_1, α_2) are those of the period 2 cycle. The Julia set J shape ceases to be of "petal-like" kind, and presents infinitely many spikes. (b) $c = 0.5$, J presents infinitely many excrescences with a "base" having a decreasing length, tending toward zero for $c \to c_{b1}$.

new situation the parameter, separating two different shapes of J, is obtained for $c = \underline{c}_k^j$, $c_{(k)_0}^j < \underline{c}_k^j < c_k^j(S = 0)$, which corresponds to a multiplier $0 < S < 1$.

For $c = \underline{c}_k^j$ the Julia set J is made up of infinitely many separated concave continuous closed curves $(\check{C})$, constituting a fractal set, separating "petal-like" shapes from shapes with "spikes". The existence of such curves $(\check{C})$, appears numerically from successive enlargements, with precision increase. The mathematical proof seems very difficult to establish. Among these curves there are k curves $(\check{C}_k^j)$ invariant by T_Z^k. The intersection $J \cap (y = 0) = J \cap ([q_1^{-1}, q_1])$ is a fractal set made up of all the repelling cycles generated on the x-axis for $c < c_{(k)0}^j$ (the lower boundary of the box Ω_k^j), their limit set, the subset of the increasing rank preimages of all these points, located on $y = 0$. Clearly for $y \neq 0$ this fractal structure is reproduced in all the preimages $T_Z^{-n}([q_1^{-1}, q_1])$ for $n > 0$, and $T_Z^{-n}(\check{C}_k^j)$.

The above properties can be illustrated for the interval $c_{(3)_0}^1 < c < c_{3b1}^1$ of the $\omega_3^1 \subset \Omega_3^1$ spectrum. When $S = 0$, $c = c_{S=0} \simeq 1.7548776662$, a rough numerical simulation without a sufficient enlargement might lead to think that the immediate basin boundary of the superstable period three cycle is made up of three circles. This is wrong as shown in Fig. 13 obtained from a strong enlargement, which indicates that the boundary contains infinitely many "spikes". A more elaborated simulation shows that smooth concave closed curves $(\check{C})$ (enlargement of Fig. 14(e)) are obtained for $\underline{c}_3^1 \simeq 1.7545313$ with a cycle multiplier $S \simeq 0.037$. Smooth concave closed curves not intersecting $y = 0$ cannot be clearly seen in Fig. 14(a), but they appear in the enlargement of Figs. 14(b)–14(d).

Consider the boundary $\partial D_0^n(k; j)$ of the immediate basins $D_0^n(k; j)$, $n = 1, 2, \ldots, k$, of the k stable fixed points of T_Z^k, which are n J subsets. In the interval $c_{(k)_0}^j < c < \underline{c}_k^j$, each $\partial D_0^n(k; j)$ is of

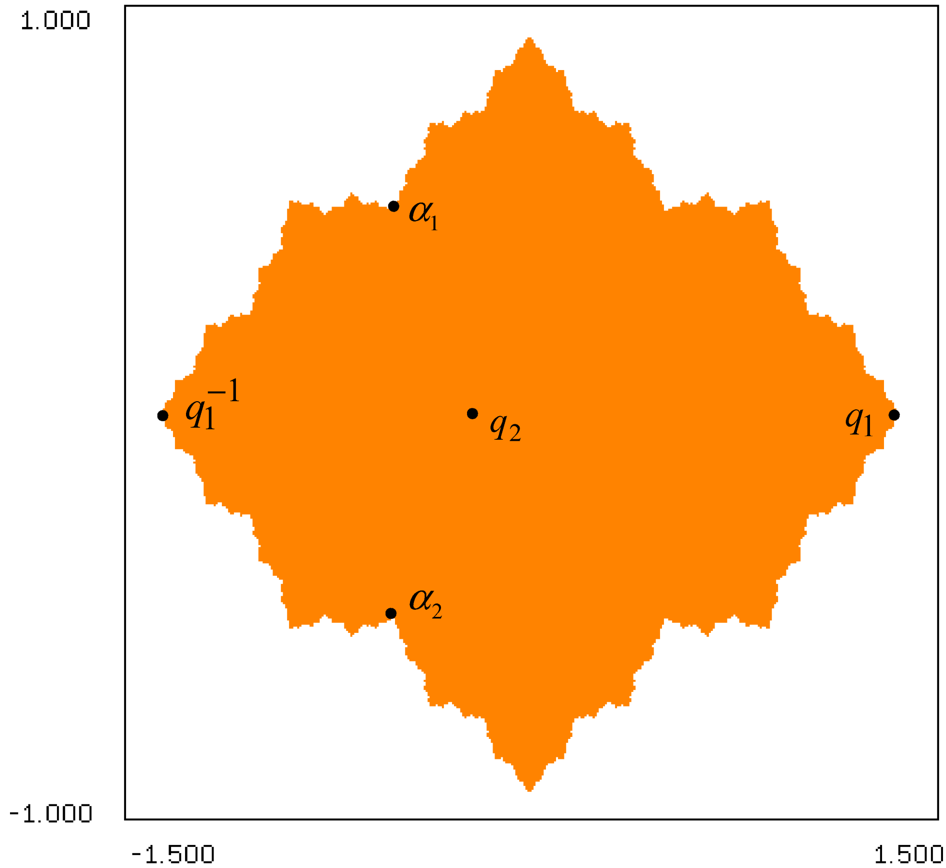


Fig. 12. Filled Julia set for $c = 0.5$. The filled Julia set is a basin toward the stable fixed point q_2 $(-1 < S < 0)$. The points (α_1, α_2) are those of the period 2 cycle. The shape of the Julia set J presents infinitely many excrescences with a "base" of decreasing length, which tends toward zero for $c \to c_{b1} = 3/4$ (cf. Figs. 2 and 8).

petal-like type. When $c \to \underline{c}_k^j$ the $\partial D_0^n(k;j)$ bumps progressively become less and less pronounced, and disappear at $c = \underline{c}_k^j$, for which $\partial D_0^n(k;j)$ is made up of k smooth concave closed curves $(\check{C}_k^j)$. For $\underline{c}_k^j < c < c_{kb1}^j$, with c increasing values $\partial D_0^n(k;j)$ is made up of k subsets of J, first made up of infinitely many "spikes" in a fractal configuration, and then $\partial D_0^n(k;j)$ presents infinitely many excrescences with a "base" having a decreasing length, which tends toward zero when $c \to c_{kb1}^j$. So according to the c-value of the interval $c_{(k)_0}^j < c < c_{kb1}^j$, $J = \mathcal{C}_l(\bigcup_{r\geq 0} T_Z^{-r}[\partial D_0^n(k;j)]$ locally will present outlines either of petal-like type, or smooth concave closed curve, or "spike", or excrescences types.

The increasing rank preimages of $\partial D_0^n(k;j)$ intersect the x-axis at the boundaries of the nonconnected parts of the total basin of the map restricted to the x-axis, and their limit points defined in Sec. 2, i.e. infinitely many Cantor like sets Cs (cf. Sec. 2.3) born for $c < c_{(k)0}^j$. All the points of J with $y \neq \emptyset$ consist of the points of all the other repelling cycles existing in the plane and still outside the x-axis, and their limit points. All such repelling cycles belonging to J will enter the x-axis at higher values of the parameter c, that is at the other bifurcations occurring for $c > c_{(k)0}^j$.

For $c_{(1)0} < c < c_{b1}$, $J \cap (y \neq 0)$ contains all the unstable cycles generated in the box Ω_1. In a same way, the k J subsets $\partial D_0^n(k;j)$ are such that $\bigcup_{n=1}^k[\partial D_0^n(k;j)] \cap (y \neq 0)$ contains all the unstable cycles generated inside the box Ω_k^j $(c_{(k)0}^j \leq c \leq c_k^{*j})$. The cycles generated for $c > c_k^{*j}$ occupy other places on J, in particular as limit points $(y \neq 0)$ of the increasing rank preimages of $\partial D_0^n(k;j)$. The same property occurs at any c-value of the rank-a box $\Omega_{k_1,\ldots,k_a}^{j_1,\ldots,j_a}$.

The fact that the circle situation obtained for $c = 0$, related to a multiplier $S = 0$, cannot occur for a $(k;j)$ cycle, is easily explained. Indeed, consider the map T, $x' = x^2 - c$, restricted to the x-axis, and the arcs of T^k in the $(x;x')$ plane, defining the

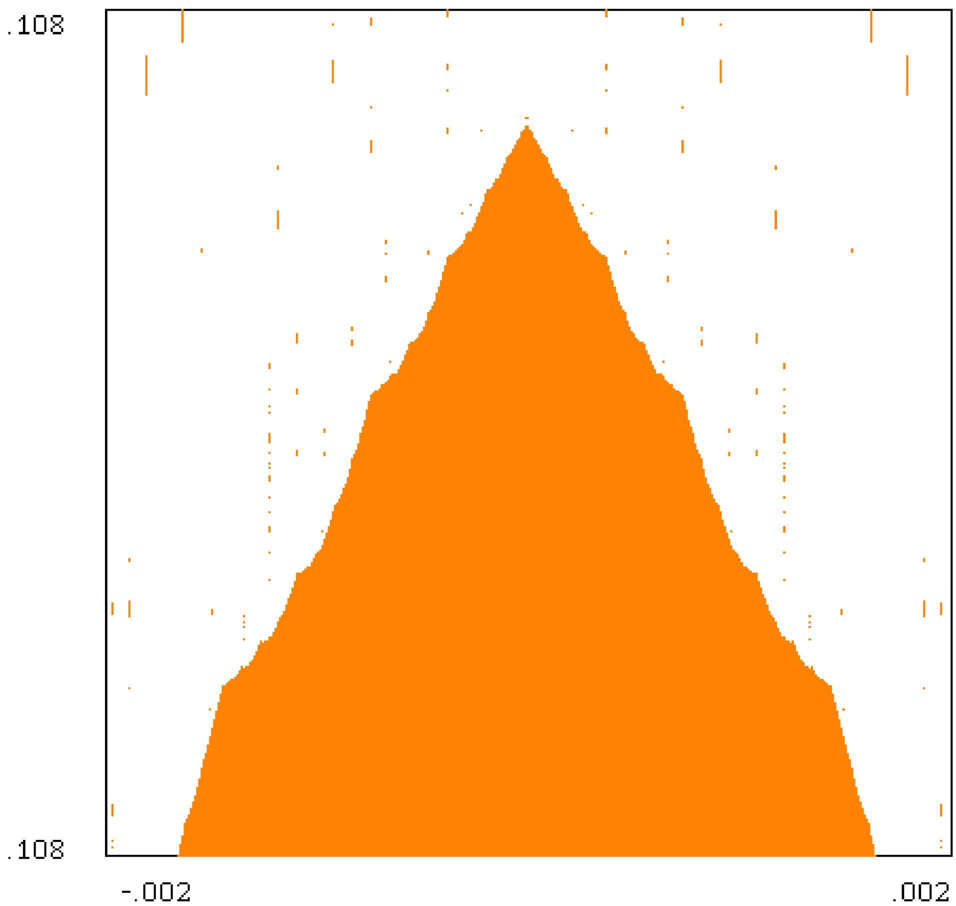


Fig. 13. Enlargement of the filled Julia set part containing the point $(x = y = 0)$ of the superstable $(S = 0)$ period 3 cycle $(\alpha_1, \alpha_2, \alpha_3)$, $c \simeq 1.7548776662$, with the framework $-0.002 < x < 0.002$, $0.1075 < y < 0.1077$. The corresponding immediate basin boundary contains infinitely many "spikes" in a fractal configuration.

cycle pair $(k; j)$. The arc on both sides of $x = 0$, abscissa of one of points of the superstable $(k; j)$ cycle, is symmetric with respect to this line $x = 0$. It is not the case for the other $k-1$ arcs, not having such a symmetry with respect to the other points of the superstable cycle (for example with $k = 3$ see Fig. 2.6 in [Mira *et al.*, 1996]), which cannot be centers of circles generated by T_Z.

5.4. *Fourth type of Julia set*

This Julia set type is obtained for each c-value of the interval $c_{bm} < c < c_{b(m+1)}$ of the ω_1 spectrum, $m = 1, 2, \ldots$. The interval $c^j_{kbn} < c < c^j_{kb(n+1)}$ of the ω^j_k spectrum has equivalent properties.

Julia set type of class A (Proposition 2a) $c_{bm} < c < c_{b(m+1)}$.

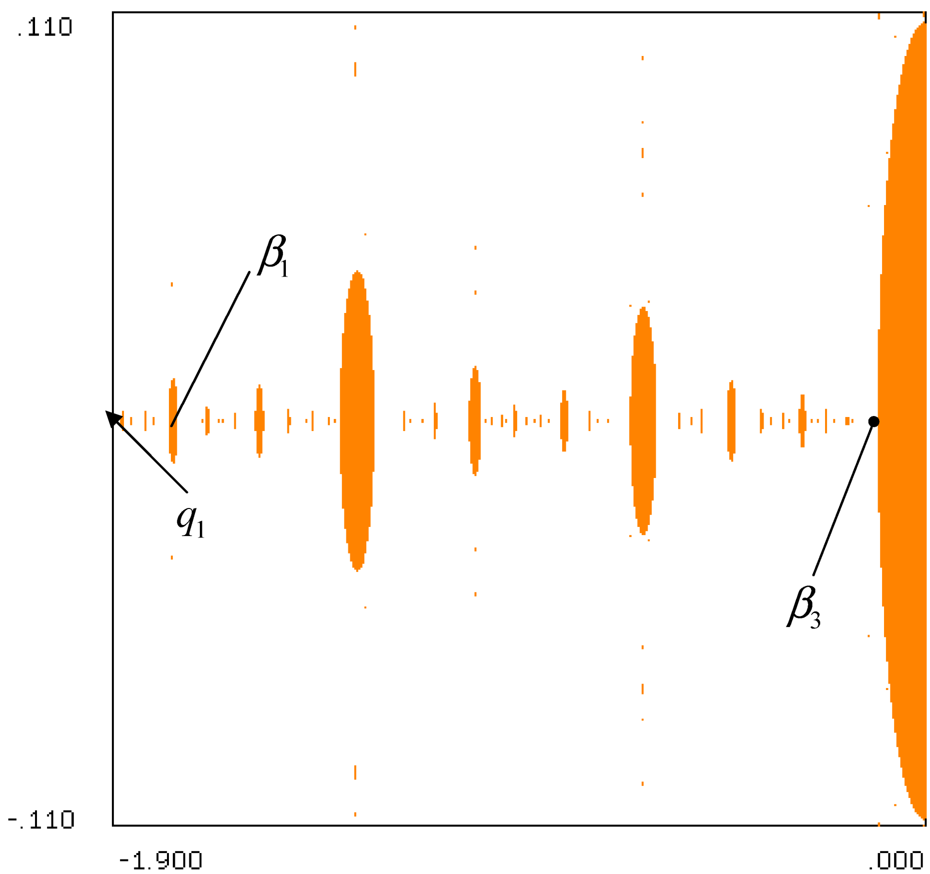
The interval $c_{bm} < c < c_{b(m+1)}$, belongs to the ω_1 spectrum, and the Julia set structure is given by Proposition 2a. In particular $J \cap (y = 0) = \mathcal{C}_l(\bigcup_{r\geq 0} T^{-r}(E^m_1))$, where $E^m_1 \in [q_1^{-1}, q_1]$ is the (finite) set of the m repelling cycles of the x-axis of period 2^p, $0 \leq p < m$, created by the period doubling bifurcations for $c < c_{bm}$, $\mathcal{C}_l$ is the set closure. The cycle of period 2^0 is the fixed point q_2. The Julia set J is the boundary $\partial D(2^m; p_m)$ of the nonconnected basin $D(2^m; p_m)$ of the attracting period 2^m cycle on the x-axis. The set $\partial D_0(2^m; p_m)$ is the boundary of the immediate basin $D_0(2^m; p_m)$ of the attracting period 2^m cycle. *The unstable period 2^{m-1} cycle $(y = 0)$ belongs to the boundary $\partial D_0(2^m; p_m)$. All the other unstable period 2^h cycles $(y = 0)$, $h = 0, 1, 2, \ldots, n-2$, belong to $\partial D(2^m; p_m)$, and are limit of a subset of increasing rank preimages of the unstable period 2^{m-1}* cycle. The basin of the point at infinity is simply connected (Julia–Fatou configuration (A1)), and its boundary is the external part of J.

The situation in Sec. 5.2 $c = c_{b1} = 3/4$ $(S(q_2) = -1)$, also occurs for the parameter interval

$c_{b1} = 3/4 < c < c_{b2} = 5/4$, but now J is without tangent at any of its points. In particular J has the Julia–Fatou basic configuration (A2), with J having multiple points R_{dp1} (cf. Sec. 5.3) everywhere dense on itself. The set J is made up of the union of infinitely many curves (C^u), limiting nonconnected open areas, and $R_{dp1} = \mathcal{C}_l(\bigcup_{r\geq 0} T^{-r}(q_2)$. The J nonsmoothness gives a "spike shaped" contact between two curves (C^u). The points of R_{dp1} belong to the connected basin boundary $\partial D(2^1; p_1)$ of the stable period two cycle on $y = 0$. Each of these points *separates two bordering* (*adjacent*) *nonconnected parts of the total basin* $D(2^1; p_1)$.

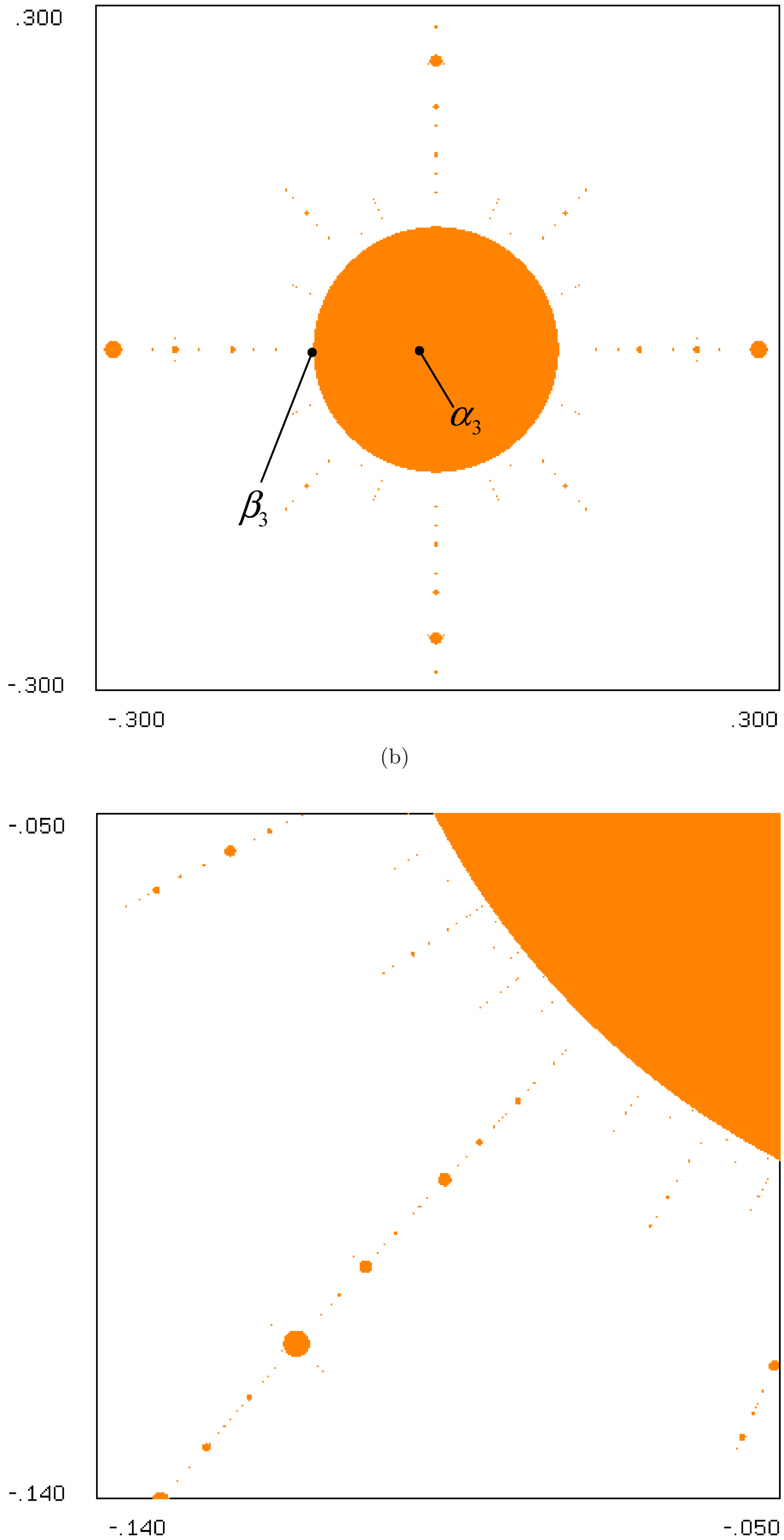
On the x-axis $J \cap (y = 0)$ is made up of q_2 and its increasing rank preimages, tending toward $q_1 \cup q_1^{-1}$. For $y \neq 0$ the structure of $J \cap (y = 0)$ is reproduced on the fractal set of arcs given by $T_Z^{-r}([q_1^{-1}, q_1])$ for $r > 0$. All the points of J with $y \neq \emptyset$ (belonging to the closure of $\bigcup_{r\geq 0} T^{-r}([q_1^{-1}, q_1]))$ consist of points of all the other repelling cycles existing in the plane and still outside the x-axis, and their limit points (all such repelling cycles will enter the x-axis at higher values of the parameter c).

When c increases in the interval $c_{b1} < c < c_{b2}$ Fig. 15 ($c = 1.24$) shows that progressively the nonconnected basin of the period 2^1 cycle (bounded by J containing the R_{dp1} set) presents infinitely many new excrescences with a "base" having a decreasing length, tending toward zero for $c \to c_{b2} = 5/4$ (Fig. 9). This situation is easily explained from the four points η_i ($y < 0$ and $y > 0$, $i = 1, \ldots, 4$) of the period 2^2 cycle of J. Indeed when $c \to c_{b2}$ this period 2^2 cycle with $y \neq 0$ tends toward the stable period 2^1 cycle (α_1, α_2) on the x-axis, and locally creates two narrow "vertical" sections in the basin



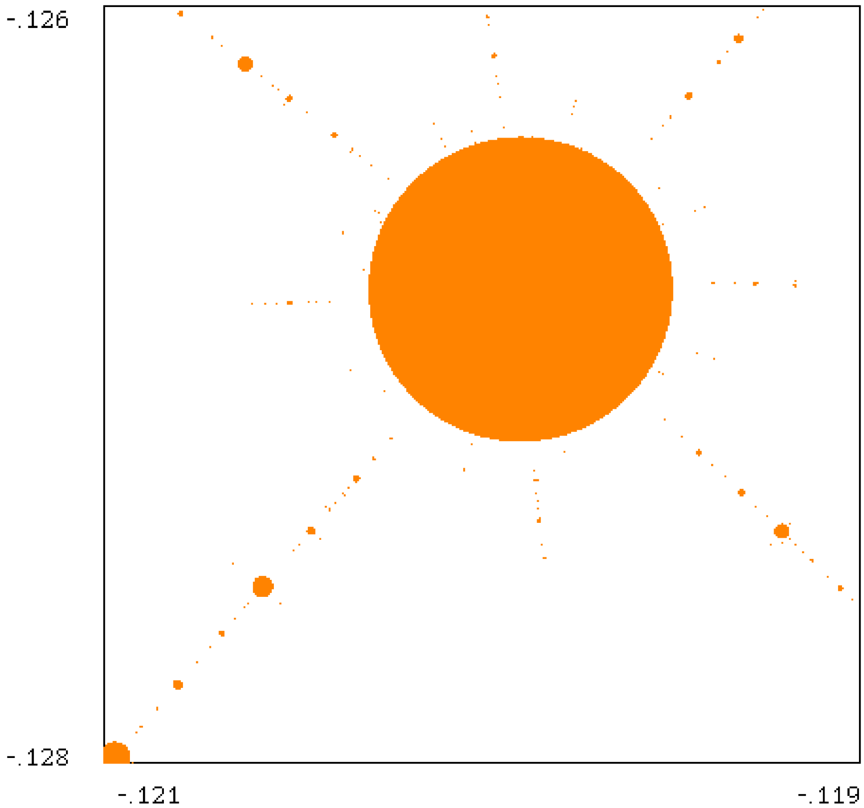
(a)

Fig. 14. (a) Filled Julia set (symmetric with respect to $x = 0$) for $c = \underline{c}_3^1 \simeq 1.7545313$. The stable period 3 cycle $(\alpha_1, \alpha_2, \alpha_3)$ has a multiplier $S \simeq 0.037$. The cycle $(\beta_1, \beta_2, \beta_3)$ is repelling and located on the immediate basin boundary of the period 3 cycle $(\alpha_1, \alpha_2, \alpha_3)$. For this parameter value a smooth period 3 concave closed curve $(\check{C})$, boundary of the immediate basin of $(\alpha_1, \alpha_2, \alpha_3)$ is obtained. This situation, which now does not occur when the multiplier is $S = 0$, separates two different forms of the Julia set, a petal-like one, and the other with "spikes". (b)–(d) Some enlargements in different regions of the $(x; y)$ plane. (e) Enlargement in the region defined by Fig. 13, with $-0.005 < x < 0.005$, $0.1075 < y < 0.1077$.

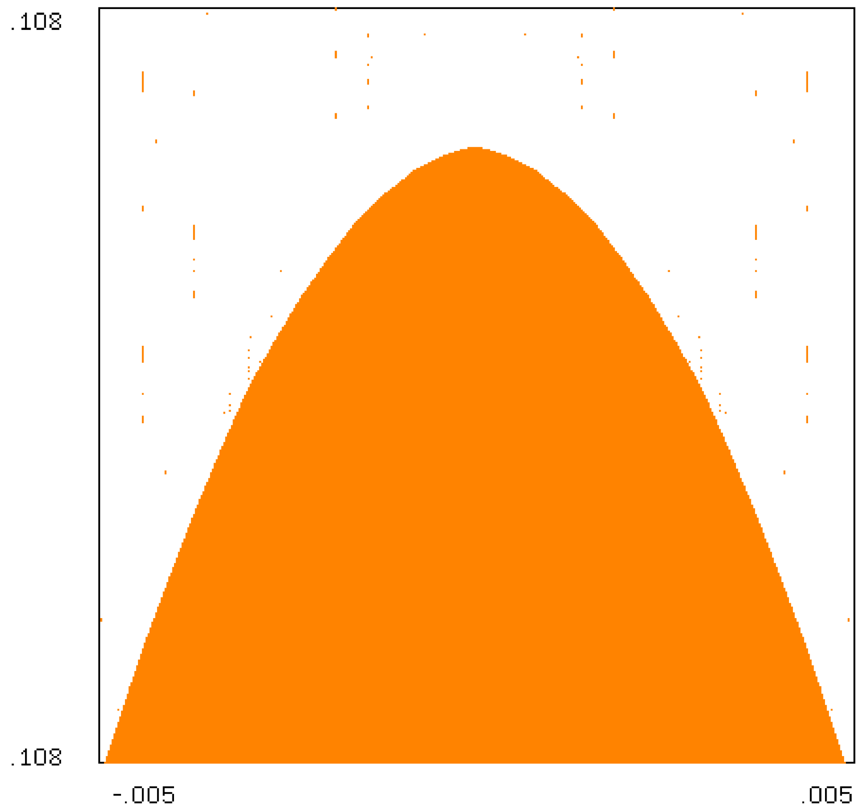


(c)

Fig. 14. (*Continued*)



(d)



(e)

Fig. 14. (*Continued*)

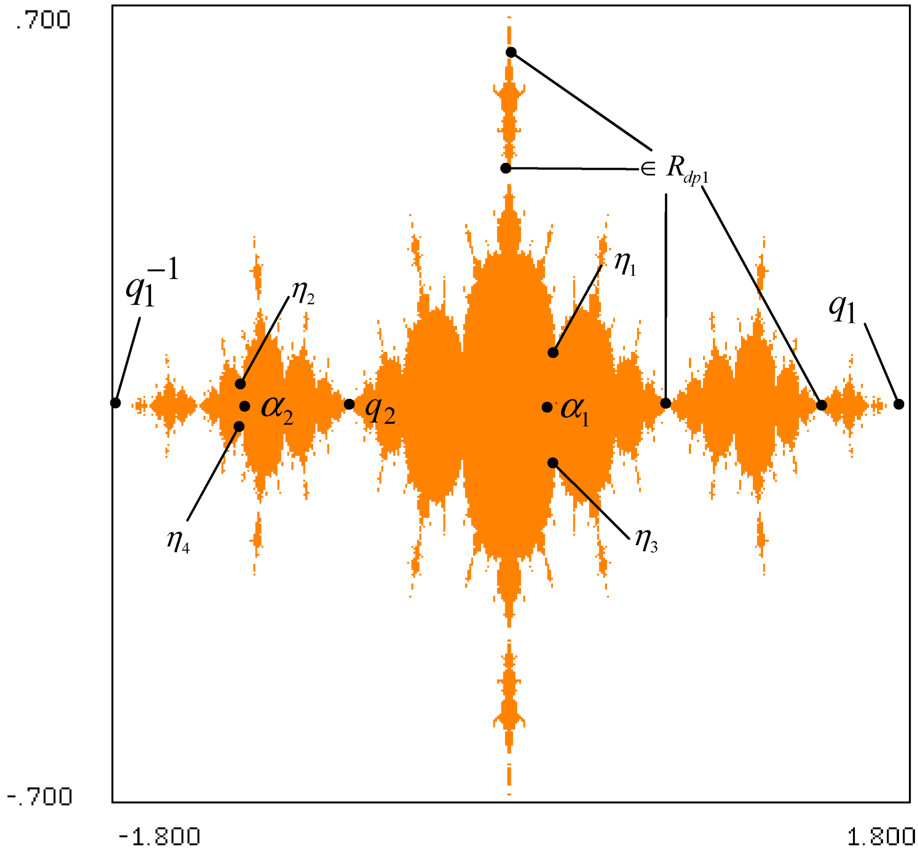


Fig. 15. Interval $c_{b1} < c < c_{b2}$, $c = 1.24$. The filled Julia set is a basin toward the stable period 2^1 cycle $\alpha_1 \cup \alpha_2$. The points $\eta_i \in J$ ($y < 0$ and $y > 0$, $i = 1, \ldots, 4$) are those of unstable period 2^2 cycle. The Julia set J contains only the set R_{dp1}. When $c \to c_{b2} = 5/4$ the period 2^2 cycle η_i with $y \neq 0$ tends toward the stable period 2^1 cycle (α_1, α_2) on the x-axis, and locally creates two narrow "vertical" sections in the basin of the stable period 2^1 cycle.

of the stable period 2^1 cycle. From this situation a new (rank-2) infinite set R_{dp2} of double points of J results when $c \geq c_{b2}$. Then for $c \geq c_{b2}$ the points of $R_{dp2} = \mathcal{C}_l(\bigcup_{r\geq 0} T^{-r}(\alpha_1 \cup \alpha_2))$ belong to the boundary $\partial D(2^2; p_2)$ of the period four cycle now located on $y = 0$, and *separate two bordering nonconnected parts of* $D(2^2; p_2)$. It is not the case of the points of $R_{dp1} \in \partial D(2^2; p_2)$, which now *do not separate two bordering nonconnected parts of* $D(2^2; p_2)$, but turn into limit points of R_{dp2}, thus are accumulation of *non connected parts of* $D(2^2; p_2)$. This last situation is due to the fact that when $c > c_{b2}$ the points of the unstable fixed point q_2 is a limit point of a subset of increasing rank preimages of the unstable period 2^1 cycle (α_1, α_2) $\in \partial D(2^2; p_2)$, immediate basin of the period 2^2 cycle.

The same behavior occurs for any interval $c_{bm} \leq c < c_{b(m+1)}$ of the ω_1 spectrum. When $c \to c_{bm}$ the 2^m points ($y < 0$ and $y > 0$) of the period 2^m cycle tend toward the stable period 2^{m-1} cycle on the x-axis, creating locally 2^{m-1} narrow "vertical" sections (base of excrescences) in the basin of the period 2^{m-1} cycle, and after a new (rank-m) infinite set R_{dpm} of double points of J appearing when $c \geq c_{bm}$. The points of R_{dpm} *separate two bordering (adjacent) nonconnected parts of* $D(2^m)$. The points of $R_{dph} \in \partial D(2^m; p_m)$, $1 \leq h < m$, *does not separate two bordering nonconnected parts of* $D(2^2)$, but are limit points of *nonconnected parts of* $D(2^p; p_p)$. This situation is due to the fact that when $c > c_{bm}$ the points of an unstable period 2^q cycle, belonging to the basin $\partial D(2^m; p_m)$ are limit points for a subset of increasing rank preimages of the period 2^h cycles $q < h$ located on the x-axis. Figures 4–6 show J as boundary of the brown region (filled Julia set) for a c-value of the interval $m = 2, 3, 4$ corresponding to the basin of the stable period 2^m cycle, J containing the m sets R_{dpm}.

When $m \to \infty$, the sets $D(2^m) \to 0$ and $\bigcup_{m=1}^{\infty} R_{dpm}$ tend toward the dendrite (cf. Sec. 5.5), obtained for $c = c_{1s}$ (Fig. 17).

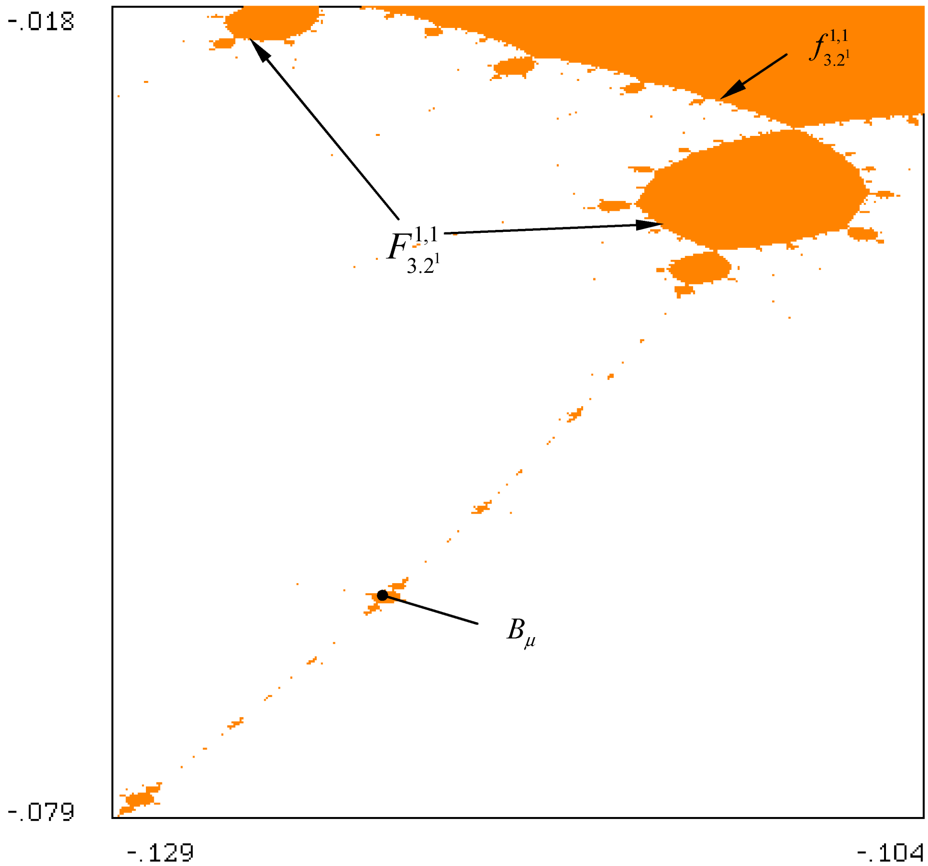


Fig. 16. Interval $c^1_{3b1} < c < c^1_{3b2}$, $c = 1.77289$. Partial view of the filled Julia set J, basin toward the stable period 6 cycle $(3.2^1; 1, 1)$, with $S \simeq 0$. The boundary $f^{1,1}_{3.2^1}$ belongs to the immediate basin boundary of the stable period 6 cycle $(3.2^1; 1, 1)$. The boundary $F^{1,1}_{3.2^1}$ is one of the three components of the period three subset of J (cf. Sec. 4 Proposition 2b), each one having the basic Julia–Fatou configuration (A2). Some of the preimages of this configuration can be seen, for example B_μ.

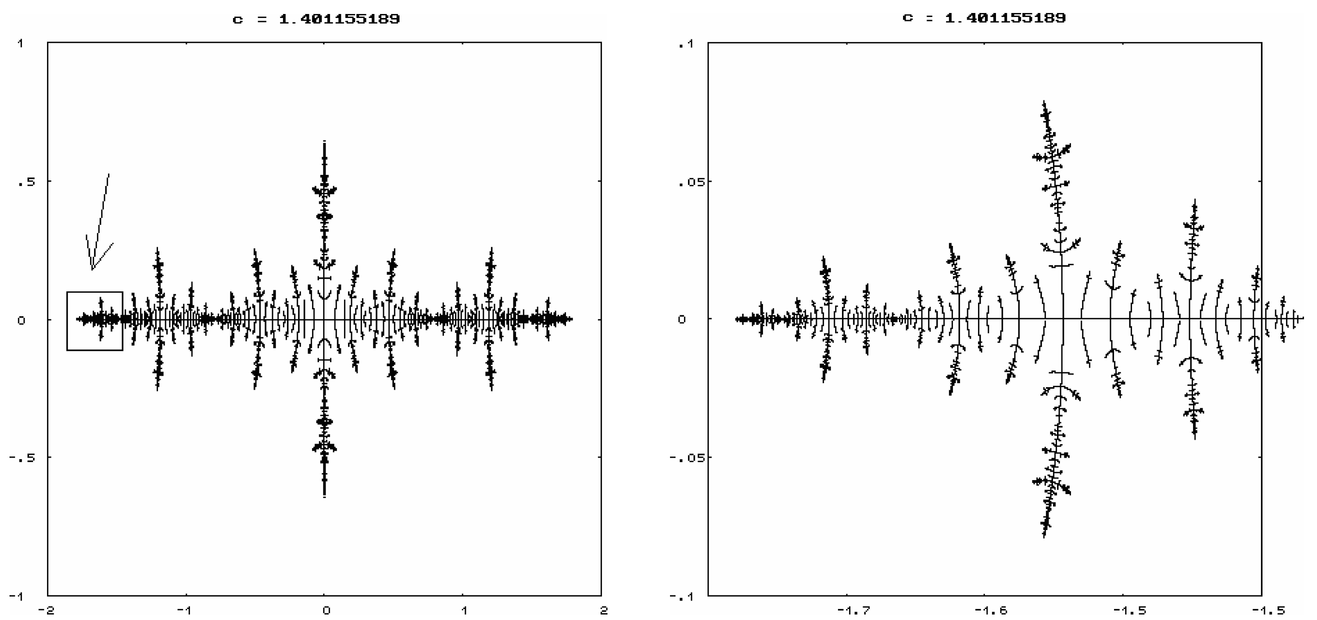
Julia sets of class B: intervals $c^j_{kbm} \le c < c^j_{kb(m+1)}$ *of the spectrum* ω^j_k.

For intervals $c^j_{kbm} \le c < c^j_{kb(m+1)}$ of the spectrum ω^j_k, *the Julia set structure is given by Proposition 2b*. Now contrarily to intervals $c_{bm} \le c < c_{b(m+1)}$ of ω_1, $J \cap (y = 0)$ contains infinitely many Cantor like sets Cs (cf. Sec. 2.3) born for $c < c^j_{(k)0}$ and the unstable cycles with their increasing rank preimages located on $(y = 0)$, born for $c^j_{(k)0} \le c < c^j_{kbm}$. So the set $J \cap (y = 0)$ is a well defined fractal set, $J \cap (y = 0) \subset \mathcal{C}_l(\bigcup_{r\ge 0} T^{-r}(E^j_{km}))$, where $E^j_{km} \in [q_1^{-1}, q_1]$ is the Cantor set of repelling cycles born for $c < c^j_{kbm}$. The basin of the stable $(k.2^m; j, p_m)$ cycle intersects the x-axis including $2^m k$ segments invariant by $T_Z^{k2^m}$, and their increasing rank preimages located on the segment $[q_1^{-1}, q_1]$. The other increasing rank preimages are located on the fractal set $T_Z^{-r}([q_1^{-1}, q_1])$ for all $r > 0$. All the points of J with $y \ne \emptyset$ consist of points of all the other repelling cycles existing in the plane and still outside the x-axis, and their limit points (which will enter the x-axis at higher values of the parameter c). The properties of the structure of J are given in Proposition 2b. Figure 16 represents an enlargement of the filled Julia set in the neighborhood of the immediate basin boundary of the stable $(k.2^m; j, p_m)$ cycle.

Similarly equivalent behaviors occur for intervals of spectra $\omega^{j_1,\ldots,j_a}_{k_1,\ldots,k_a}$ between two consecutive flip-bifurcations.

5.5. *Fifth type of Julia set. Dendrites*

Dendrites are characterized by the fact that $E_c \cap J \ne \emptyset$ (i.e. a dendrite occurs if and only if $E_c \cap J \ne \emptyset$). As we have seen in Sec. 3.3 this occurs when the attracting set A_c of the Myrberg's map is either a critical set A_{cr} (with Cantor like structure, of zero



(a) (b)

Fig. 17. (a) Situation of the Julia set J, a dendrite, at the boundary $c = c_{1s} \simeq 1.401155189$ of the ω_1 Myrberg spectrum. This dendrite is the limit of the filled Julia sets of Figs. 4–6 for an attracting period 2^m cycle with $m \to \infty$. Now the filled Julia set does not exist, in other words it reduces to J. (b) Enlargement.

Lebesgue measure), or when A_c consists of k-cyclic chaotic intervals, $k \geq 1$ as described in Sec. 3.2 (point (P′2) and related properties). Clearly at any value of c in the parameter set denoted by $\hat{c} = c_{cr} \cup c_{ch}$ at which a dendrite occurs, the structure of J is at a bifurcation situation, as conjectured in p. 73 (last paragraph of Chapter 4) in [Fatou, 1920].

The set c_{cr} includes all the values of the parameter c which are limit points of flip bifurcation cascades, i.e. one of the two boundaries of a Myrberg spectrum (A_c is a critical set), as the values c^j_{ks} and their embedded forms in all the rank-a boxes, $a \geq 1$. These cases are characterized by the fact that the trajectory of the critical point C belongs to the critical set as $C \in E'_c = A_{cr}$, so that $E_c \subset E'_c \subset J$.

The set c_{ch} includes all the values of the parameter c which are global bifurcations, at the closure of any box of first or second kind, as the values c^{*j}_k, $c^{*j}_{k2^m}$, and limit points of such bifurcations, and other global bifurcations as the values $\tilde{c}$, and their embedded forms in any rank-a boxes, $a > 1$ (Sec. 2.4). In these cases A_c consists of k-cyclic chaotic intervals ($k \geq 1$), the critical point C is either periodic or preperiodic, merging into a repelling cycle ($|S| > 1$), thus $E_c \cap J \neq \emptyset$.

As already remarked in Secs. 3.2 and 3.3, for such parameter values, say $\bar{c} \in \hat{c}$, J is not the basin boundary of the attracting set on the x-axis, but only the frontier of the basin of divergent trajectories. This situation is called a dendrite, as J is made up of a basic segment, the whole interval $[q_1^{-1}, q_1]$ of the x-axis, and all its preimages of any rank, $T_Z^{-n}([q_1^{-1}, q_1])$ for $n > 0$, as given in (5), which includes an "arborescent" sequence of infinitely many arcs belonging to J for $y \neq 0$. Clearly the basic segment $[q_1^{-1}, q_1]$ of J includes all the repelling cycles already created in the interval (for $c < \bar{c} \in \hat{c}$) and belonging to the attracting set A_c of the Myrberg's map (except for the point q_1) as well as their preimages and limit points on the x-axis, while the arborescent part of J for $y \neq 0$ includes all the remaining cycles with periodic points having $y \neq 0$, and their preimages, that will become real at higher values of c.

It is worth to note that since J is also the closure of all the repelling points (at any value of the parameter c), it follows that the points of the interval $[q_1^{-1}, q_1]$ which are not periodic, or limit points of periodic points, on the x-axis (in particular, all those of the interval $[q_1^{-1}, -c[$) are limit points of periodic points belonging to J with $y \neq \emptyset$. Also the points belonging to $T_Z^{-n}([q_1^{-1}, q_1])$ for $n > 0$ are not such periodic points, and are thus are only in the limit set of such preimages.

As a result of all the preimages of any rank, $T_Z^{-n}([q_1^{-1}, q_1])$ for $n > 0$, the "arborescent" sequence of infinitely many J arcs with $y \neq 0$ has the same qualitative shape whatever be the parameter $\hat{c} = c_{cr} \cup c_{ch}$. The related dendrites only differ by the nature of the singular sets (Sec. 2.4) located on the basic segment $[q_1^{-1}, q_1]$. Figure 17 represents the case $c = c_{1s}$. Equivalent qualitative figures, at a correctly chosen scale, can be obtained for $c = c_{ks}^{j}$, $c = c_{(k_1,\ldots,k_a)_s}^{j_1,\ldots,j_a}$, $c = c_{ch}$.

6. Conclusion

Julia and Fatou have already described the basic situations generated by a one-dimensional complex map (and it is remarkable that this was done without the help of any computer). In this Part I, this paper has shown how the bifurcations symbolism related to the box-within-a-box organization (described in Sec. 2) permits to introduce a fractal ordering in the qualitative changes of the Julia sets generated by (1) when c is real, $-1/4 \leq c \leq 2$. So it is possible to follow the evolution of the Julia set shape in this interval, and identify the subintervals giving the same qualitative shape. Moreover, as shown in Sec. 4, the structure of the Julia set, defined from the the location unstable cycles (defined by Sec. 2 symbolism) in the plane, can be identified.

We remark that the paper results, based on the box-within-a-box organization, describe the situation given from the section by the real parameter axis of the boundary of the classical Mandelbrot set. Then in the Mandelbrot parameter plane it is likely that there exist routes with c complex, $c = a \pm jb$, $j^2 = -1$, reproducing with a two parameter symbolism what occurs when c is real.

Considering now the "indirect" embedding of T_Z into the two-dimensional family of noninvertible maps $\overline{T}$ (2) (object of Part II of this paper), which depends on the two real parameters c and γ, Secs. 3–5 results define completely the map behavior in the half plane $y \leq 0$, when $\gamma = 0$. More particularly, the second part of this paper will explain bifurcation routes leading to the different configurations of the Julia sets J generated by T_Z when $\gamma \to 0$, with $\gamma \geq 0$.

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