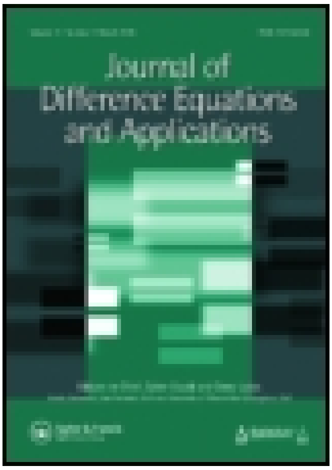

Journal of Difference Equations and Applications

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gdea20

Border collision and fold bifurcations in a family of one-dimensional discontinuous piecewise smooth maps: unbounded chaotic sets

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Published online: 16 Jun 2015.

Journal of Difference Equations and Applications, 2015
http://dx.doi.org/10.1080/10236198.2015.1045893

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(*Received 21 March 2015; accepted 18 April 2015*)

In this work we consider a class of generalized piecewise smooth maps, proposed in the study of engineering models. It is a class of one-dimensional discontinuous maps, with a linear branch and a nonlinear one, characterized by a power function with a term $x^{-\gamma}$ and a vertical asymptote. The bifurcation structures occurring in the family of maps are classified according to the invertibility or non-invertibility of the map, depending on the parameters characterizing the two branches. When the map is non-invertible we prove the persistence of chaos. In particular, the existence of robust unbounded chaotic attractors. The parameter space is characterized by intermingled regions of attracting cycles born by smooth fold bifurcations, issuing from codimension-two bifurcation points. The main result is related to the description of the relationship between two types of bifurcations, smooth fold bifurcations and border collision bifurcations (BCBs). We describe the particular role of codimension-two bifurcation points associated with these bifurcations related to cycles with the same symbolic sequences. We show that they exist related to the border collision of any admissible cycle. We prove that each BCB, each fold bifurcation and each homoclinic bifurcation is a limit set of infinite families of other BCBs. We prove that in the considered range all the unstable cycles are always homoclinic, and that an unbounded chaotic set always exists, either in an invariant set of zero measure or of full measure.

Keywords: piecewise smooth maps; border collision bifurcations; codimension-two bifurcation points; unbounded chaotic attractors

1. Introduction

The study of piecewise smooth (PWS) systems had a wide expansion in the last decade. This is due to the large number of physical and engineering systems with non-smooth vector fields, a recent survey can be found in [33]. Many applications come from power electronic circuits in electrical engineering, which gave a wide impulse to the study of piecewise defined systems, both continuous and discontinuous (see, e.g. the cases presented in [11,14]). Several kinds of bifurcations of non-smooth systems also appear in forced impact oscillators [46,59], in mechanical engineering [42,43], in economics and social sciences [13,47,48,57,58].

Sometimes, simplifying assumptions on systems lead to piecewise linear (PWL) models, and since many years particular studies have been devoted to PWL systems, both continuous and discontinuous (see, e.g. [2,10,11,14,23]).

However, many applications in engineering may include specific nonlinearities in the map, as power functions. In particular, much attention has been devoted to the square-root

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singularities in impact oscillators, following the works of Nordmark [42,43]. The PWS system already considered by many authors is given by

$$x \mapsto f_\mu(x) = \begin{cases} f_L(x) = ax + \mu & \text{if } x \le 0 \\ f_R(x) = bx^z + \mu & \text{if } x > 0, \end{cases} \quad (1)$$

where a, b, z and μ are real parameters. In particular, the PWL case $z = 1$ leads to the continuous skew-tent map, whose dynamics are now well known (see, e.g. [30–32,54]). The case $z = 1/2$ is related to the square-root nonlinearity typical in Nordmark systems and grazing bifurcations. Also the power $z = 3/2$ was considered in [19] in analysing the stick-slip motion with dry friction. In [17] the normal-form mapping of sliding bifurcations is derived, leading to map (1) with power $z = 3/2$, $z = 2$ and $z = 3$, related to different cases of sliding bifurcations. Other examples of grazing and sliding bifurcations with nonlinear leading-order terms occur in power converters and in non-smooth sliding-mode controls [1,15,16].

System (1) with $z = 2$ is a particular case of the linear-logistic map considered in [52,53]. A generalization of system (1) in the case $z > 0$ is also considered in [8], by using the smooth function $f_R(x) = bx^z + cx + \mu$ on the right side, which introduces a second critical point in the map. While system (1) in the case $z = 1/2$ but with different offsets (and thus a discontinuous system) was considered in [20].

The characteristic feature of PWS maps which recently attracted the interest of many scholars is the occurrence of border collision bifurcations (BCBs). This term was introduced by Nusse and Yorke [44,45], and since then it is generally used to denote a bifurcation caused by a fixed point or periodic point of a cycle colliding (or merging) with a kink point or a border point of a map, crossing which the system changes its definition. This kind of bifurcation leads to dynamic behaviours which may be completely different from those occurring in smooth systems. In PWSsystems it is often very difficult to predict the dynamic behaviour occurring after a collision, which is one of the main goals in the analysis of applied systems. In one-dimensional PWS continuous maps, the linear case (skew-tent map) may be used as a normal form to predict the effect of a BCB, at least in generic cases (codimension-one cases), some applications can be found in [22,55]. Codimension-two cases are considered in [18].

The classification of the possible different results of a border collision in discontinuous one-dimensional maps is still to be investigated. In the references cited above, particular cases have already been studied in detail, by using the discontinuous PWL map in normal form. However, its possible use as a normal form is still to be clarified. The main point is that in PWL discontinuous maps the only possible bifurcations are related to border collision, while in PWS systems also standard bifurcations peculiar of smooth systems are involved. Thus, it is necessary to investigate the interactions between these two kinds of bifurcations (smooth bifurcations and BCBs) in order to understand the dynamical behaviours of PWS systems, as remarked in [8]. Among other properties and results, it is exactly this particular inter-connection which attracted our interest in the present work.

System (1) was investigated also in [49] where, besides the cases $z > 0$ mentioned above, the authors extend the analysis to the discontinuous case with $z < 0$. This leads to particular maps in which the discontinuity point is also a vertical asymptote for the function $f_R(x)$ defined on the right branch. The particular case with $z = -1/2$ is also considered in [50]. The existence of a vertical asymptote is not new in the engineering applications, an interesting application can be found in [29] and for experimental occurrence we refer to [51]. A peculiarity of such systems is that unbounded chaotic attractors can exist (see [12]), and in this work we prove their occurrence.

Differently from PWL systems, where cycles may appear/disappear via BCB [14,23] and attracting cycles may be organized in the period incrementing structures or in the period adding structure (infinitely many non-overlapping periodicity regions exist, and the rotation numbers of the related cycles follow a Farey summation rule [2,4,6,25]), in PWS systems, both continuous and discontinuous, the appearance of cycles may occur via a smooth bifurcation, and the existing cycles may persist (also changing the local stability/instability) or may disappear by BCB [8,26,28,49,56].

The system considered in [49] is a discontinuous one, and the occurrence of BCBs and smooth fold bifurcations associated with basic cycles was shown. However, the complex bifurcation structure occurring in the system is left to study. A detailed analysis, although still far from complete, of that system is the goal of a series of works recalled below. Let us set $z = -\gamma$ in map (1), and we consider positive values of γ. Moreover, we restrict our investigation to the system in which the right function $f_R(x)$ is increasing, considering the case $b < 0$, so that the parameter space of interest is as follows:

$$a \in \mathbb{R}, \quad b < 0, \quad \gamma > 0. \tag{2}$$

Preliminary results are reported in [34], where it is shown that for the study of the dynamic behaviours of the system it is sufficient to consider only the three values $\mu = 0$, $\mu = +1$ and $\mu = -1$. In this work we are interested in the dynamics of the map with $\mu > 0$ as in this case the system is relevant for applications to engineering [49], and for any $\mu > 0$ the transformation $(x, a, b, \mu) \rightarrow (x/\mu, a, b\mu^{-\gamma-1}, 1)$ leads from (1) to the map

$$x \mapsto f(x) = \begin{cases} f_L(x) = ax + 1 & \text{if } x \leq 0 \\ f_R(x) = \frac{b}{x^\gamma} + 1 & \text{if } x > 0 \end{cases} \tag{3}$$

and the other parameters are as defined in (2).

For this PWS system, the change of definition occurs at the discontinuity point $x = 0$ that is also a vertical asymptote for the function on the right side. As it is often used in PWS, the dynamical properties are studied making use of the symbolic notation based on the letters L and R corresponding to the two disjoint partitions

$$I_L = (-\infty, 0], \quad I_R = (0 + \infty). \tag{4}$$

To each trajectory we associate its itinerary by using the letter L when a point belongs to I_L and R when a point belongs to I_R. A cycle is represented by its finite symbolic sequence. For example, a cycle with symbolic sequence RL^n (corresponding to a basic cycle) has one periodic point on the right partition and n on the left one.

A graphical representation suggests the richness of the dynamics of the system (3). Figure 1 shows the two-dimensional bifurcation diagram in the parameter space $(a, S(b))$ at the fixed value $\gamma = 0.5$. In order to consider the complete range for the parameter b, which means $-\infty < b < 0$, following [4] we consider the nonlinear transformation $S(y) = \arctan(y)$ for the parameter b, so that $S(b) \in (-(\pi/2), 0)$. In particular, in the case $b = -1$ we have $S(-1) = -(\pi/4)$ which is evidenced in Figure 1. The coloured regions represent sets of values of the parameters in which the map has an attracting cycle, different colours are associated with different periods. White points represent parameter sets at which there exists an unbounded chaotic attractor. As it is visible, infinitely many

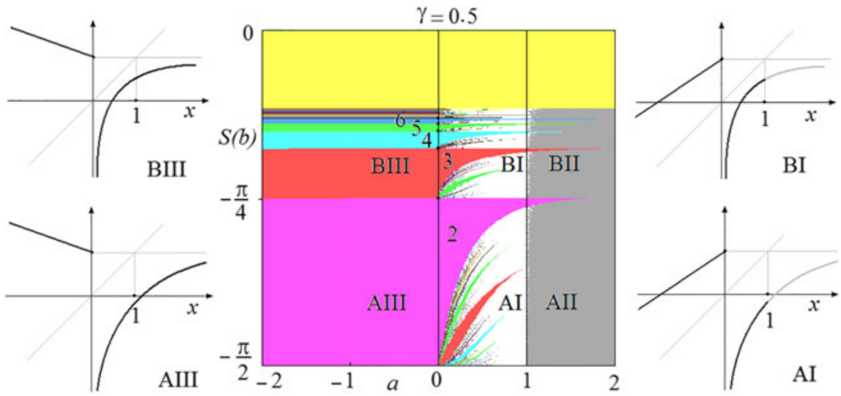

Figure 1. Two-dimensional bifurcation diagram in the parameter space $(a; S(b))$ where $S(b) = \arctan(b)$ for $a \in (-2, 2), S(b) = (-(\pi/2), 0)$, at the fixed value $\gamma = 0.5$. In the vertical axis $-(\pi/4)$ corresponds to $b = -1$.

periodicity regions are issuing from particular points (which are codimension-two bifurcation points). Grey regions existing for $a > 1$ represent parameter sets at which the system has divergent trajectories (which may or may not coexist with an attracting cycle).

Depending on the sign of the parameters, the qualitative shape of the map changes, and correspondingly changes the dynamic properties. The range related to $a < 0$ is associated with an invertible discontinuous map. In that case, a peculiar periodic incrementing structure has been proved to exist and, as shown in [36], related to different border collision and flip bifurcations, depending on $0 < \gamma < 1$, $\gamma = 1$ and $\gamma > 1$, both in the range $-\infty < b \leq -1$ and $-1 < b < 0$ marked by AIII and BIII in Figure 1, respectively. Also the particular case related to $a = 0$ is considered in that paper.

In the range $a > 0$ the map is non-invertible. This leads to peculiar dynamic properties for $0 < a \leq 1$ and $-\infty < b \leq -1$ (region AI in Figure 1), which are the object of this work. The other ranges $a > 1$ and $-\infty < b \leq -1$ (marked as region AII in Figure 1), as well as $0 < a \leq 1$ and $-1 < b < 0$ (region BI) and $a > 1$ and $-1 < b < 0$ (region BII), are considered in a companion paper [35]. In this work and in [35] several properties are shown and proved. However, there are still open problems, which deserve further investigations.

In this work, rigorous proofs are given in several cases. To prove the existence of chaotic sets we make use of homoclinic orbits. It is well known that in one-dimensional non-invertible maps, homoclinic orbits of a repelling k−cycle, $k \geq 1$ ($k = 1$ corresponds to a fixed point), exist when it is a snap-back repellor (SBR), following the definition given by Marotto in [38,39]. Recall that a repelling cycle may become a SBR via a critical homoclinic orbit, associated with critical homoclinic explosions, or Ω-explosions, as shown in [21] for smooth continuous systems, and in [27] for generic PWL and PWS systems, continuous and discontinuous. Moreover, besides invariant unbounded chaotic repellors, which always exist in the range here considered, we show the occurrence of unbounded chaotic attractors, by using the results in [37], which are also robust according to the definition given in [9].

In the parameter space, particular codimension-two bifurcation points, called organizing centres or big bang bifurcation points (following the definition used in [4]), are clearly visible in Figure 1, which are issuing points of infinite families of bifurcation curves. These are codimension-two points related to fold bifurcations and BCBs of cycles with different symbolic sequences, and the structure of the bifurcation curves issuing from those points is still to be understood. We shall see that it is much richer than the period adding bifurcation structure (which is included as a subset) and also not related to the U-sequence [40] well known in continuous 1D maps.

Codimension-two points related to fold bifurcations and BCBs of cycles with the same symbolic sequence do not behave as organizing centres. However, we shall prove that they are limit sets of infinite families of BCBs of other cycles. Starting from the properties of basic cycles, we generalize the properties to the BCB curve of any admissible cycle.

The plan of the work is as follows. In Section 2 we introduce the bifurcations of basic cycles, both BCB and fold bifurcations, their codimension-two points and their role in the bifurcation sequences. To prove the dynamic properties related to these bifurcations, as well as the properties related to all the admissible cycles, we make use of the first return map, defined in Section 3. In Section 4 we prove that in our system all the unstable cycles are always homoclinic, and that unbounded chaotic set always exists. Moreover, in several full measure regions of the parameter space the existing attractor is robust, full measure and unbounded (the interval $(-\infty, 1]$). In Section 5 we generalize the properties of basic cycles to any admissible cycle, showing that the BCB curves may be in pair with fold bifurcation curves, and that a codimension-two point exists on each BCB curve. The dynamic properties on their crossing are the same as for basic cycles. In Section 6 we illustrate the possible sequences of bifurcations existing between two consecutive BCBs of basic cycles, making use of two examples, one related to fold bifurcation curves and the other to the chaotic range. We shall see that by crossing periodicity regions of attracting cycles the adding structure may be observed but only as a strict subset. The full bifurcation structure is much more rich. In Section 7 we show that each BCB curve is a limit set of infinite families of other BCB curves. The same property holds for all the homoclinic bifurcation curves as well as for the fold bifurcation curves (from one side only). Section 8 concludes, evidencing that many open problems on the dynamics of our system are still to be investigated.

2. Codimension-two points on BCB curves of basic cycles

As stated in the Introduction, we are here interested in the dynamic properties of the system (3) for parameters in region AI shown in Figure 1:

$$0 < a \leq 1, \quad -\infty < b \leq -1, \quad \gamma > 0. \tag{5}$$

In the Introduction we have mentioned that for $a < 0$ the map is invertible, thus no chaotic set can exist. Differently, for $a > 0$ the map is non-invertible, and chaos may exist. What is peculiar here is that unbounded chaotic sets necessarily exist. What may change is the measure of the chaotic set, which, as we shall see, can be of zero measure or of full measure.

In the range we are interested in the map is without fixed points. In fact, the left branch $f_L(x)$ is increasing and without fixed points for $0 < a \leq 1$, while the right one, $f_R(x)$, is such that any point $x > 1$ is mapped in one iteration to a point smaller than 1, it has horizontal asymptote in 1, and since

$$f'_R(x) = \frac{-b\gamma}{x^{\gamma+1}} > 0, \quad f''_R(x) = \frac{b\gamma(\gamma+1)}{x^{\gamma+2}} < 0 \tag{6}$$

$f_R(x)$ is increasing and concave. From $f_R(1) = 1 + b \leq 0$ in the considered range, we have that no fixed point can exist (indeed, a fold bifurcation of $f_R(x)$ occurs for $-1 < b < 0$). Moreover, $f_R(x)$ intersects the x-axis in the point

$$x = (-b)^{1/\gamma} =: O_R^{-1} \geq 1 \tag{7}$$

from which we can immediately see that $b = -1$ is the equation in the parameter space of the BCB curve of a 2−cycle, as $f_L(0) = 1$ and $f_R(1) = 0$ holds, independently of the values of the other parameters a and γ.

We recall that any cycle of map f may undergo a BCB: this happens when a periodic point of the cycle collides with $x = 0$ from the left side and thus its image is a periodic point on the right side $(0, 1]$ colliding with $x = 1$. As described above, when $b = -1$ a cycle of period two undergoes a border collision as

$$f_R \circ f_L(0) = b + 1 = 0. \tag{8}$$

However, we still have not clarified if this cycle exists for values of b smaller or larger than -1, and if the 2−cycle undergoing the border collision is the unique 2−cycle or not, attracting or repelling. We shall clarify these properties in the next sections, describing the dynamics in this range.

From the properties of the functions $f_L(x)$ and $f_R(x)$ described above, we can consider the interval $(-\infty, 1]$ (range of the map f). Moreover, for $b < -1$ any point of $(0, 1]$ is mapped into I_L in one iteration. That is, in the itinerary of any trajectory the symbol R is necessarily followed by L (at least one L). Thus the only possible basic cycles are those with the symbolic sequence RL^n, and they all exist for any $n \geq 1$, in suitable parameter regions. Indeed, let $0 < x_0 < 1$ be a point of I_R, then when b is very small it is $f_R(x_0) \ll 0$, and it takes many iterations by f_L for the trajectory in order to reach the right side I_R again. At $b = -1$ a 2−cycle RL exists, and as b tends to $-\infty$ the cycles RL^n must exist as well, for any $n \geq 1$.

A first property is immediate:

Property 1. *The appearance/disappearance of cycles of map f in the considered range is either due to a BCB or to a smooth bifurcation related to the eigenvalue* $+1$.

Proof. Since f consists of two disjoint increasing branches, the eigenvalue of any cycle is necessarily positive, and since $f_L(x)$ is affine and $f_R(x)$ concave, a bifurcation with eigenvalue $+1$ can only be related to a fold type. □

In order to detect the bifurcation curves associated with a cycle having symbolic sequence RL^n for $n > 1$, let us recall that for the repeated application of the linear part we can take advantage of the following formula:

$$f_L^n(x) = a^n x + a^{n-1} + a^{n-2} + \cdots + 1 = a^n x + \frac{1 - a^n}{1 - a} \tag{9}$$

(in the case $a = 1$ the indetermined ratio $(1 - a^n)/(1 - a)$ is obviously to be substituted by n), and considering $0 < x_0 \leq 1$ a cycle with symbolic sequence RL^n can be detected as a fixed point of the composite function

$$F_{RL^n}(x) := f_L^n \circ f_R(x) = \frac{a^n b}{x^\gamma} + \frac{1 - a^{n+1}}{1 - a} \tag{10}$$

so that the equation related to the fixed points becomes

$$\frac{a^n b}{x^\gamma} + \frac{1 - a^{n+1}}{1 - a} = x. \tag{11}$$

It is worth to note that this equation gives the fixed points of the smooth function $F_{RL^n}(x)$ but such points define true cycles of map f in (3) only when the solutions belong to the interval $(0, 1]$.

In general, the Equation (11) cannot be solved in explicit form, and it may also have two admissible solutions. Indeed, when this happens it is associated with a smooth fold bifurcation. The eigenvalue of a basic cycle RL^n is given by

$$F'_{RL^n}(x_0) = \frac{-b\gamma}{x_0^{\gamma+1}} a^n, \tag{12}$$

where x_0 is the periodic point on the right side, and taking into account that at a fold bifurcation two fixed points are merging in one point, say $x^*_{RL^n}$, and that $F'_{RL^n}(x^*_{RL^n}) = 1$, from (12) we obtain the condition

$$x^*_{RL^n} = (-b\gamma a^n)^{1/(\gamma+1)}. \tag{13}$$

By substituting this expression into (11), the equation of the fold bifurcation of the function F_{RL^n} (a curve in the parameter plane (a, b)) is obtained, given by

$$\Phi_{RL^n} : b = -\frac{1}{\gamma a^n} \left(\frac{1 - a^{n+1}}{1 - a} \frac{\gamma}{\gamma + 1} \right)^{\gamma+1} \tag{14}$$

which is a true fold bifurcation for a cycle of f only when $x^*_{RL^n}$ in (13) belongs to the interval $(0, 1]$, that is, only for

$$b \geq -\frac{1}{\gamma a^n}. \tag{15}$$

For any $\gamma > 0$ the curve Φ_{RL^n} in (14) is increasing and concave, as well as the curve of equation $b = -1/\gamma a^n$ related to (15). The two curves are intersecting in a particular point, say $(\bar{a}_n, \bar{b}_n)$, where $\bar{a}_n$ is the solution of the equation

$$\frac{1 - a^{n+1}}{1 - a} = 1 + \frac{1}{\gamma} \tag{16}$$

or, equivalently,

$$a \frac{1 - a^n}{1 - a} = \frac{1}{\gamma} \tag{17}$$

and

$$\bar{b}_n = -\frac{1}{\gamma \bar{a}_n}. \tag{18}$$

It follows that the curve Φ_{RL^n} in (14) is related to a fold bifurcation of our map f only for $a \le \bar{a}_n$ (as for $a > \bar{a}_n$ the fold bifurcation does not take place, as used in PWS systems we can say that it is virtual).

A cycle with symbolic sequence RL^n can undergo a border collision when the periodic point on the right side (which must satisfy Equation (11)) collides with $x = 1$. By substituting $x = 1$ into (11), the equation of the bifurcation $f_L^n \circ f_R(1) = 1$ leads, for any $n \ge 1$, to the explicit expression for the BCB curve B_{RL^n} as follows[3]:

$$B_{RL^n} : b = -\frac{1 - a^n}{a^{n-1}(1 - a)}. \tag{19}$$

We can notice that the two curves of equation Φ_{RL^n} in (14) (only for $a \le \bar{a}_n$) and the BCB curve B_{RL^n} in (19) have a common point in $(\bar{a}_n, \bar{b}_n)$. In fact, it is easy to see that the curve of equation $b = -1/\gamma a^n$ intersects B_{RL^n} in (19) when the condition in (17) holds. It follows that the point $(\bar{a}_n, \bar{b}_n)$ is a codimension-two point at which a cycle RL^n undergoes simultaneously a fold bifurcation and a BCB bifurcation, and the merging fixed points and colliding point satisfy $x^*_{RL^n} = (-b\gamma a^n)^{1/(\gamma+1)} = 1$ in which $F'_{RL^n}(1) = 1$ holds.

We can so prove that for $a < \bar{a}_n$ a fold bifurcation curve is always 'below' the related BCB curve, when the parameters (a, b) belong to the BCB curve B_{RL^n} then a periodic point is merging with $x = 1$, and it is $F'_{RL^n}(1) = -b\gamma a^n = \gamma a(1 - a^n)/(1 - a)$. As shown above at $a = \bar{a}_n$ it holds $F'_{RL^n}(1) = 1$, while:

(i.1) for $0 < a < \bar{a}_n$ it is $F'_{RL^n}(1) < 1$, which means that the colliding cycle is attracting (and, as we shall see, the fold bifurcation curve Φ_{RL^n} at the same value of a must have been occurred before, at a smaller value of b);

(i.2) for $\bar{a}_n < a \le 1$ it is $F'_{RL^n}(1) > 1$, which means that the colliding cycle is repelling (the cycle does not exist at smaller values of b as for $a > \bar{a}_n$ the fold bifurcation does not occur, it is virtual).

Some more properties on the codimension-two points can be obtained from the equation in (17) considering $1 - a^n/1 - a = a^{n-1} + \cdots + a + 1$ that leads to the equation

$$a^n + a^{n-1} + \cdots + a - \frac{1}{\gamma} = 0 \tag{20}$$

from which it follows that increasing n the solutions are decreasing values (i.e. $\bar{a}_{n+1} < \bar{a}_n$). While from (17) considering $a - a^{n+1} = (1 - a)/\gamma$ we have the equation

$$a^{n+1} - a\left(1 + \frac{1}{\gamma}\right) + \frac{1}{\gamma} = 0 \tag{21}$$

from which it follows that increasing n the decreasing solutions have as limit value a constant a_∞ which can be obtained from (21) as $n \to \infty$, leading to

$$a_\infty = \frac{1}{\gamma + 1}. \tag{22}$$

We have so proved the following.

Proposition 1 (codimension-two point). *Let $b \leq -1$, $\gamma > 0$. For any $n \geq 1$ the BCB curve B_{RL^n} in* (19) *and the fold bifurcation curve Φ_{RL^n} in* (14) *have a* (*codimension-two*) *contact point in* $(\bar{a}_n, \bar{b}_n)$ where $\bar{b}_n = -1/\gamma\bar{a}_n$ and $\bar{a}_n$ *satisfies the equation in* (17). *For any $n > 1$, the inequalities*

$$a_\infty = \frac{1}{\gamma+1} < \bar{a}_{n+1} < \bar{a}_n < \bar{a}_1 = \frac{1}{\gamma} \tag{23}$$

hold. For $a > \bar{a}_n$ the fold bifurcation in (14) *does not occur.*

For the BCB curve of the 2−cycle $RL\,(n = 1)$, at $b = -1$, the condition in (17) leads to the codimension-two point at

$$\bar{a}_1 = \frac{1}{\gamma}, \quad \bar{b}_1 = -1 \tag{24}$$

while for $n = 2$, related to the BCB curve of the 3−cycle RL^2, from (17) we obtain

$$\bar{a}_2 = \frac{1}{2}\left(-1 + \sqrt{1 + \frac{4}{\gamma}}\right), \quad \bar{b}_2 = -\frac{1}{\gamma\bar{a}_2}.$$

In the particular case $\gamma = 0.5$, which we use in many figures, this leads to $\bar{a}_2 = 1$ and $\bar{b}_2 = -2$.

In [50], for $\gamma = 0.5$, the contact point $(\bar{a}_n, \bar{b}_n)$ is called 'cross point', but the two curves Φ_{RL^n} and B_{RL^n} are not crossing (we have numerical evidence that they are tangent in the codimension-two point $(\bar{a}_n, \bar{b}_n)$), in any case the fold bifurcation Φ_{RL^n} does not occur for $a > \bar{a}_n$. In the following sections we shall clarify the role of such a particular point in terms of the dynamics. Moreover, we shall generalize a similar property occurring for the BCB curve of any admissible cycle of f.

We can see that the equations of the BCB curves B_{RL^n} do not depend on the parameter γ, differently from the fold bifurcation curves Φ_{RL^n}. It is shown in Figure 2 that for small values of γ there are many periodicity regions of attracting cycles, see also Figure 3(a), at a smaller value of γ ($\gamma = 0.1$). While only a few are visible for large values of γ, see Figure 3(c). The BCB curves B_{RL^n} (in black) are reported on the right side and are unchanged in Figure 3(b),(c), while the fold bifurcation curves Φ_{RL^n} (in red) change very much, and the limit value a_∞ given in (22) of the codimension-two points decreases as γ increases. The fold bifurcation curves Φ_{RL^j} for $j = 1, \ldots, 10$ shown in Figure 3(a) all have the codimension-two points which are in the region $a > 1$, while for $\Phi_{RL^{11}}$ it occurs for $a < 1$. Differently, for $\gamma > 1$ it is $\bar{a}_1 = 1/\gamma < 1$ so that all the codimension-two points $\bar{a}_n$ belong to the range $0 < a < 1$.

We can state the following.

Proposition 2 (crossing the BCB curves of basic cycles). *Let $\gamma > 0$. At a fixed value of $a \in (0, 1]$, as b increases from $-\infty$ to -1, all the bifurcation curves of basic cycles RL^n are crossed for decreasing values of n as follows:*

If $0 < a \leq a_\infty = 1/(\gamma + 1)$, then for any $n \geq 1$ a fold bifurcation curve Φ_{RL^n} is crossed first, leading to a pair of basic cycles RL^n, followed by a BCB curve B_{RL^n} crossing which the attracting cycle RL^n disappears while the repelling one persists.

If $a_\infty = 1/\gamma + 1 < a < \bar{a}_1 = 1/\gamma$, then a suitable integer $m \geq 1$ exists such that for any $n < m$ the basic cycles RL^n appear repelling crossing a BCB B_{RL^n}, while for $m \leq$

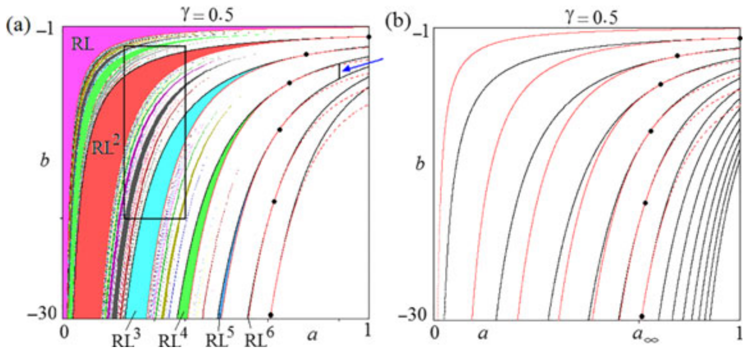

Figure 2. Two-dimensional bifurcation diagram in the $(a,\, b)$ parameter plane at $\gamma = 0.5$. The periodicity regions of the basic cycles RL^n are evidenced. The lower boundary (in red) is a fold bifurcation curve Φ_{RL^n} while the upper boundary (in black) is a BCB curve B_{RL^n}. In (b) only the bifurcation curves Φ_{RL^n} and B_{RL^n} are drawn by using the equations given in (14) and (19), respectively. The codimension-two points $(\bar{a}_n, \bar{b}_n)$ are marked with black circles, $\bar{a}_1 = 2, \bar{a}_2 = 1, \ldots, a_\infty = 2/3$. The region evidenced in (a) with a rectangle will be investigated below, as well as the segment evidenced by the blue arrow at $a = 0.9$.

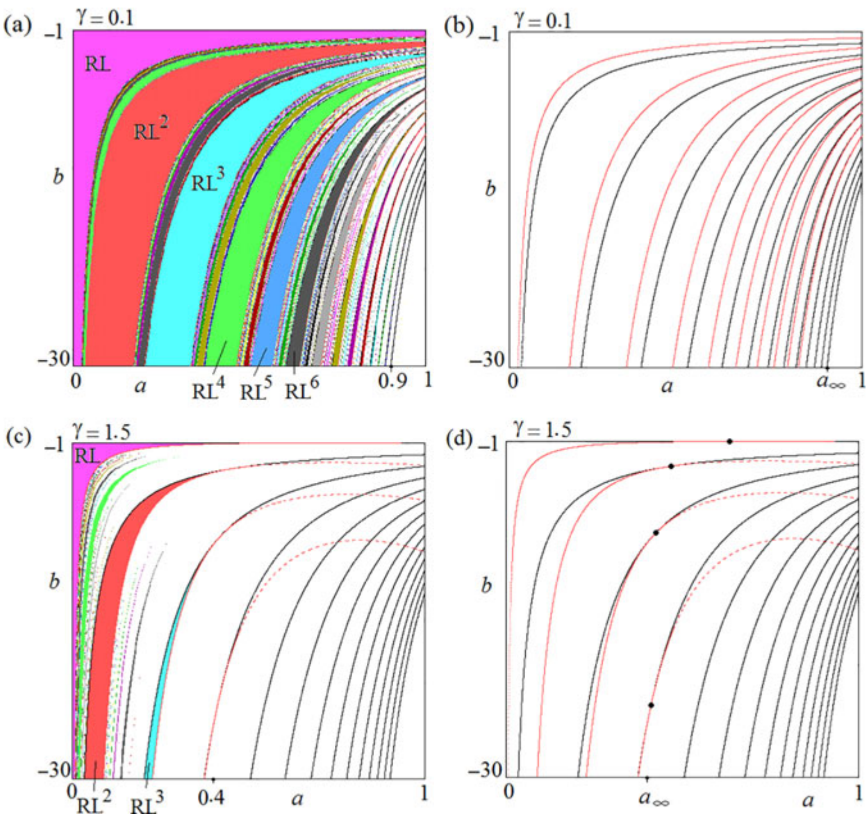

Figure 3. Two-dimensional bifurcation diagram in the $(a,\, b)$ parameter plane at $\gamma = 0.1$ in (a) and (b), $\bar{a}_1 = 10$ and $a_\infty = 0.9$. At $\gamma = 1.5$ in (c) and (d), $\bar{a}_1 = 0.\bar{6}$ and $a_\infty = 0.4$. In red are the fold bifurcation curves Φ_{RL^n} while in black are the BCB curves B_{RL^n}.

$n \leq 1$ a fold bifurcation curve Φ_{RL^n} is crossed first, leading to a pair of basic cycles RL^n, followed by a BCB B_{RL^n} crossing which the attracting cycle RL^n disappears. Once created, all the repelling basic cycles persist.

If $a > \bar{a}_1 = 1/\gamma$, then for any $n \geq 1$ the basic cycles RL^n appear repelling crossing a BCB B_{RL^n} and persist.

Proof. From the equations of the BCB curves of basic cycles B_{RL^n} in (19) it follows that at a fixed value of $a \in (0, 1]$, as b increases from $-\infty$ to -1 all the BCB curves B_{RL^n} are necessarily crossed, while for the fold bifurcation curves Φ_{RL^n} we have to take into account the codimension-two point $(\bar{a}_n, \bar{b}_n)$. From Proposition 1 we have that the values $\bar{a}_n$ belong to a decreasing sequence with limit value a_∞, so that at fixed value of a, as b increases from $-\infty$ to -1, different cases may occur. When $a < \bar{a}_n$ the fold bifurcation curve Φ_{RL^n} is crossed and since $F'_{RL^n}(1) < 1$ (from point (i.1) above) then the BCB occurring crossing B_{RL^n} involves an attracting cycle. When $a > \bar{a}_n$ the fold bifurcation curve Φ_{RL^n} is not crossed and since $F'_{RL^n}(1) > 1$ (from point (i.2) above) then the BCB occurring crossing B_{RL^n} involves a repelling cycle. The proof of this proposition, showing the order in which the bifurcations occur increasing b and the persistence with b, will be completed in Section 3 (see Properties 7 and 8). □

To end the proof of Proposition 2, as well as to prove other properties related to the codimension-two point on the BCB of any cycle, we make use of the first return map of f in a suitable interval, often useful in PWS systems [24], whose existence and construction are given in the next section.

3. First return map

From the properties of map f described in the previous section we can prove the following.

PROPOSITION 3 (MAP $F_r(x)$). *Let $0 < a \leq 1$, $b < -1$, $\gamma > 0$. The dynamics of map f in* (3) *can be described by using the first return map $F_r(x)$ in the interval $I = [0, 1]$. $F_r(x)$ is a discontinuous map with infinitely many branches defined as follows:*

$$F_r(x) := \begin{cases} F_{RL^{\bar{n}}}(x) = f_L^n \circ f_R(x) & \text{if } \xi_{\bar{n}+1} \leq x \leq 1 \\ F_{RL^{\bar{n}+1}}(x) = f_L^{\bar{n}+1} \circ f_R(x) & \text{if } \xi_{\bar{n}+2} \leq x < \xi_{\bar{n}+1} \\ \vdots & \vdots \\ F_{RL^{\bar{n}+j}}(x) = f_L^{\bar{n}+j} \circ f_R(x) & \text{if } \xi_{\bar{n}+j+1} \leq x < \xi_{\bar{n}+j}, \\ \vdots & \vdots \end{cases} \tag{25}$$

where $\bar{n} \geq 0$ is the smallest integer for which

$$f_L^{\bar{n}} \circ f_R(1) \in [0, 1) \tag{26}$$

with

$$F_{RL^m}(x) = \frac{a^m b}{x^\gamma} + \frac{1 - a^{m+1}}{1 - a} \tag{27}$$

and the discontinuity points are preimages of the origin given by

$$\xi_{m+1} = f_R^{-1} \circ f_L^{-m}(0) = \left(\frac{-b}{(a^m - 1/a^m(a-1)) + 1} \right)^{1/\gamma} \tag{28}$$

which have as limit value, for $m \to \infty$, the point $x = 0$. For any $m \geq \bar{n} + 1$, $F_{RL^m}(\xi_{m+1}) = 0$ and $F_{RL^m}(\xi_m) = 1$ hold, while the rightmost branch satisfies $F_{RL^{\bar{n}}}(\xi_{\bar{n}+1}) = 0$. The case

$$f_L^{\bar{n}} \circ f_R(1) = 0 \tag{29}$$

corresponds to the BCB of a basic cycle with symbolic sequence $RL^{\bar{n}+1}$.

Proof. Since for $b < -1$ it is $f_R(1) = 1 + b < 0$, it follows that a point $x \in (0, 1]$ on the right side is mapped by f_R in the left side, and in a finite number of iterations by f_L the trajectory is mapped into the right side again. Thus, it is possible to define the first return map of $f(x)$ in the interval $[0, 1]$. Recall that the first return map $F_r(x)$ is defined as the function which associates with any point $x > 0$ associates the first non-negative value of the trajectory of x, that is, the first value satisfying $f^n(x) \geq 0$, which in our case necessarily satisfies $f^n(x) \in [0, 1)$. We also notice that when a point ξ satisfies $f^n(\xi) = 0$, then it is also $f_L \circ f^n(\xi) = 1$. So, given a value of $b < -1$, let $\bar{n} \geq 0$ be the smallest integer for which (26) holds.

In the generic case at which $f_L^{\bar{n}} \circ f_R(1) \in (0, 1)$, since the function $f_L^{\bar{n}} \circ f_R(x)$ is increasing we have that decreasing x from 1 the value of $f_L^{\bar{n}} \circ f_R(x)$ decreases as well, so that the first return map must be defined as $F_r(x) = f_L^{\bar{n}} \circ f_R(x)$ for all the points of the interval $[\xi_{\bar{n}+1}, 1]$ where the point $\xi_{\bar{n}+1}$ is such that

$$f_L^{\bar{n}} \circ f_R(\xi_{\bar{n}+1}) = 0 \tag{30}$$

that is, $\xi_{\bar{n}+1}$ is a preimage of the origin of rank $(\bar{n} + 1)$, as taking the inverses in (30) we have $\xi_{\bar{n}+1} = f_R^{-1} \circ f_L^{-\bar{n}}(0)$. By applying f_L on both sides in (30), we also have that

$$f_L^{\bar{n}+1} \circ f_R(\xi_{\bar{n}+1}) = 1. \tag{31}$$

It follows that in a left neighbourhood of the point $\xi_{\bar{n}+1}$ the first return map must be defined as $F_r(x) = f_L^{\bar{n}+1} \circ f_R(x)$, up to a point $\xi_{\bar{n}+2}$ in which it holds $f_L^{\bar{n}+1} \circ f_R(\xi_{\bar{n}+2}) = 0$, and so on. We can state that, for any $j \geq 0$, the first return map is defined by branches of this kind:

$$F_{RL^{\bar{n}+j}}(x) = f_L^{\bar{n}+j} \circ f_R(x) \tag{32}$$

separated by discontinuity points (preimages of the origin).

The number of branches is necessarily infinite. In fact, as described above, we have to consider the preimages of the origin obtained, for any $j \geq 0$, as follows:

$$\xi_{\bar{n}+j+1} = f_R^{-1} \circ f_L^{-(\bar{n}+j)}(0). \tag{33}$$

Considering the inverse functions

$$f_R^{-1}(y) = \left(\frac{b}{y-1} \right)^{1/\gamma}, \quad f_L^{-1}(y) = \frac{y-1}{a} \tag{34}$$

the iterative application of the inverse on the left side leads to

$$f_L^{-k}(y) = \frac{y}{a^k} - \frac{a^k - 1}{a^k(a-1)} \tag{35}$$

so that from (33), by using (34), we have explicitly

$$\xi_{\bar{n}+j+1} = \left(\frac{-b}{(a^{\bar{n}+j} - 1)/(a^{\bar{n}+j}(a-1)) + 1} \right)^{1/\gamma}. \tag{36}$$

The points $f_L^{-(\bar{n}+j)}(0)$ exist on the left side for any $j \geq 0$. This is because the affine function f_L is increasing with slope $a \leq 1$, so that as $j \to \infty$ the points $f_L^{-(\bar{n}+j)}(0)$ tend to $-\infty$ and thus $f_R^{-1} \circ f_L^{-(\bar{n}+j)}(0)$ exist for any $j \geq 0$ and tend to 0.

The first return map is thus defined by infinitely many branches separated by discontinuity points, preimages of the origin of rank $(\bar{n}+j)$, denoted by $\xi_{\bar{n}+j}$. Namely by $F_r(x) = F_{RL^{\bar{n}}}(x) = f_L^{\bar{n}} \circ f_R(x)$ for $\xi_{\bar{n}+1} \leq x \leq 1$, and $F_r(\xi_{\bar{n}+1}) = 0$; by $F_r(x) = F_{RL^{\bar{n}+1}}(x) = f_L^{\bar{n}+1} \circ f_R(x)$ for $\xi_{\bar{n}+2} \leq x < \xi_{\bar{n}+1}$ which is a continuous increasing branch from 0 to 1, as $F_r(\xi_{\bar{n}+2}) = f_L^{\bar{n}+1} \circ f_R(\xi_{\bar{n}+2}) = 0$ and $F_r(\xi_{\bar{n}+1}) = f_L^{\bar{n}+1} \circ f_R(\xi_{\bar{n}+1}) = 1$, and so on. This holds for any integer. That is, for any j, $F_r(x) = F_{RL^{\bar{n}+j}}(x) = f_L^{\bar{n}+j} \circ f_R(x)$ is a continuous increasing branch for $\xi_{\bar{n}+j+1} \leq x < \xi_{\bar{n}+j}$, taking values from 0 to 1, as $F_r(\xi_{\bar{n}+j+1}) = f_L^{\bar{n}+j} \circ f_R(\xi_{\bar{n}+j+1}) = 0$ and $F_r(\xi_{\bar{n}+j}) = f_L^{\bar{n}+j} \circ f_R(\xi_{\bar{n}+j}) = 1$.

In the particular case in which the condition in (26) occurs as $f_L^{\bar{n}} \circ f_R(1) = 0$, we also have

$$F_{RL^{\bar{n}+1}}(x) = f_L^{\bar{n}+1} \circ f_R(1) = 1 \tag{37}$$

and thus it corresponds to the BCB of a cycle of period $(\bar{n}+2)$ with symbolic sequence $RL^{\bar{n}+1}$. We define $F_r(1) = f_L^{\bar{n}} \circ f_R(1) = 0$ in the single point $\xi_{\bar{n}+1} = 1$ and then $F_r(x) = f_L^{\bar{n}+1} \circ f_R(x)$ in $[\xi_{\bar{n}+2}, \xi_{\bar{n}+1})$. Notice that in this case the range of $F_{RL^{\bar{n}+1}}(x) = f_L^{\bar{n}+1} \circ f_R(x)$ in $[\xi_{\bar{n}+2}, 1]$ is exactly $[0, 1]$, and similarly, in all the other branches of $F_r(x)$ which are defined as above. □

An example is shown in Figure 4. For $x = 1$ we have $f_L^4 \circ f_R(1) > 0$ so that $\bar{n} = 4$. The preimages of the origin on the left side are infinitely many and a few of the infinitely many branches of $F_r(x)$ can be seen in the enlargement.

In terms of the preimages of the origin the condition in (29) also corresponds to

$$1 = f_R^{-1} \circ f_L^{-\bar{n}}(0) \tag{38}$$

that is, by using the definition in (33) with $j = 0$,

$$\xi_{\bar{n}+1} = 1. \tag{39}$$

Obviously, the equation of the border collision $\xi_{\bar{n}+1} = 1$ from (36) with $j = 0$ coincides with the BCB curve detected in (19) for $n = \bar{n}+1$. In fact, considering $n = \bar{n}+1$ in (36) we have

$$1 = \left(\frac{-b}{(a^n - 1)/(a^n(a-1)) + 1} \right)^{1/\gamma}$$

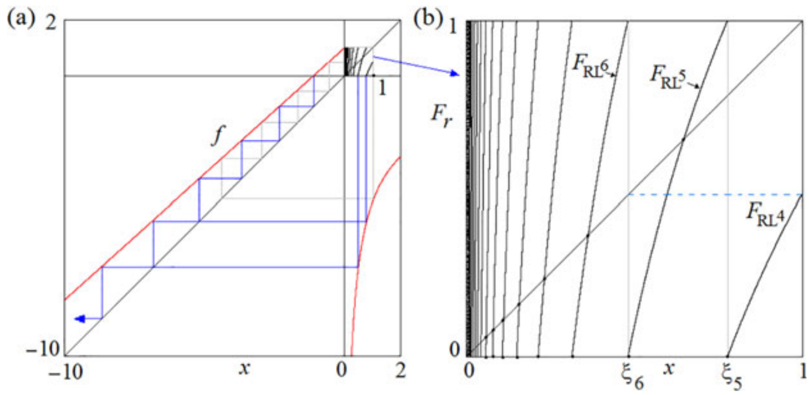

Figure 4. Map f at $\gamma = 0.5$, $a = 0.9$, $b = -5.5$, for which it is $\bar{n} = 4$. In the enlargement its first return map $F_r(x)$.

and we can see that the dependence on the parameter γ can be ignored, as it is equivalent to

$$-b = \frac{a^n - 1}{a^n(a-1)} + 1$$

and rearranging we obtain the equation in (19).

For the example shown in Figure 4 the BCB related to $F_{RL^4}(1) = f_L^4 \circ f_R(1) = 0$ (BCB of the cycle with symbolic sequence RL^5) occurs at a smaller value of b, and it is shown in Figure 5(a) (from (19) with $a = 0.9$ and $n = 5$ we obtain $b = -6.24$). It can be seen that $\xi_5 = 1$ and increasing b, ξ_5, decreases and one more branch appears in the definition of the first return map $F_r(x)$, given by the function $F_{RL^4}(x) = f_L^4 \circ f_R(x)$ (as shown in Figure 4 (b)). As b is further increased the value $F_{RL^4}(1) = f_L^4 \circ f_R(1)$ of the rightmost branch of F_r increases, and when $F_{RL^4}(1) = f_L^4 \circ f_R(1) = 1$, from (37) the BCB of the cycle with symbolic sequence RL^4 occurs, as shown in Figure 5(b) (from (19) with $a = 0.9$ and $n = 4$ the bifurcation value $b = -4.72$ is obtained).

From the definition of the first return map we can have immediately some properties:

PROPERTY 2. *The itinerary of any point for the map f consists of sequences associated with the symbols RL^j for $j \geq \bar{n}$.*

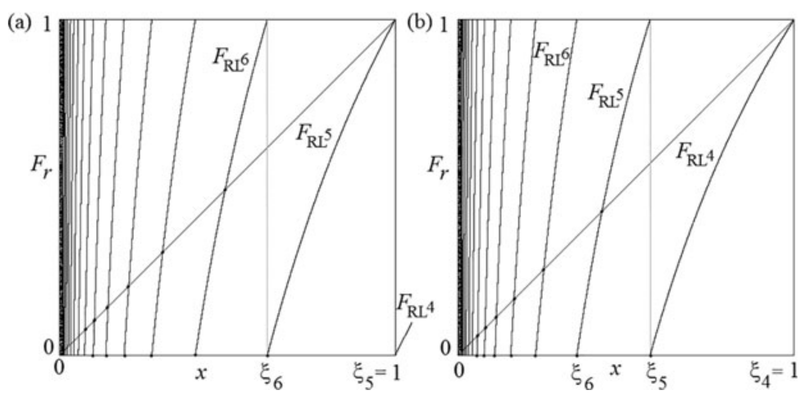

Figure 5. First return map $F_r(x)$ at $\gamma = 0.5$, $a = 0.9$. In (a) $b = -6.24$, BCB of the basic cycle RL^5. In (b) $b = -4.72$ BCB of the basic cycle RL^4.

Proof. This is because the trajectories of f are related one-to-one to those of the first return map $F_{\rm r}(x)$, and thus to the symbolic sequences RL^j related to the branches F_{RL^j} for $j \geq \bar{n}$ defining $F_{\rm r}(x)$. □

Property 3. *At a fixed value of a, the number of discontinuity points of the first return map* $F_{\rm r}$, *preimages* ξ_j *of the origin in the interval* $(0, 1]$, *increases as b increases from* $-\infty$ *to* -1.

Proof. This is because the smallest integer $\bar{n}$ from its definition decreases as b increases. □

Property 4. *Each component of the first return map* $F_{\rm r}$, $F_{RL^n}(x)$, *is continuous in the proper domain, increasing and concave*.

Proof. In fact, for $x > 0$ it is $F'_{RL^n}(x) > 0$ and $F''_{RL^n}(x) < 0$, which follows immediately from the explicit expressions:

$$F'_{RL^n}(x) = a^n f'_R(x) = \frac{-b\gamma}{x^{\gamma+1}} a^n > 0, \quad F''_{RL^n}(x) = \frac{b\gamma(\gamma+1)}{x^{\gamma+1}} a^n < 0$$

and the same properties hold for any composition of the functions $F_{RL^n}(x)$. □

Property 5. *All the branches* $F_{RL^n}(x)$ *have range from* 0 *to* 1, *except at most the first branch to the right side (called rightmost branch), for which* $F_{RL^{\bar{n}}}(1)$ *increases with b*.

Proof. This follows by construction of the first return map $F_{\rm r}(x)$, the range of $F_{RL^n}(x)$ for any $n > \bar{n}$ extends from 0 to 1, while from $F_{RL^{\bar{n}}}(1) = f_L^{\bar{n}} \circ f_R(1) = a^{\bar{n}} b + (1 - a^{\bar{n}+1})/(1 - a)$ we have that the endpoint $F_{RL^{\bar{n}}}(1)$ increases with b. □

Property 6. *Infinitely many repelling basic cycles of f necessarily exist.*

Proof. In fact, for any $j > \bar{n}$ all the branches $F_{RL^j}(x)$ have repelling fixed points x_{RL^j}. The fact that each fixed point must be repelling comes from Properties 4 and 5 above. □

While in the rightmost branch different cases may occur. That is, for $0 < F_{RL^{\bar{n}}}(1) < 1$ either no fixed point exists, or a fold bifurcation with two merging fixed points, or a pair of fixed points exists, one attracting and one repelling.

In the example given in Figure 4, where $\bar{n} = 4$, the rightmost branch of $F_{\rm r}$ is defined by $F_{RL^4}(x) = f_L^4 \circ f_R(x)$. All the branches defined by $F_{RL^{4+k}} = f_L^{4+k} \circ f_R(x)$ exist for any $k \geq 1$ and intersect the diagonal, leading to the existence of repelling fixed points, which are cycles of f of periods $(5 + k)$ for any $k \geq 1$, and there are no other fixed points (i.e. no other basic cycles).

Property 7. *Once a repelling cycle of the first return map* $F_{\rm r}(x)$ *exists, it persists for any larger value of b*.

Proof. In fact, the discontinuity points ξ_j of the first return map $F_{\rm r}$ increasing b all persist, approaching $x = 0$, and thus also the branches all persist (and new ones may enter), leading to persistence of all the repelling fixed points of $F_{\rm r}$. The same property holds also for any repelling k–cycle ($k > 1$) of the first return map $F_{\rm r}$, as we can reason similarly for the k−th iterate $F_r^k(x)$. □

Property 8. *As b increases from* $-\infty$ to -1 *the BCB curves are crossed with decreasing order in the period of the basic cycles, and if a fold bifurcation* Φ_{RL^n} *occurs it must be crossed before the BCB curve* B_{RL^n}.

Proof. This is because (by construction) the smallest integer $\bar{n}$ decreases as b increases, thus the period of the smallest repelling fixed point of the first return map F_r decreases. Moreover, when $a < \bar{a}_n$ then increasing b the BCB B_{RL^n} (occuring when $F_{RL^n}(1) = 1$ holds) is crossed at the left side of the codimension-two point, and involves an attracting cycle (since $F'_{RL^n}(1) < 1$), which means that the fold bifurcation curve Φ_{RL^n} must be crossed at a smaller value of b. While when $a > \bar{a}_n$ the BCB curve is crossed (when $F_{RL^n}(1) = 1$ holds) at the right side of the codimension-two point so that the fold bifurcation curve Φ_{RL^n} is not crossed, and since $F'_{RL^n}(1) > 1$ the BCB involves a repelling cycle. □

The Properties 7 and 8 lead to a complete proof of Proposition 2 given in the previous section.

We have already stated that the codimension-two point $(\bar{a}_n, \bar{b}_n)$ on a BCB curve B_{RL^n} of a basic cycle (occuring when $f_L^n \circ f_R(1) = 1$) leads to a difference in the dynamic behaviour, and this also follows from the properties listed above, in particular Propery 8. Let us illustrate this difference in detail, by using the first return map.

Case $a \geq \bar{a}_n$. Considering the crossing of a BCB curve B_{RL^n} at a point (a, b) at the right side of the codimension-two point $(\bar{a}_n, \bar{b}_n)$, or at the point itself, that is for $a > \bar{a}_n$ or $a = \bar{a}_n$ and the value of b given in (19), we have that the slope of the branch $F_{RL^n}(x) = f_L^n \circ f_R(x)$ of $F_r(x)$ in the point $x = 1$ at the bifurcation value is larger than $1, F'_{RL^n}(1) > 1$, or $F'_{RL^n}(1) = 1$, which means that before the bifurcation (at a smaller value of b) we have the rightmost branch $F_r(x) = F_{RL^n}(x)$ which satisfies $F'_{RL^n}(1) > 1$. Increasing b this branch is approaching the diagonal from below and at the bifurcation the fixed point of $F_{RL^n}(x)$ appears, which did not exist before (repelling on its left side). The fixed point x_{RL^n} of the branch $F_r(x) = F_{RL^n}(x)$ for $\xi_{n+1} \leq x < \xi_n$ persists, repelling, after the BCB (and for any larger value of b), while the rightmost branch is given by $F_{RL^{n-1}}(x) = f_L^{n-1} \circ f_R(x)$ for $\xi_n \leq x \leq 1$ and is not intersecting the diagonal, for values of b close to the BCB value.

The case associated with the example in Figure 5(a), related to the crossing of the curve B_{RL^5}, is shown in Figure 6. In Figure 6(a)–(c) we can see the shape of the rightmost branch of $F_r(x)$ before, at (when $F_{RL^5}(x) = f_L^5 \circ f_R(1) = 1$, $F_r(1) = f_L^4 \circ f_R(1) = 0$) and after the bifurcation value.

In the particular case in which a BCB is crossed at its codimension-two point, say $a = \bar{a}_n$, from (23) we can state that increasing b from $\bar{b}_n$, each BCB curve B_{RL^j} for $n < j < 1$ is crossed on the left of the codimension-two point (and thus crossing also the fold bifurcation curve Φ_{RL^j}), while decreasing b from $\bar{b}_n$, each BCB curve B_{RL^j} for any

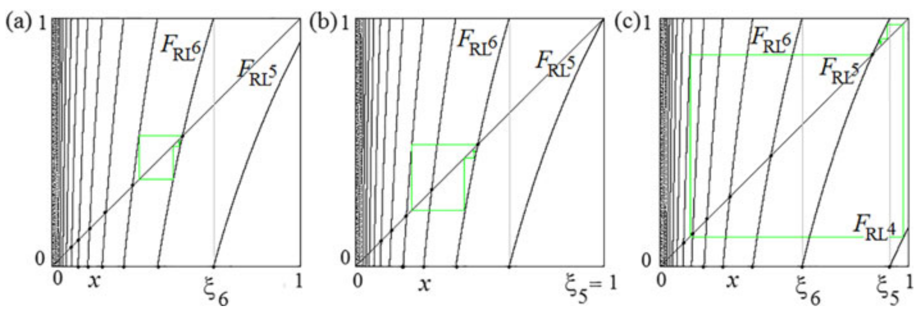

Figure 6. First return map $F_r(x)$ at $\gamma = 0.5$, $a = 0.9$. In (a) $b = -6.4$, before the BCB B_{RL^5}. In (b) $b = -6.24$, BCB of the basic cycle RL^5. In (c) $b = -6$ after the BCB. In green are evidenced some homoclinic orbits.

$j < n$ is crossed on the right of the related codimension-two point (and thus the fold bifurcation curve Φ_{RL^j} is not crossed).

Case $a < \bar{a}_n$. When a BCB curve B_{RL^n} is crossed on the left of its codimension-two point $(\bar{a}_n, \bar{b}_n)$, that is for $a < \bar{a}_n$ and the value of b given in (19), it must be considered in pair with a smooth fold bifurcation curve Φ_{RL^n}. In fact, in such a case at the bifurcation we have that the slope of the branch $F_{RL^n}(x) = f_L^n \circ f_R(x)$ of $F_r(x)$ in the point $x = 1$ satisfies $F'_{RL^n}(1) < 1$, which means that the colliding cycle is attracting. Before the bifurcation (at a smaller value of b) by continuity an attracting fixed point $x^s_{RL^n}$ must exist, and necessarily also a repelling one $x_{RL^n} < x^s_{RL^n}$, in the rightmost branch $F_r(x) = F_{RL^n}(x)$, which satisfies $F'_{RL^n}(1) < 1$. Increasing b the attracting cycle approaches $x = 1$, merging with it at the bifurcation value. After the bifurcation the attracting cycle no longer exists, while the repelling fixed point x_{RL^n} persists for any larger value of b. This proves that a smooth fold bifurcation of the increasing and concave rightmost branch $F_r(x) = F_{RL^n}(x)$ for $\xi_{n+1} \leq x \leq 1$ must occur at a smaller value of b (before the BCB), leading to the appearance of the two fixed points in the rightmost branch of $F_r(x)$.

An example associated with the crossing of the curve Φ_{RL^5} at $a = 0.6$ is shown for increasing b in Figure 7 and in Figure 8 the crossing of the BCB curve B_{RL^5} (at the same value $a = 0.6$). In Figure 8(a)–(c), are shown the cases before, at and after the BCB, respectively.

One more example in Figure 9 shows the crossing of the BCB curve B_{RL^3} at $a = 0.3$. We can see that increasing b the periodic point $x^s_{RL^3}$ of the attracting cycle RL^3 approaches

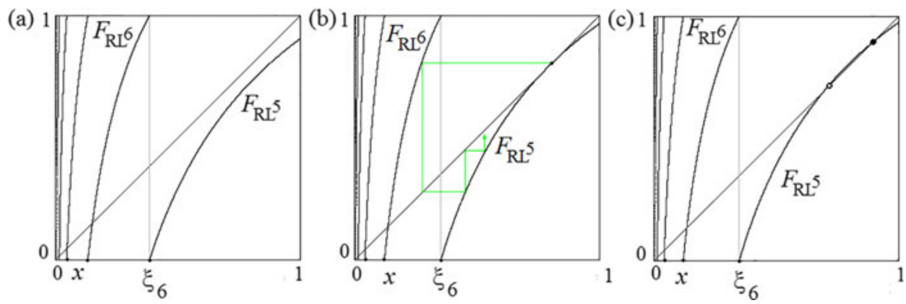

Figure 7. First return map $F_r(x)$ at $\gamma = 0.5$, $a = 0.6$. In (a) $b = -19$, before the fold bifurcation Φ_{RL^5}. In (b) $b = -18.21$, fold bifurcation of the basic cycle RL^5. A homoclinic orbit is shown in green. In (c) $b = -18.1$ after the fold bifurcation and two cycle RL^5 exist.

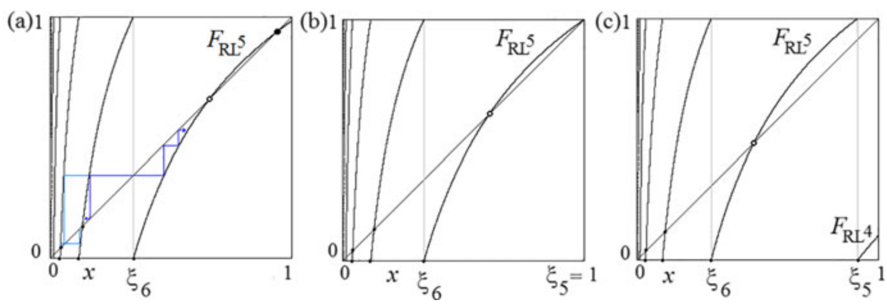

Figure 8. First return map $F_r(x)$ at $\gamma = 0.5$, $a = 0.6$. In (a) $b = -18$, before the BCB B_{RL^5}. In (b) $b = -17.79$, at the BCB. In (c) $b = -17$ after the BCB B_{RL^5}.

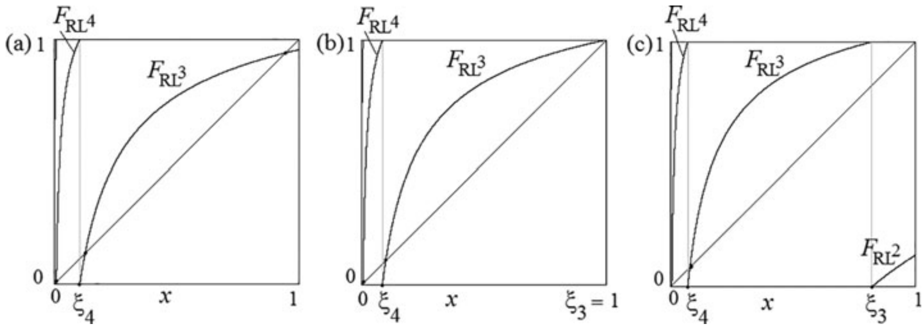

Figure 9. First return map $F_r(x)$ at $\gamma = 0.5$, $a = 0.3$. In (a) $b = -17$, before the BCB B_{RL^3}. In (b) $b = -15.4$, at the BCB. In (c) $b = -14$ after the BCB B_{RL^3}.

1 and disappears, while the repelling fixed point x_{RL^3} persists for any larger value of b, and even if it is difficult to see, the infinitely many branches of F_r in Figure 9 all exist ($F_{RL^j}(x)$, $j \geq 3$) leading to the existence of all the repelling cycles x_{RL^j} with symbolic sequence RL^j for any $j \geq 3$.

4. Homoclinc orbits, unbounded chaotic repellors and unbounded chaotic attractors, related to basic cycles

In this section we prove that map f is always chaotic in the range here considered, and that robust unbounded chaotic attractors may exist. Let us first prove the following.

Proposition 4 (unbounded chaotic sets).

(*a*) *All the repelling fixed points of the first return map* $F_r(x)$ *are homoclinic, and* f *has always an unbounded chaotic set in the interval* $(-\infty, 1]$.
(*b*) *Let* $F_{RL^n}(x)$ *be the rightmost branch of the first return map. If* $F'_{RL^n}(1) = -b\gamma a^n \geq 1$ *then* f *has* $(-\infty, 1]$ *as unbounded chaotic attractor.*

Proof. The proof is different depending on a generic case, in which the rightmost branch of the first return map $F_r(x)$, $F_{RL^n}(x)$, satisfies $F_{RL^n}(1) \in (0, 1)$, or the bifurcation case at which $F_{RL^n}(1) = 1$ holds.

Let us consider first a generic case, so that the parameters are not at the BCB of a basic cycle (case that is considered below). As already remarked (Property 6), independently of the shape of the rightmost branch $F_{RL^n}(x)$, we have that for any $j > n$ the functions $F_{RL^j}(x)$ intersect the diagonal, leading to repelling fixed points x_{RL^j}, corresponding to repelling basic cycles of f having the symbolic sequence RL^j, of which x_{RL^j} is the periodic point belonging to the right side. From the construction of the first return map F_r we know that between any pair of consecutive fixed points $x_{RL^{j+1}}$ and x_{RL^j} there exists a discontinuity point ξ_{j+1} (at which $F_{RL^j}(\xi_{j+1}) = 0$ and $F_{RL^{j+1}}(\xi_{j+1}) = 1$). Let us consider separately the fixed points x_{RL^j} for $j > n$ and $j = n$.

For any $j > n$, the fixed points x_{RL^j} are all homoclinic (i.e. are SBR), on both sides. In fact, from any fixed point x_{RL^j} a homoclinic orbit on the left side can be immediately obtained considering one preimage on its left side followed by infinitely many preimages on the right side (as shown in Figure 6(a)). That is,

$$F_{RL^j}^{-k} \circ F_{RL^{j+1}}^{-1}(x_{RL^j}),\ k \geq 1$$

is a homoclinic orbit of x_{RL^j} with homoclinic points on its left side. A similar reasoning applies also on the right sides of the fixed points $x_{RL^{j+1}}$ for any $j > n$, considering that

$$F^{-k}_{RL^{j+1}} \circ F^{-1}_{RL^j}(x_{RL^{j+1}}), \quad k \geq 1$$

is a homoclinic orbit of $x_{RL^{j+1}}$ with homoclinic points on its right side. While regarding the fixed point $x_{RL^{n+1}}$ of the branch $F_{RL^{n+1}}(x)$, we can consider m large enough such that $F^{-1}_{RL^m}(x_{RL^{n+1}})$ belongs to the range of $F_{RL^n}(x)$, defined for $\xi_{n+1} \leq x \leq 1$, so that

$$F^{-k}_{RL^{n+1}} \circ F^{-1}_{RL^n} \circ F^{-1}_{RL^m}(x_{RL^{n+1}}), \quad k \geq 1,$$

is a homoclinic orbit of $x_{RL^{n+1}}$ with homoclinic points on its right side (as shown in Figure 6(c)).

The existence of infinitely many SBR cycles proved so far shows that at least a chaotic set $\Lambda' \subset [0, 1]$ always exists. Moreover, as the homoclinic fixed points x_{RL^j} have $x = 0$ as limit point, homoclinic points exist which are as close as we want to the right side of the discontinuity point of f, $x = 0$, i.e. the origin is a limit point of homoclinic points. In terms of the map f this corresponds to an *unbounded chaotic set* Λ in the interval $(-\infty, 1]$.

For $j = n$ we have to consider the rightmost branch $F_{RL^n}(x)$, and (to show that all the repelling fixed points are homoclinic) the comments now differ depending on the possible cases for $F_{RL^n}(x)$.

(i) If $F'_{RL^n}(1) = -b\gamma a^n \geq 1$ then as x decreases from 1 in the interval $\xi_{n+1} \leq x \leq 1$, the first derivative $F'_{RL^n}(x) = -b\gamma a^n / x^{\gamma+1}$ increases, so that the slope is larger than 1 in all the points $\xi_{n+1} \leq x < 1$, and the rightmost branch $F_{RL^n}(x)$ does not intersect the diagonal. In this case the map $F_r(x)$ is chaotic in the interval $[0, 1]$, as proved in [37], and f is chaotic in the unbounded interval $(-\infty, 1]$, which is a chaotic attractor. We notice that the first return map $F_r(x)$ (and thus f) has all repelling cycles, which are SBR and homoclinic on both sides. The preimages of the fixed points x_{RL^j}, for any $j > n$, as well as of any repelling cycle, are dense in the interval $[0, 1]$. An example is shown in Figure 5(a).
(ii) If $F'_{RL^n}(1) = -b\gamma a^n < 1$ then as x decreases from 1 in the interval $\xi_{n+1} \leq x \leq 1$, the first derivative $F'_{RL^n}(x) = -b\gamma a^n / x^{\gamma+1}$ increases, so that it is possible to have points with slope 1. In this case, the rightmost branch $F_{RL^n}(x)$ may be below the diagonal, tangent to it or crossing it in two more fixed points x_{RL^n} and $x^s_{RL^n}$, with $\xi_{n+1} < x_{RL^n} < x^s_{RL^n} < 1$. An example is shown in Figure 7. We consider the different cases separately:
(ii.1) When the branch is below the diagonal there are no further fixed points (and the existing ones are homoclinic).
(ii.2) When the branch is tangent to the diagonal, two fixed points are merging in one point, $x^*_{RL^n}$, which is locally attracting on the right side, while repelling and already SBR on its left side. In fact, for example

$$F^{-k}_{RL^n} \circ F^{-1}_{RL^{n+1}}(x^*_{RL^n}), \quad k \geq 1$$

is a homoclinic orbit of $x^*_{RL^n}$ with homoclinic points on its left side (an example is shown in Figure 7(b)). Moreover, the interval $[x^*_{RL^n}, 1]$ is the immediate stable set of the fixed point $x^*_{RL^n}$ which is a Milnor attractor (or attractor in Milnor sense) [41]. That is, although not attracting all the points in a neighbourhood U of $x^*_{RL^n}$, the stable set S of $x^*_{RL^n}$, set of initial conditions whose trajectories converge to $x^*_{RL^n}$, given by all the preimages of any rank of the immediate stable set:

$$S = \bigcup_{k=0}^{\infty} F_r^{-k}([x^*_{RL^n}, 1])$$

has positive measure, and consists of intervals forming a fractal structure dense in $[0, 1]$. Its frontier ∂S (an invariant set) is the chaotic repellor $\Lambda' \subset [0, x^*_{RL^n}]$.

(ii.3) When the branch crosses the diagonal, two further fixed points exist in the rightmost branch of $F_r(x) = F_{RL^n}(x)$ for $\xi_{n+1} \le x \le 1$, x_{RL^n} and $x^s_{RL^n}$, with $\xi_{n+1} < x_{RL^n} < x^s_{RL^n} < 1$. Then the repelling fixed point x_{RL^n} is SBR on its left side, as for example

$$F_{RL^n}^{-k} \circ F_{RL^{n+1}}^{-1}(x_{RL^n}), \quad k \ge 1,$$

is a homoclinic orbit of x_{RL^n} with homoclinic points on its left side, while the interval $(x_{RL^n}, 1]$ is the immediate basin of the attracting fixed point $x^s_{RL^n}$. In this case the parameters belong to the periodicity region of an attracting basic cycle (as in the examples shown in Figure 8 and in Figure 9). Similarly to the previous case, the total basin B is given by all the preimages of any rank of the immediate basin

$$B = \bigcup_{k=0}^{\infty} F_r^{-k}((x_{RL^n}, 1]) \tag{40}$$

so that almost all the initial conditions have a trajectory which is converging to the attracting cycle, but due to the existence of infinitely many SBR cycles, the basin B consists of intervals forming a fractal structure dense in $[0, 1]$. The frontier ∂B of B includes all the repelling cycles of F_r in $[0, 1]$, and their limit points, that is, the invariant set ∂B is the chaotic repellor $\Lambda' \subset [0, x_{RL^n}]$.

Besides the generic case considered so far, let us now assume that $F_{RL^n}(1) = 1$. We know that the BCB of a cycle RL^n occurs, and we have already shown that if $F'_{RL^n}(1) \ge 1$ then the colliding cycle is repelling and homoclinic on its left side, and we can argue as in case (i) above (the first return map $F_r(x)$ consists of infinitely many branches, from 0 to 1, and all expanding, thus it is a full shift with infinitely many branches chaotic in $[0, 1]$). While if $F'_{RL^n}(1) < 1$ then the colliding cycle is attracting on its left side, and we can argue as in case (ii.3) above. □

Notice that in the case (ii.1) considered in the proof of Proposition 4, the dynamics may be further investigated, and it is possible that f is chaotic in the unbounded interval $(-\infty, 1]$, or a cycle may exist, attracting almost all the points, and thus f has an unbounded chaotic set Λ of zero measure in the interval $(-\infty, 1]$.

From Proposition 4 we have, in particular, the dynamic behaviour of the map at the BCB of a basic cycle RL^n, which differs depending on the point on the bifurcation curve. In fact, let us assume that $F_{RL^n}(1) = 1$ and let a parameter point (a, b) be fixed, and $b = b(B_{RL^n})$ at the bifurcation value given in (19). Then the interval $(-\infty, 1]$ is an unbounded chaotic attractor for any $a \ge \bar{a}_n$ and any $b \le b(B_{RL^n})$, so that the unbounded chaotic attractor is strongly persistent, i.e. robust (examples are shown in Figure 5(a),(b)). While for $a < \bar{a}_n$ the attracting fixed point $x^s_{RL^n}$ is merging with $x = 1$, disappearing for larger values of b (examples are shown in Figures 8(b) and 9(b)).

5. BCB and homoclinic bifurcations of a generic cycle of f (not basic)

The properties commented up to now for the basic cycles of f, which are fixed points of the first return map F_r, may be extended to other cycles of f, which clearly are also related to cycles of the first return map F_r.

A BCB of some cycle of f occurs whenever a preimage of the origin merges with the point $x = 1$, or, equivalently, when an iterate of the point $x = 1$ merges with the origin. In terms of the first return map, recall that the discontinuity points ξ_j of F_r are preimages of the origin. Thus, considering the rightmost branch of the first return map, say $F_r(x) = F_{RL^m}(x)$ for $x \in [\xi_{m+1}, 1]$, a BCB corresponds to the value in $x = 1$, $F_r(1) = F_{RL^m}(1)$, which merges with ξ_j or with a preimage of some rank of ξ_j, for $j \geq (m+1)$. So we can prove the following.

PROPOSITION 5 (BCB OF NOT BASIC CYCLES). *Any BCB involving a not basic cycle of f is represented by the condition*

$$F_r(1) = \xi_j^{-k}, \tag{41}$$

where ξ_j^{-k} is a preimage of some rank $k \geq 0$ of one discontinuity point of $F_r(x)$ (for $k = 0$ it is $\xi_j^{-k} = \xi_j$).

Let D be the first derivative of the function $F_{RL^j} \circ F_r^{k+1}(x)$ in the point $x = 1$. *Then the BCB leads to the appearance of a repelling cycle or to the disappearance of an attracting cycle depending on $D \geq 1$* or *$D < 1$, respectively. The case $D = 1$ corresponds to the codimension-two point at which the BCB occurs simultaneously with a fold bifurcation of cycles having the same symbolic sequence.*

Proof. As recalled above, by definition of F_r it must hold $F_r(1) = F_{RL^m}(1)$ for some integer $m \geq 1$, and thus $j \geq (m+1)$. Then let us consider separately the cases $k = 0$ and $k > 0$ in (41).

If $k = 0$ in (41) we have

$$F_{RL^m}(1) = \xi_j$$

and thus, by applying F_{RL^j} on both sides (being $F_{RL^j}(\xi_j) = 1$), we have

$$F_{RL^j} \circ F_{RL^m}(1) = 1$$

representing the BCB of a 2−cycle of the first return map F_r, and thus a cycle of f with symbolic sequence $RL^m RL^j$. For example, the value of the parameters used in Figure 4(a) is such that $F_r(1) = F_{RL^4}(1) = \xi_6$, thus the parameters belong to the BCB of equation $F_{RL^6} \circ F_{RL^4}(1) = 1$, representing the BCB of the 2−cycle of F_r between the two branches given by F_{RL^6} and F_{RL^4}, and a cycle of f with symbolic sequence RL^4RL^6.

If $k \geq 1$ in (41) let

$$\xi_j^{-k} = F_r^{-k}(\xi_j) = F^{-1}_{RL^{n_k}} \circ \cdots \circ F^{-1}_{RL^{n_1}}(\xi_j),$$

where $F^{-1}_{RL^{n_i}}$ represent the involved branches of the first return map, with $n_i \geq (m+1)$ for $i = 1, \ldots, k$. From

$$F_{RL^m}(1) = F_r^{-k}(\xi_j) = F^{-1}_{RL^{n_k}} \circ \cdots \circ F^{-1}_{RL^{n_1}}(\xi_j)$$

we can write

$$F_{RL^{n_1}} \circ \cdots \circ F_{RL^{n_k}} \circ F_{RL^m}(1) = \xi_j$$

and thus, by applying F_{RL^j} on both sides, we have

$$F_{RL^j} \circ F_{RL^{n_1}} \circ \cdots \circ F_{RL^{n_k}} \circ F_{RL^m}(1) = 1$$

representing the BCB of a cycle of the first return map F_r (of period $(k+2)$), and a cycle of f with symbolic sequence

$$RL^m RL^{n_k} \ldots RL^{n_1} RL^j. \tag{42}$$

We remark that for the first return map the BCB in (41) also corresponds to

$$F_r^{k+1}(1) = \xi_j$$

(that is, the trajectory of $x = 1$ for the first return map F_r is mapped to a discontinuity point of F_r), so that by using $F_{RL^{j-1}}(\xi_j) = 0$ and $F_{RL^j}(\xi_j) = 1$, it can also be written as

$$F_{RL^{j-1}} \circ F_r^{k+1}(1) = 0 \tag{43}$$

or equivalently

$$F_{RL^j} \circ F_r^{k+1}(1) = 1. \tag{44}$$

Moreover, from the above condition (43) or (44), we can notice that at each BCB the properties of the iterate of order $(k+2)$ of the first return map are similar (with obvious changes) to those commented in the previous sections for the BCB of a basic cycle of f, i.e. of a fixed point of F_r. That is, a suitable iterate of the first return map, $F_r^{k+2}(x)$, has the rightmost branch which is increasing from 0 to 1. Then what matters is the first derivative of that rightmost branch of the function F_r^{k+2} in the point $x = 1$. We recall that all the composite functions consist of branches which are monotone increasing and concave, so that the first derivative exists and is necessarily positive and decreasing. So, considering the first derivative D of the function $F_{RL^j} \circ F_{RL^{n_1}} \circ \cdots \circ F_{RL^{n_k}} \circ F_{RL^m}(x)$ in the point $x = 1$:

$$D = \frac{d}{dx}(F_{RL^j} \circ F_{RL^{n_1}} \circ \cdots \circ F_{RL^{n_k}} \circ F_{RL^m})(x)|_{x=1}, \tag{45}$$

we can state that increasing b the BCB curve is crossed on the right side, at or on the left side of its codimension-two point depending on $D > 1$, $D = 1$ and $D < 1$, respectively, and:

(i1) if $D \geq 1$ then it leads to the *appearance of a repelling cycle* (having the symbolic sequence given in (42));
(i2) if $D < 1$ then it leads to the *disappearance of an attracting cycle* (having the symbolic sequence given in (42)), leaving a repelling cycle with the same symbolic sequence, which exists for larger values of b, and this also means that at smaller values of b a fold bifurcation of cycles with this symbolic sequence must occur, leading to their existence. □

In the example shown in Figure 4(a) the parameters belong to the BCB of equation $F_{RL^6} \circ F_{RL^4}(1) = 1$, it is $F'_{RL^4}(1) > 1$ and $F'_{RL^6}(\xi_6) > 1$, thus $D = F'_{RL^6}(\xi_6)F'_{RL^4}(1) > 1$, so that a repelling cycle with symbolic sequence RL^4RL^6 appears as b increases. In this example we have $F'_r(x) > 1$ in all the points different from the discontinuity points, thus the first return map $F_r(x)$ is chaotic in the whole interval $[0, 1]$ and the dynamics of f is chaotic in $(-\infty, 1]$, as proved in [37].

Differently, when $D < 1$, we can argue as in Propostion 4 for the basic cycles: the suitable iterate $F_r^{k+2}(x)$ of the first return map has the point $x = 1$ which is attracting from its left side, with immediate basin $(x_\sigma, 1]$ where $\sigma = RL^m RL^{n_k} \dots RL^{n_1} RL^j$, and x_σ is homoclinic on its left side, while all the other repelling cycles are homoclinic on both sides. Thus, almost all the points are converging to 1, all the points except for those of a chaotic repellor in $[0, x_\sigma]$ for the first return map F_r, or in $(-\infty, x_\sigma]$ for map f.

From Proposition 5 we have that whenever the iterate of the point $x = 1$ in the first return map F_r merges with a preimage of a discontinuity point ξ_j then a BCB occurs. Similarly, let us now prove that whenever the iterate of the point $x = 1$ in the first return map F_r merges with a repelling periodic point, say x^*, then another homoclinic bifurcation (also called $\Omega-$explosion) of that cycle occurs, as stated in the following.

Proposition 6. *Whenever*

$$F_r^k(1) = x^* \tag{46}$$

for some $k \geq 1$, where x^* is a repelling periodic point of F_r, then another homoclinic bifurcation of that cycle occurs.

Proof. Since $1 = f_L(0)$ is a critical point of f (and also a critical point of F_r being the left value in all the discontinuity points of F_r)[4], whenever the point $F_r(1)$ is mapped into the periodic point of a repelling cycle, a homoclinic bifurcation of that cycle *may occur* (see [21,27]). However, due to the property of our system, that all the branches are monotone increasing and concave, it is necessarily true, that is a homoclinic bifurcation must necessarily occur. In fact, let us consider the value $b = b^*$ at which (46) holds. Thus at $b < b^*$ (close to b^*), it is $F_r^k(1) < x^*$ and we are before the bifurcation, while at $b > b^*$ (close to b^*), it is $F_r^k(1) > x^*$ so that we are after the bifurcation, and in this case new preimages of x^* appear (which did not exist before) considering $F_r^{-k}(x^*)$, leading to an explosion of new homoclinic orbits of the same cycle, which did not exist before (for $F_r^k(1) < x^*$). □

6. Bifurcations between two BCBs

As we have already remarked (Proposition 2), for any fixed value of a, $0 < a \leq 1$ (and $\gamma > 0$), as b increases from $-\infty$ to -1 all the BCB curves of the basic cycles RL^n are crossed. Thus, let us consider two consecutive BCB values of b, associated with two consecutive basic cycles, say $b(B_{RL^n})$ and $b(B_{RL^{n+1}})$, showing that for any b in between, $b(B_{RL^{n+1}}) < b < b(B_{RL^n})$, infinitely many other BCB values exist, and this is always true independently of the stability or instability of the colliding cycles. Depending on the value of the parameters, the BCB curves of not basic cycles may be associated with attracting or repelling cycles, depending on the value of the derivative D as defined above in (45) (Proposition 5).

Indeed, from the periodicity regions reported in Figures 2 and 3, it can be seen that not only the basic cycles with symbolic sequence RL^n have a stability region, but also many other regions exist, associated with attracting cycles with different symbolic sequence, especially at small values of γ.

In the next two sections we shall describe two different sequences, one between two consecutive BCB values associated with attracting basic cycles, and the other between two consecutive BCB values associated with repelling basic cycles.

6.1 BCBs of attracting basic cycles

Let us consider a value $a < \bar{a}_n$ and $b(B_{RL^{n+1}}) < b < b(B_{RL^n})$. Other attracting cycles (not basic) may exist for values of b in this interval, and the attracting cycle can only be created in pair with a companion repelling one having the same symbolic sequence, by a fold bifurcation of cycles of the first return map. Clearly, such a bifurcation may be also associated with fixed points of a suitable iterate F_r^m of the first return map, where $m > 1$ is the period of the cycle of F_r. When a pair of cycles appears by fold bifurcation, then the BCB curve of the related cycle, which necessarily occurs by increasing b, is associated with the merging of the rightmost periodic point of the attracting cycle with $x = 1$, leaving only the repelling cycle after the crossing.

As already remarked in the previous section, the crossing of a BCB related to a cycle of any symbolic sequence leads to different dynamical properties depending on where it occurs, on the left side, at, or on the right side of the related codimension-two point, which can be deduced from the value of the related derivative D in (45), $D < 1$, $D = 1$ or $D > 1$, respectively. Crossing a BCB curve at which $D \geq 1$ occurs, a repelling cycle is created which persists for any larger value of b, while crossing a BCB curve with $D < 1$ an attracting cycle (born by fold bifurcation at a smaller value of b) disappears, leaving the repelling one (with the same symbolic sequence) for any larger value of b.

In order to show the existence of infinitely many BCBs let us consider, for clarity of exposition, the parameter b which varies from the BCB value on B_{RL^3} to the BCB value on B_{RL^2}, on the vertical segment at $a = 0.3$ shown in Figure 10, enlarged part of the rectangle shown in Figure 2(a), that is for $-15.4 = b(B_{RL^3}) < b < b(B_{RL^2}) = -4.3$.

At $b = b(B_{RL^3}) = -15.4$ the BCB curve B_{RL^3} is crossed, leading to the disappearance of the attracting 4−cycle RL^3, so that increasing b the attracting 4−cycle RL^3 no longer exists, while the repelling one, x_{RL^3}, persists increasing b. The new branch F_{RL^2} is now in

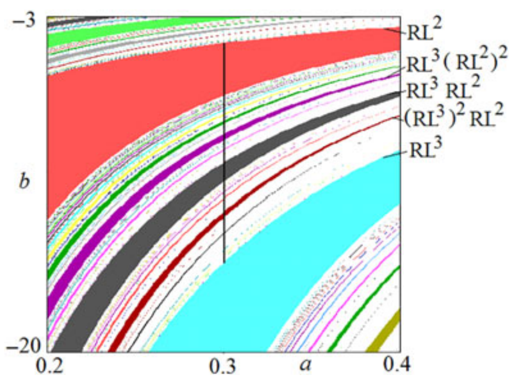

Figure 10. Enlarged part of the rectangle shown in Figure 2(a) and a segment at $a = 0.3$ is marked. The lower boundary of each coloured periodicity region is a fold bifurcation curve Φ_σ while the upper boundary is a BCB curve B_σ.

the first return map F_r on the right side of the discontinuity point ξ_3 (and we know that all the other preimages ξ_j, $j > 3$, also exist). As b increases from $b(B_{RL^3})$ to $b(B_{RL^2})$ the value of the rightmost branch of the first return map in $x = 1$, $F_r(1) = F_{RL^2}(1)$, increases from 0 to 1. In this interval, infinitely many BCB curves are crossed, each one leading to the disappearance of an attracting cycle (previously born by a fold bifurcation) leaving the repelling one, or leading to the appearance of a repelling cycle. From Proposition 5, a BCB of some cycle of f occurs whenever the point $x = 1$, $F_r(1)$, is mapped into a preimage of the origin, or, equivalently $F_r(1) = F_{RL^2}(1)$ merges with a preimage of some rank of a discontinuity point ξ_j, $j \geq 3$.

The first return map $F_r(x)$ shown in Figure 11 has infinitely many branches F_{RL^n}, for any $n > 2$, and thus all the repelling fixed points x_{RL^n} for any $n > 2$ exist and all are homoclinic.

Let us consider the value of b at which

$$F_{RL^2}(1) = x_{RL^3}$$

occurs, that is a homoclinic bifurcation of the repelling fixed point x_{RL^3} of F_r, as shown in Figure 11(a), at $b = b(x_{RL^3}) = -14.45$. To clarify the term $\Omega-$explosion, consider a slight increase of the value of b, i.e. after this homoclinic bifurcation, so that it is $F_{RL^2}(1) > x_{RL^3}$ and infinitely many new homoclinic orbits of x_{RL^3} can be found (which did not exist before) considering the newly appeared preimage on the right side $F_{RL^2}^{-1}(x_{RL^3})$.

Moreover, let us show via this example that a homoclinic bifurcation is a limit set of infinitely many BCB curves (the generic results are given in the next section). Consider, for example, that the fixed point x_{RL^3} is the limit point of preimages $F_{RL^3}^{-k}(\xi_3)$ for any $k \geq 0$ (see Figure 11(a)), and as b increases also $F_r(1) = F_{RL^2}(1)$ increases, so that all such values $F_{RL^3}^{-k}(\xi_3)$ are reached and crossed by $F_{RL^2}(1)$, say at values b_k, which means that all the BCB curves due to

$$F_{RL^2}(1) = F_{RL^3}^{-k}(\xi_3)$$

are crossed.

As noticed above, from

$$F_{RL^3}^k \circ F_{RL^2}(1) = \xi_3$$

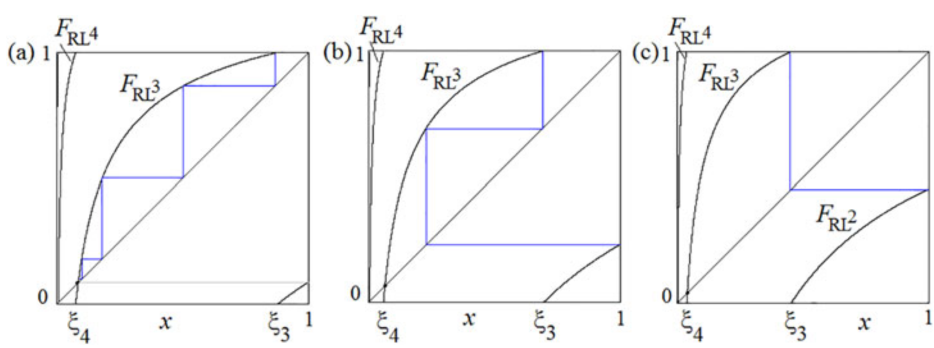

Figure 11. First return map $F_r(x)$ at $\gamma = 0.5$, $a = 0.3$. In (a) $b = -14.45$. In (b) $b = -12.87$. In (c) $b = -10.385$.

considering that

$$F_{RL^3}(\xi_3) = 1$$

we can state that there must exist values of b, say b_k for any $k \geq 0$, such that at $b = b_k$ the BCB curves having the following equations are crossed:

$$F_{RL^3}^{k+1} \circ F_{RL^2}(1) = 1, \quad k \geq 0 \tag{47}$$

which are the BCB curves of cycles of f having symbolic sequence $RL^2(RL^3)^{k+1}$ for any $k \geq 0$. Since for $k \to \infty$ the values b_k tend to $b(x_{RL^3})$ these border collisions have as limit set the homoclinic bifurcation value. The BCB corresponding to the case $k = 1$ ($b_1 = -12.87$) is shown in Figure 11(b) and the case $k = 0$ ($b_0 = -10.385$), i.e. merging with ξ_3, is shown in Figure 11(c).

It is not easy to evaluate the first derivative of the function $F_{RL^3}^{k+1} \circ F_{RL^2}(x)$ in the point $x = 1$. However, from Figure 10 we have that the BCB of the 11−cycle $RL^2(RL^3)^2$ and that of the 7−cycle RL^2RL^3 both occur after a fold bifurcation as they are the BCBs of an attracting cycle (as attracting cycles with these symbolic sequences are numerically detected, leading to the coloured periodicity regions).

Notice that we can find infinitely many families of BCB curves which occur in the interval of values of b considered up to now, $b(B_{RL^3}) < b < b_0$. For example, for any $j > 3$ we can consider also the preimages $F_{RL^j}^{-k}(\xi_3)$ for any $k \geq 1$ which have as limit sets the repelling fixed points x_{RL^j} and thus there must exist values of b such that

$$F_{RL^2}(1) = F_{RL^j}^{-k} \circ F_{RL^3}^{-1}(1)$$

that is

$$F_{RL^3} \circ F_{RL^j}^{k} \circ F_{RL^2}(1) = 1, \quad k \geq 1, \ j > 3 \tag{48}$$

are BCB curves which must have been crossed at suitable b values, related to cycles having symbolic sequence $RL^2(RL^j)^k RL^3$ for any $k \geq 1$ and any $j > 3$.

Also we can consider that in the interval $-15.4 = b(B_{RL^3}) < b < b(x_{RL^3}) = -14.45$ the value $F_{\mathrm{r}}(1) = F_{RL^2}(1)$ is increasing from 0 and thus there must exist the homoclinic bifurcation values $b(x_{RL^j})$ for any $j > 3$ in which $F_{\mathrm{r}}(1)$ is mapped into the repelling fixed point x_{RL^j} for any $j > 3$,

$$F_{RL^2}(1) = x_{RL^j}, \quad j > 3$$

and must satisfy

$$b(B_{RL^3}) < \cdots < b(x_{RL^{j+1}}) < b(x_{RL^j}) < \cdots < b(x_{RL^3}).$$

For each homoclinic bifurcation at $b(x_{RL^j})$ we can reason as above considering the preimages of discontinuity points ξ_m for any $m \geq 3$ placed on the right side of x_{RL^j}, for $j \geq 4$. For example, let $j = 4$ which is observable in Figure 11, then $F_{RL^4}^{-k}(\xi_4)$ for any $k \geq 0$ exist, as well as $F_{RL^4}^{-k}(\xi_3)$ for any $k \geq 0$, and both sequences have as limit set the repelling fixed point x_{RL^4}. For $j = 5$ then also $F_{RL^5}^{-k}(\xi_5)$, $F_{RL^5}^{-k}(\xi_4)$, $F_{RL^5}^{-k}(\xi_3)$ for any $k \geq 0$ exist which have as limit sets the repelling fixed point x_{RL^5} and so on. Thus, we have that there must exist values of b such that

$$F_{RL^2}(1) = F^{-k}_{RL^j}(\xi_m), \quad k \geq 0,\ j \geq 4,\ 3 \leq m \leq j$$

that is, considering $\xi_m = F^{-1}_{RL^m}(1)$,

$$F_{RL^2}(1) = F^{-k}_{RL^j} \circ F^{-1}_{RL^m}(1)$$

leading to

$$F_{RL^m} \circ F^{k}_{RL^j} \circ F_{RL^2}(1) = 1, \quad \text{for any } k \geq 0,\ j \geq 4,\ 3 \leq m \leq j \tag{49}$$

which are BCB curves crossed at suitable b values, related to cycles having symbolic sequence

$$RL^2(RL^j)^k RL^m$$

for any $k \geq 0$, $j \geq 4$ and $3 \leq m \leq j$.

For values of b smaller than $b_0 = -10.385$ (value at which the BCB of the $7-$cycle occurs, as shown in Figure 11(c), at which $F_r(1) = F_{RL^2}(1)$ is merging with ξ_3), the symbolic sequence of cycles which undergo a BCB cannot have the symbols $(RL^2)^k$ for $k > 1$, i.e. two consecutive applications of the rightmost branch F_{RL^2} cannot occur. While for larger values of b they can occur. For $b > b_0$ we can easily detect BCBs of cycles having symbolic sequence $(RL^2)^k RL^3$ for any $k > 1$. In fact, as b increases from b_0, the inverse of ξ_3 on the right side, $F^{-1}_{RL^2}(\xi_3)$, appears (which did not exist before). Moreover, also many other preimages of ξ_j on the right side, $F^{-1}_{RL^2}(\xi_j)$, for any $j > 3$ may now be crossed, which could not be possible before, and so on with many other preimages.

As b increases from b_0, at $b = -8.98$ we have that $F^2_{RL^2}(1)$ is merging with ξ_3 (see Figure 12(a)), at $b = -7.2$ we have that $F^{12}_{RL^2}(1)$ is merging with ξ_3, see Figure 12(b). It is clear that before the fold bifurcation of the branch F_{RL^2} (shown in Figure 12(c), at $b = b(\Phi_{RL^2}) = -7.008$), this must occur for any integer. That is, for any $k > 1$ the condition

$$F^k_{RL^2}(1) = \xi_3$$

must occur, leading (from $\xi_3 = F^{-1}_{RL^3}(1)$) to the BCB curves

$$F_{RL^3} \circ F^k_{RL^2}(1) = 1, \quad k \geq 2 \tag{50}$$

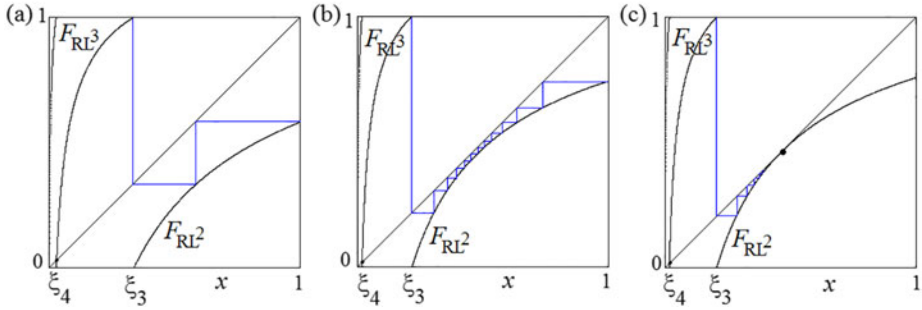

Figure 12. First return map $F_r(x)$ at $\gamma = 0.5$ $a = 0.3$ In (a) $b = -8.98$ In (b) $b = -7.2$ In (c) $b = -7.01$.

which have as limit set the fold bifurcation curve Φ_{RL^2}. From Figure 10 it seems that this family of cycles with symbolic sequence $(RL^2)^k RL^3$ for any $k > 1$ is associated with the BCB of attracting cycles.

Clearly we can find many more families of BCB curves also above the periodicity region of the 7−cycle. For example, we can easily see that the above reasoning can be repeated for the inverse of ξ_j on the right side, $F_{RL^2}^{-1}(\xi_j)$, for any $j > 3$. That is, for $b > b_0$, we must have suitable b values such that, for $k \geq 1$,

$$F_{RL^2}^{k}(1) = F_{RL^2}^{-1}(\xi_j)$$

leading (by using $\xi_j = F_{RL^j}^{-1}(1)$) to the BCB curves of attracting cycles with symbolic sequence $(RL^2)^k RL^j$ for any $k \geq 2$ and any $j > 3$,

$$F_{RL^j} \circ F_{RL^2}^{k}(1) = 1, \quad k \geq 2, \ j > 3 \tag{51}$$

which also have as limit set the fold bifurcation curve Φ_{RL^2}.

A one-dimensional bifurcation diagram of the example here considered at $a = 0.3$ and b in the range $-17 < b < -5$ is shown in Figure 13. We can argue (as we shall prove in the next section) that each BCB and each fold bifurcation is a limit set of infinitely many other BCBs, as well as each homoclinic bifurcation. The homoclinic bifurcation occurring at $b(x_{RL^3})$, limit set of BCB of cycles with symbolic sequence $(RL^2)(RL^3)^k$ for any $k > 1$ and of many other families as well can be better observed in the enlargement in Figure 14 (a), marked by a red arrow. While in the enlargement of Figure 14(b) we can appreciate the cascade of attracting cycles with symbolic sequence $(RL^2)^k RL^3$ for any $k > 1$, occurring for $b > b_0$. At all the values of b in which an attracting cycle does not exist, the first return map seems chaotic in the whole interval $[0, 1]$, and thus the map f seems chaotic in the whole unbounded interval $(-\infty, 1]$ (now the first return map is not expanding, so that we cannot make use of the result in [37]).

The short description given above, between the two BCB curves B_{RL^3} and B_{RL^2}, can obviously be repeated between any pair of BCB curves $B_{RL^{n+1}}$ and B_{RL^n} in parameter ranges not including the related codimension-two points and in which the fold bifurcation curves $\Phi_{RL^{n+1}}$ and Φ_{RL^n} exist.

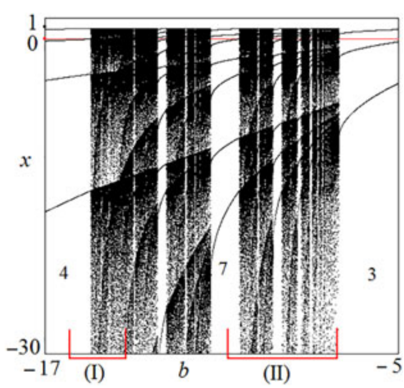

Figure 13. One-dimensional bifurcation diagram at $\gamma = 0.5$, $a = 0.3$, along the vertical segment shown in Figure 10. The intervals (I) and (II) are enlarged in Figure 14.

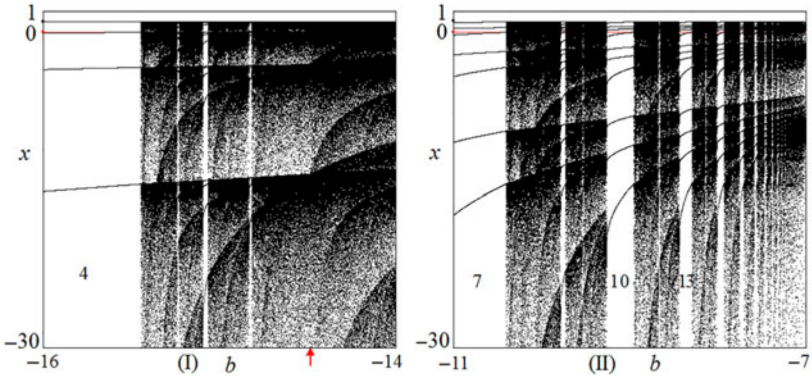

Figure 14. One-dimensional bifurcation diagrams at $\gamma = 0.5$, $a = 0.3$, enlargements of the parts (I) and (II) in Figure 13. The arrow in (I) shows the homoclinic bifurcation occurring at $b(x_{RL^3})$.

However, it is worth to note that the true bifurcation structure occurring in this range is still to be understood. Considering the description of the cycles involved in the BCB curves by using the concatenation of the symbolic sequences, in the example described above we have seen that between two BCBs of stable cycles RL^3 and RL^2 the families $(RL^3)^m RL^2$ and $RL^3(RL^2)^m$ for any $m \geq 1$ exist, but their limit sets are unusual (they are not BCB curves). The limit set of the family $RL^3(RL^2)^m$ as $m \to \infty$ is the fold bifurcation curve Φ_{RL^2} while the limit set of the family $(RL^3)^m RL^2$ as $m \to \infty$ is the homoclinic bifurcation of x_{RL^3}, i.e. of the unstable basic cycle RL^3. Thus we can see that the sequence of BCBs related to the period adding structure exists, but not only. Here we have the existence of many other families, and not related to the U-sequence occurring in continuous unimodal one-dimensional maps, because we cannot have here the cascades of period doubling bifurcations (existing in the U-sequence, see [40]), as flip bifurcations cannot occur in our system.

6.2 BCBs of repelling basic cycles

Let us now turn to consider an interval between two BCB curves $B_{RL^{n+1}}$ and B_{RL^n} in parameter ranges not including the related fold bifurcation curves $\Phi_{RL^{n+1}}$ and Φ_{RL^n}. That is, consider a value $a > \bar{a}_n$ and $b(B_{RL^{n+1}}) < b < b(B_{RL^n})$.

In Figure 2 we have evidenced a small segment at $a = 0.9 > \bar{a}_4$, between the BCB curves B_{RL^5} and B_{RL^4}, in what can be called a chaotic regime, as in this interval of b values, $-6.24 = b(B_{RL^5}) < b < b(B_{RL^4}) = -4.72$, no attracting cycle can exist, as we can prove that the first return map F_{r} has a first derivative which is larger than 1 in all the points, and thus (from Proposition 4) we have that F_{r} is chaotic in the whole interval $[0, 1]$, and the map f has $(-\infty, 1]$ as robust chaotic attractor. In fact, at $b = b(B_{RL^5})$ it is $F'_{RL^5}(1) = -b(B_{RL^5})\gamma a^5 > 1$, crossing which the branch of F_{RL^4} enters in the definition of the first return map F_{r}. From $F'_{RL^4}(1) = F'_{RL^5}(1)/a > F'_{RL^5}(1) > 1$ at $b = b(B_{RL^5})$, the rightmost branch of F_{r} enters with slope larger than 1, and since at $b = b(B_{RL^4})$ by assumption it is $F'_{RL^4}(1) > 1$ it follows $F'_{RL^4}(1) > 1$ for any value of b in the considered interval. Moreover, from this we also have that at any value of b it must be $F'_{RL^4}(x) > 1$ in all the points of definition of the function F_{RL^4} of the first return map, that is for $\xi_5 \leq x \leq 1$ (as in fact decreasing x from 1 the slope $F'_{RL^4}(x)$ must increase, being F_{RL^4} increasing and concave).

The BCB occurring at $b = b(B_{RL^5})$ is shown in Figure 5(a), the one occurring at $b = b(B_{RL^4})$ is shown in Figure 5b, while at a value of b inside the interval it is shown in Figure 4.

Similarly we can reason in a generic case. So, no attracting cycle can exist and the first return map F_r is chaotic in [0, 1]. However, as b increases in the range $b(B_{RL^{n+1}}) < b < b(B_{RL^n})$ the point $F_r(1) = F_{RL^n}(1)$ increases from 0 to 1 (being $F_{RL^n}(1) = 1$ at the BCB related to $b = b(B_{RL^n})$), *thus crossing infinitely many preimages of the discontinuity points (which indeed are dense in the interval* $[0, 1]$*), which means the crossing infinitely many other BCBs leading to the appearance of repelling cycles, as well as crossing infinitely many preimages of repelling cycles (which also are dense in the interval* $[0, 1]$*) and thus crossing infinitely many homoclinic bifurcations.*

For each value of b in the considered interval we can classify what occurs considering the trajectory of the point $x = 1$. We know that the dynamics are chaotic in [0, 1] thus we can list the following cases:

(i) if $F_r^k(1)$ is mapped into a discontinuity point at some k, then b is the value of the BCB of a cycle;
(ii) if $F_r^k(1)$ is mapped into a repelling periodic point at some k, then b is the value of a homoclinic bifurcation of the cycle;
(iii) if $F_r^k(1)$ is aperiodic (and dense in $[0, 1]$), then b is not a bifurcation value.

Families of BCB curves can be easily identified which are crossed in this interval (among the infinitely many which we know to occur), considering the preimages of discontinuity points, and reasoning as in the previous case. That is, all the families of BCBs previously described occur also now (with obvious changes), with the only difference that now they involve only repelling cycles (i.e. no fold bifurcation curve can be crossed). For example, considering the discontinuity points ξ_j for $j \geq 5$, and the repelling fixed points x_{RL^k} for $k \geq 5$, we can list several families of BCBs and homoclinic bifurcations.

For each discontinuity point ξ_j for $j \geq 5$ we can consider the preimages $F_{RL^m}^{-k}(\xi_j)$ for any $k > 0$ and any $m \geq j$. The sequences $F_{RL^m}^{-k}(\xi_j)$ for any $k > 0$ have as limit set the repelling fixed points x_{RL^m} $(m \geq j)$. Thus each homoclinic bifurcation at which, for $m \geq 5$,

$$F_{RL^4}(1) = x_{RL^m}$$

occurs, is a limit set of infinitely many other BCBs. For example, let $b(x_{RL^m})$ be the bifurcation value at which $F_{RL^4}(1) = x_{RL^m}$ occurs, then, as b is further increased, BCB values at which

$$F_{RL^4}(1) = F_{RL^m}^{-k}(\xi_j)$$

must occur for any $j \geq 5$, $m \geq j$ and $k > 0$. The limit value as $k \to \infty$ is $b(x_{RL^m})$. Thus, the following BCB values must be crossed:

$$F_{RL^4}(1) = F_{RL^m}^{-k} \circ F_{RL^j}^{-1}(1)$$

that is

$$F_{RL^j} \circ F_{RL^m}^{k} \circ F_{RL^4}(1) = 1$$

which are BCB curves of cycles with symbolic sequence $RL^4(RL^m)^k RL^j$, $j \geq 5$ and $m \geq j$, and as $k \to \infty$ they accumulate to a homoclinic bifurcation curve.

7. Each bifurcation is a limit set of families of BCBs

The examples shown in the previous section suggest that all the bifurcation curves, border collision, fold and homoclinic bifurcation curves are limit sets of families of BCB, as qualitatively represented in Figure 15. This is indeed what we prove in the following.

PROPOSITION 7. *Any BCB is a limit set of infinitely many BCBs, on both sides when related to a repelling cycle, only from above when related to an attracting cycle.*

Proof. Let us first analyse this property related to the BCBs of basic cycles of f, then, considering the fact that at each BCB a suitable iterate of the first return map has qualitatively the same structure in the rightmost branch, the result holds in general for the BCB of any cycle. The main point not yet completely commented is the role played by the codimension-two point in each BCB. However, from the examples already described its role is clarified. In fact, in all the cases we have shown that a BCB of a basic cycle is a limit set of other BCB values from above. In fact, at any value of a, after the BCB, increasing the parameter b, the new branch entering the definition of the first return map leads to infinitely many new BCBs, specially related to the crossing of the preimages of the discontinuity points which are accumulated to $x = 0$. More rigorously, consider the BCB of a basic cycle of f, say B_{RL^m}, then at $b = b(B_{RL^m})$ it is $F_{RL^m}(1) = 1$, and for values of b slightly larger than $b(B_{RL^m})$ the new rightmost branch of the first return map F_r is $F_{RL^{m-1}}(x)$ for $\xi_m \leq x \leq 1$, and $F_r(1) = F_{RL^{m-1}}(1)$ increases from the value 0. Thus considering the discontinuity points ξ_j, $j > m$, of the first return map, which have as limit value for $j \to \infty$ the point $x = 0$, we have that as soon as $F_{RL^{m-1}}(1) > 0$ infinitely many of them belong to the range of $F_{RL^{m-1}}$, and thus BCBs must occur satisfying

$$F_{RL^{m-1}}(1) = \xi_j$$

that is (considering $\xi_j = F^{-1}_{RL^j}(1)$)

$$F^{-1}_{RL^j} \circ F_{RL^{m-1}}(1) = 1$$

at bifurcation values say $b = b_j > b(B_{RL^m})$ and such that the limit value of b_j for $j \to \infty$ is $b(B_{RL^m})$.

This occurs independently of the crossing of the BCB curve B_{RL^m} below, at or above its codimension-two point $a = \bar{a}_m$. What makes difference, and thus leads to a particular role of the codimension-two point, is the following:

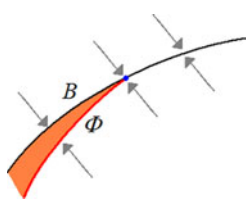

Figure 15. Qualitative representation of the curves which are limit sets of other bifurcation curves. B denotes a BCB curve while Φ a fold bifurcation curve.

(j.1) crossing the BCB curve B_{RL^m} at a point with $a < \bar{a}_m$ and varying b the bifurcation point is a limit set of other BCB values only from above;
(j.2) crossing the BCB curve B_{RL^m} at $a \geq \bar{a}_m$ and varying b the bifurcation point is a limit set of other BCBs both from below and from above.

This difference is immediately shown. In fact, as we know, for $a < \bar{a}_m$ the colliding fixed point is attracting and also a repelling one exists, x_{RL^m}, with the same symbolic sequence. The immediate basin given by the interval $(x_{RL^m}, 1]$ and no BCB can involve points of this interval, so that for smaller values of b, before the bifurcation, there cannot be other BCB values (see the example in Figure 8(a),(b) or in Figure 9(a),(b).

Differently, in the case $a \geq \bar{a}_m$ (see the example in Figure (6a)) at the BCB value $b = b(B_{RL^m})$ we can consider each discontinuity point ξ_j for $j \geq m$ and the preimages $F^{-k}_{RL^m}(\xi_j)$ for any $k > 0$. For any $j \geq m$ the sequences $F^{-k}_{RL^m}(\xi_j)$ have as limit value for $k \to \infty$ the point $x = 1$. Thus BCB values for b, say $b_{j,k}$, must have been crossed satisfying

$$F_{RL^{m-1}}(1) = F^{-k}_{RL^m}(\xi_j)$$

that is (considering $\xi_j = F^{-1}_{RL^j}(1)$)

$$F_{RL^j} \circ F^k_{RL^m} \circ F_{RL^{m-1}}(1) = 1$$

and for any $j \geq m$ the limit for $k \to \infty$ of the values $b_{j,k}$ is (from below) $b(B_{RL^m})$ □

Proposition 8. *Each homoclinic bifurcation curve is a limit set of infinitely many border collisions, both from below and above.*

Proof. Let us consider a homoclinic bifurcation associated with a fixed point of $F_r(x)$, then for any $k-$cycle of F_r the reasoning is the same by using the kth iterate of the first return map.

Let the rightmost branch of $F_r(x)$ be $F_r(x) = F_{RL^m}(x)$ for some integer $m \geq 1$, and thus ξ_j, $j \geq (m+1)$, are the discontinuity points, and x_{RL^j} for any $j \geq (m+1)$ are repelling fixed points (all SBR, as already shown). Then there are infinitely many sequences of preimages of the discontinuity points having as limit set a repelling fixed point from below and from above. Thus any value b_j at which a homoclinic bifurcation $F_r(1) = x_{RL^j}$ for some j occurs, is a limit set from below and from above of b values at which BCB curves related to the equations $F_r(1) = \xi_i^{-k}$ must have been crossed before, at smaller values of b having b_j as limit, or will be crossed increasing b. □

Proposition 9. *Each fold bifurcation is a limit set of infinitely many homoclinic bifurcations and border collisions, only from below.*

Proof. Let us consider a fold bifurcation associated with a pair of fixed points of $F_r(x)$ (i.e. basic cycles of f), then for any $k-$cycle of F_r the reasoning is the same by using the kth iterate of the first return map.

At the fold bifurcation the rightmost branch of $F_r(x)$, say $F_r(x) = F_{RL^m}(x)$, leads to a fixed point $x^*_{RL^m}$ which is repelling and homoclinic from its left side, and attracting from its right side. Thus, the result follows by the same arguments used above as $x^*_{RL^m}$ is the limit set from below of sequences of preimages of the existing discontinuity points as well as of sequences of preimages of all the existing repelling fixed points. □

Proposition 10. *Chaotic bands do not occur.*

Proof. In PWS systems, it is common to have chaos occurring in a finite number of intervals, called chaotic bands, which are bounded by the images of the critical point(s) of a map [3,5,7]. However, in many applications it is preferable to have the chaotic system in one unique band, or interval. In the case of PWL systems, as shown in [7], it is also proved that when chaotic bands exist, then a kink point or discontinuity point must belong to a chaotic band. This result is true also for the PWS map f in (3) with a vertical asymptote. In fact, as we have evidenced in this section, the origin $x = 0$ is necessarily a limit point of repelling cycles and homoclinic points, and thus must be a limit point of the chaotic set. But map f has a vertical asymptote on the right side of $x = 0$, and this implies that the chaotic set can never be bounded. It follows that chaos is either an unbounded chaotic repellor (when the chaotic set belongs to the boundary of a stable set or of a basin of attraction), or the dynamics are chaotic in the whole unbounded interval $(-\infty, 1]$. □

Differently, in the region with $a > 1$, which we consider in the Part II [35], the chaotic sets, when existing, are necessarily bounded in a finite interval, and only in the form of chaotic repellors (persisting the property that chaotic bands cannot exist).

8. Conclusions and outlook

In this work we have investigated some properties of the discontinuous map given in (3) for the parameter range given in (5) and defined as AI in Figure 1. Besides the introduction of the contact points between the BCB curves and the fold bifurcation curves of basic cycles and their role, we have proved that a similar codimension-two point exists on any BCB curve of an admissible cycle. We have explained their role in terms of the bifurcations occurring on their crossing. The main results are obtained making use of the first return map defined in Section 3. Moreover, we have proved that chaos always exists, as in our system all the unstable cycles are homoclinic, and that the chaotic set is unbounded, of zero measure or of full measure, and robust (persistent under variation of the parameters). The dynamic properties on the crossing of the BCB curves when fold bifurcations are also crossed have been illustrated via examples, showing that the structure is much richer than that occurring in other systems (for example in comparison with the period adding structure or the U-sequence). In our system, any BCB curve as well as any fold bifurcation curve and any homoclinic bifurcation curve is a limit set of infinite families of BCB curves, as proved in Section 7. However, the understanding of the bifurcation structure related to the occurrence of fold bifurcations and BCBs as b increases is left for further investigation. Another open problem is related to the particular role of codimension-two bifurcation points, a BCB and a fold bifurcation related to cycles with different symbolic sequences, in which case the codimension-two points act as organizing centres and are issuing points of infinitely many families of bifurcation curves, both of fold type and of border collision type, as clearly visible in Figure 1. The open problems here evidenced are neither considered in [35], in which the other ranges marked as regions AII, BI and BII in Figure 1 are considered.

Acknowledgements

The first author thanks the Department DESP of the University of Urbino for the hospitality during her stay in Italy. The work of L. Gardini has been done within the activities of the GNFM (National Group of Mathematical Physics, INDAM Italian Research Group), and of the COST action IS1104 ‘The EU in the new economic complex geography: models, tools and policy evaluation’ www.gecomplexitycost.eu.

Disclosure statement

No potential conflict of interest was reported by the authors.

Notes

1. Email: r_makrooni@sbu.ac.ir
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3. We recall that for the particular case $a = 1$, the bifurcation curves are obtained by replacing $1 - a^{n+1}/1 - a$ with $n + 1$ and $1 - a^n/1 - a$ with n in the expressions given above.
4. Recall that in a discontinuity point of a map, the two limiting values of the map on both sides of the discontinuity are both called critical points, as both may be involved in border collision bifurcations.

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