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# Robust chaos in a credit cycle model defined by a one-dimensional piecewise smooth map

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ABSTRACT

We consider a family of one-dimensional continuous piecewise smooth maps with monotone increasing and monotone decreasing branches. It is associated with a credit cycle model introduced by Matsuyama, under the assumption of the Cobb-Douglas production function. We offer a detailed analysis of the dynamics of this family. In particular, using the skew tent map as a border collision normal form we obtain the conditions of abrupt transition from an attracting fixed point to an attracting cycle or a chaotic attractor (cyclic chaotic intervals). These conditions allow us to describe the bifurcation structure of the parameter space of the map in a neighborhood of the boundary related to the border collision bifurcation of the fixed point. Particular attention is devoted to codimension-two bifurcation points. Moreover, the described bifurcation structure confirms that the chaotic attractors of the considered map are robust, that is, persistent under parameter perturbations.



## 1. Introduction

The one-dimensional (1D for short) piecewise smooth (PWS for short) map considered in the present paper defines an important credit cycle model first introduced by Matsuyama in [20]. This model generates *endogenous fluctuations* of borrower net worth and aggregate investment, following the same trend as several micro-founded, dynamic general equilibrium models of financial frictions, in which the steady state is *unstable*, and persistent fluctuations occur *without exogenous shocks* (see, for example, [1,3,21]). Such an approach differs from the basic ideas of a vast majority of the macroeconomics literature on financial frictions that follows the seminal works [6] and [18], and continues to study amplification effects of financial frictions within a setting that ensures the existence of a *stable steady state* toward which the economy would gravitate in the absence of recurring exogenous shocks. In fact, the idea that market mechanisms are *inherently dynamically unstable* can be traced back at least to Goodwin [12]. Recent events have also renewed interest in the hypothesis that financial frictions are responsible not only for amplifying the effects of exogenous shocks but also for causing macroeconomic instability (see, e.g., [17] and [25]).

A detailed description of the Matsuyama model can be found in [20] and [22] (see also [23]). It is defined by a 1D map which consists of upward, downward, and flat branches. Furthermore, as discussed in [23], when the production function is Cobb-Douglas, the map depends on four parameters. The bifurcation structure of the parameter space of this map significantly depends on whether the constant branch is involved into asymptotic dynamics or not. In our companion paper [32] we study in detail the case where all three branches are involved, demonstrating that it is characterized by periodicity regions related to superstable cycles existing due the constant branch, and that these regions are ordered according to the well known *U-sequence* distinctive for unimodal maps (first described in [24], see also [13]), which is adjusted to the considered map.

In the present paper we analyze the dynamics of the map when the constant branch does not participate in the asymptotic dynamics. Such a map belongs to a class of 1D PWS continuous unimodal maps possessing quite complicated dynamics which, depending on the parameters, is characterized by attracting cycles of any pe-

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riod, as well as cyclic chaotic intervals. The mechanisms governing the transitions between such attractors under parameter variation are already described in our paper [23]. The main purpose of the present work is to give detailed proofs of the related results and to describe the overall bifurcation structure of the parameter space of the map, evidencing the role of codimension-two bifurcation points.

From the point of view of nonlinear dynamics theory the main feature of the considered map is its *non smoothness*. In fact, as we mentioned above, the map is given by two different smooth functions whose definition regions are separated by a *border point* at which the system function is not differentiable. As a result, under variation of a parameter it is possible to observe not only bifurcations typical for 1D smooth maps (such as, for example, flip bifurcation of a fixed point related to its eigenvalue crossing $-1$, or homoclinic bifurcation related to a contact of a stable and unstable sets of a repelling fixed point), but *border collision bifurcations* (BCB for short) as well, which are characteristic of nonsmooth systems (see [5,14,15,26]). Recall that a BCB occurs when an invariant set, for example, a fixed point or cycle, collides with a border point. The result of such a bifurcation can be a direct transition from an attracting fixed point to a chaotic attractor that is impossible in smooth systems. Such an abrupt transition to chaos in a 1D PWS map can be observed also due to a degenerate bifurcation which is related to the eigenvalue of a fixed point (or cycle) crossing 1 or $-1$ in presence of a particular degeneracy of the system function. For example, a *degenerate flip bifurcation* (DFB for short) of a fixed point occurs when its eigenvalue crosses $-1$ and the related branch of the function at the bifurcation value is linear or linear fractional (see [31]). Note that a general bifurcation theory for nonsmooth dynamical systems has not yet such a complete form as the one established for smooth systems. As an important advancement towards such a theory we refer to the books [34], [10]. Examples of PWS models coming from economic applications can be found in [7,9,11,15,28], to cite a few.

As one of the main contributions of the present paper we give the conditions under which abrupt transitions via a BCB from an attracting fixed point to an attracting cycle or to a chaotic attractor are observed. Such conditions are obtained by using a 1D piecewise linear map defined by two linear functions, called *skew tent map*. The dynamics of the skew tent map are completely described depending on the slopes of the linear branches (see [16,19]) that makes it possible to use this map as a *border collision normal form* ([5,27,29,30]).

The skew tent map is used to classify not only the BCB of the fixed point, mentioned above, but BCBs of the attracting $n$-cycles as well, $n \geq 3$. More precisely, we show that one boundary of the periodicity region related to an attracting $n$-cycle is associated (at least in a certain neighbourhood) with the so-called *fold BCB*. The crossing of this boundary leads to the appearance of a couple of $n$-cycles, one attracting and one repelling. This bifurcation is to some extent similar to the smooth fold bifurcation, being, however, not related to an eigenvalue equal to 1. Another boundary of the $n$-periodicity region is related to the smooth flip bifurcation, sub- or supercritical.

It is known that one more distinctive feature of PWS maps is associated with *robust chaotic attractors* (see [4]), that means that in the parameter space of a PWS map an open region may exist, called chaotic domain, related to chaotic attractors persistent under parameter perturbations. Considering a chaotic attractor which consists of $n$ cyclic intervals, $n \geq 1$, under parameter variation inside a chaotic domain bifurcations can be observed at which the number of intervals constituting the chaotic attractor changes. In particular, a *merging bifurcation* is related to the transition from $2n$- to $n$-cyclic chaotic attractor. It is caused by the first homoclinic bifurcation of a repelling cycle with negative eigenvalue, located at the immediate basin boundary of the attractor. An *expansion bifurcation* occurs when a chaotic attractor abruptly increases in size filling the complete absorbing interval due to the first homoclinic bifurcation of a repelling cycle with positive eigenvalue (see [2] for details). By using the skew tent map we get the conditions of the homoclinic bifurcations leading to merging and expansion bifurcations in the considered map.

The paper is organized as follows. In Section 2 we describe the map, its fixed points and the conditions of their stability. The parameter region we are interested in is confined by three boundaries. One of them is related to a contact of the absorbing interval with the border point (crossing this boundary the constant branch becomes involved into asymptotic dynamics), and two other boundaries are related to the bifurcations of a fixed point associated with the downward branch of the map. Namely, crossing one of such boundaries a BCB of this fixed point occurs, whose possible results are listed in Section 3 (see Proposition 1) and proved using the skew tent map as a border collision normal form. The second boundary is related to the flip bifurcation described in Section 4 (see Proposition 2). In Section 5 it is discussed the overall bifurcation structure of the parameter space of the considered map, emphasizing the role of codimension-two bifurcation points. Section 6 concludes.

## 2. Description of the map, its fixed points and their bifurcations

We consider a 4-parameter family of 1D piecewise smooth maps defined as

$$T : w \mapsto T(w) = \begin{cases} T_L(w) = w^{\alpha} & \text{if } 0 < w < w_c, \\ T_M(w) = \left[\dfrac{1}{\mu\beta}\left(1 - \dfrac{w}{m}\right)\right]^{\frac{\alpha}{1-\alpha}} & \text{if } w_c < w < w_\mu, \\ T_R(w) = \beta^{\frac{\alpha}{\alpha-1}} & \text{if } w \geq \max\{w_c, w_\mu\}, \end{cases} \tag{1}$$

where $\alpha$, $\beta$, $\mu$ and $m$ are real parameters such that

$$0 < \alpha, \mu < 1, \quad \beta \equiv B\frac{1-\alpha}{\alpha} > 0, \quad 1 < m < \frac{1}{1-\alpha}, \tag{2}$$

$w_c$ and $w_\mu$ are the border points defined by

$$w_c^{1-\alpha} = \frac{1}{\mu\beta} \max\left\{1 - \frac{w_c}{m}, \mu\right\}, \qquad w_\mu = m(1-\mu). \tag{3}$$

Map $T$ describes the dynamic trajectory of the entrepreneur *net worth* $w$ in a credit cycle model, first introduced in [20], under the additional assumption that the aggregate production function is Cobb-Douglas (see [23,32]).

In the simplest case map $T$ is defined only by the branches $T_L(w)$ and $T_R(w)$ with the border point $w_c = (w_B)^{1/\alpha}$. The boundary in the parameter space defined by

$$\beta = (m(1-\mu))^{\alpha-1} \tag{4}$$

is related to the appearance of the middle branch in the definition of $T$. Namely, for $\beta > (m(1-\mu))^{\alpha-1}$ map $T$ can be written in the following form:

$$T : w \mapsto T(w) = \begin{cases} T_L(w) = w^{\alpha} & \text{if } 0 \leq w \leq w_c, \\ T_M(w) = \left[\dfrac{1}{\mu\beta}\left(1 - \dfrac{w}{m}\right)\right]^{\frac{\alpha}{1-\alpha}} & \text{if } w_c < w < w_\mu, \\ T_R(w) = w_B & \text{if } w > w_\mu. \end{cases} \tag{5}$$

Note that $T$ maps (0, 1] into itself, so that we restrict $T$ on (0, 1] from now on.

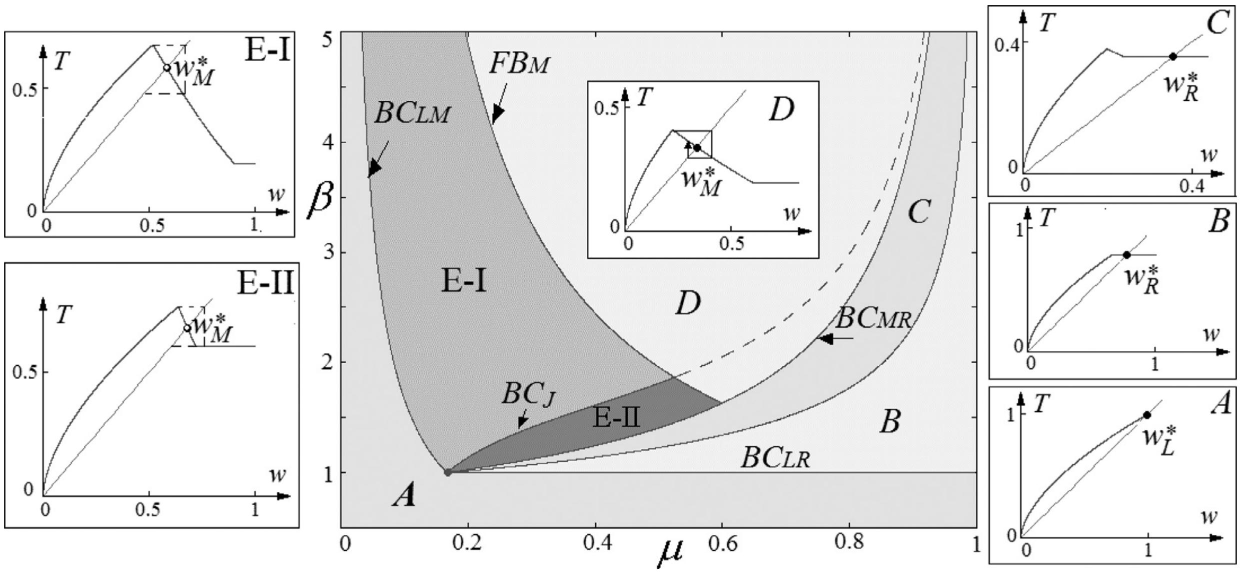


**Fig. 1.** Basic bifurcation curves of map $T$ in $(\mu, \beta)$-parameter plane at $m = 1.2$, $\alpha = 0.6$. Examples of map $T$ in different parameter regions are also shown.

Let us first recall the simplest bifurcation conditions (see [32] and [23]) related to existence and stability of the fixed points of map $T$. We illustrate the corresponding regions and bifurcation curves in Fig. 1 which shows also examples of map $T$ associated with different parameter regions.

The fixed points related to the upward, downward and flat branches of map $T$ are denoted $w_L^*$, $w_M^*$ and $w_R^*$, respectively. The fixed point $w_L^* = 1$ exists and is globally attracting for the parameter values belonging to the region

$$\mathbf{A}: \quad \beta \le \max\left\{\frac{1}{\mu}\left(1-\frac{1}{m}\right), 1\right\}, \tag{6}$$

two boundaries of which correspond to BCBs of $w_L^*$, namely, for

$$BC_{LM}: \quad \beta = \frac{1}{\mu}\left(1-\frac{1}{m}\right), \tag{7}$$

we have $w_L^* = 1 = w_M^*$, and for

$$BC_{LR}: \quad \beta = 1, \tag{8}$$

the equality $w_L^* = 1 = w_R^*$ holds. The fixed point $w_R^* = w_B$ (which is obviously superstable) exists for the parameter region

$$1 < \beta < (m(1-\mu))^{1-\frac{1}{\alpha}}.$$

At the boundary $\beta = 1$ (denoted as $BC_{LR}$) we have $w_R^* = w_L^* = 1$. If the parameter point crosses $BC_{LR}$ we observe a border collision leading from the superstable fixed point $w_R^*$ to the stable fixed point $w_L^*$.[1] The region of existence of $w_R^*$ is divided by the boundary given in (4) in two subregions:

$$\mathbf{B}: \quad 1 < \beta < (m(1-\mu))^{\alpha-1},$$

$$\mathbf{C}: \quad (m(1-\mu))^{\alpha-1} < \beta < (m(1-\mu))^{1-\frac{1}{\alpha}},$$

(see Fig. 1). While at the boundary

$$BC_{MR}: \quad \beta = (m(1-\mu))^{1-\frac{1}{\alpha}} \tag{9}$$

we have $w_R^* = w_\mu = w_M^*$, so that $BC_{MR}$ is related to one more border collision of $w_R^*$. The fixed point $w_M^*$ exists if $w_c \le w_M^* \le w_\mu$ that holds for

$$\beta \ge \max\left\{\frac{1}{\mu}\left(1-\frac{1}{m}\right), (m(1-\mu))^{1-\frac{1}{\alpha}}\right\}. \tag{10}$$

Both boundaries of this parameter region are related to the border collision of $w_M^*$, namely, at the boundary $BC_{LM}$ (see (7)) $w_M^* = 1 = w_L^*$, as already mentioned. The possible results of this BCB are described in Proposition 1 below. While at the boundary $BC_{MR}$ (see (9)) we have $w_M^* = w_\mu = w_R^*$. Crossing $BC_{MR}$ in the generic case we observe either a persistence border collision, or a flip BCB[2] (see [32]).

The fixed point $w_M^*$ may become unstable via a standard flip bifurcation (see Proposition 2 below). The flip bifurcation curve of $w_M^*$ is given by

$$FB_M: \quad \beta = \frac{\alpha}{\mu}(m(1-\alpha))^{1-\frac{1}{\alpha}}. \tag{11}$$

So, for parameter values belonging to the region

$$\mathbf{D}: \quad \beta > \max\left\{\frac{\alpha}{\mu}(m(1-\alpha))^{1-\frac{1}{\alpha}}, (m(1-\mu))^{1-\frac{1}{\alpha}}\right\}$$

(see Fig. 1) there exists the locally attracting fixed point $w_M^*$.

We have the following two possibilities for an invariant absorbing interval $J$ of map $T$:

(1) In the absorbing interval $J$ only the functions $T_L(w)$ and $T_M(w)$ are defined, that holds for parameter values belonging to the region

$$\textbf{E-I}: \begin{cases} \beta < \dfrac{\alpha}{\mu}(m(1-\alpha))^{1-\frac{1}{\alpha}}, \\ \beta > \max\left\{\dfrac{1}{\mu}\left(1-\dfrac{1}{m}\right), 1-\dfrac{1}{\mu}+\dfrac{1}{\mu}(m(1-\mu))^{1-\frac{1}{\alpha}}\right\} \end{cases} \tag{12}$$

In such a case $J = [T^2(w_c), T(w_c)]$.

(2) All the three functions, $T_L(w)$, $T_M(w)$ and $T_R(w)$, are involved in $J$, that holds in the region

$$\textbf{E-II}: \begin{cases} \beta > (m(1-\mu))^{1-\frac{1}{\alpha}}, \\ \beta < \min\left\{1-\frac{1}{\mu}+\frac{1}{\mu}(m(1-\mu))^{1-\frac{1}{\alpha}},\ \frac{\alpha}{\mu}(m(1-\alpha))^{1-\frac{1}{\alpha}}\right\} \end{cases} \tag{13}$$

In such a case $J = [T(w_\mu), T(w_c)] = [w_B, T(w_c)]$.

The boundary between the two regions corresponds to the contact of $J$ with the border point $w_\mu$, occurring when the condition

[1] We say that *persistence border collision* occurs if neither the kind nor the stability properties of the colliding invariant set change after the collision.

[2] The border collision of a fixed point due to which the fixed point changes stability while a 2-cycle emerges from the border point is called *flip BCB*. Similarly to the smooth flip bifurcation a flip BCB can be sub- or supercritical. Note, however,

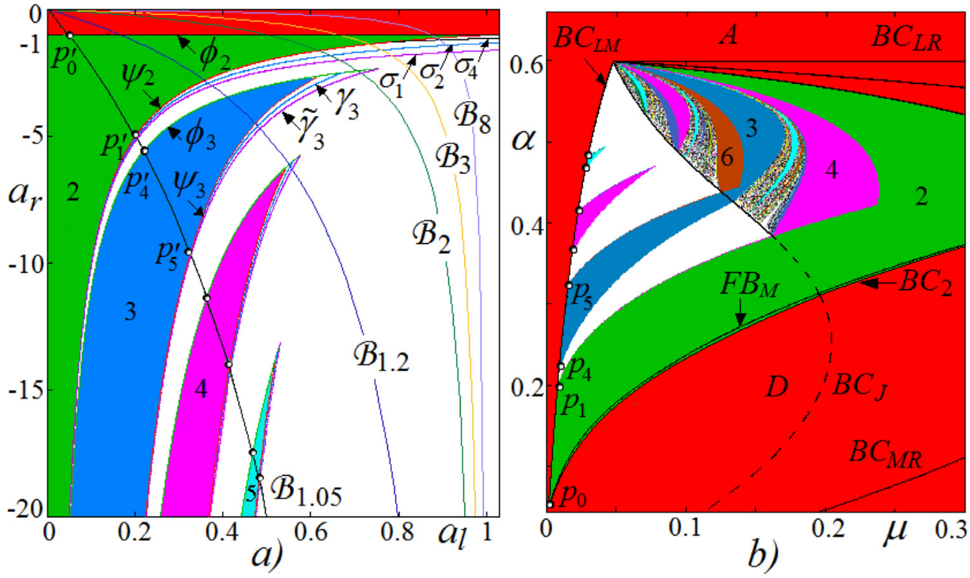


**Fig. 2.** *(a)* Bifurcation structure of the $(a_l,\ a_r)$-parameter plane of the skew tent map, where the border collision curves $\mathcal{B}_m$ are shown for $m = 1.05,\ 1.2,\ 2,\ 3$ and $8$; *(b)* Bifurcation structure of the $(\mu,\ \alpha)$-parameter plane of the map $T$ at $m = 1.05,\ B = 1.5$.

$T(w_c) = w_\mu$ is satisfied, leading to the curve $BC_J$ having the following equation:

$$BC_J: \quad \beta = 1 - \frac{1}{\mu} + \frac{1}{\mu}(m(1-\mu))^{1-\frac{1}{\alpha}}. \tag{14}$$

The bifurcation structure of the region $E - II$ formed by the periodicity regions related to superstable cycles of map $T$ (existing due to its flat branch) is described in [32]. In the following we first describe the border collision and flip bifurcations of the fixed point $w_M^*$ in detail and then we discuss the overall bifurcation structure of the region $E - I$.

## 3. Crossing the curve $BC_{LM}$: BCB of the fixed point

Consider first the BCB of the fixed point $w_M^*$, occurring when a parameter point crosses the boundary $BC_{LM}$ given in (7) of the region $E - I$. To describe the possible results of this BCB we can use the skew tent map defined by

$$q: x \mapsto q(x) = \begin{cases} a_l x + \varepsilon & \text{if } x \le 0, \\ a_r x + \varepsilon & \text{if } x > 0, \end{cases} \tag{15}$$

as a border collision normal form. This approach is based on the following statement (see [5,27,30]): *For a family of 1D piecewise smooth continuous maps $g$: $x \mapsto g(x, c)$ depending smoothly on a parameter $c$ and having a border point $x = d$, suppose that*

$$g(d, c^*) = d \tag{16}$$

*and let*

$$a_l^* = \lim_{x \uparrow d} \frac{d}{dx} g(x, c^*), \quad a_r^* = \lim_{x \downarrow d} \frac{d}{dx} g(x, c^*). \tag{17}$$

*Then in the generic case the border collision occurring in the map $g$ as $c$ varies through $c^*$ is of the same kind as the one occurring in the skew tent map (15) as $\varepsilon$ varies through 0 at $(a_l, a_r) = (a_l^*, a_r^*)$.*

Clearly, this statement refers to the border collision of a fixed point $x = x^*$ of the map $g$ (its existence before or/and after the collision follows from the conditions of the statement).[3] Generic case means that at $c = c^*$ the fixed point $x = x^*$ of the map $g$ undergoes only one bifurcation, i.e. a codimension-one BCB. An example of codimention-two bifurcation is when a border collision and a flip bifurcation occur simultaneously at the same point in the parameter space (in fact, this can happen also in map $T$, as we discuss later). For the detailed classification of the possible BCBs in the skew tent map and explanation how to use this map as a border collision normal form we refer to [30].

Let us recall in short the equations of the curves forming the bifurcation structure in the $(a_l,\ a_r)$-parameter plane of the skew tent map given in (15) for any $\varepsilon > 0$. Let $q_n$ denote a cycle of period $n$, $n \ge 2$, of the skew tent map. The stability region of $q_n$ is bounded from above by the curve $\phi_n$ and from below by the curve $\psi_n$ defined as

$$\phi_n: \quad a_r = -\frac{1 - a_l^{n-1}}{(1 - a_l)a_l^{n-2}}, \tag{18}$$

$$\psi_n: \quad a_r = \frac{-1}{a_l^{n-1}}, \tag{19}$$

(see Fig. 2(a)). The curve $\phi_n$ is related to the fold BCB[4] leading to the appearance of the basic cycle[5] $q_n$ and its complementary cycle[6] $\tilde{q}_n$. The curve $\psi_n$ is related to the degenerated flip bifurcation (DFB) of $q_n$ leading to $2n$-cyclic chaotic intervals $Q_{n,\,2n}$, $n \ge 3$, where the first index $n$ means that this chaotic attractor is born due to a DFB of the $n$-cycle, while $2n$ indicates that the chaotic intervals constituting the attractor are $2n$-cyclic. The transitions $Q_{n,\,2n} \Rightarrow Q_{n,\,n}$ (merging bifurcation) and $Q_{n,\,n} \Rightarrow Q_1$ (expansion

that it is not related to an eigenvalue passing through −1. Moreover, it may result in a chaotic attractor that is impossible for a smooth flip bifurcation.

[3] The skew tent map can be also used as a border collision normal form for a BCB of an $n$-cycle of the map $g$, in which case the statement has to be applied to the map $g^n$ and its fixed point corresponding to the periodic point of $g$ colliding with the border point.

[4] *Fold BCB* is a border collision at which two fixed points (one attracting and one repelling, or both repelling) simultaneously collide with the border point and disappear after the collision. It is worth to emphasize that a fold BCB is not associated with an eigenvalue passing through 1.

[5] For a 1D piecewise smooth map defined on two partitions, $L$ and $R$, an $n$-cycle with symbolic sequence $LR^{n-1}$ or $RL^{n-1}$ for any $n \ge 2$ is called basic. The basic cycle $q_n$ of the skew tent map (15) for $\varepsilon > 0$ has symbolic sequence $RL^{n-1}$. It can be shown that only such cycles can be stable (see [30]).

[6] The symbolic sequences of two complementary cycles differ by one symbol. The symbolic sequence of the cycle $\tilde{q}_n$ which is complementary to the basic cycle $q_n$ is $RL^{n-2}R$.

bifurcation) take place crossing the curves $\gamma_n$ and $\widetilde{\gamma}_n$, respectively, whose equations are given by

$$\gamma_n: \quad a_l^{2(n-1)}a_r^3 - a_r + a_l = 0, \tag{20}$$

$$\widetilde{\gamma}_n: \quad a_l^{n-1}a_r^2 + a_r - a_l = 0. \tag{21}$$

For the description of merging and expansion bifurcations we refer to [2]. The curves $\gamma_n$ and $\widetilde{\gamma}_n$ are related to the first homoclinic bifurcation of the cycles $q_n$ and $\widetilde{q}_n$, respectively. There is also a set of curves $\sigma_{2^i}, i \geq 0$, given by

$$\sigma_{2^i}: \quad \left(a_l^{\delta_i}a_r^{\delta_{i+1}}\right)^2 + (a_l/a_r)^{(-1)^{i+1}} - 1 = 0, \tag{22}$$

where $\delta_i = (2^i - (-1)^i)/3$. The curve $\sigma_{2^i}$ for $i \geq 1$ corresponds to the first homoclinic bifurcation of harmonic $2^i$-cycle, causing the merging bifurcation $Q_{2,2^{i+1}} \Rightarrow Q_{2,2^i}$, and the curve $\sigma_1$ $(i = 0)$ is related to the first homoclinic bifurcation of the fixed point leading to the merging bifurcation $Q_{2,\,2} \Rightarrow Q_1$. The curves $\sigma_{2^i}$ for $i \to \infty$ are accumulating to the point $(a_l, a_r) = (1, -1)$ (see Fig. 2(a)).

To construct a normal form for the border collision occurring in map $T$ when its fixed point collides with the border point $w_c$ (in which case $w_M^* = w_L^* = w_c = 1$) we have to evaluate the left- and right-side derivatives of $T$ at $w = 1$ for the parameter values belonging to the boundary $BC_{LM}$ given in (7):

$$a_l^* = \lim_{w\uparrow 1}\frac{d}{dx}T(w) = \alpha, \quad a_r^* = \lim_{w\downarrow 1}\frac{d}{dx}T(w) = -\frac{\alpha}{(1-\alpha)(m-1)}. \tag{23}$$

The relation between a point belonging to $BC_{LM}$ and the parameters $a_l$, $a_r$ of the skew tent map is given by

$$(a_l, a_r) = \left(\alpha, -\frac{\alpha}{(1-\alpha)(m-1)}\right),$$

so, if a parameter point moves along the boundary $BC_{LM}$ the related point in the $(a_l,\ a_r)$-parameter plane moves along the curve denoted $\mathcal{B}_m$:

$$\mathcal{B}_m: \quad a_r = -\frac{a_l}{(1-a_l)(m-1)}. \tag{24}$$

Recall that the curve $BC_{LM}$ is valid for $\beta = B\frac{1-\alpha}{\alpha} > 1$, i.e., for $\alpha < \frac{B}{B+1}$. Moreover, $\alpha > 1 - \frac{1}{m}$ (see (2)). So, the curve $\mathcal{B}_m$ is valid in the range

$$1 - \frac{1}{m} < a_l < \frac{B}{B+1}, \quad \text{or} \quad \frac{-B}{m-1} < a_r < -1, \tag{25}$$

which is nonempty for $B > m - 1$.

Using the bifurcation curves of the skew tent map we can state the following

**Proposition 1.** *Consider map $T$ given in (5) for some fixed parameter values satisfying (2), and let $\beta = (1 - 1/m)/\mu$ (the boundary $BC_{LM}$). Consider the bifurcation structure of the $(a_l,\ a_r)$-parameter plane of the skew tent map given in (15) for $\varepsilon > 0$, defined by the curves (18)-(22), and let $(a_l, a_r) = (a_l^*, a_r^*)$ as defined in (23). Then the BCB occurring in map $T$ when its parameter point crosses transversely the boundary $BC_{LM}$ leads from the attracting fixed point $w_L^*$ to the following attractor:*

- *$n$-cycle $g_n$, $n \geq 2$, if $(a_l^*, a_r^*)$ is below the BCB curve $\phi_n$ and above the flip bifurcation curve $\psi_n$;*
- *$2n$-cyclic chaotic intervals $G_{n,\,2n}$, $n \geq 3$, if $(a_l^*, a_r^*)$ is below the BCB curve $\phi_n$, the flip bifurcation curve $\psi_n$, and above the merging bifurcation curve $\gamma_n$;*
- *$n$-cyclic chaotic intervals $G_{n,\,n}$, $n \geq 3$, if $(a_l^*, a_r^*)$ is below the BCB curve $\phi_n$, the merging bifurcation curve $\gamma_n$ and above the expansion bifurcation curve $\widetilde{\gamma}_n$;*
- *$2^i$-cyclic chaotic intervals $G_{2,2^i}$, $i \geq 1$, if $(a_l^*, a_r^*)$ is below the BCB curve $\phi_2$, the flip bifurcation curve $\psi_2$, the merging bifurcation curve $\sigma_{2^i}$ and above the merging bifurcation curves $\sigma_{2^{i-1}}$;*
- *Otherwise, the attractor is chaotic interval $G_1 = [T^2(w_c), T(w_c)]$.*

To illustrate this proposition we present in Fig. 2(a) the bifurcation structure of the $(a_l,\ a_r)$-parameter plane of the skew tent map together with the curves $\mathcal{B}_m$ for different values of $m$, and in Fig. 2(b) it is shown the 2D bifurcation diagram in the $(\mu,\ \alpha)$-parameter plane for $m = 1.05$, $B = 1.5$, where the curve $BC_{LM}$ corresponds to the curve $\mathcal{B}_{1.05}$.

Let us associate the regions which are crossed by the curve $\mathcal{B}_{1.05}$ (see Fig. 2(a) and Eqs. (18)–(22)) with the attractors which appear when the curve $BC_{LM}$ is crossed (see Fig. 2*(b)*). First note that due to (25) the curve $\mathcal{B}_{1.05}$ is valid for $-30 < a_r < -1$. Starting from the point $p_0'$ of $\mathcal{B}_{1.05}$ with $a_r = -1$, the curve $\mathcal{B}_{1.05}$ intersects (moving from above to below) the curve $\psi_2$ at the point $p_1'$, the curves $\sigma_2$ and $\sigma_1$ at the points $p_2', p_3'$, the curve $\phi_3$ at the point $p_4', \psi_3$ at $p_5'$, $\gamma_3$ at $p_6', \widetilde{\gamma}_3$ at $p_7'$, and so on, up to the intersection with the curve $\widetilde{\gamma}_5$ at the point $p_{15}'$. It can be checked that $\mathcal{B}_{1.05}$ does not intersect any other bifurcation curve. Substituting (24) to the related Eqs. (18)–(22), we obtain the $a_l$-coordinates of the intersection points, that is, $a_l = \alpha \equiv \alpha_j$, $j = 0, \ldots, 15$, which then can be substituted to (7) (recall that $\beta = B\frac{1-\alpha}{\alpha}$). In such a way we obtain the corresponding points $p_i$ of the curve $BC_{LM}$ (see Fig. 2(b)). Namely, the $\alpha$-coordinates of the points $p_j$ are the following: $\alpha_0 = 0.047619$, $\alpha_1 \approx 0.199961$, $\alpha_2 \approx 0.201786$, $\alpha_3 \approx 0.203248$, $\alpha_4 \approx 0.218205$, $\alpha_5 \approx 0.322973$, $\alpha_6 \approx 0.324797$, $\alpha_7 \approx 0.326245$, and so on. The intersection point of $BC_{LM}$ and $BC_{LR}$ is $(\mu, \alpha) = (0.047619, 0.6)$ related to the end point of $\mathcal{B}_{1.05}$ with $a_r = -30$.

Let $BC_{LM}|_{p_j}^{p_{j+1}}$ denote an open arc of the curve $BC_{LM}$ bounded by the points $p_j$ and $p_{j+1}$. Now we can state, for example, that if the parameter point crosses the arc $BC_{LM}|_{p_0}^{p_1}$ then an attracting 2-cycle $g_2$ is born due to this BCB, because the related arc $\mathcal{B}_{1.05}|_{p_0'}^{p_1'}$ belongs to the stability region of the 2-cycle of the skew tent map. Similarly we can conclude that crossing $BC_{LM}|_{p_1}^{p_2}$, $BC_{LM}|_{p_2}^{p_3}$ and $BC_{LM}|_{p_3}^{p_4}$ leads to chaotic intervals $G_{2,\,4}$, $G_{2,\,2}$ and $G_1$, respectively, while crossing $BC_{LM}|_{p_4}^{p_5}$ leads to an attracting 3-cycle $g_3$, and so on.

Analyzing Fig. 2(a) one can conclude also that for larger values of $m$ less periodicity regions are intersected by $\mathcal{B}_m$. For example, the curve $\mathcal{B}_2$ intersects only the 2-periodicity region (which is in fact intersected by $\mathcal{B}_m$ for any $m$), thus, besides an attracting 2-cycle only chaotic attractors can appear due to the BCB. It is clear also that for fixed $B$ the interval of valid values of $\alpha$ (see (25)) decreases for increasing $m$.

## 4. Crossing the curve $FB_M$: flip bifurcation of the fixed point

Let us consider now the flip bifurcation of the fixed point $x_M^*$ which occurs if the parameter point crosses the boundary of the region **D**, the curve $FB_M$ given in (11). As we show below, this bifurcation can be supercritical, subcritical or degenerate as illustrated in Fig. 3 by means of 1D bifurcation diagrams.

Namely, in Fig. 3*(a)* one can see that decreasing $\mu$ a pair of 2-cycles ($g_2$ attracting and $\widetilde{g}_2$ repelling) are born due to a fold BCB before the subcritical flip bifurcation of the fixed point. So, in the interval between these two bifurcations the attracting fixed point $w_M^*$ coexists with the 2-cycles $g_2$ and $\widetilde{g}_2$. Then, if we continue to decrease $\mu$, at the subcritical flip bifurcation the fixed point $w_M^*$ loses stability merging with $\widetilde{g}_2$ so that after the bifurcation the map $T$ has the attracting 2-cycle $g_2$ and the repelling fixed point. The DFB of $w_M^*$ illustrated in Fig. 3*(b)* also leads to an attracting 2-cycle $g_2$, but the characteristic feature of this bifurcation is that at the bifurcation value any point of the interval $[w_c,\ T(w_c)]$, except for the fixed point $w_M^*$, is 2-periodic, including the end points of this interval. Thus, we have $T^2(w_c) = w_c$, that is, the BCB of the 2-cycle $g_2$ occurs simultaneously with the DFB of $w_M^*$. As for the

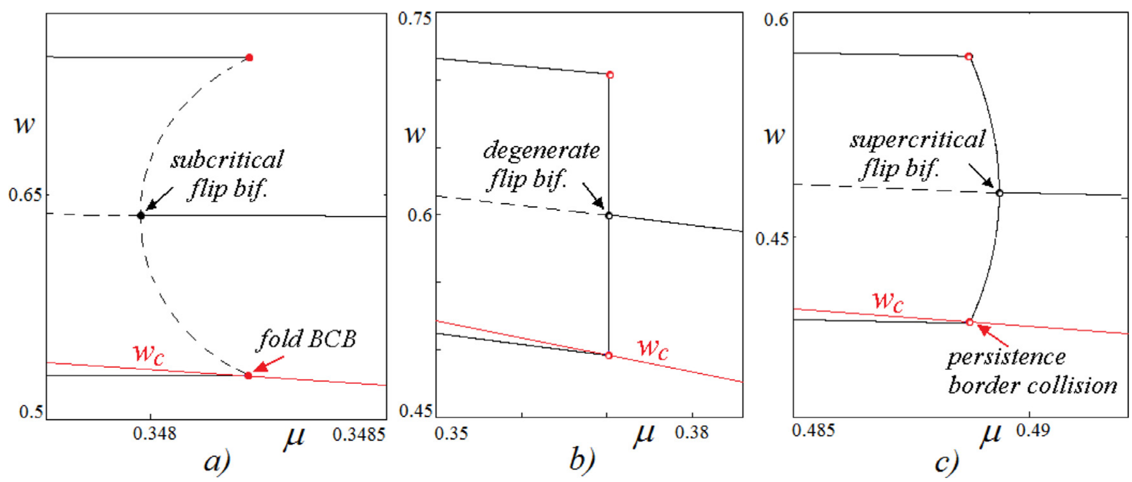


**Fig. 3.** 1D bifurcation diagrams illustrating subcritical *(a)*, degenerate *(b)* and supercritical *(c)* flip bifurcation of the fixed point $w_M^*$. Here $m = 1.2$ and $\alpha = 0.47,\ \beta = 2.25$ in *(a)*, $\alpha = 0.5,\ \beta = 2.25$ in *(b)*,$\alpha = 0.6,\ \beta = 2$ in *(c)*.

supercritical flip bifurcation (see Fig. 3*(c)*) note that soon after this bifurcation the attracting 2-cycle $g_2$ changes its symbolic sequence, from *MM* to *LM*, due to a persistence border collision. That is, one periodic point of the 2-cycle crosses the boundary $w_c$ (from the region *M* to the region *L*) so that a border collision occurs, but the attractor is a 2-cycle before the bifurcation with symbolic sequence *MM* and persists as a 2-cycle after the bifurcation, with symbolic sequence *LM*.

The conditions of degenerate, sub- and supercritical flip bifurcations of $w_M^*$ are stated in the following

**Proposition 2.** *The flip bifurcation of the fixed point $w_M^*$ of the map T defined in (5) occurs for parameter values satisfying (2) and (10) at $\beta = \alpha(m(1-\alpha))^{1-\frac{1}{\alpha}}/\mu$ (the boundary $FB_M$). The flip bifurcation of $w_M^*$ is supercritical for $\alpha > 0.5$, subcritical for $\alpha < 0.5$ and degenerate for $\alpha = 0.5$.*

To prove this proposition we have to check the sign of $(T_M^2)'''(w)$ evaluated at the fixed point $w_M^*$ for the bifurcation parameter value, namely, if we have $(T_M^2)'''(w_M^*) < 0$ then the flip bifurcation is supercritical, while for $(T_M^2)'''(w_M^*) > 0$ it is subcritical (see, e.g., [33]). In the case of a DFB (when it is $(T_M^2)'''(w_M^*) = 0$), it is enough to show that $T_M^2(w) \equiv w$ occurs in an interval around $w_M^*$ (see [31]).

In order to simplify the calculations let us introduce a change of variable, $x := (1 - w/m)$, and let also $\gamma = \alpha/(1-\alpha)$, $C = (\mu\beta)^\gamma/m$. Now the middle branch $T_M$ of map $T$ has the form $t(x) = 1 - Cx^\gamma$, and its fixed point satisfies $x_M^* = 1 - C(x_M^*)^\gamma$. It is easy to see that at the flip bifurcation value we have $x_M^* = \alpha$. Using this equality after some algebraic computations and rearrangements we get

$$(t^2)'''(x_M^*) = (\gamma C)^2(1-\gamma)(x_M^*)^{2(\gamma-2)}(1+\gamma),$$

so that the sign of this expression depends on $\gamma$, namely, $(t^2)'''(x_M^*) < 0$ for $\gamma > 1$, and $(t^2)'''(x_M^*) > 0$ for $\gamma < 1$. Coming back to the map $T$ and the original parameters we conclude that for $\alpha > 0.5$ we have $(T_M^2)'''(w_M^*) < 0$, thus, the flip bifurcation is supercritical, while for $\alpha < 0.5$ the inequality $(T_M^2)'''(w_M^*) > 0$ holds, so that the flip bifurcation is subcritical. For $\alpha = 0.5$ corresponding to $\gamma = 1$ we have $C = 1$, so that

$$t^2(x) = 1 - C(1 - Cx^\gamma)^\gamma|_{C=1,\gamma=1} \equiv x.$$

Thus, the flip bifurcation is degenerate. For map $T$ this means that any point of the absorbing interval, except for the fixed point $w_M^*$, is 2-periodic. The absorbing interval in such a case is $J = [w_c, T(w_c)]$ for the parameter region $E - I$, and $J = [w_B, T(w_B)]$ for the region $E - II$.

As we can see in Fig. 3, all the bifurcation sequences associated with the flip bifurcation of the fixed point $w_M^*$ include a border collision of a 2-cycle. Let us consider it in more details. The condition which is to be satisfied is

$$T_M \circ T_L(w_c) = w_c$$

and the related boundary in the parameter space is denoted $BC_2$:

$$BC_2:\ \left[\frac{1}{\mu\beta}\left(1 - \frac{w_c^\alpha}{m}\right)\right]^{\frac{\alpha}{1-\alpha}} = w_c. \tag{26}$$

(See, for example, the curve $BC_2$ shown in case of subcritical flip bifurcation of $w_M^*$ in Fig. 2*(b)*). To see the result of this bifurcation we can use the skew tent map as a normal form for the border collision of the related fixed point of the map $T^2$. For this we need to evaluate the left- and right-side derivatives of $T^2$ at $w = w_c$ for the parameter values belonging to $BC_2$. Obviously, $a_l^* = (T_M \circ T_L)'(w_c) < 0$ and $a_r^* = (T_M^2)'(w_c) > 0$, and the skew tent map (15) with $\varepsilon < 0$ can be used as a normal form. However, it is easy to show that bifurcation structure of the $(a_l,\ a_r)$-parameter plane for $\varepsilon < 0$ is symmetric with respect to $a_l = a_r$ to the one for $\varepsilon > 0$. Thus, we can use the results related to dynamics of the skew tent map presented in the previous section considering the symmetric point $(a_l, a_r) = (a_r^*, a_l^*)$. In particular, one can check that $a_l^* = (T_M \circ T_L)'(w_c) > -1$ for

$$w_c^\alpha\left(1 + \frac{\alpha^2}{1-\alpha}\right) < m \tag{27}$$

and $a_r^* = (T_M^2)'(w_c) > 1$ for $\alpha < 0.5$. The point $(a_l, a_r) = (a_r^*, a_l^*)$ with $a_l > 1$ and $0 < a_r < 1$ belongs to the region at which the skew tent map has an attracting and repelling fixed points (in Fig. 2*(a)* a small part of this region can be seen), and a fold BCB occurs in the skew tent map if $\varepsilon$ passes through 0. Thus, in the map $T^2$ also a fold BCB occurs. For map $T$ this means that the border collision occurring at $BC_2$ is also a fold BCB leading to a pair of 2-cycles, an attracting $g_2$ and a repelling $\widetilde{g}_2$, with symbolic sequences *LM* and *MM*, respectively. We can check also that crossing $BC_2$ for $\alpha = 0.5$ always leads to one attracting 2-cycle. To see this, note that the curve $FB_M$ at $\alpha = 0.5$ is defined by

$$FB_M|_{\alpha=0.5}: \qquad \mu\beta = \frac{1}{m},$$

and the branches of map $T$ are $T_L(w) = \sqrt{w}$ and $T_M(w) = m - x$ with the border point $w_c = (-1 + \sqrt{1+4m})^2/4$. We have

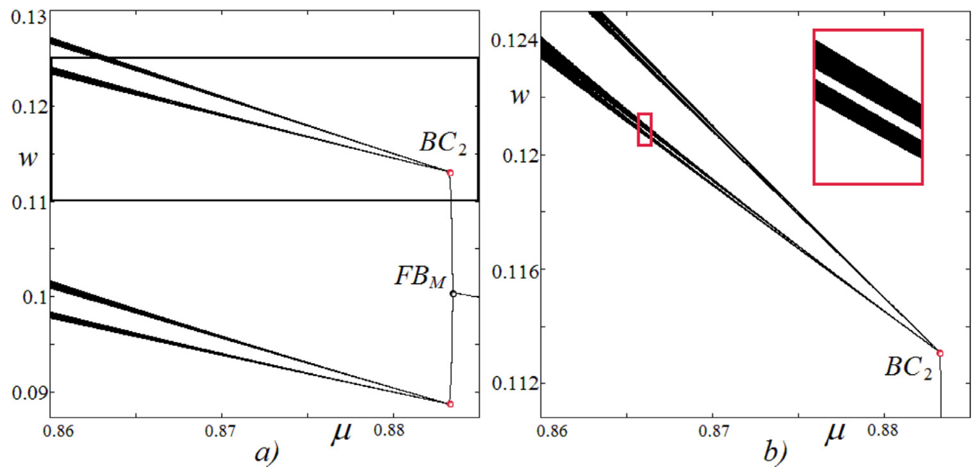


**Fig. 4.** 1D bifurcation diagram in the map $T$ for $\alpha = 0.9, m = 1.005, \beta = 1.315, \mu \in [0.86, 0.885]$ is shown in *a)*,and its enlargements are in *b)*. Here the BCB of the 2-cycle leads to 8-cyclic chaotic intervals.

$(T_M^2)'(w_c) = 1$, while $(T_M \circ T_L)'(w_c) > -1$, where the last inequality holds for $m > 3/4$, that is always true given that $m > 1$. Thus, the 2-cycle born due to this bifurcation (with symbolic sequence $LM$) is attracting. For $\alpha > 0.5$ we have $(T_M^2)'(w_c) < 1$ and $(T_L \circ T_M)'(w_c) > -1$ (for the parameter values satisfying (27)), so that due to collision with $w = w_c$ the 2-cycle remains attracting and only changes its symbolic sequence from $MM$ to $LM$ (persistence border collision). If the condition (27) does not hold, that is, if $(T_L \circ T_M)'(w_c) < -1$, then the crossing of the curve $BC_2$ leads to two repelling 2-cycles and to a chaotic attractor. An example of such a bifurcation is shown in Fig. 4.

Suppose that map $T$ has an attracting 2-cycle $g_2 = \{w_1, w_2\}$ with symbolic sequence $LM$. Let us obtain the condition of its flip bifurcation. First, from $T_M \circ T_L(w_1) = w_1$ we get that $w_1 = [(1 - w_1^\alpha/m)/\mu\beta]^{\frac{\alpha}{1-\alpha}}$. Then, from $(T_M \circ T_L)'(w)|_{w=w_1} = -1$ we get $w_1^\alpha = m(1-\alpha)/(\alpha^2 - \alpha + 1)$, so that the flip bifurcation of $g_2$ occurs for

$$FB_2: \quad \mu\beta = \frac{\alpha^2}{\alpha^2 - \alpha + 1}\left(\frac{(\alpha^2 - \alpha + 1)}{m(1-\alpha)}\right)^{\frac{1-\alpha}{\alpha^2}}. \tag{28}$$

Note that for $\alpha = 0.5$ the curve $FB_2$ is given by

$$FB_2|_{\alpha=0.5}: \qquad \mu\beta = \frac{3}{4m^2}.$$

## 5. Overall bifurcation structure of the region E-I

In this section we discuss the overall bifurcation structure of the region $E-I$ defined in (12). The bifurcation structure of the region $E-II$ defined in (13) is studied in detail in [32]. Recall that the region $E-I$ is confined by the boundaries $BC_{LM}$ (7), $FB_M$ (11) and $BC_J$ (14). Using Proposition 1 which describes the dynamics of map $T$ in a neighborhood of the curve $BC_{LM}$ we can state which bifurcation curves issue from this boundary, namely, from the points $p_j$, $j = 0, \dots, l$ (where $l$ depends on the parameters). Recall that these points correspond to the intersection points of the curve $\mathcal{B}_m$ (24) with the bifurcation curves (18)–(22) of the skew tent map.

Note that all the points $p_j$ are codimention-two bifurcation points, for which, as we have already mentioned, the skew tent map does not help to state precisely which attractor appears after the BCB. Consider, for example, the codimension-two bifurcation point $p_0$, at which the BCB of the fixed point occurs simultaneously with its flip bifurcation, that is, the fixed point is (one-side) nonhyperbolic. Such a point is called *border-flip* codimention-two bifurcation point. It is shown in [8], focusing, in particular, on the geometric shapes of the bifurcation curves around a border-flip point, that in general three bifurcation curves are issuing from such a point, among which one is a curve related to the smooth bifurcation and the other two curves are BCB curves. In fact, in Fig. 2*(b)* we see that besides the curve $BC_{LM}$ two more curves issue from the border-flip point $p_0$, namely, the curve $FB_M$ corresponding to the subcritical flip bifurcation of the fixed point $w_M^*$ and the curve $BC_2$ related to the fold BCB of the 2-cycle. Clearly, if the curve $BC_{LM}$ is crossed at the point $p_0$, then the parameter point can enter to the narrow region bounded by the curves $BC_2$ and $FB_M$, where an attracting 2-cycle coexists with the attracting fixed point. Such a coexistence obviously cannot be classified using only the skew tent map. In fact, any border-flip point of $BC_{LM}$ corresponding to the intersection of the BCB curve $\mathcal{B}_m$ and DFB curve $\psi_n$, $n \geq 2$ (as, e.g., the points $p_1$ and $p_5$ indicated in Fig. 2*(b)*), is an issuing point of two curves, namely, a flip bifurcation curve $FB_n$ and a border collision curve $BC_{2n}$.

Let us suppose that the curve $\mathcal{B}_m$ crosses an $n$-periodicity region of the skew tent map, for $n \geq 3$, that is, there is an arc $\mathcal{B}_m|_{p'_j}^{p'_{j+1}}$ belonging to this region (as shown in Fig. 2*(a)* for several values of $m$). A neighborhood of the curve $BC_{LM}$ in such a case is shown schematically in Fig. 5. According to Proposition 1 in the one-side neighborhood of the arc $BC_{LM}|_{p_j}^{p_{j+1}}$ there must be a region related to an attracting $n$-cycle $g_n$ of map $T$ (to simplify, the region related to the attracting cycle $g_n$ is denoted in Fig. 5 in the same way as the cycle, that is, $g_n$. Similar notations are used for the regions related to other attractors). Its boundary issuing from the point $p_j$ is related to the fold BCB satisfying the condition

$$BC_n: \quad T_L^{n-2} \circ T_M \circ T_L(w_c) = w_c.$$

Note that due to continuity of map $T$ at $w = w_c$ an equivalent condition of $BC_n$ is $T_L^{n-2} \circ T_M^2(w_c) = w_c$. Crossing the boundary $BC_n$ (from the right to the left in Fig. 5) two $n$-cycles are born, an attracting cycle $g_n$ and a repelling cycle $\widetilde{g}_n$. The cycle $g_n$ has a periodic point $w_n$ which satisfies $T_L^{n-1} \circ T_M \circ T_L(w_n) = w_n$, while the cycle $\widetilde{q}_n$ has a periodic point $\widetilde{w}_n$ satisfying $T_L^{n-2} \circ T_M^2(\widetilde{w}_n) = \widetilde{w}_n$.

The boundary of the $n$-periodicity region issuing from the point $p_{j+1}$ is related to the flip bifurcation of $g_n$ defined by the condition

$$FB_n: \quad \left(T_L^{n-2} \circ T_M \circ T_L\right)'(w_n) = -1. \tag{29}$$

As already mentioned, one more bifurcation curve issues from $p_{j+1}$, namely, the curve $BC_{2n}$ related to the border collision of a $2n$-cycle $g_{2n}$ (as show in [8], it is tangent to the flip bifurcation

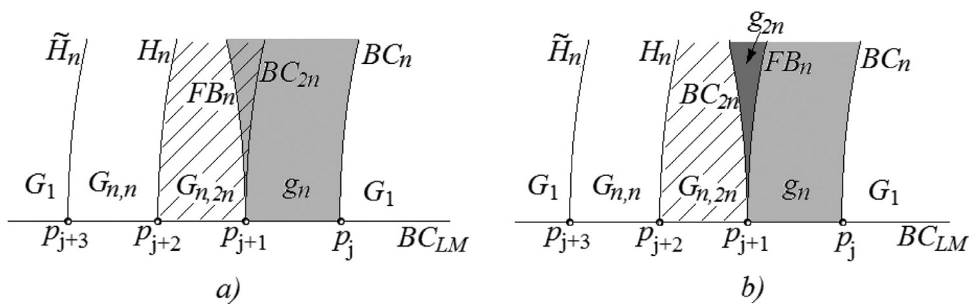


**Fig. 5.** A neighborhood of the curve $BC_{LM}$ shown schematically in case when the BCB curve $\mathcal{B}_m$ given in (24) related to $BC_{LM}$ crosses an $n$-periodicity region of the skew tent map. The flip bifurcation at $FB_n$ is subcritical in *a)* and supercritical in *b)*. The point $p_{j+1}$ is a border-flip codimention-two bifurcation point.

curve). The curve $BC_{2n}$ satisfies the condition

$$BC_{2n}:\quad \left(T_L^{n-2}\circ T_M\circ T_L\right)^2(w_c)=w_c. \tag{30}$$

Given that the arc $\mathcal{B}_m|_{p'_{j+1}}^{p'_{j+2}}$ belongs to the region related to a $2n$-cyclic chaotic intervals $Q_{n,\,2n}$ of the skew tent map, in the one-side neighborhood of the arc $BC_{LM}|_{p_{j+1}}^{p_{j+2}}$ there is a region related to $2n$-cyclic chaotic intervals $G_{n,\,2n}$ (see the dashed region in Fig. 5). There are two possibilities: if the flip bifurcation $FB_n$ is subcritical, as in Fig. 5(a), then in the region between $FB_n$ and $BC_{2n}$ an attracting $n$-cycle $g_n$ coexists with a chaotic attractor $G_{n,\,2n}$, while if the flip bifurcation $FB_n$ is supercritical, as in Fig. 5(b), the region between $BC_{2n}$ and $FB_n$ is related to an attracting $2n$-cycle $g_{2n}$. More precisely, in Fig. 5(a) the curve $BC_{2n}$ belongs to the stability region of $g_n$, and the bifurcation occurring at $BC_{2n}$ is a fold BCB leading to a pair of repelling $2n$-cycles, $g_{2n}$, $\widetilde{g}_{2n}$, and to a chaotic attractor $G_{n,\,2n}$ coexisting with the $n$-cycle $g_n$ (in fact, as we illustrate in Fig. 8(b), or Fig. 9(b), the cycle $\widetilde{g}_{2n}$ separates the basins of $G_{n,\,2n}$ and $g_n$, while the cycle $g_{2n}$ belongs to $G_{n,\,2n}$). Then, moving from the right to the left the curve $FB_n$ is crossed at which the repelling cycle $\widetilde{g}_{2n}$ merges with the attracting cycle $g_n$ due to a subcritical flip bifurcation, so that after this bifurcation the attractor is $G_{n,\,2n}$. In case of supercritical flip bifurcation, the crossing of the curve $BC_{2n}$ leads from an attracting cycle $g_{2n}$ to a chaotic attractor $G_{n,\,2n}$ (see Fig. 5(b)).

Next, we can state that the one-side neighborhood of the arc $BC_{LM}|_{p_{j+2}}^{p_{j+3}}$ (see Fig. 5) is related to $n$-cyclic chaotic intervals $G_{n,\,n}$ of map $T$ because the related arc $\mathcal{B}_m|_{p'_{j+2}}^{p'_{j+3}}$ belongs to the region of $n$-cyclic chaotic intervals $Q_{n,\,n}$ of the skew tent map. Its boundary issuing from the point $p_{j+2}$ is related to the first homoclinic bifurcation of the cycle $g_n$, which satisfies the conditions

$$H_n:\quad \begin{cases}\left(T_L^{n-2}\circ T_M\circ T_L\right)^2(w_c)=w_n,\\ T_L^{n-2}\circ T_M\circ T_L(w_n)=w_n.\end{cases} \tag{31}$$

So, crossing the curve $H_n$ we observe the merging bifurcation $G_{n,\,2n}\Rightarrow G_{n,\,n}$. See, for example, the curve $H_3$ in Fig. 6 and the related merging bifurcation $G_{3,6}\overset{H_3}{\Rightarrow}G_{3,3}$ in Fig. 9*(a)*. The boundary issuing from the point $p_{j+3}$ corresponds to the first homoclinic bifurcation of the cycle $\widetilde{g}_n$ and satisfies the conditions

$$\widetilde{H}_n:\quad \begin{cases}T_L^{n-2}\circ T_M\circ T_L(w_c)=\widetilde{w}_n,\\ T_L^{n-2}\circ T_M^2(\widetilde{w}_n)=\widetilde{w}_n.\end{cases} \tag{32}$$

Thus, crossing the curve $\widetilde{H}_n$ an expansion bifurcation $G_{n,\,n}\Rightarrow G_1$ occurs. An example of the curve $\widetilde{H}_3$ is shown in Fig. 6, and the related expansion bifurcation $G_{3,3}\overset{\widetilde{H}_3}{\Rightarrow}G_1$ is illustrated in Fig. 9*(a)*.

As we have seen, the curve $\mathcal{B}_m$ may not intersect the $n$-periodicity regions for $n\geq 3$, of the skew tent map (see Fig. 2*(a)*).

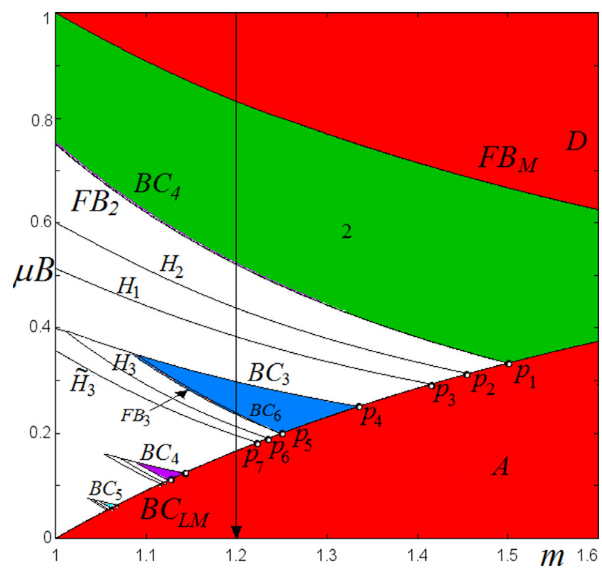


**Fig. 6.** 2D bifurcation diagram in the $(m,\,\mu B)$-parameter plane at $\alpha=0.5$. 1D bifurcation diagram at $m=1.2$ and its enlargements are shown in Fig. 8 and Fig. 9.

The description presented above can be easily adjusted to such a case. However, the 2-periodicity region is intersected for any $m$, and this case differs from the one described above. In fact, we know that from the border-flip point $p_0$ of the curve $BC_{LM}$ the boundaries $FB_M$ and $BC_2$ issue related to the flip bifurcation of the fixed point $w_M^*$ and border collision of the 2-cycle $g_2$, as we show schematically in Fig. 7. Differently from the generic case we have three possibilities as stated in Proposition 2 (see also Fig. 3):

(1) if the flip bifurcation is subcritical, that holds for $\alpha<0.5$, then the curve $BC_2$ is related to a fold BCB leading to a pair of 2-cycles, an attracting one ($g_2$) and a repelling one ($\widetilde{g}_2$), in which case the region between $BC_2$ and $FB_M$ is related to coexisting attractors, the fixed point $w_M^*$ and the 2-cycle $g_2$ (see Fig. 7*(a)*);

(2) if the flip bifurcation is supercritical, that holds for $\alpha>0.5$, then the curve $BC_2$ is a persistence border collision curve crossing which the 2-cycle $g_2$ born before due to supercritical flip bifurcation just changes its symbolic sequence, remaining attracting (see Fig. 7*(b)*);

(3) if the flip bifurcation is degenerate that holds for $\alpha=0.5$, we have $FB_M=BC_2$, so that crossing this boundary one attracting cycle $g_2$ appears (with symbolic sequence $LM$).

Thus, in the one-side neighborhood of the arc $BC_{LM}|_{p_0}^{p_1}$ there is a region related to an attracting 2-cycle $g_2$ of map $T$. From the border-flip point $p_1$ the boundaries $FB_2$ and $BC_4$ originate related to the flip bifurcation of $g_2$ and BCB of $g_4$. The next point $p_2$ corresponds to the intersection of $\mathcal{B}_m$ with the curve $\sigma_{2i}$ (22) for some $i\geq 1$. From $p_2$ a curve denoted $H_{2i}$ issues (see Fig. 7), related to the

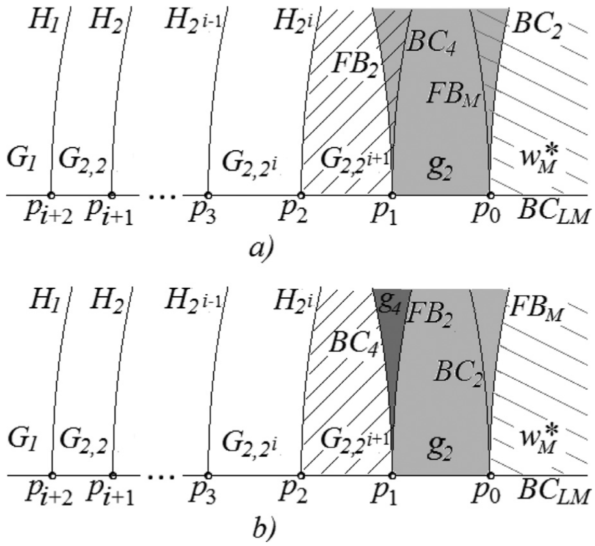


**Fig. 7.** A neighborhood of the curve $BC_{LM}$ shown schematically near the border-flip point $p_0$. The flip bifurcation at $FB_M$ is subcritical in *a)* and supercritical in *b)*. The point $p_1$ is also a border-flip codimention-two bifurcation point.

first homoclinic bifurcation of the harmonic $2^i$-cycle of the map $T$. For the skew tent map the crossing of the curve $\sigma_{2^i}$ leads to the merging bifurcation $Q_{2,2^{i+1}} \Rightarrow Q_{2,2^i}$. Thus, in the one-side neighborhood of the arc $BC_{LM}|_{p_1}^{p_2}$ there is a region related to $2^{i+1}$-cyclic chaotic intervals $G_{2,2^{i+1}}$, and the crossing of $BC_4$ leads to a chaotic attractor $G_{2,2^{i+1}}$. Similarly, the point $p_3$ is an issuing point for the curve $H_{2^{i-1}}$ related to the first homoclinic bifurcation of the harmonic $2^{i-1}$-cycle of map $T$, and so on, up to the point $p_{i+2}$ which is an issue point of the curve $H_1$ related to the first homoclinic bifurcation of the fixed point $w_M^*$ (see Fig. 7). For example, from the point $p_{i+1}$ of the curve $BC_{LM}$ related to the intersection of $\mathcal{B}_m$ with the curve $\sigma_2$ (see (22) for $i = 1$), the curve $H_2$ issues which corresponds to the first homoclinic bifurcation of the cycle $g_2$, satisfying the conditions

$$H_2\colon \quad \begin{cases} (T_M \circ T_L)^2(w_c) = w_2, \\ T_M \circ T_L(w_2) = w_2. \end{cases} \tag{33}$$

The crossing of this curve leads to the merging bifurcation $G_{2,4} \overset{H_2}{\Rightarrow} G_{2,2}$ (see, e.g., Fig. 8*(a)* and the curve $H_2$ in Fig. 6 issuing from the point $p_2$). From the point $p_{i+2}$ the curve $H_1$ issues corresponding to the first homoclinic bifurcation of the fixed point $w_M^*$, satisfying the conditions

$$H_1\colon \quad \begin{cases} T_L \circ T_M \circ T_L(w_c) = w_M^*, \\ T_M(w_M^*) = w_M^*. \end{cases} \tag{34}$$

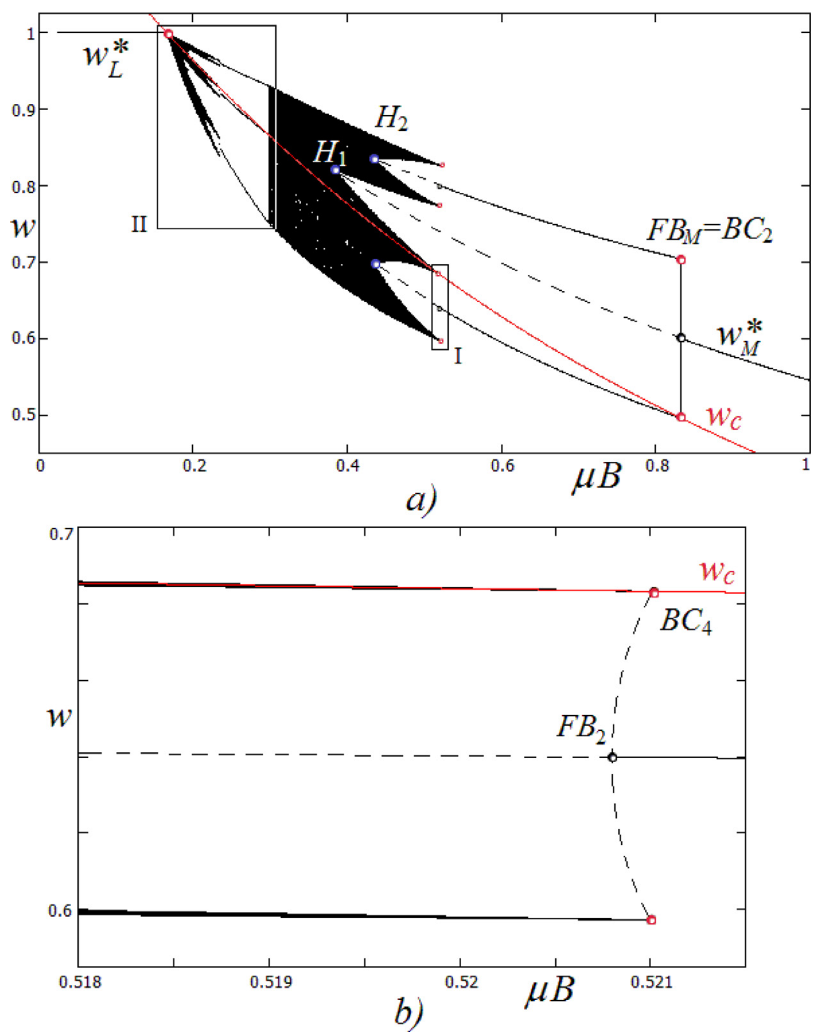


**Fig. 8.** In *(a)* 1D bifurcation diagram of the map $T$ is shown for $\alpha = 0.5$, $m = 1.2$ and $\mu B \in [0, 1]$ related to the vertical line with an arrow in Fig. 6. In *(b)* the window I indicated in *a)* is shown enlarged.

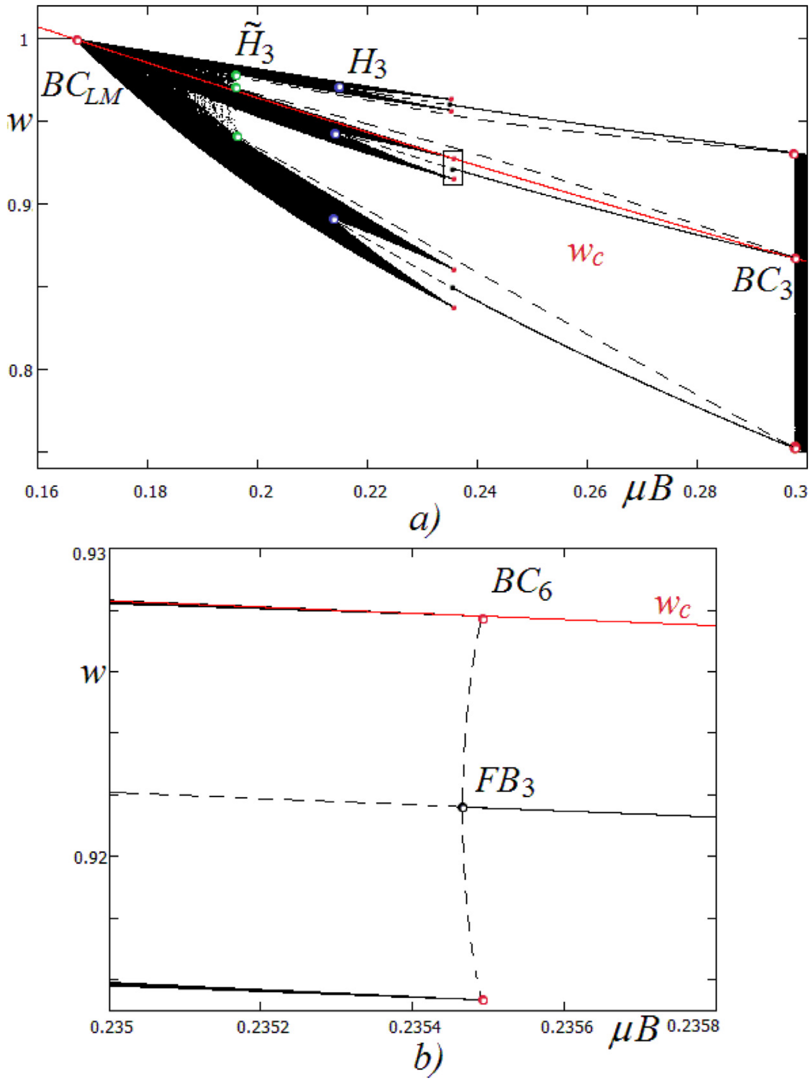


**Fig. 9.** In *(a)* an enlargement of window II indicated in Fig. 8(a) is shown, and in *(b)* the window indicated in *(a)* is enlarged.

The crossing of this curve leads to the merging bifurcation $G_{2,2} \overset{H_1}{\Rightarrow} G_1$ (see, e.g., Fig. 8*(a)* and the corresponding curve $H_1$ in Fig. 6 issuing from the point $p_3$).

The bifurcation structure described above is illustrated in Fig. 6 in the $(m, \mu B)$-parameter plane at $\alpha = 0.5$. The curve $BC_{LM}$ in such a case is defined by

$$BC_{LM}|_{\alpha=0.5}: \quad \mu B = 1 - \frac{1}{m}$$

(note that for $\alpha = 0.5$ we have $B = \beta$). The curve $\mathcal{B}_m$ (24) in the $(a_l, a_r)$-parameter plane of the skew tent map represents a vertical line $a_l = 0.5$ where $\frac{-B}{m-1} < a_r < -1$ (see (25)):

$$\mathcal{B}_m|_{\alpha=0.5}: \quad a_l = 0.5, \quad a_r = -\frac{1}{m-1}. \tag{35}$$

Using the Eqs. (18)–(22) we can obtain the points $p'_j$, $j = 0, \ldots, 15$, related to the intersection of $\mathcal{B}_m|_{\alpha=0.5}$ with the bifurcation curves of the skew tent map. Then, substituting the related values $a_r$ into (35) we obtain the $m$-coordinates of the point $p_j$ of the curve $BC_{LM}$ (see Fig. 6). The curves issuing from the points $p_j$ in Fig. 6 are obtained numerically using the related conditions (29)–(34).

To illustrate the bifurcations (29)–(34) we present in Fig. 8*(a)* a 1D bifurcation diagram related to the vertical line with an arrow indicated in Fig. 6. Enlargements of this diagram are shown in Fig. 8*(b)* and Fig. 9. The sequence of observed bifurcations for decreasing $\mu B$ can be summarized as follows:

$$w^*_M \overset{FB_M=BC_2}{\Rightarrow} g_2 \overset{BC_4}{\Rightarrow} \{g_2, G_{2,4}\} \overset{FB_2}{\Rightarrow} G_{2,4} \overset{H_2}{\Rightarrow} G_{2,2} \overset{H_1}{\Rightarrow} G_1$$
$$\overset{BC_3}{\Rightarrow} g_3 \overset{BC_6}{\Rightarrow} \{g_3, G_{3,6}\} \overset{FB_3}{\Rightarrow} G_{3,6} \overset{H_3}{\Rightarrow} G_{3,3} \overset{\tilde{H}_3}{\Rightarrow} G_1 \overset{BC_{LM}}{\Rightarrow} w^*_L$$

## 6. Conclusion

In the present paper we have studied the dynamics of a credit cycle model introduced in [20], under the additional assumption that the production function is Cobb-Douglas. In the generic case this model is defined by a 4-parameter family of 1D piecewise smooth maps with upward, downward and flat branches. We have considered the cases for which the flat branch is not involved in the asymptotic dynamics, that correspond to the region $E - I$ given in (12).

The bifurcation structure of the region $E - I$ is described in detail. It is formed by the boundaries related to border collision bifurcations characteristic for nonsmooth systems, as well as flip bifurcations and homoclinic bifurcations (causing merging and expansion of the chaotic attractors). These boundaries separate regions corresponding to different attractors of the map, namely, attracting cycles and chaotic attractors (cyclic chaotic intervals). In particular, possible results of a BCB of the fixed point are classified in Proposition 1 using skew tent map as a border collision normal form. The conditions are obtained under which this BCB leads directly to an attracting cycle of period $n$, or to an $n$-cyclic chaotic attractor, $n \geq 1$. The skew tent map helps also to describe the overall bifurcation structure of the region $E - I$ in a neighborhood of the BCB boundary. Proposition 2 states that the flip bifurcation of the fixed point is supercritical for $\alpha > 0.5$, subcritical for $\alpha < 0.5$ and degenerate for $\alpha = 0.5$. It is shown that an attracting 2-cycle which appears due to the supercritical flip bifurcation soon after collides with the border point. In fact, a cascade of flip bifurcations characteristic for smooth unimodal maps is not realized in the considered map. The subcritical flip bifurcation is characterized by bistability related to coexistence of an attracting fixed point and an attracting 2-cycle which is born, together with a repelling 2-cycle, due to a fold BCB before the flip bifurcation. From an economic point of view this implies corridor stability, i.e., the steady state of the economy is stable against small shocks but unstable against large shocks. Furthermore, when the steady state loses its stability via such a subcritical flip bifurcation, the effect is catastrophic and irreversible in that restoring the stability of the steady state by reversing the parameter change is not enough for the economy to return to the steady state. Examples of an attracting cycle coexisting with a cyclic chaotic attractor are also presented. It is important to emphasize that chaotic attractors of the considered map are robust, that is, they are persistent under parameter perturbations.

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