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Chaos, Solitons and Fractals 21 (2004) 403–412
CHAOS SOLITONS & FRACTALS
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Tongues of periodicity in a family of two-dimensional discontinuous maps of real Möbius type

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Accepted 4 December 2003

Abstract

In this paper we consider a two-dimensional piecewise-smooth discontinuous map representing the so-called "relative dynamics" of an Hicksian business cycle model. The main features of the dynamics occur in the parameter region in which no fixed points at finite distance exist, but we may have attracting cycles of any periods. The bifurcations associated with the periodicity tongues of the map are studied making use of the first-return map on a suitable segment of the phase plane. The bifurcation curves bounding the periodicity tongues in the parameter plane are related with saddle-node and border-collision bifurcations of the first-return map. Moreover, the particular "sausages structure" of the bifurcation tongues is also explained.

1. Introduction

The present article reconsiders the Hicksian multiplier-accelerator model of business cycles. It was introduced by Samuelson [8] in 1939, and was based on two interacting principles: consumers that spend a fraction c of past income, $C_t = cY_{t-1}$, and investors that maintain a stock of capital K_t in given proportion a to the income Y_t. With an additional time lag for the construction period of capital equipment, net investment, by definition the change in capital stock, $I_t = K_t - K_{t-1}$, becomes $I_t = a(Y_{t-1} - Y_{t-2})$. As income is generated by consumption and investments, i.e. $Y_t = C_t + I_t$, a simple feed back mechanism $Y_t = (c+a)Y_{t-1} - aY_{t-2}$ was derived.

Hicks [3] in 1950 further developed this model. As it stands, it just produces damped or explosive oscillatory motion. Hicks preferred to model a system producing sustained limited amplitude oscillations and gave an economic explanation for how the model should be changed to give this result: in a depression phase of the business cycle we have $I_t = a(Y_{t-1} - Y_{t-2}) < 0$, and it can even happen that income decreases at a pace so fast that more capital can be dispensed with than disappears through natural wear. As nobody actively destroys capital, there is a lower limit to disinvestment, called the "floor", and fixed at the (negative) net investment when no worn out capital is replaced at all. So the investment function is changed to $I_t = \max(a(Y_{t-1} - Y_{t-2}), -I^f)$, where I^f is the absolute value of the floor disinvestment. (Hicks also suggested that there be a "ceiling" at full employment, when income could not be expanded any further, but we do not consider this at present.)

To the complete model, which Hicks never formulated mathematically, also belong exponentially growing "autonomous expenditures", which produce a growth trend, around which the business cycles, produced by the model, provide the fluctuations. To make this type of model suitable for analysis, the floor and the ceiling must then be

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0960-0779/$ - see front matter
doi:10.1016/j.chaos.2003.12.012

assumed to be growing too, and even at the same rate as the autonomous expenditures. This seems to have been Hicks's own tacit assumption, however, all these equal growth rates look fairly arbitrary.

In a previous paper [7], the authors tried their hands at a slight reformulation, through actually relating the floor to the stock of capital, putting $I_t^f = rK_t$, where K_t is capital stock and r is the rate of depreciation. Making this change to the model, we get the benefit that the growing trend need not be exogenously introduced. It results *within* the model through capital accumulation and hence *explains* both the growth trend and the fluctuations around it. Growth is something economists regard as a good feature, but it is no good for the use of standard mathematical methods, as all variables explode to infinity at an exponential rate. To make the model suitable for analysis, we focus on relative dynamics, the rate of growth of income, and the actual capital/output ratio.

Before this reduction to relative dynamics, we, however, have to state the complete model. As noted, the consumption function is $C_t = cY_{t-1}$, and the investment function is $I_t = \max(a(Y_{t-1} - Y_{t-2}), -rK_{t-1})$, where c, a and r are real parameters. As the fraction of income spent is positive but less than unity, $0 < c < 1$. Further, the capital output ratio obviously is a positive number, so $a > 0$. Finally, the capital depreciation is a small positive number, so we definitely have $0 < r < 1$. As before, the income formation equation reads $Y_t = C_t + I_t$, and we now also add an updating equation for capital stock $K_t = K_{t-1} + I_t$. This completes the model. We can eliminate C_t and I_t, and thus obtain a recurrence map in the income and capital variables alone: $Y_t = cY_{t-1} + \max(a(Y_{t-1} - Y_{t-2}), -rK_{t-1})$, and $K_t = K_{t-1} + \max(a(Y_{t-1} - Y_{t-2}), -rK_{t-1})$.

In order to obtain the relative dynamics, define: $x_t := K_t/Y_{t-1}$ and $y_t := Y_t/Y_{t-1}$ These new variables are the actual capital/output ratio, and the relative change of income from one period to the next, i.e. the growth factor. Using these variables results in the following iterated map:

$$x_t = \frac{x_{t-1}}{y_{t-1}} + a\left(1 - \frac{1}{y_{t-1}}\right), \quad y_t = c + a\left(1 - \frac{1}{y_{t-1}}\right)$$

if

$$x_{t-1}(a(y_{t-1} - 1) + rx_{t-1}) \geqslant 0$$

and

$$x_t = (1 - r)\frac{x_{t-1}}{y_{t-1}}, \quad y_t = c - r\frac{x_{t-1}}{y_{t-1}}$$

if

$$x_{t-1}(a(y_{t-1} - 1) + rx_{t-1}) < 0.$$

It is obvious that the domain of definition for this map is not the entire phase plane (x, y), but this plane with exclusion of the line of nondefinition, $y = 0$, as well as of all its preimages of any rank.

In the next sections we shall investigate the dynamic properties of this map. We shall see that an attracting fixed point may exist. However, the more interesting features occur in a parameter region in which no attracting fixed point exists, whereas we can have attracting cycles of any period. The bifurcation diagram in the (a, c)-parameter plane shows a structure qualitatively similar to that occurring at the Neimark bifurcation. However, there is no Neimark bifurcation in our model. The main purpose of the present paper is to explain the bifurcation structure associated with such tongues of periodicity. To perform this study we construct a one-dimensional "first-return map" on a suitable segment. That is we reduce the degree of our map by using a suitable Poincaré section on a well defined segment which is necessarily visited by the trajectories.

The plan of the work is as follows. After this introduction, Section 2 describes the main characteristics of the two maps which are involved in our model, showing a two-dimensional bifurcation diagram at a fixed value of r. We only use one value of r in this paper as any other value of this parameter in its admitted range gives bifurcation diagrams having a qualitatively similar structure. In Section 3 we introduce the Poincaré section and show how the bifurcation curves may be detected by using this "first-return map". Section 4 illustrates some more properties of the bifurcation curves, related to the "sausages structure", and Section 5 is the conclusion.

2. Description of the model

As introduced in the previous section, we are interested in a family of two-dimensional nonlinear discontinuous maps $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by two maps F_1 and F_2 defined in the regions R_1 and R_2, respectively:

$$F : (x, y) \mapsto \begin{cases} F_1(x, y), & \text{if } (x, y) \in R_1; \\ F_2(x, y), & \text{if } (x, y) \in R_2; \end{cases} \tag{1}$$

where

$$F_1 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x/y + a(1 - 1/y) \\ c + a(1 - 1/y) \end{pmatrix}, \quad R_1 = \{(x, y) : x(a(y - 1) + rx) \geqslant 0\};$$

$$F_2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x(1 - r)/y \\ c - x/y \end{pmatrix}, \quad R_2 = \{(x, y) : x(a(y - 1) + rx) < 0\}.$$

As we recall, a, c and r are real parameters such that $a > 0$, $0 < c < 1$, $0 < r < 1$.

One can see that with this definition of the map F the (x, y)-phase plane is separated into four regions by the straight lines $x = 0$ and $y = 1 - rx/a$, so that the map F_1 is defined in $R_1 = \{x \geqslant 0, y \geqslant 1 - rx/a\} \cup \{x \leqslant 0, y \leqslant 1 - rx/a\}$ and F_2 is defined in $R_2 = \{x < 0, y > 1 - rx/a\} \cup \{x > 0, y < 1 - rx/a\}$. We call these straight lines *critical lines* and denote them LC_{-1} and LC'_{-1}:

$$LC_{-1} = \{(x, y) : y = 1 - rx/a\};$$

$$LC'_{-1} = \{(x, y) : x = 0\}.$$

The map F is continuous on LC_{-1}. Its image by F is the straight line

$$LC = \{(x, y) : y = c - rx/(1 - r)\}.$$

The map F is discontinuous on LC'_{-1}, the image of which, using the map F_1, is the straight line

$$LC' = \{(x, y) : y = x + c\},$$

while by using F_2 we get just a point $(0, c)$.

Studying the map F numerically we get an interesting two-dimensional bifurcation diagram in the (a, c)-parameter plane (see Fig. 1) with tongues of periodicity which look like the Arnol'd tongues that usually appear due to the Neimark bifurcation. But for the map F this is not the case, as there are no fixed point with complex eigenvalues on the bifurcation curve. So, the purpose of the present paper is to consider the origin and structure of these tongues.

Let us first show that the maps F_1 and F_2 alone have rather simple dynamics.

The map F_1 is triangular: the variable y is mapped, independently of x, by a one-dimensional map f:

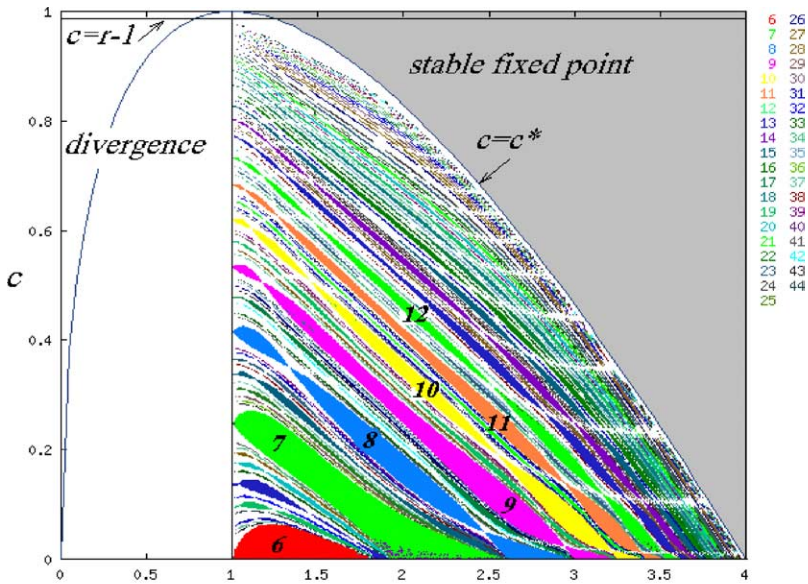

Fig. 1. Bifurcation diagram of the map F at $r = 0.01$.

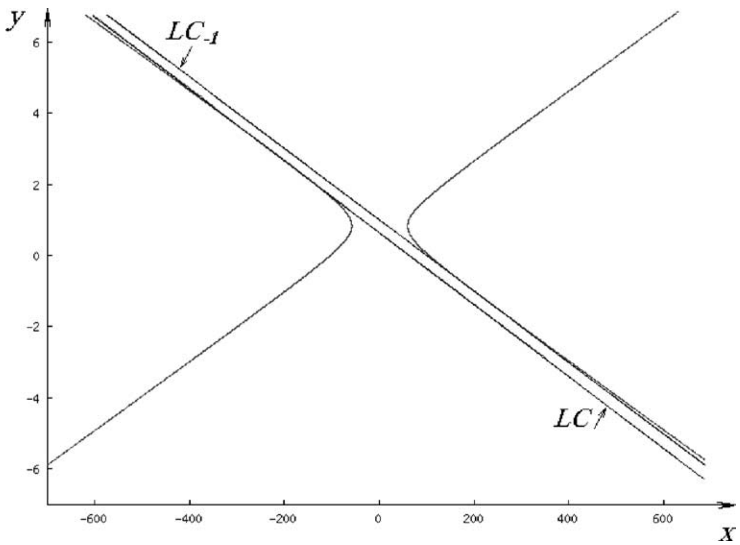

Fig. 2. A trajectory of the map F for $r = 0.01$, $a = 1$ and $c = 0.65$. The trajectory is tangent to the boundary of R_1.

$$f : y \mapsto \frac{(c+a)y - a}{y} \tag{2}$$

which is a so-called real Möbius map. [1]It has two fixed points denoted y_+ and y_-

$$y_{\pm} = \frac{c + a \pm \sqrt{(c+a)^2 - 4a}}{2} \tag{3}$$

which have real values for

$$c \geqslant c^* \stackrel{\text{def}}{=} 2\sqrt{a} - a. \tag{4}$$

At $c = c^*$, these fixed points appear due to a saddle-node bifurcation. For $0 < c < c^*$ the map f has neither fixed points, nor cycles of any period (due to the fact that for Möbius maps the solutions to the equation $f^k(y) = y$, $k > 1$, are the same as those to $f(y) = y$). Thus, for $0 < c < c^*$ any trajectory of f, and of F_1 as well, is diverging.

Let us check the stability of y_+ and y_- when they exist. An eigenvalue of f can be written $\lambda_1(y) = a/y^2$. For the parameter range considered $0 < \lambda_1(y_+) < 1$ and $\lambda_1(y_-) > 1$. Thus, the fixed point y_+ is attracting and y_- is repelling.

The corresponding fixed points of the map F_1 are (a, y_-) and (a, y_+). The second eigenvalue of F_1 is $\lambda_2(y) = 1/y$. For $c < 2 - a$ we have $\lambda_2(y_+) > 1$ and $\lambda_2(y_-) < 1$, while for $c > 2 - a$ the inequalities $\lambda_2(y_+) < 1$ and $\lambda_2(y_-) < 1$ hold. Thus, taking (4) into account, we conclude that for $a > 1$, $c^* < c < 1$ the fixed point (a, y_+) is an attracting node and (a, y_-) a saddle, whereas, for $0 < a < 1$, $c^* < c < 1$, (a, y_+) is a saddle and (a, y_-) is a repelling node. To summarize, we can state the following

Proposition 1. *For $0 < c < c^*$ any trajectory of F_1 is diverging. For $0 < a < 1$, $c^* < c < 1$ any trajectory (except those with initial points at (a, y_+) and (a, y_-), or their preimages) is diverging as well.*

Consider now the map F_2. Any straight line (t, mt), $t \in R$, of slope m is mapped by F_2 into one point $(x', y') = ((1 - r)/m, c - r/m)$ belonging to the critical line LC. The dynamic behavior of F_2 is thus reduced to a

[1] A real rational map $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f : x \mapsto \frac{ax + b}{cx + d}, \quad \text{where } ad - cb \neq 0,$$

is called real Möbius map.

one-dimensional map on the straight line LC. If x is the first coordinate of a point $(x, y) \in LC$, then its image by F_2 on LC is given by a one-dimensional map g:

$$g : x \mapsto g(x) = \frac{x(1-r)^2}{c(1-r) - rx}, \tag{5}$$

which again is a Möbius map. It has two fixed points: $x_1 = 0$ and $x_2 = (1-r)(c-1+r)/r$. If $c < 1-r$, then x_1 is repelling and x_2 is attracting, while if $c > 1-r$, then x_2 is repelling and x_1 is attracting. At $c = 1-r$ these fixed points merge, i.e. $x_1 = x_2 = 0$. The corresponding fixed points of the map F_2 are $p_1(x_1, c)$ and $p_2(x_2, (1-r))$.

We come back to the map F defined in (1). It is now obvious that only if both maps F_1 and F_2 are applied, we can get attracting cycles corresponding to the tongues of periodicity shown in Fig. 1. Numerical analysis shows that for $a < 1$ a generic trajectory of the map F, after some transient, belongs only to R_1 where the map F_1 is applied and, according to Proposition 1, is diverging. (See Fig. 2 where a transient part of a trajectory of the map F is shown for $a = 1$ when this trajectory is just tangent to the boundary of R_1.) Thus, we restrict our considerations to the case $a > 1$.

3. First-return map

Let the following inequalities hold: $a > 1$, $c < c^*$ and $c < 1-r$.

Let $[AB]$ denote a segment of LC which belongs to R_1, that is $[AB] = LC \cap R_1$, where $A = LC \cap LC_{-1}$ and $B = LC \cap LC'_{-1}$ (see Fig. 3a). Assume that a trajectory has a point $(x_k, y_k) \in R_2$. Then, as it was shown, its image by F_2 belongs to LC, i.e. $(x_{k+1}, y_{k+1}) \in LC$. We have that either $(x_{k+1}, y_{k+1}) \in [AB]$, where the map F_1 applies, or we have to apply the one-dimensional map g given in (5). One can easily check, that, for the parameter range considered, the attracting fixed point x_2 of the map g belongs to $[AB]$ while the repelling fixed point of g is just the point B (see Fig. 3b). Thus, approaching x_1, the trajectory must enter the segment $[AB]$, i.e. there exists an integer $s > 0$ such that $(x_{k+1+s}, y_{k+1+s}) \in [AB]$. It follows that we can describe the essential features of the two-dimensional map F by studying a one-dimensional return map on the segment $[AB]$.

Definition 1. The first-return map $\varphi : [AB] \to [AB]$ is a map φ through which the x-coordinate of a given point $(x, y) \in [AB]$ is mapped to the point $\varphi(x)$ which is the x-coordinate of the "first" point satisfying $F^k(x, y) \in [AB]$.

Clearly, the "second" return map of a point of $[AB]$ to the segment $[AB]$ is given by the map φ^2, and generally, the kth return on $[AB]$ is given by φ^k. This will be used to characterize the dynamics of the map F completely. Moreover, the following proposition is immediate, and its proof is left as an exercise:

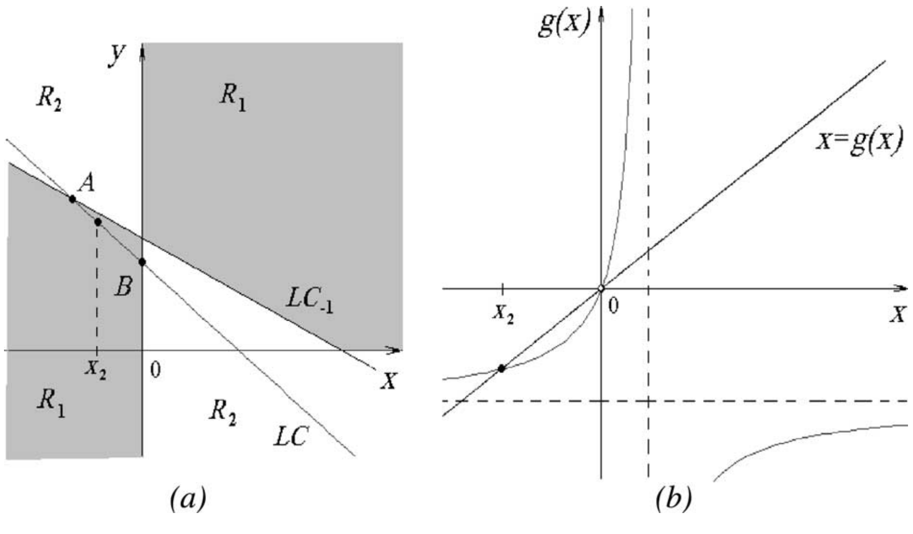

Fig. 3. In (a) it is shown the segment $[AB]$ suitable for the first-return map. In (b) it is shown the one-dimensional map $g(x)$, restriction of F_2 to LC.

Proposition 2. *Let C be an attracting (repelling) cycle of the map F of period $k > 1$ having $p\ (\geqslant 1)$ periodic points in the segment $[AB]$. Then it corresponds to an attracting (repelling) fixed point of the p-return map φ^p.*

We recall that the map F is discontinuous on the y-axis and thus also the map φ is discontinuous. As a matter of fact, we cannot write down the analytical expression for φ, but it is easy to compute it numerically. In the following examples the map φ is used to explain the structure of the periodicity tongues which are shown in Fig. 1.

Consider the point $(a, c) = (1.5, 0.32)$, which in Fig. 1 is above the 8-tongue, and then decrease c. Our objective is to describe the bifurcations which characterize the upper and lower boundary of the tongue.

Let c_1^* denote the value of c corresponding to the upper boundary of the 8-tongue, and c_2^* denote the value corresponding to the lower boundary. The first-return map φ for $c = c_1^* \approx 0.3174$ is shown in Fig. 4, from which we can see that at this parameter value a *saddle-node bifurcation* occurs, giving rise to two fixed points of φ existing for $c_2^* < c < c_1^*$ (see Fig. 5 where $c = 0.3$). This bifurcation is also a border-collision bifurcation for piecewise-smooth maps, because the

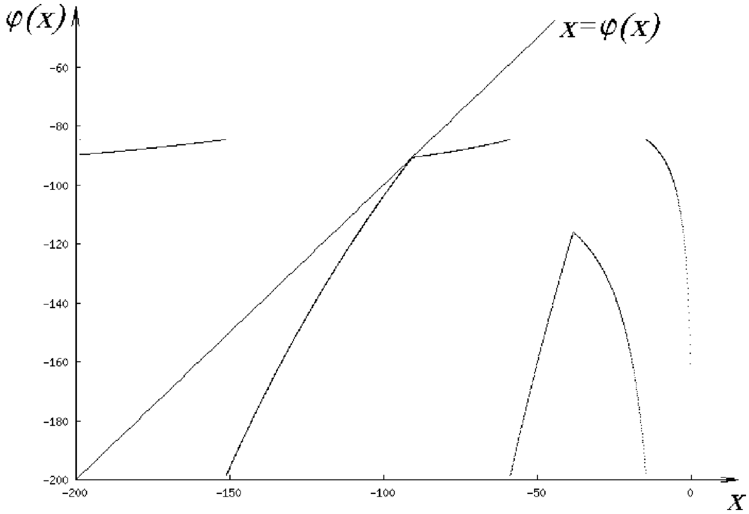

Fig. 4. Saddle-node bifurcation of the first-return map $\varphi(x)$ at $r = 0.01$, $a = 1.5$, $c = 0.3174$.

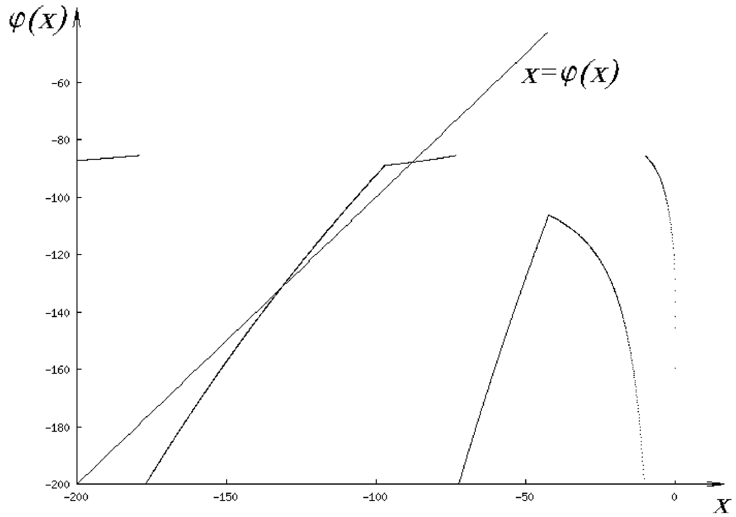

Fig. 5. First-return map $\varphi(x)$ at $r = 0.01$, $a = 1.5$, $c = 0.3$. Two fixed points are clearly visible, one stable and one unstable.

saddle-node bifurcation is not a tangent bifurcation as it is defined for different maps on the right and on the left of the contact point.

Clearly, just knowing the shape of φ we cannot deduce the period of the corresponding orbit of F. We can get this information only through iterating F. In this example the period of the two cycles (one attracting and one saddle) is 8.

As c decreases, the graph of φ is modified so that the distance between the two fixed points increases and they both approach the discontinuity points of φ. At the bifurcation value $c = c_2^* \approx 0.2868$ (the point of the lower boundary of the 8-tongue), the fixed points of φ merge with the discontinuity points (see Fig. 6) and disappear (for $c < c_2^*$). Thus, the lower bifurcation curve corresponds to the *border-collision bifurcation*.

Similar bifurcations occur in all other "main" tongues in which the rotation number of the q-periodic orbit of F is $1/q$.

The order of the tongues with different periodicity in the bifurcation diagram of Fig. 1 follows the usual Farey rule which applies in the similar bifurcation diagram associated with the Neimark bifurcation. This means that between two tongues corresponding to the cycles with rotation numbers p_1/q_1 and p_2/q_2 a tongue exists which corresponds to a cycle with rotation number $(p_1 + p_2)/(q_1 + q_2)$ (see [1,4–6]).

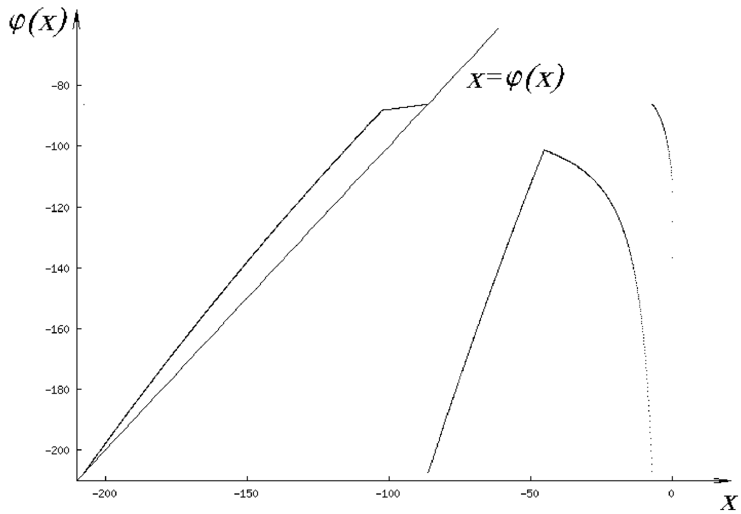

Fig. 6. First-return map $\varphi(x)$ at $r = 0.01$, $a = 1.5$, $c = 0.2868$. Border-collision bifurcation.

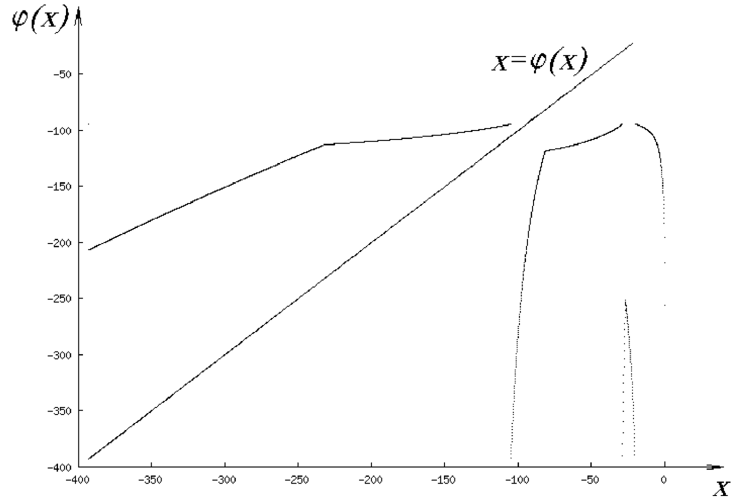

Fig. 7. First-return map $\varphi(x)$ at $r = 0.01$, $a = 1.26$, $c = 0.149$.

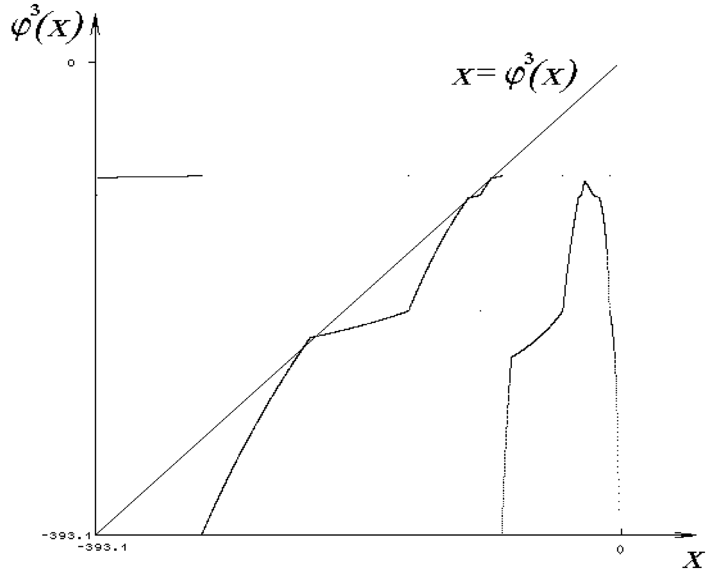

Fig. 8. Three-return map $\varphi^3(x)$ at $r = 0.01$, $a = 1.26$, $c = 0.149$. Three stable and three unstable fixed points are clearly visible.

In our case a tongue associated with a rotation number p/q corresponds to a q-periodic orbit of the map F having p points in the segment $[AB]$, and thus the cycles are represented as fixed points of the p-return map φ^p. As an example, Fig. 7 shows the graph of φ at $a = 1.26$, $c = 0.149$, which has no fixed points. But the corresponding map F for such parameter values has a cycle of period $q = 20$ with $p = 3$ points in the segment $[AB]$. The corresponding three-return map φ^3 is shown in Fig. 8.

From the properties described above we know that the dynamics of the points belonging to the segment $[AB]$ are representative of the dynamics of the map F, and we can state the following

Proposition 3. *Let $a > 1$, $c < c^*$ and $c < 1 - r$. Then an invariant attracting set of the map F belongs to the closure of the set $A = \cup_{n \geqslant 0} F^n([AB])$.*

4. Sausages structure of the periodicity tongues

In this section we give reason of the particular shape of the tongues of periodicity which can clearly be seen in Fig. 1. This shape is due to the piecewise definition of F, and a similar shape in a two-dimensional parameter plane was already described in [1,2,9,10]. We recall here the main features. Let us consider a point $(a, c) = (1.25, 0.365)$ which belongs to the first area of the sausages structure associated with the tongue of periodicity 8 in Fig. 1. At this parameter value (as for any other value in this area), of the periodic points of the 8-cycle of F two belong to region R_2 and six belong to R_1 (see Fig. 9). At the transition point between the first and the second area of the same tongue, one of the periodic points belongs to the critical line LC_{-1}. For $(a, c) = (1.5, 0.3)$, belonging to the second area (as well as for any other point in this second area) of the tongue of period 8 the 8-cycle of F has three points in region R_2 and five in region R_1 (see Fig. 10). For (a, c), belonging to the third area of the same tongue the 8-cycle has four points in region R_1 and four in region R_2. It follows that inside the different "sausage" portions of the areas associated with a tongue, the period is the same, whereas changes occur in the sequence in which the maps (F_1 and F_2) are applied to give the cycle. The "waist" points of such a structure correspond to a border-collision bifurcation of the periodic orbit, which changes only the structure of the cycle (but not its period).

We close this section with a final remark on the dynamics of F associated with points outside the periodicity tongues shown in Fig. 1. It is well known that quasiperiodic orbits, associated with the Neimark bifurcation, correspond to irrational rotation numbers, and also that chaotic regimes may exist between the tongues. For the map F it is not easy to give a definition of an "irrational rotation number". However, in such a case we numerically observe an invariant attracting set which consists of two curves (crossing the line of nondefinition $y = 0$), approaching infinity, whose shape is similar to the one shown in Fig. 2. Moreover, we conjecture that chaotic dynamics cannot occur for the map F.

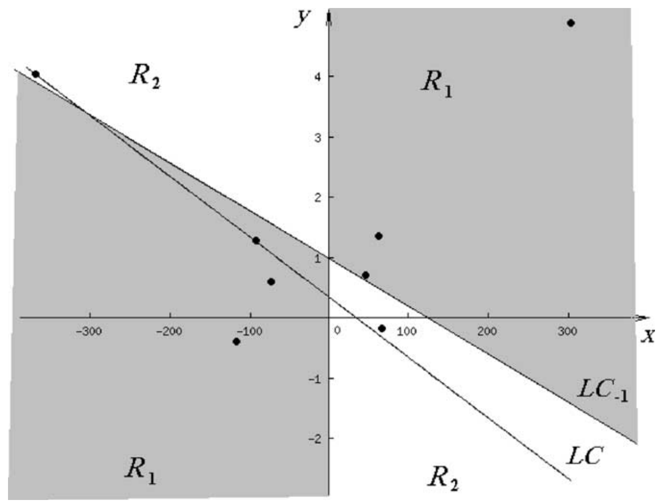

Fig. 9. Attracting 8-cycle of F in the phase plane at $r = 0.01$ and $(a, c) = (1.25, 0.365)$ belonging to the first area of the sausage structure of the period-8 tongue. Two periodic points belong to the region R_2 and six belong to R_1.

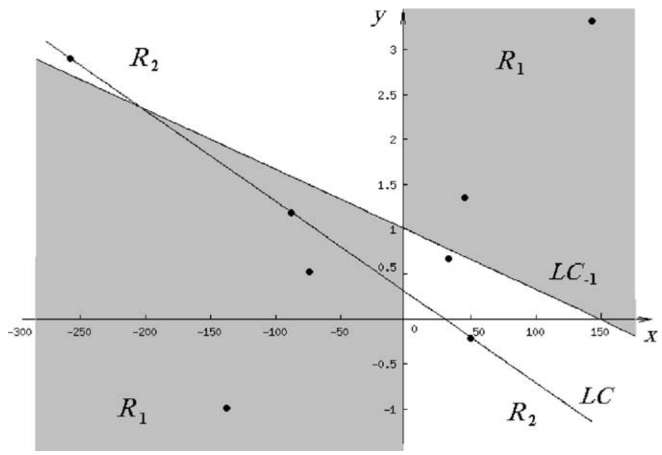

Fig. 10. Attracting 8-cycle of F in the phase plane at $r = 0.01$ and $(a, c) = (1.5, 0.3)$ belonging to the second area of the sausage structure of the period-8 tongue. Three periodic points belong to the region R_2 and five belong to R_1.

A numerical computation of the Lyapunov exponents gives one exponent equal to 0 and the second one with a negative value.

5. Conclusion

In this paper we have considered a two-dimensional piecewise-smooth discontinuous map F representing the so-called "relative dynamics" of the Hicksian business cycle model proposed in [7]. The main features of the dynamics related to this map occur in the parameter region in which no fixed points at finite distance exist, but we may have attracting cycles of any periods. The bifurcations associated with the periodicity tongues of the map have been studied, making use of the first-return map on a suitable segment of the phase plane, belonging to an invariant attracting set of the map. We have thus explained the bifurcation curves bounding the periodicity tongues shown in Fig. 1 and the related "sausages" structure.

The peculiarity of this bifurcation diagram is that it looks like (and possesses the same properties as) a bifurcation diagram associated with the Neimark bifurcation of a fixed point. However, no fixed point at finite distance is involved on the bifurcation line $a = 1$ of that picture, so that it is not a Neimark bifurcation. Thus, the starting points of the tongues, issuing from the line $a = 1$, are still an open problem for our map F, even if we can explicitly obtain their values (described in [7]). The parameter values of the starting point of a tongue of periodicity of p/q are given by $a = 1$ and $c = 2\cos(2\pi p/q) - 1$. But their relation to the two-dimensional map F is still an open question.

Acknowledgements

This work has been performed under the activity of the national research project ''Nonlinear models in economics and finance: complex dynamics, disequilibrium, strategic interactions'', MIUR, Italy, and in the framework of Joint Research Grant (0382): ''Reconsideration of economic dynamics from a new perspective of nonlinear theory'', Chuo University, Tokyo. Iryna Sushko acknowledges the support of the Italian GNFM (Gruppo Nazionale di Fisica Matematica).

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