Center Bifurcation for Two-Dimensional Border-Collision Normal Form

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Abstract

In this work we study some properties associated with the border-collision bifurcations in a two-dimensional piecewise linear map in canonical form, related to the case in which a fixed point of one of the linear maps has complex eigenvalues and undergoes a *center* bifurcation when its eigenvalues pass through the unit circle. This problem is faced in several applied piecewise smooth models, such as switching electrical circuits, impacting mechanical systems, business cycle models in economics, etc. We prove the existence of an invariant region in the phase space for parameter values related to the center bifurcation and explain the origin of a closed invariant attracting curve after the bifurcation. This problem is related also to particular border-collision bifurcations leading to such curves which may coexist with other attractors. We show how periodicity regions in the parameter space differ from Arnold tongues occurring in smooth models in case of the Neimark-Sacker bifurcation, how so-called dangerous border-collision bifurcations may occur, as well as multistability. We give also an example of a *subcritical* center bifurcation which may be considered as a piecewise-linear analogue of the subcritical Neimark-Sacker bifurcation.

Keywords: Border Collision Bifurcations, Border Collision Normal Forms, Center Bifurcation, Piecewise-Linear 2D maps.

1 Introduction

Recently, more and more works have been published showing a rich variety of applied models ultimately described by continuous piecewise smooth or piecewise linear maps which are not differentiable in a subset (of zero measure) of the state space. See, among others, Hommes [1991], Hommes and Nusse [1991], Nusse and Yorke [1992, 1995], Maistrenko et al. [1993, 1995], Yuan [1997], Banerjee and Grebogi [1999], Banerjee et al. [2000a,b], Fournier-Prunaret et al. [2001], Taralova-Roux and Fournier-Prunaret [2002], Sushko et al. [2005, 2006]. For such maps particular kinds of bifurcations may occur, different from those studied in smooth models, which since Nusse and Yorke [1992] are denoted as border collision bifurcations (BCB for short henceforth). This quite recent research subject (although the first works by Feigin date back to the 70th, and were rediscovered only a few years ago, see Di Bernardo et al. [1999]), has been mainly studied by using one- and two-dimensional canonical forms, proposed in Nusse and Yorke [1992], which are piecewise linear maps defined by two linear functions, being this analysis at the basis also of the BCB occurring in piecewise smooth systems. The two-dimensional canonical form has been mainly considered in *dissipative* cases associated with *real* eigenvalues of the point which undergoes the BCB. Among the effects studied up to now are uncertainty about the occurrence after the BCB (see e.g. Kapitaniak and Maistrenko [1998], Dutta et al. [1999]), multistability and unpredictability of the number of coexisting attractors (see e.g. in Zhusubaliyev et al. [2006]), as well as the so-called dangerous BCB (Hassouneh et al. [2004], Ganguli and Banerjee [2005]), related to the case in which a fixed point is attracting before and after the BCB, while at the bifurcation value the dynamics are divergent.

However, in the last years the problem of BCB associated with points having *complex* eigenvalues, was raised in several applied models, see e.g. a sigma-delta modulation model in Feely et al. [2000], several physical and engineering models in Zhusubaliyev and Mosekilde [2003], a dc-dc converter in Zhusubaliyev et al. [2007], business cycles models in economics as in Gallegati et al. [2003], Sushko et al. [2003], Gardini et al. [2006a,b]. The so-called *center bifurcation*, first described in Sushko et al. [2003], associated with the transition of a fixed point to an unstable focus and the appearance of an attracting closed invariant curve, in piecewise linear maps is completely new with respect to the theory existing for smooth maps, known as Neimark-Sacker bifurcation, although, as we shall see, there is a certain analogy: For example, the closed invariant curve made up by the saddle-node connections of a pair of cycles (a saddle and a node) is clearly similar to those occurring in smooth maps, however such a curve is not smooth but made up of finitely or infinitely many (depending on the type of noninvertibility of the map) segments and corner points. While similarly to the Arnold tongues in the smooth case, the periodicity regions in the piecewise linear case may be classified with respect to the rotation numbers, the boundaries of these periodicity regions, issuing from the center bifurcation line at points associated with rational rotation numbers, are BCB curves, instead of saddle-node bifurcation curves issuing from the Neimark-Sacker bifurcation curve. Moreover, while the emanating point from the Neimark-Sacker curve of an Arnold tongue is a cusp point (except for the strong resonance cases $1 : n$, $n = 1, 2, 3, 4$), in the piecewise linear case the periodicity regions are issuing with a nonzero opening angle. In this respect, uncertain is the existence of curves in the parameter space (having zero measure) which we know existing for smooth maps, issuing from the Neimark-Sacker bifurcation curve at points associated with irrational rotation numbers, and corresponding to parameter values for which the closed invariant curve has no cycles but quasiperiodic trajectories (which fill in densely the invariant curve). Some authors (see e.g. Zhusubaliyev et al. [2006], Zhusubaliyev et al. [2007]) refer to the dynamics occurring on closed invariant curves in piecewise linear maps with the same term, namely, speaking of quasiperiodic trajectories. On our opinion this problem deserves a more detailed investigation. In fact, besides the different structure of the tongues in the angle issuing from the bifurcation curves, one more difference with respect to the smooth case has to be emphasized: In a piecewise linear map a closed invariant curve must intersect the boundary separating the regions of different definitions of the system. A point of intersection is mapped into a corner point on the curve. But given that a quasiperiodic orbit is everywhere dense on the curve, and the images of the corner point are generally corner points as well, the corner points are expected to be also everywhere dense. In such a case the closed invariant curve is a nowhere differentiable set, in contrast with the smooth case. Anyhow, we are not going to consider this problem in the present paper: we mainly are interested in the periodicity regions (i.e. the so called "resonant cases") and some of the related bifurcations.

The plan of the work is as follows. In Section 2 we shall introduce a two-dimensional piecewise linear map which is a normal form to study BCB in piecewise smooth two-dimensional maps, describing the case in which one of the fixed points considered is a focus which undergoes a center bifurcation. This is the object of our studies: The dynamics at the center bifurcation value is considered in Section 3, showing how to detect an invariant region in the phase space filled with closed invariant curves on which the dynamics are either quasiperiodic or converging towards a periodic orbit, depending on the rotation number (irrational or rational, respectively). In Section 4 we shall consider the dynamics 'after' the center bifurcation, which gives birth to an invariant attracting closed curve, and the possible effects of the BCB associated with a repelling focus. Here several examples of two-dimensional bifurcation diagrams are presented and the structure of the periodicity regions is explained, related with different kinds of noninvertibility of the map. However, the main characteristic features of the periodicity regions are the same in all the cases: The so-called 'sausages' structure of the regions, their overlapping (associated with the multistability phenomena), the BCB-boundaries of periodicity regions issuing from the center bifurcation line, as well as the mechanism of homoclinic tangle of the stable and unstable sets destroying the closed invariant curve. In Subsection 4.1 we obtain analytically the equation of the BCB boundaries of the main periodicity regions in the parameter space, using which we show how an issuing point of a periodicity region is not a cusp point. In Subsection 4.2 we consider in detail the boundaries of the periodicity region associated with the rotation number 1/3 while in Subsection 4.3 we illustrate overlapping periodicity regions representing coexistence of several stable cycles both before and after the center bifurcation. We give also an example of the dangerous BCB (the term was introduced in Hassouned et al. [2004]) related to the case in which the fixed point is attracting before and after the collision with the border, and divergent trajectories also exist, so that at the BCB value the basin of attraction of the fixed point shrinks to zero size and all the orbits with non zero initial conditions diverge to infinity. Finally, in Section 5 we present the example which may be considered as a piecewise-linear analogue of the *subcritical* Neimark-Sacker bifurcation for smooth maps. Section 6 gives some conclusion.

2 Border-Collision Normal Form

As it was proposed in Nusse and Yorke [1992], the normal form for the border-collision bifurcation in a 2D phase space, a real plane, is represented by a family of two-dimensional piecewise linear maps $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by two linear maps F_1 and F_2 which are defined in two half planes L and R:

$$F : (x, y) \mapsto \begin{cases} F_1(x, y),\ (x, y) \in L; \\ F_2(x, y),\ (x, y) \in R; \end{cases} \tag{1}$$

where

$$F_1 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \tau_L x + y + \mu \\ -\delta_L x \end{pmatrix},\ L = \{(x, y) : x \le 0\}; \tag{2}$$

$$F_2 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \tau_R x + y + \mu \\ -\delta_R x \end{pmatrix},\ R = \{(x, y) : x > 0\}. \tag{3}$$

Here τ_L, τ_R are traces and δ_L, δ_R are determinants of the Jacobian matrix of the map F in the left and right halfplanes, i.e., in L and R, respectively, $\mathbb{R}^2 = L \cup R$.

The straight line $x = 0$ separating the regions L and R, and its images (backward by F^{-1} and forward by F) are called *critical lines* of the corresponding rank, that is, $LC_{-1} = \{(x, y) : x = 0\}$ is called basic critical line separating the definition regions of the two maps; $LC = F(LC_{-1}) = \{(x, y) : y = 0\}$ is the critical line (of rank 1) and $LC_i = F^i(LC)$ is the critical line of rank i. For convenience of notation we shall identify LC_i, $i = 0$, with LC. Note that due to continuity of the map F the first image of the straight line $x = 0$ by either F_1 or F_2 is the same straight line $y = 0$, i.e., $F_1(LC_{-1}) = F_2(LC_{-1}) = LC_0$, while LC_i, $i > 0$, is in general a broken line.

Property 1. The map F is invertible for $\delta_L\delta_R > 0$, noninvertible of (Z_0-Z_2)-type for $\delta_L\delta_R < 0$, noninvertible of $(Z_0 - Z_\infty - Z_1)$-type for $\delta_L = 0$, $\delta_R \ne 0$ or $\delta_R = 0$, $\delta_L \ne 0$ and noninvertible of $(Z_0 - Z_\infty - Z_0)$-type for $\delta_L = 0$, $\delta_R = 0$.

To check this property it is enough to consider images of the regions L and R, i.e., $F_1(L)$ and $F_2(R)$. Let $\delta_R \ne 0$, $\delta_L \ne 0$. Then the map F is invertible if $F_1(L) \cap F_2(R) = LC$, i.e., L and R are mapped into two different halfplanes, that is true for $\delta_L\delta_R > 0$. The map F is noninvertible if $F_1(L) = F_2(R)$, i.e., if L and R are mapped into the same halfplane, so that the image of the plane is folded into a halfplane, in each part of which F has two distinct preimages. The map F is noninvertible of $(Z_0 - Z_2)$-type. It is easily to check that this is true for $\delta_L\delta_R < 0$.

If one of the two determinants is 0 then the related halfplane is mapped into the straight line LC, that is any point of LC has an infinity of preimages (a whole halfline), one of the two halfplanes separated by LC has no preimages, and another has one preimage, so that we have $(Z_0 - Z_\infty - Z_1)$-noninvertibility. In such a case the asymptotic dynamics of F are often reduced to a one-dimensional subspace of the phase space (examples may be found in Hommes and Nusse [1991], Kowalczyk [2005], Sushko and Gardini [2006]). In the case in which both the determinants are 0 we have two halfplanes mapped into LC. The map F on LC is reduced to a one-dimensional piecewise linear map

$$f : x \mapsto f(x) = \begin{cases} \tau_L x + \mu, \ x \leq 0, \\ \tau_R x + \mu, \ x \geq 0, \end{cases} \tag{4}$$

representing a border-collision normal form for one-dimensional piecewise smooth continuous maps. The dynamics of the map f have been studied by many authors. See, e. g., Maistrenko et al. [1993], Nusse and Yorke [1995], Banerjee et al. [2000a], Sushko et al. [2006] and references therein, where one can find a full classification of border-collision bifurcations, depending on left- and right-sides derivatives of the map evaluated at the border-crossing fixed point (i.e., depending on the parameters τ_L and τ_R in (4)).

Following Banerjee and Grebogi [1999] we denote by L^* and R^* the fixed points of F_1 and F_2 determined, respectively, by

$$\left(\frac{\mu}{1 - \tau_i + \delta_i}, \frac{-\delta_i \mu}{1 - \tau_i + \delta_i} \right), \ i = L, R.$$

Obviously, L^* and R^* become fixed points of the map F if they belong to the related regions L and R. Namely, L^* is the fixed point of the map F if $\mu/(1 - \tau_L + \delta_L) \leq 0$, otherwise it is a so-called virtual fixed point which we denote by $\overline{L}^*$. Similarly, R^* is the fixed point of F if $\mu/(1 - \tau_R + \delta_R) \geq 0$, otherwise it is a virtual fixed point denoted by $\overline{R}^*$. Clearly, if the parameter μ varies through 0, the fixed points (actual or/and virtual) cross the border LC_{-1}, so that the collision with it occurs at $\mu = 0$, value at which L^* and R^* merge with the origin $(0, 0)$.

Let μ vary from a negative to a positive value. As it was noted in Banerjee and Grebogi [1999], if some bifurcation occurs for μ increasing through 0, then the same bifurcation occurs also for μ decreasing through 0 if we interchange the parameters of the maps F_1 and F_2, i.e., there is a symmetry of the bifurcation structure with respect to $\tau_R = \tau_L$, $\delta_R = \delta_L$ in the $(\tau_R, \tau_L, \delta_R, \delta_L)$-parameter space. Thus, it is enough to consider μ varying from negative to positive.

We consider the parameter values such that the fixed point of the map F is attracting for $\mu < 0$, i.e., before the border-collision. For $\mu < 0$, and $1 - \tau_L + \delta_L > 0$, the point L^* is a fixed point of F. Its stability is defined by the eigenvalues $\lambda_{1,2(L)}$ of the Jacobian matrix of the map F_1, which are

$$\lambda_{1,2(L)} = \left(\tau_L \pm \sqrt{\tau_L^2 - 4\delta_L} \right) /2. \tag{5}$$

The triangle of stability of L^*, say S_L, is defined as follows:

$$S_L = \{(\tau_L, \delta_L) : 1 + \tau_L + \delta_L > 0, 1 - \tau_L + \delta_L > 0, 1 - \delta_L > 0\}. \quad (6)$$

Thus, let $(\tau_L, \delta_L) \in S_L$.

At $\mu = 0$ we have $L^* = R^* = (0, 0)$, i.e., the fixed points collide with the border line LC_{-1}. For $\mu > 0$ (i.e., after the border-collision) and for $1-\tau_R+\delta_R > 0$ the point R^* is the fixed point of F. The eigenvalues $\lambda_{1,2(R)}$ of the Jacobian matrix of the map F_2, and the triangle of stability S_R of R^* are defined as in (5) and (6), respectively, putting the index R instead of L.

Our main purpose is to describe the bifurcation structures of the (δ_R, τ_R) - parameter plane depending on the parameters $(\tau_L, \delta_L) \in S_L$ at some fixed $\mu > 0$. Such a bifurcation diagram reflects the possible results of the border-collision bifurcation occurring when the attracting fixed point of F crosses the border $x = 0$ while μ passes through 0. A classification of the different types of border-collision bifurcation depending on the parameters of F is presented in Banerjee and Grebogi [1999], Banerjee et al. [2000b], but related only to the case in which this map is dissipative on both sides of the border, i.e., for $|\delta_L| < 1$, $|\delta_R| < 1$.

We are interested in a different case, with $|\delta_L| < 1$, $\delta_R > 1$, related, in particular, to a specific type of border-collision bifurcations, giving rise to closed invariant attracting curves. A similar problem is posed in Zhusubaliyev et al. [2006] where among other results there is a descriptive analysis of the bifurcation structure of the (τ_L, τ_R)-parameter plane (called there as a chart of dynamical modes) for some fixed $\delta_R > 1$.

Our approach to investigate the dynamical modes in the (δ_R, τ_R)-parameter plane gives the advantage of discussing the origin of the periodicity regions, namely to connect this problem to the center bifurcation occurring for $\delta_R = 1$, $-2 < \tau_R < 2$, $\mu > 0$. The center bifurcation was first described in Sushko et al. [2003] for a two-dimensional piecewise linear map coming from an economic application. Later it was analyzed in more details in Sushko and Gardini [2006] for some generalized piecewise linear map. In the following section we describe the center bifurcation for the map F.

3 Center bifurcation ($\delta_R = 1$)

Without loss of generality we can fix $\mu = 1$ in the following consideration. Indeed, one can easily see that $\mu > 0$ is a scale parameter: Due to linearity of the maps F_1 and F_2 with respect to x, y and μ, any invariant set of F contracts linearly with μ as μ tends to 0, collapsing to the point $(0, 0)$ at $\mu = 0$.

For $(\tau_L, \delta_L) \in S_L$, $(\tau_R, \delta_R) \in S_R$, $\mu = 1$, the map F has the stable fixed point R^* and the virtual fixed point $\overline{L}^*$. For $\tau_R^2 - 4\delta_R < 0$ the fixed point R^* is an attracting focus. If the (τ_R, δ_R)-parameter point leaves the stability triangle S_R crossing the straight line $\delta_R = 1$, then the complex-conjugate eigenvalues $\lambda_{1,2(R)}$ cross the unit circle, i.e., the fixed point R^* becomes a repelling focus.

At $\delta_R = 1$ the fixed point $R^* = (x^*, y^*)$, $x^* = 1/(2-\tau_R)$, $y^* = -x^*$, is locally a center. What is the phase portrait of the map F in such a case? Note that at $\delta_R = 1$ the map F_2 is defined by a rotation matrix characterized by a rotation number which may be rational, say m/n, or irrational, say ρ. Obviously, there exists some neighborhood of the fixed point in which the behavior of F is defined only by the linear map F_2, i.e., there exists an invariant region included in R filled with invariant ellipses, each point of which is either periodic of period n (in case of a rational rotation number m/n, and we recall that the integer n denotes the period of the periodic orbit while m denotes the number of tours around the fixed point which are necessary to get the whole orbit), or quasiperiodic (in case of an irrational rotation number ρ).

Let F_2 be defined by a rotation matrix with an irrational rotation number ρ, which holds for $\delta_R = 1$, and

$$\tau_R = \tau_{R,\rho} \stackrel{def}{=} 2\cos(2\pi\rho). \tag{7}$$

Then any point from some neighborhood of the fixed point is quasiperiodic, and all the points of the same quasiperiodic orbit are dense on the invariant ellipse to which they belong. In such a case an invariant region Q exists in the phase space, bounded by an invariant ellipse $\mathcal{E}$ of the map F_2, tangent to LC_{-1}, and, thus, also tangent to LC_i, $i = 0, 1, \ldots$. The equation of an invariant ellipse of F_2 with the center (x^*, y^*) through (x_0, y_0) is given by:

$$x^2 + y^2 + \tau_{R,\rho}xy - x + y = x_0^2 + y_0^2 + \tau_{R,\rho}x_0y_0 - x_0 + y_0. \tag{8}$$

In order to obtain an ellipse tangent to LC_{-1}, we first get a tangency point

$$(\overline{x}, \overline{y}) = (0, -1/2), \tag{9}$$

which is *the same for any rotation number*. Then we write the equation of the ellipse (8) through $(\overline{x}, \overline{y})$, that gives us the equation of $\mathcal{E}$:

$$x^2 + y^2 + \tau_{R,\rho}xy - x + y = -1/4. \tag{10}$$

Thus, we can state the following

Proposition 1. *Let $\delta_R = 1$, $\tau_R = \tau_{R,\rho}$ given in (7). Then in the phase space of the map F there exists an invariant region Q, bounded by the invariant ellipse $\mathcal{E}$ given in (10). Any initial point $(x_0, y_0) \in Q$ belongs to a quasiperiodic orbit dense in the corresponding invariant ellipse (8).*

Let now F_2 be defined by the rotation matrix with a rational rotation number m/n, which holds for $\delta_R = 1$, and

$$\tau_R = \tau_{R,m/n} \stackrel{def}{=} 2\cos(2\pi m/n). \tag{11}$$

Then any point in some neighborhood of the fixed point R^* is periodic with rotation number m/n, and all the points of the same periodic orbit are located on an invariant ellipse of F_2. As before, the invariant region we are looking for includes obviously the region bounded by an invariant ellipse, say $\mathcal{E}_1$, tangent to LC_{-1}, given by

$$x^2 + y^2 + \tau_{R,m/n} xy - x + y = -1/4. \tag{12}$$

But in the case of a rational rotation number the invariant region is wider: There are other periodic orbits belonging to R. To see this, note that there exists a segment $S_{-1} = [a_0, b_0] \subset LC_{-1}$, which we call *generating segment*, such that its end points a_0 and b_0 belong to the same m/n-cycle located on an invariant ellipse of F_2 which crosses LC_{-1}, denoted $\mathcal{E}_2$ (note that $\mathcal{E}_2$ is not invariant for the map F). Obviously, the generating segment S_{-1} and its images by F_2, that is, the segments $S_i = F_2(S_{i-1})$, $S_i \subset LC_i = F_2(LC_{i-1})$, $i = 0, ..., n-2$, form a boundary of an invariant polygon denoted by P, with n sides, completely included in the region R. The polygon P is inscribed by $\mathcal{E}_1$ and circumscribed by $\mathcal{E}_2$ (see Fig.1 where such a polygon is shown in the case $m/n = 3/11$).

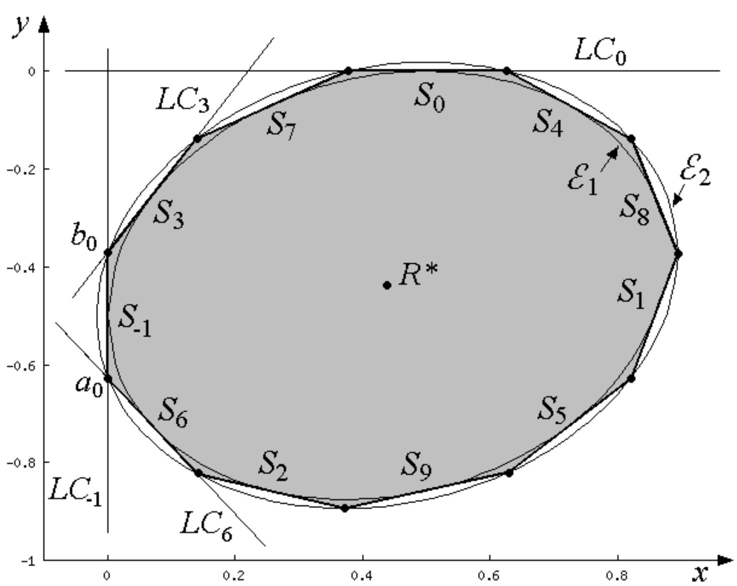

Figure 1: The invariant polygon P of the map F at $\delta_R = 1$, $\tau_R = 2\cos(2\pi m/n)$, $m/n = 3/11$.

The case $m/n = 1/n$ is the simplest one: It can be easily shown that the point $LC_{-1} \cap LC_0 = (0, 0)$ and its $n-1$ images form a cycle of period n, all points of which are in R. The ellipse $\mathcal{E}_2$ through $(0, 0)$ is given by

$$x^2 + y^2 + \tau_{R,1/n} xy - x + y = 0,$$

and the generating segment S_{-1} for any n has the end points $a_0 = (-1, 0)$ and $b_0 = (0, 0)$.

The case m/n for $m \neq 1$ is more tricky. To clarify our exposition we use the example of the rotation number $m/n = 3/11$ (see Fig.1). The end points of the generating segment S_{-1} are obtained as intersection points of LC_{-1} with two critical lines of proper ranks. We first obtain an equation for the image of LC_{-1} of any rank i by F_2 (for convenience, in this section we denote these images by LC_i, as in the general case, but recall that in the general case the images by F_1 have to be also considered so that LC_i is indeed a broken line). Let A denote the matrix defining F_2, i.e.,

$$A = \begin{pmatrix} \tau_{R,m/n} & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2\cos(2\pi m/n) & 1 \\ -1 & 0 \end{pmatrix}.$$

For any integer $0 < i < n$ we can write down

$$A^i = \frac{1}{\sin(2\pi m/n)} \begin{pmatrix} \sin(2\pi(i+1)m/n) & \sin(2\pi i m/n) \\ -\sin(2\pi i m/n) & -\sin(2\pi(i-1)m/n) \end{pmatrix}. \tag{13}$$

(Note that for $i = n$ we get an identity matrix). Making a proper change of coordinates and using (13) we get the following equation for the straight line LC_i for $0 \leq i < n$:

$$LC_i : \qquad y = -\frac{\sin(2\pi i m/n)}{\sin(2\pi(i+1)m/n)}x + \frac{\tan(\pi(i+1)m/n)}{2\tan(\pi m/n)} - \frac{1}{2}.$$

The point of intersection of LC_{-1} and LC_i has the following coordinates:

$$\left(0, \frac{\tan(\pi(i+1)m/n)}{2\tan(\pi m/n)} - \frac{1}{2}\right). \tag{14}$$

Now we need to determine the proper rank k_1 such that the side $S_{k_1} \subset LC_{k_1}$ of the polygon P is an upper adjacent segment of the generating segment S_{-1}. The number n which is the period of the m/n-cycle, can be written as $n = rm + l$, where an integer $r = \lfloor n/m \rfloor$ is the number of periodic points visited for one turn around the fixed point, and an integer $l < m$ is the rest. For our example $m/n = 3/11$ we have $r = 3$ and $l = 2$. Following some geometrical reasoning, which we omit here, one can get that if $(m-1)/l$ is an integer, then

$$k_1 = \frac{(m-1)r}{l}, \tag{15}$$

so that the coordinates of the point b_0 are determined through m and n by substituting $i = k_1$ into (14). It can be easily shown that the coordinates of the other end point a_0 of S_{-1} are determined by substituting $i = k_2$ into (14), where

$$k_2 = n - 2 - k_1. \tag{16}$$

For the example shown in Fig.1 we have $k_1 = 3$ and $k_2 = 6$, so that the end points of S_{-1} are $a_0 = LC_{-1} \cap LC_6$ and $b_0 = LC_{-1} \cap LC_3$, whose coordinates are obtained by substituting $m/n = 3/11$ and, respectively, $i = 6$ and $i = 3$ into (14).

If $(m-1)/l$ is not an integer number, then we use a numerical algorithm to determine k_1 as the rank of the critical line whose intersection with LC_{-1} is m/n-periodic point; k_2 is determined by (16) as before.

Obviously, such a polygon P can be constructed for any rotation number m/n. Summarizing we can state the following

Proposition 2. *Let $\delta_R = 1$, $\tau_R = \tau_{R,m/n}$ given in (11). Then in the phase space of the map F there exists an invariant polygon P with n edges whose boundary is made up by the generating segment $S_{-1} \subset LC_{-1}$ and its $n-1$ images $S_i = F_2(S_{i-1}) \subset LC_i$, $i = 0, ..., n-2$. Any initial point $(x_0, y_0) \in P$ is periodic with rotation number m/n.*

Up to now we have not discussed the behavior of a trajectory with an initial point (x_0, y_0) not belonging to the invariant region (either P or Q), which obviously depends on the parameters δ_L, τ_L of the map F_1. Such a behavior can be quite rich, even in the case we are restricted to, that is for $(\delta_L, \tau_L) \in S_L$ in which the fixed point $\overline{L}^*$ of F_1 is attracting being virtual for F. Without going into a detailed description we give here different examples: A trajectory initiated outside P or Q can be

- attracted to a periodic or quasiperiodic trajectory belonging to the boundary of the invariant region (as, for example, for $\delta_L = 0.3$, $\tau_L = -0.4$, when F is invertible, $\overline{L}^*$ is a focus);

- mapped inside the invariant region (it is possible if F is $(Z_0 - Z_2)$ - noninvertible, like, for example, for $\delta_L = -0.5$, $\tau_L = 0.3$; $\overline{L}^*$ is a flip node);

- mapped to the boundary of the invariant region (it is possible for $(Z_0 - Z_\infty - Z_1)$ - noninvertibility, for example, at $\delta_L = 0$, $\tau_L = -0.3$; $\overline{L}^*$ is a flip node);

- attracted to some other attractor, regular, i.e., periodic or quasiperiodic (e.g., to a periodic attractor for $\tau_R = 0.25$, $\delta_L = 0.9$, $\tau_L = -0.7$) or chaotic (e.g., for $\tau_R = -1.5$, $\delta_L = 0.1$, $\tau_L = 0.63$), coexisting with the invariant region (for both examples $\overline{L}^*$ is a focus);

- divergent (e.g., for $\tau_R = -1.5$, $\delta_L = 0.9$, $\tau_L = -0.7$; $\overline{L}^*$ is a focus).

In the following sections we investigate dynamics of the map F 'after' the center bifurcation, that is for $\delta_R > 1$. Among all the infinitely many invariant curves filling the invariant region (P or Q) at $\delta_R = 1$, only the boundary of it survives, being modified, after the bifurcation, that is for $\delta_R = 1 + \varepsilon$ at some sufficiently small $\varepsilon > 0$. Roughly speaking, the boundary of the former invariant region is transformed into an attracting closed invariant curve $\mathcal{C}$ on which the map F is reduced to a rotation. Similar to the Neimark-Sacker bifurcation occurring for smooth maps, we can use the notion of rotation numbers: In case of a rational rotation number m/n two cycles of period n with rotation number m/n are born at the center bifurcation, one attracting and one saddle, and the closure of the unstable set of the saddle cycle approaching points of the attracting cycle forms the curve $\mathcal{C}$. In the piecewise linear case such a curve is not smooth, but a piecewise linear set, which in general has infinitely many corner points accumulating at the points of the attracting cycle. Differently from the smooth case such a curve is born not in a neighborhood of the fixed point: Obviously, its position is defined by the distance of the fixed point from the critical line LC. Description of such a curve, born due to the center bifurcation for some piecewise linear maps, as well as proof of its existence in particular cases, can be found in Gallegati et al. [2003], Sushko et al. [2003], Zhusubaliyev et al. [2006], Sushko and Gardini [2006]. A rigorous proof of the existence of an invariant attracting closed curve $\mathcal{C}$ for the map F given in (1) for $(\delta_L, \tau_L) \in S_L$, $-2 < \tau_R < 2$, $\delta_R = 1 + \varepsilon$, $\mu > 0$, is the subject of a work in progress and we don't discuss it in the present paper. Our main interest here is related to the bifurcation structure of the (δ_R, τ_R)-parameter plane, namely, to the periodicity regions corresponding to the attracting cycles born due to the center bifurcation.

4 (δ_R, τ_R)-parameter plane

Before entering into some general considerations we present three examples of the 2D bifurcation diagram in the (δ_R, τ_R)-parameter plane for different values of δ_L and τ_L giving some comments on the bifurcation structure of the parameter plane. Note that each of these examples deserves more detailed investigation being quite rich in a sense of possible bifurcation scenarios. Some properties of similar bifurcation diagrams for piecewise linear and piecewise smooth dynamical systems were described, e. g., in Hao Bai-Lin [1989], Sushko et al. [2003], Zhusubaliyev and Mosekilde [2003], Sushko and Gardini [2006]. Referring to these papers, we recall here a few properties using our examples.

For all the bifurcation diagrams, presented in Fig.s 2-4, the parameter regions corresponding to attracting cycles of different periods n, $n \leq 32$, are shown in different colors (note that the periodicity regions related to attracting cycles with the same period n, but different rotation numbers, say m_1/n and m_2/n, are shown by the same color). If one takes the (δ_R, τ_R)-parameter point belonging to some m/n-periodicity region, denoted by $P_{m/n}$, then the corresponding map F has an attracting cycle of period n, which in general may be not the unique attractor. Some periodicity regions are marked also by the corresponding rotation numbers. White region on these figures is related either to higher periodicities, or to chaotic trajectories. Gray color corresponds to divergent trajectories.

Fig.2 presents the 2D bifurcation diagram for $\delta_L = 0.25$, $\tau_L = 0.5$. In such a case the map F is invertible (given that we consider $\delta_R > 1$, see Property 1); $\overline{L}^*$ is an attracting focus. In Fig.3 we show the (δ_R, τ_R)-parameter plane at parameter values $\delta_L = 0$, $\tau_L = 0.5$, for which F is noninvertible of $(Z_0 - Z_\infty - Z_1)$-type; $\overline{L}^*$ is a superstable node. While Fig.4 presents the bifurcation diagram at $\delta_L = -0.5$, $\tau_L = 0.3$: For such parameter values F is noninvertible, of $(Z_0 - Z_2)$-type; $\overline{L}^*$ is an attracting flip node (i.e. one negative eigenvalue exists). Recall that such bifurcation diagrams representing qualitatively different dynamic regimes, reflect also possible results of the border-collision bifurcation of the attracting fixed point of the map F occurring when μ changes from a negative to a positive value. For example, if we fix $\delta_L = 0.25$, $\tau_L = 0.5$, $\delta_R = 4$, $\tau_L = 0.5$ (the parameter point is inside the region $P_{1/5}$ on Fig.2), then for $\mu < 0$ the map F has the attracting focus L^* which at $\mu = 0$ undergoes the border-collision bifurcation resulting (for $\mu > 0$) in the attracting and saddle cycles of period 5.

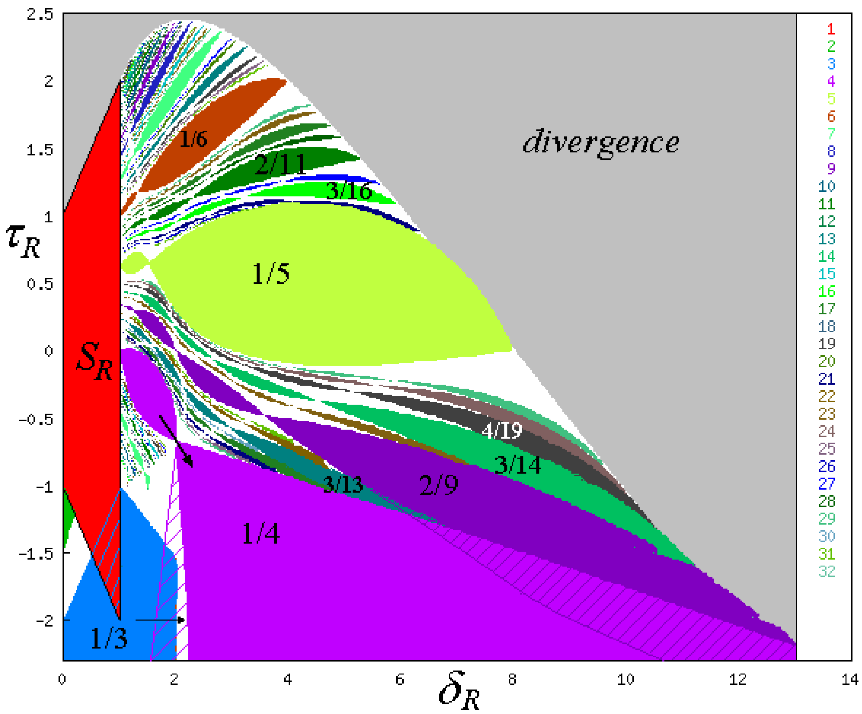

Figure 2: Two-dimensional bifurcation diargam of the map F in the (δ_R, τ_R)-parameter plane for $\delta_L = 0.25$, $\tau_L = 0.5$. F is invertible and $\overline{L}^*$ is an attracting focus.

First of all we recall that an issuing point for the periodicity region $P_{m/n}$ is $(\delta_R, \tau_R) = (1, \tau_{R,m/n})$, where $\tau_{R,m/n}$ is given in (11). In the vicinity of the bifurcation line $\delta_R = 1$ the periodicity regions are ordered in a way similar to that of the Arnold tongues associated with the Neimark-Sacker bifurcation occurring for smooth maps. In short, the periodicity regions follow a summation rule, or Farey sequence rule, holding for the related rotation numbers (see, e.g., Mira [1987], Maistrenko et al. [1995]). In particular, according to this rule if, for example, $r_1 = m_1/n_1$ and $r_2 = m_2/n_2$ are two rotation numbers, associated at $\delta_R = 1$ with $\tau_R = \tau_{R,r_1}$ and $\tau_R = \tau_{R,r_2}$, $\tau_{R,r_1} < \tau_{R,r_2}$, then there exists also a value $\tau_R = \tau_{R,r_3}$, $\tau_{R,r_1} < \tau_{R,r_3} < \tau_{R,r_2}$, related to the rotation number $r_3 = (m_1 + m_2)/(n_1 + n_2)$, so that $(\delta_R, \tau_R) = (1, \tau_{R,r_3})$ is an emanating point

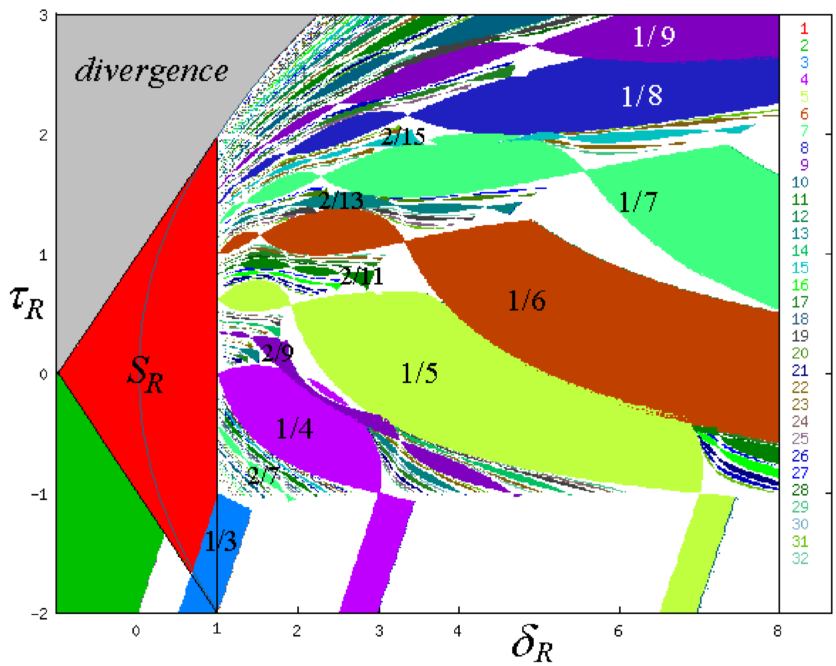

Figure 3: Two-dimensional bifurcation diargam of the map F in the (δ_R, τ_R)-parameter plane for $\delta_L = 0$, $\tau_L = 0.5$. F is noninvertible, of $(Z_0 - Z_\infty - Z_1)$-type, and $\overline{L}^*$ is a superstable node.

for the region P_{r_3}. To illustrate the summation rule some periodicity regions are marked in Fig.s 2-4 by the rotation numbers of the related cycles.

The kind of bifurcations associated with the boundaries of the periodicity regions differs from the smooth case: It is known that the boundaries of the Arnold tongues issuing from the Neimark-Sacker bifurcation curve are related to saddle-node bifurcations, and the other boundaries correspond to stability loss of the related cycle. While for piecewise linear maps the boundaries of the periodicity regions issuing from the center bifurcation line correspond to so-called border-collision pair bifurcations (a piecewise linear analogue of the saddle-node bifurcation), which we shall consider in detail in the next section.

Note also that differently from the smooth case the periodicity regions can have a ‘sausage’ structure (see Fig.s 2 and 3) with several subregions, first described in Hao Bai-Lin [1989], which is typical for piecewise smooth and piecewise linear systems (see also Zhusubaliyev and Mosekilde [2003], Sushko et al. [2003]). In fact, different subregions of the same periodicity region for the considered map F are related to different compositions of the maps F_1 and F_2 applied to get the corresponding cycle (attracting or saddle). It can be shown

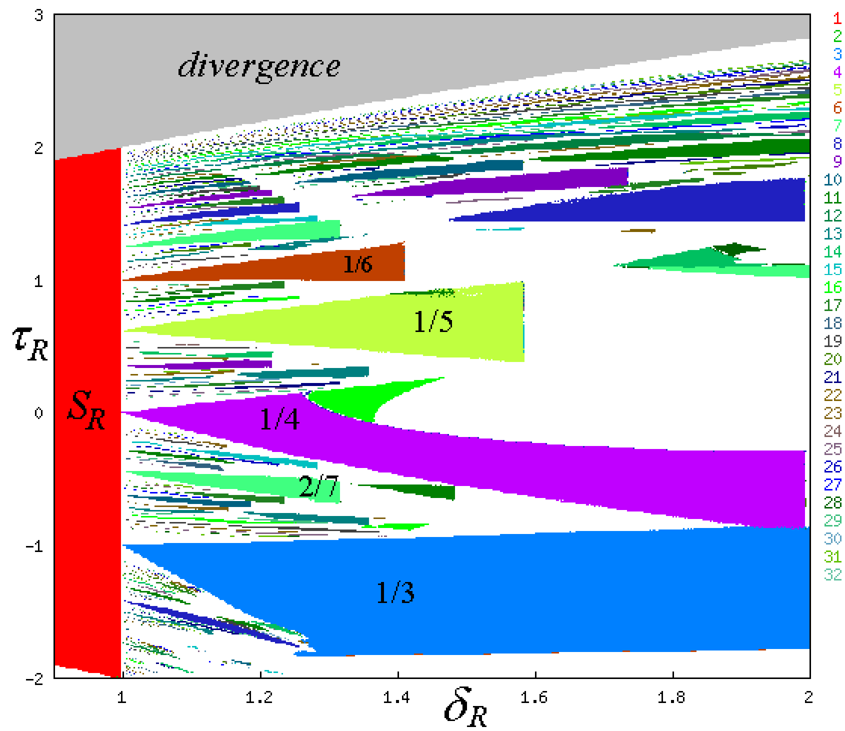

Figure 4: Two-dimensional bifurcation diargam of the map F in the (δ_R, τ_R)-parameter plane for $\delta_L = -0.5$, $\tau_L = 0.3$. F is noninvertible, of $(Z_0 - Z_2)$-type, and $\overline{L}^*$ is an attracting flip node.

that the first (leftmost) subregion of the m/n-periodicity region, denoted by $P^1_{m/n}$, is related to an attracting m/n-cycle with two periodic points located in L and $n-2$ points in R, that is, the related composition can be written as $F^n = F_1^2 \circ F_2^{n-2}$ for $m = 1$, and $F^n = F_1 \circ F_2^i \circ F_1 \circ F_2^{n-2-i}$, for $m \neq 1$, where $i > 1$ depends on m and n. The corresponding saddle m/n-cycle for any m for parameters from $P^1_{m/n}$ has one periodic point in L and $n-1$ points in R, that is, for such a cycle $F^n = F_1 \circ F_2^{n-1}$.

The 'waist' points separating subregions are related to a particular border-collision bifurcation at which points of the attracting and saddle cycles exchange their stability colliding with the border: Namely, after the collision the former attracting cycle becomes a saddle one while the saddle cycle becomes attracting (for details see Di Bernardo et al. [1999], Sushko et al. [2003]). To illustrate such a border-collision bifurcation we have chosen the waist point $(\delta_R, \tau_R) \approx (2, -0.6666)$ of the region $P_{1/4}$ at $\delta_L = 0.5$, $\tau_L = 0.25$ (see Fig.2). Fig.5 presents a bifurcation diagram for $\delta_R \in [1.9, 2.05]$, $\tau_R = -1.6665\delta_R + 2.66635$, so that the parameter point moves from the first subregion $P^1_{1/4}$, to the second one, denoted $P^2_{1/4}$ (the related parameter path is shown by the thick straight line with an arrow in Fig.2). On this diagram the points of the attracting and saddle cycles are shown in red and blue, respectively. Three (x, y)-planes shown in gray represent a part of the phase portrait of the system: Before the bifurcation, i.e., for $(\delta_R, \tau_R) \in P^1_{1/4}$; at the moment of the border-collision related to the waist point $(\delta_R, \tau_R) \approx (2, -0.6666)$; and after the bifurcation, i.e., for $(\delta_R, \tau_R) \in P^2_{1/4}$. Comparing the phase portrait related to the subregion $P^2_{1/4}$ with the one related to $P^1_{1/4}$, one can see that the number of periodic points in L is increased: Now three points of the attracting cycle are in L and one in R, and for the saddle cycle we have two points in L and two in R.

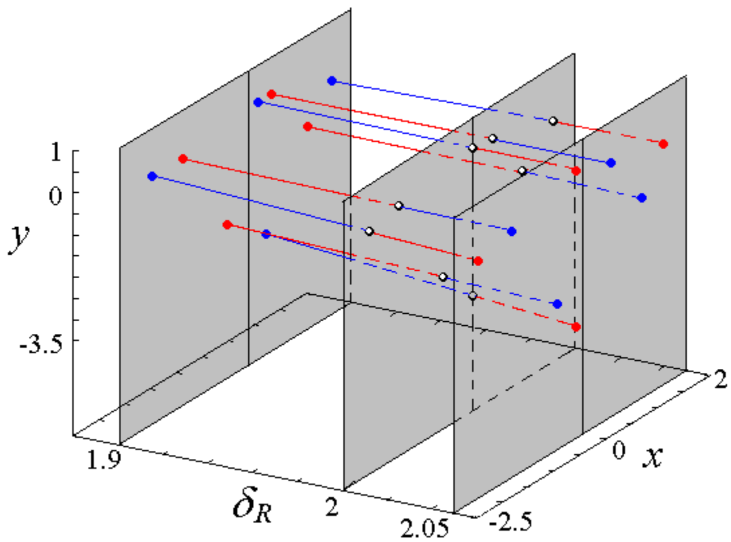

Figure 5: Bifurcation diagram for $\delta_R \in [1.9, 2.05]$, $\delta_L = 0.5$, $\tau_L = 0.25$, related to the parameter path shown in Fig.2 by the thick straight line with an arrow. The points of the attracting and saddle cycles are shown in red and blue, respectively.

To end this section let us comment the overlapping of periodicity regions, which corresponds to multistability (as an example, see Fig.2 on which several multistability regions are dashed, related with the periodicity regions $P_{1/3}$ and $P_{1/4}$. Some other overlapping zones can be seen in the same figure, as well as in Fig.s 3 and 4). Recall that considering the initial problem of the BCB of the fixed point of F, we have that in the case of multistability, varying μ through 0, the fixed point bifurcates into several attractors. As it was already mentioned, any invariant set of F contracts linearly with μ as μ tends to 0 collapsing to the origin at $\mu = 0$. Among such invariant sets we have the basins of attraction of coexisting attractors which shrink to 0 as well. Thus, one cannot answer a priori to which attractor the initial point will be attracted after the bifurcation. This gives a source of unpredictability of the results of the BCB. This problem was posed first in Kapitaniak and Maistrenko [1998], see also Dutta et al.[1999]. To give an example, we fix $\delta_L = 0.25$, $\tau_L = 0.5$, $\tau_R = -2$ and will increase the value of δ_R starting from $\delta_R = 1.5$, when the map F has attracting and saddle cycles of period 3 (see the arrow in Fig.2). At $\delta_R \approx 1.64$ a border-collision pair bifurcation occurs giving birth to attracting and saddle cycles of period 4, i.e., the parameter point enters the bistability region. Fig.6 presents a part of the phase portrait of the system at $\delta_R = 1.65$ when there are coexisting attracting cycles of period 3 and 4 whose basins of attraction, separated by the stable set of the period 4 saddle, are shown in yellow and green, respectively. The unstable set (shown in blue) of the saddle 3-cycle, approaching points of the attracting 3-cycle, forms a saddle-node connection which is wrinkled due to two negative eigenvalues of the attracting 3 cycle. With further increasing δ_R the

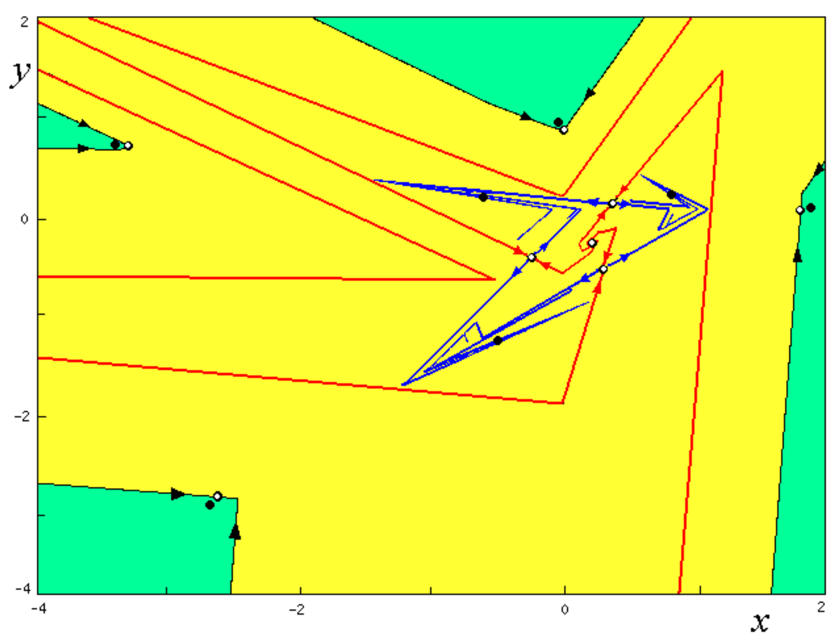

Figure 6: Attracting cycles of periods 3 and 4 with their basins of attraction (shown in yellow and green, respectively) separated by the stable set of the 4-saddle; The unstable set (in blue) of the 3-saddle forms a saddle-node connection which is near to be destroyed by homoclinic tangency with the stable set (in red). Here $\delta_L = 0.25$, $\tau_L = 0.5$, $\delta_R = 1.65$, $\tau_R = -2$.

stable set of the period 3 saddle (shown in red) tends to get a tangency with its unstable set. Indeed, at $\delta_R \approx 1.68$ a homoclinic bifurcation occurs after which the saddle-node connection is destroyed. Another qualitative change of the phase space occurs when the attracting 3-cycle undergoes a ‘flip’ bifurcation (an eigenvalue passing through -1) resulting in a cyclic chaotic attractor of period 6. After pairwise merging of the pieces of the attractor it becomes a 3-piece cyclic chaotic attractor shown in Fig.7 (for further details related to the ‘flip’ bifurcation in a piecewise linear map see Maistrenko et al. [1998]). Note that the boundary separating the basins of attraction is no longer regular as in Fig.6 but fractal. Such a basin transformation is a result of the homoclinic bifurcation of the saddle 4-cycle. A contact with the fractal basin boundary leads to the disappearance of the chaotic attractor at $\delta_R \approx 2.25$. Thus, in the considered sequence of bifurcations, the attracting 4-cycle coexists first with the attracting 3-cycle, then with the 6-piece chaotic attractor and finally with the 3-piece chaotic attractor. To illustrate the border-collision bifurcation of the fixed point of F in a case of multistability we present in Fig.8 a bifurcation diagram for $\mu \in [-0.2 : 1]$, related to Fig.7. The problem of multiple attractors and the role of homoclinic bifurcation is discussed in Zhusubaliyev et al. [2006], Sushko and Gardini [2006].

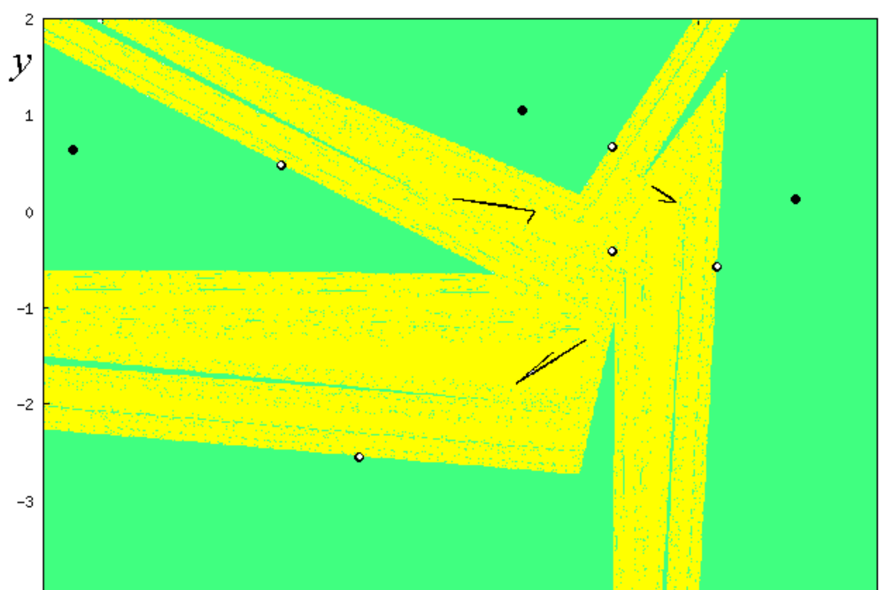

Figure 7: Basins of attraction of the 3-piece cyclic chaotic attractor and 4-cycle are shown in yellow and green, respectively. Here $\delta_L = 0.25$, $\tau_L = 0.5$, $\delta_R = 2.2$, $\tau_R = -2$.

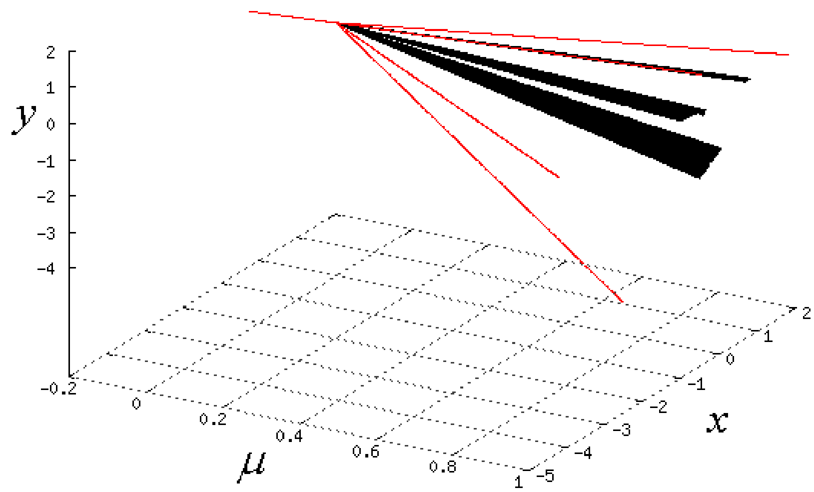

Figure 8: Phase space (x, y) as μ changes in the interval $\mu \in [-0.2 : 1]$ at $\delta_L = 0.25$, $\tau_L = 0.5$, $\delta_R = 2.2$, $\tau_R = -2$. After the BCB at $\mu = 0$ the attractor in red is the attracting 4-cycle, while the attractor in black is the 3-piece chaotic attractor.

The next Subsection is devoted to a more detailed investigation of the main periodicity regions related to the rotation number $1/n$, namely, to the border-collision pair bifurcations which define the boundaries of the $1/n$ periodicity regions.

4.1 $1/n$ periodicity regions and their BCB boundaries

Let us consider the first subregion, denoted by $P^1_{1/n}$, of the main periodicity region $P_{1/n}$. For such regions in the parameter space we can get the analytic representations for their boundaries related to the BCB, that is, the two boundaries of the regions issuing from the center bifurcation line, which we shall call BC boundaries for short. Note that in general any periodicity region has two BC boundaries and may have also other boundaries which are related to the stability loss of the corresponding attracting cycle. Note that (similarly to the smooth case) inside a periodicity region it is not guaranteed the existence of a closed invariant curve, which can be destroyed in several ways (for a list of mechanisms of destruction of a closed invariant attracting curve in the piecewise linear case see Sushko and Gardini, [2006]).

So let us consider a periodicity region $P_{1/n}$ and let $(\delta_R, \tau_R) \in P^1_{1/n}$. Denote the related attracting and saddle cycles by $p = \{p_0, ..., p_{n-1}\}$ and $p' = \{p'_0, ..., p'_{n-1}\}$, respectively. Let $p_0, p_{n-1} \in L$ and $p_1, ..., p_{n-2} \in R$. As for the saddle cycle, let $p'_0 \in L$ and $p'_1, ..., p'_{n-1} \in R$. We shall see what happens with these cycles if the (δ_R, τ_R)-parameter point crosses the two BC boundaries of $P^1_{1/n}$. To illustrate our consideration we use an example shown in Fig.9a for $\delta_L = 0.25$, $\tau_L = 0.5$, i.e., $(\delta_R, \tau_R) \in P^1_{1/5}$ (see Fig.2).

We consider a fixed value $\delta_R = \delta^*_R$ inside the periodicity tongue, such that the qualitative position of the periodic points in the (x, y) phase plane is presented in Fig.9a. Now let us increase the value of τ_R, then the point p_{n-1} of the cycle moves towards the critical line LC_{-1}, so that at some $\tau_R = \tau^*_R$ we have $p_{n-1} \in LC_{-1}$ (and, as a consequence, $p_0 \in LC_0$) which indicates a BCB. It occurs not only for the attracting cycle: Indeed, also the saddle cycle undergoes the BCB, namely, at $\tau_R = \tau^*_R$ we have $p'_{n-1} \in LC_{-1}$, moreover, $p'_{n-1} = p_{n-1}$, as well as all the other points of the cycles p and p' are pairwise merging on the critical lines of the proper ranks (see Fig9c). In such a way the 'saddle-node' BCB occurs (not related to an eigenvalue equal to 1). The value $\tau_R = \tau^*_R$ corresponds to the (δ_R, τ_R)-parameter point crossing the upper boundary of $P^1_{1/n}$, which we denote by $BC_{1/n(1)}$. While if at the fixed $\delta_R = \delta^*_R$ the value τ_R is decreased, then p_0 and p'_1 move towards the critical line LC_{-1}, so that at some $\tau_R = \tau^{**}_R$ we have $p_0 = p'_1 \in LC_{-1}$, thus one more 'saddle-node' BCB occurs (see Fig9b), related to the (δ_R, τ_R)-parameter point crossing the lower boundary of $P^1_{1/n}$, denoted by $BC_{1/n(2)}$.

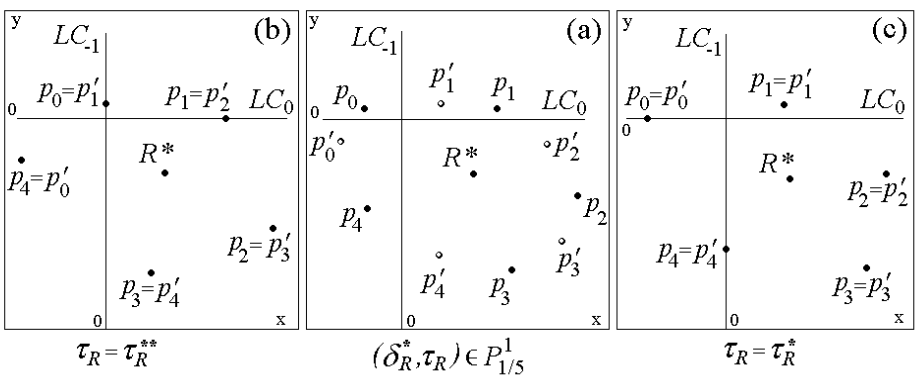

Figure 9: Examples of the 'saddle-node' BCB for (δ_R, τ_R)-parameter points crossing the boundaries of $P^1_{1/5}$ at $\delta_L = 0.25$, $\tau_L = 0.5$: (a) $(\delta^*_R, \tau_R) = (1.25, 0.65) \in P^1_{1/5}$; (b) $(\delta^*_R, \tau^{**}_R) \in BC_{1/5(2)}$ where $\tau^{**}_R \approx 0.575$; (c) $(\delta^*_R, \tau^*_R) \in BC_{1/5(1)}$ where $\tau^*_R \approx 0.726$.

Note that in the above consideration the value δ_R was fixed and the value of τ_R was varying (increasing or decreasing) in order to rich the BC boundaries of $P^1_{1/n}$. Indeed, it is not a general case: As we shall see, there are examples of the periodicity regions $P^1_{1/n}$ such that to reach the two BC boundaries from inside $P^1_{1/n}$ we can fix τ_R and vary the value of δ_R, increasing it, or decreasing. Related examples can be seen in Fig.11. One more example is the region $P_{1/3}$ shown in

Fig.3, which is bounded by the BC curves from above and on the left, while its right boundary is a curve denoted $Fl_{1/3}$, related to the 'flip' bifurcation of the attracting cycle of period 3 (in the next Subsection we shall consider again the region $P_{1/3}$, describing the general case).

Independently on the way the parameters τ_R and δ_R are varying, the two conditions for the BCB of the cycle p (at this moment we say nothing about its stability before the bifurcation), are $p_0 \in LC_{-1}$ and $p_0 \in LC_0$, or, more precisely,

$$BC_{1/n(1)} \qquad (x_0, 0) = F_2^{n-1} \circ F_1(x_0, 0), \tag{17}$$

$$BC_{1/n(2)} \qquad (0, y_0) = F_1 \circ F_2^{n-1}(0, y_0), \tag{18}$$

where (x_0, y_0) are coordinates of the point p_0.

Let the matrix defining the map F_2 be denoted by A, that is

$$A = \begin{pmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{pmatrix}.$$

It is not difficult to note that A^i, $i > 1$, can be written as follows:

$$A^i = \begin{pmatrix} a_i & a_{i-1} \\ -\delta_R a_{i-1} & -\delta_R a_{i-2} \end{pmatrix}, \tag{19}$$

where a_i is a solution of the second order difference equation

$$a_i - \tau_R a_{i-1} + \delta_R a_{i-2} = 0 \tag{20}$$

with the initial conditions

$$a_0 = 1, \; a_1 = \tau_R. \tag{21}$$

We know that the eigenvalues of the corresponding characteristic equation of (20) are complex-conjugate: $\lambda_{1,2(R)} = \left(\tau_R \pm \sqrt{\tau_R^2 - 4\delta_R}\right)/2$, where $\tau_R^2 < 4\delta_R$, so the general solution of (20) with the initial conditions (21) can be written as

$$a_i = \left(\sqrt{\delta_R}\right)^i \left(\cos(2\pi i/n) + \frac{\tau_R}{\sqrt{4\delta_R - \tau_R^2}} \sin(2\pi i/n)\right).$$

For example, $a_2 = \tau_R^2 - \delta_R$, $a_3 = \tau_R^3 - 2\tau_R\delta_R$, and so on.

Now, to get the condition in (17) in terms of the parameters of the system, we first shift the coordinate system so that the origin becomes the fixed point of F_2, that is we make a change of variables: $x' = x - x^*$, $y' = y - y^*$. Note that $y^* = -\delta_R x^*$. Then, in the new variables the maps F_1 and F_2, say $\widetilde{F}_1$ and $\widetilde{F}_2$, become

$$\widetilde{F}_1 \; : \; \begin{pmatrix} x' \\ y' \end{pmatrix} \mapsto \begin{pmatrix} \tau_L(x' + x^*) + y' + y^* + 1 - x^* \\ -\delta_L(x' + x^*) - y^* \end{pmatrix}, \; x' \leq -x^*;$$

$$\widetilde{F}_2 \; : \; \begin{pmatrix} x' \\ y' \end{pmatrix} \mapsto \begin{pmatrix} \tau_R x' + y' \\ -\delta_R x' \end{pmatrix}, \; x' \geq -x^*.$$

The equality (17) in the new variables is

$$(x'_0 - x^*, \delta_R x^*) = \widetilde{F}_2^{n-1} \circ \widetilde{F}_1(x'_0 - x^*, \delta_R x^*). \qquad (22)$$

Note that $\widetilde{F}_2^i$ can be written as

$$\widetilde{F}_2^i : \begin{pmatrix} x' \\ y' \end{pmatrix} \mapsto A^i \begin{pmatrix} x' \\ y' \end{pmatrix},$$

where A^i is given in (19). So, substituting (19) with $i = n-1$ into (22) and equating the two expressions for x'_0, we get the equality

$$\frac{\delta_R a_{n-1} - a_n + 1}{\delta_L a_{n-2} - \tau_L a_{n-1} + 1} = \frac{\delta_R a_{n-2} - a_{n-1} + 1}{\delta_L a_{n-3} - \tau_L a_{n-2}}$$

which can be also written as

$$BC_{1/n(1)}: \qquad \frac{a_{n-1} + a_{n-2} + ... + a_1 + 1}{\delta_L a_{n-2} - \tau_L a_{n-1} + 1} = \frac{a_{n-2} + a_{n-3} + ... + a_1 + 1}{\delta_L a_{n-3} - \tau_L a_{n-2}}. \quad (23)$$

Similarly, the equality in (18) in the new variables (x', y') is written as

$$(-x^*, y'_0 + \delta_R x^*) = \widetilde{F}_1 \circ \widetilde{F}_2^{n-1}(-x^*, y'_0 + \delta_R x^*),$$

from which we get the equality

$$BC_{1/n(2)}: \quad \frac{\delta_L(a_{n-1} - 1) + \delta_R}{\delta_L a_{n-2} + 1} = \frac{\tau_L(a_{n-1} - 1) - \delta_R a_{n-2} + \tau_R - 1}{\tau_L a_{n-2} - \delta_R a_{n-3}}. \quad (24)$$

For fixed values of the parameters δ_L and τ_L, the equalities (23) and (24) represent, in an implicit form, two curves in the (δ_R, τ_R)-parameter plane. As an example, in Fig.10 the curves $BC_{1/n(1)}$ and $BC_{1/n(2)}$ are plotted for $n = 3, ..., 9$, where $\delta_L = 0$, $\tau_L = 0.5$. Obviously, only particular arcs of the curves given in (23) and (24) are related to the BCB of the attracting cycle. The end points of such arcs are the waist points mentioned in Section 4, being two intersection points of (23) and (24), and one of them belongs, obviously, to the center bifurcation line, i.e., $\delta_R = 1$, $\tau_R = \tau_{R,1/n} = 2\cos(2\pi/n)$ (see (11)).

For example, let us consider in more details the region $P^1_{1/4}$ at $\delta_L = 0$, $\tau_L = 0.5$ (see Fig.10). The BC boundaries of $P^1_{1/4}$ are given by

$$BC_{1/4(1)}: \qquad \tau_R - \delta_R - \tau_L\delta_R + \tau_L\tau_R\delta_R + \tau_R^2 + \tau_L\tau_R^2 + \tau_L\delta_R^2 + 1 = 0, \quad (25)$$

$$BC_{1/4(2)}: \qquad -\tau_L\tau_R - \tau_L + \delta_R - 1 - \tau_L\tau_R^2 + \tau_L\delta_R + \delta_R\tau_R = 0. \qquad (26)$$

For $\tau_L = 0.5$ we can easily obtain the waist points, which are $(\delta_R, \tau_R) = (1, 0)$ and $(\delta_R, \tau_R) = (3, -1)$. We can also check that for the curve $BC_{1/4(1)}$ the derivative of τ_R with respect to δ_R, evaluated at $(\delta_R, \tau_R) = (1, 0)$ is

$$\tau'_R|^{(1)}_{(\delta_R,\tau_R)=(1,0)} = \frac{1 - \tau_L}{1 + \tau_L},$$

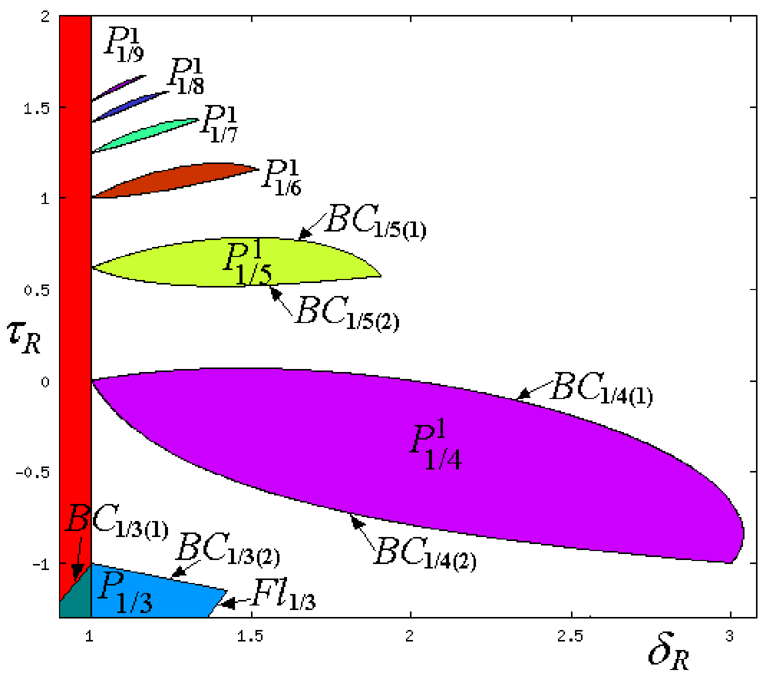

Figure 10: The border-collision bifurcation curves $BC_{1/n(1)}$ and $BC_{1/n(2)}$, $n = 3, ..., 9$; $\delta_L = 0$, $\tau_L = 0.5$.

while for the curve $BC_{1/4(2)}$ we have

$$\tau'_R|^{(2)}_{(\delta_R,\tau_R)=(1,0)} = \frac{\tau_L + 1}{\tau_L - 1} = \frac{1}{\tau'_R|^{(1)}_{(\delta_R,\tau_R)=(1,0)}}$$

These two derivatives are not equal (in effect they are reciprocal), thus the point $(\delta_R,\ \tau_R) = (1, 0)$, which is an issuing point for the region $P^1_{1/4}$, is not a cusp point.

4.2 1/3 periodicity region

Let us consider now in more details the region $P_{1/3}$ in the (δ_R, τ_R)-parameter plane for $(\delta_L, \tau_L) \in S_L$. Let $p = \{p_0, p_1, p_2\}$ be a cycle of period 3 of the map F such that $p_0, p_1 \in L$ and $p_2 \in R$. Substituting $n = 3$ to (23) and (24) we get the equations for the BC boundaries of $P_{1/3}$, which are the straight lines in the (δ_R, τ_R)-parameter plane:

$$BC_{1/3(1)} : \quad \begin{cases} \tau_R = (\delta_R - \delta_R\delta_L - 1 - \tau_L)/(\delta_L + \tau_L), & \text{for } \delta_L \neq -\tau_L; \\ \delta_R = 1, & \text{for } \delta_L = -\tau_L; \end{cases} \qquad (27)$$

$$BC_{1/3(2)}: \quad \begin{cases} \tau_R = (-\delta_R\delta_L - \delta_R\tau_L + \delta_L - 1)/(1+\tau_L), & \text{for } \tau_L \neq -1; \\ \delta_R = 1, & \text{for } \tau_L = -1. \end{cases} \tag{28}$$

We can also obtain the equations defining the boundaries of the triangle of stability of the cycle p. Indeed, the map F^3 corresponding to the considered cycle is $F^3 = F_2 \circ F_1^2$, for which the related eigenvalues $\eta_{1,2}$ are less then 1 in modulus for

$$\begin{cases} \begin{cases} \tau_R > (\delta_R(\tau_L - \delta_L^2) - 1 + \delta_L\tau_L)/(\tau_L^2 - \delta_L), \\ \tau_R > (\delta_R(\tau_L + \delta_L^2) + 1 + \delta_L\tau_L)/(\tau_L^2 - \delta_L), \end{cases} & \text{for } \tau_L^2 > \delta_L; \\ \begin{cases} \tau_R < (\delta_R(\tau_L - \delta_L^2) - 1 + \delta_L\tau_L)/(\tau_L^2 - \delta_L), \\ \tau_R < (\delta_R(\tau_L + \delta_L^2) + 1 + \delta_L\tau_L)/(\tau_L^2 - \delta_L), \end{cases} & \text{for } \tau_L^2 < \delta_L; \\ \qquad \delta_R < \frac{1}{\delta_L^2}, \end{cases} \tag{29}$$

so that the 'flip' bifurcation line denoted by $Fl_{1/3}$ and related to $\eta_2 = -1$, is given by

$$Fl_{1/3}: \quad \begin{cases} \tau_R = (\delta_R(\tau_L - \delta_L^2) - 1 + \delta_L\tau_L)/(\tau_L^2 - \delta_L), & \text{for } \tau_L^2 \neq \delta_L; \\ \delta_R = (\delta_L\tau_L - 1)/(\delta_L^2 - \tau_L), & \text{for } \tau_L^2 = \delta_L; \end{cases} \tag{30}$$

the bifurcation line related to $\eta_1 = 1$, denoted by $T_{1/3}$ (a particular "transcritical" bifurcation in our examples, as we shall see), is given by

$$T_{1/3}: \quad \begin{cases} \tau_R = (\delta_R(\tau_L + \delta_L^2) + 1 + \delta_L\tau_L)/(\tau_L^2 - \delta_L), & \text{for } \tau_L^2 \neq \delta_L; \\ \delta_R = -(\delta_L\tau_L + 1)/(\delta_L^2 + \tau_L), & \text{for } \tau_L^2 = \delta_L; \end{cases} \tag{31}$$

and by $C_{1/3}$ we denote the center bifurcation line (related to $\left|\eta_{1,2}\right| = 1$ for the complex-conjugate $\eta_{1,2}$), which is given by

$$C_{1/3}: \quad \delta_R = \frac{1}{\delta_L^2}, \; \delta_L \neq 0. \tag{32}$$

Thus, in the (δ_R, τ_R)-parameter plane we have 5 straight lines such that two of them, namely, $BC_{1/3(1)}$, $BC_{1/3(2)}$ are necessarily the boundaries of $P_{1/3}$, while three others depend on δ_L and τ_L : All the three lines may be involved as boundaries of $P_{1/3}$, or only two of them, or only one. Note that it may also happen that $P_{1/3} = \varnothing$, as well as $P_{1/3}$ may be an unbounded set (as, for example, in the case shown in Fig.10, in which the straight line $BC_{1/3(1)}$ is parallel to the straight line $Fl_{1/3}$). All the above cases can be classified depending on the values of δ_L and τ_L.

Thus, coming back to the initial problem of the BCB of the attracting fixed point of F occurring for μ varying through 0 at some fixed values of the other parameters of the normal form (1), one can check analytically, using (27)-(32), whether an attracting cycle of period 3 is born due to the bifurcation.

4.3 Overlapping of 1/n periodicity regions with the triangle of stability

In this section we discuss several examples of the bifurcation structure of the (δ_R, τ_R)-parameter plane where the periodicity regions $P_{1/n}$ have intersection with the triangle of stability S_R of the fixed point of F. We have already seen that the region $P_{1/3}$ can overlap S_R (see Fig.s 2, 3 and Fig.11). In terms of the BCB of the fixed point of F when μ is varying through 0, such a case means that the fixed point remains attracting after the collision with the border: Let us consider fixed all the parameters $(\delta_L, \tau_L) \in S_L$, $(\delta_R, \tau_R) \in S_R$ and let μ vary, then for $\mu < 0$ the fixed point is stable on the left side, it is on the boundary at $\mu = 0$ and for $\mu > 0$ it is again stable but on the right side. If (δ_R, τ_R) belongs to the region S_R overlapping with other periodicity regions, then for $\mu > 0$ the attracting fixed point coexists with other pairs of attracting and saddle cycles, born due to the BCB. For example, if $(\delta_R, \tau_R) \in (P_{1/3} \cap S_R)$, shown in Fig.2, then for $\mu > 0$ the attracting fixed point coexists with a cycle of period 3 born due to the BCB. Indeed, not only the region $P_{1/3}$ can overlap S_R (in fact this does not occur in Fig.4), while we may also have cases with overlapping regions of other periodicities: As an example, Fig.11 presents a bifurcation diagrams in the (δ_R, τ_R)-parameter plane for $\delta_L = 0.8, \tau_L = -0.7$. In this figure the BC

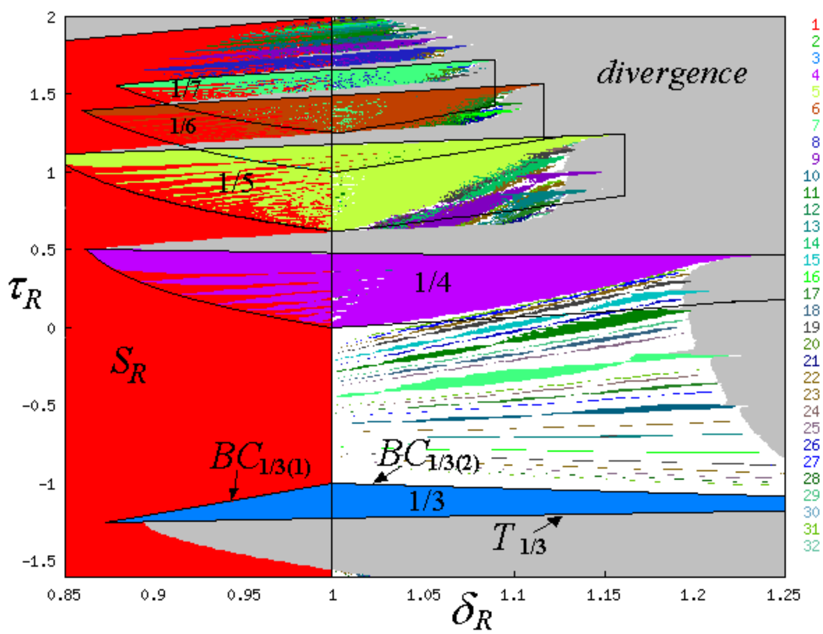

Figure 11: Two-dimensional bifurcation diagram in the (δ_R, τ_R)-parameter plane for $\delta_L = 0.8, \tau_L = -0.7$.

boundaries of the $1/n$-periodicity regions for $n = 3, ...7$ are plotted using their analytical expressions given in (23) and (24). The lower boundary in the region $P_{1/3}$ and the upper boundaries in the other periodicity regions, are the bifurcation curves associated with an eigenvalue equal to $+1$ which, in the considered cases, are related to a particular 'transcritical' bifurcation of the corresponding attracting cycles. We have to clarify the mechanism of 'transcritical' bifurcation of a cycle here occurring, which is particular because it involves an "exchange of stability" with a cycle at infinity, belonging to the Poincaré equator: A stable cycle approaches infinity and becomes unstable, while on the Poincaré equator an unstable cycle becomes attracting. The rightmost boundaries are the center bifurcation lines defined by $\delta_L^2 \delta_R^{n-2} = 1$. The regions of coexisting attractors are clearly seen: For example, at $\delta_R = 0.999$, $\tau_R = 1.04$ the attracting fixed point coexists with attracting cycles of period 5, 6 and 11 (in Fig.12 these attractors are shown together with their basins of attraction).

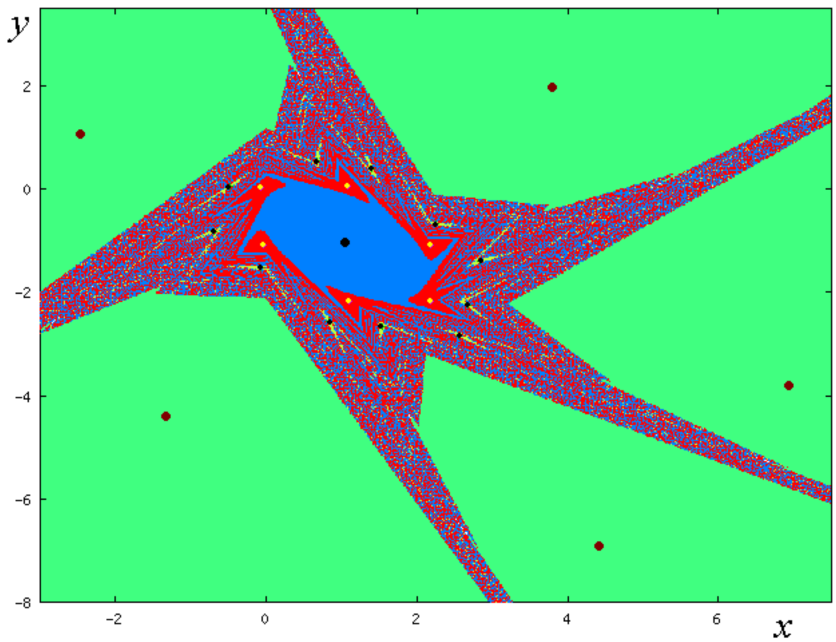

Figure 12: Basins of attraction of coexisting fixed point and cycles of period 5, 6 and 11 at $\delta_L = 0.8$, $\tau_L = -0.7$, $\delta_R = 0.999$, $\tau_R = 1.04$.

It is worth to note that if the parameter point belongs to periodicity regions overlapping S_R, then the bifurcation occurring at $\mu = 0$ may be related to a wide uncertainty: It is unpredictable to which attractor an initial point will be attracted after the bifurcation. It is also important to note in Fig.11 that there are periodicity regions overlapping the parameter regions related to the divergent trajectories. *For such parameter values at the bifurcation occurring at $\mu = 0$ all the related basins of attraction reduce to the fixed point, except the basin of infinity.* Thus, a small noise on the initial point may lead the trajectory to sudden divergence at the bifurcation, as well as before and after, and this also may be considered as dangerous (although this term is used to denote divergence at the BCB $\mu = 0$ while a stable fixed point exists both before and after the bifurcation, see Hassouneh et al. [2004], Ganguli and Banerjee [2005]).

5 An example of subcritical center bifurcation

The center bifurcation of the fixed point of the map F described in the previous sections may be considered of supercritical type. A natural question arises whether a subcritical center bifurcation may also occur in piecewise linear systems. In the present paper we don't study this problem in detail but just give an example of such a bifurcation. For this purpose one more bifurcation diagram is presented in Fig.13 where $\delta_L = 0.9, \tau_L = 0.7$. The dashed region in this figure indicates the 1/4-periodicity region related to an attracting cycle of period 4 coexisting with other attractors. The periodicity region $P_{1/4}$ is bounded by two BC curves plotted using the equations in (23) and (24) and the center bifurcation line given by $\delta_R^2\delta_L^2 = 1$, that is, $\delta_R = 1/\delta_L \approx 1.1111$. Let us investigate stability loss of the 4-cycle occurring when the parameter point crosses the center bifurcation line.

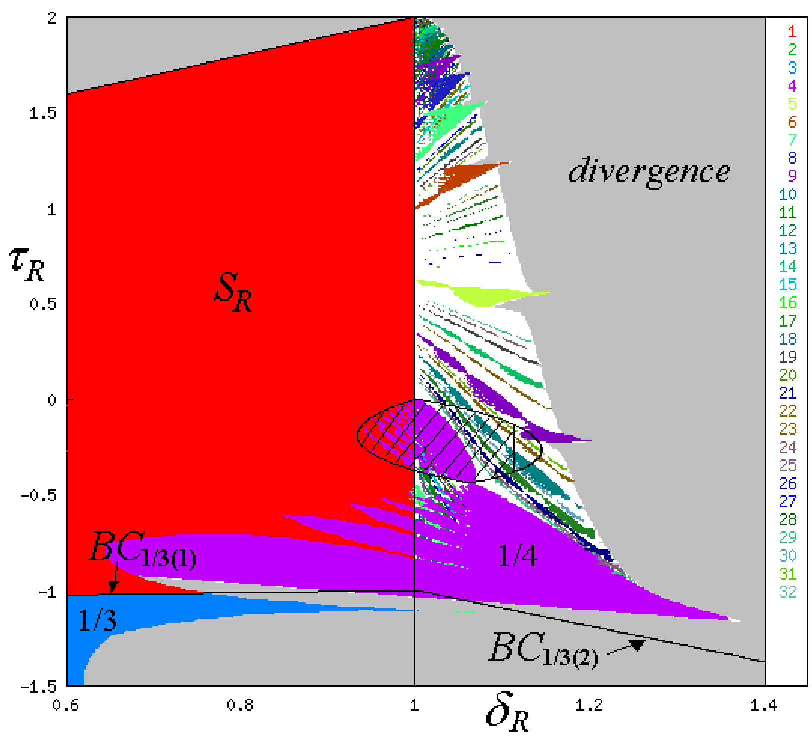

Figure 13: Two-dimensional bifurcation diagram in the (δ_R, τ_R)-parameter plane for $\delta_L = 0.9, \tau_L = 0.7$.

First we fix $(\delta_R, \tau_R) \in P_{1/4}$ as $\delta_R = 1.057$, $\tau_R = -0.2$: Fig.14 shows the basins of coexisting attracting cycles of period 4 and 25; The unstable set of the saddle 25-cycle approaching the points of the attracting 25-cycle form a closed invariant curve of the 'saddle-focus' type. The basin of attraction of the 4-cycle is bounded by the stable set of the saddle 4-cycle.

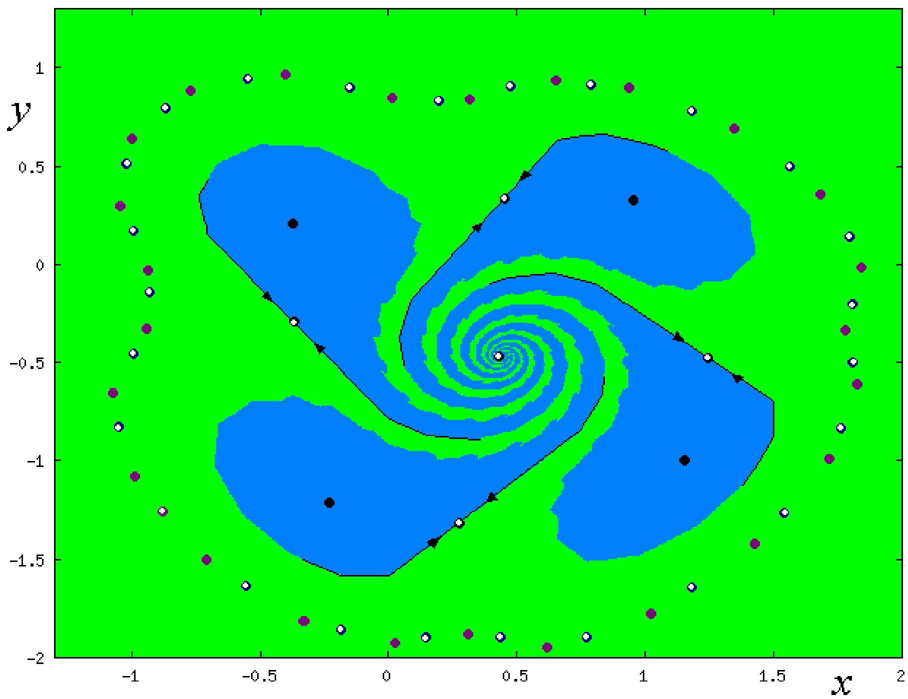

Figure 14: $\delta_L = 0.9$, $\tau_L = 0.7$, $\delta_R = 1.057$, $\tau_R = -0.2$. Basins of coexisting attracting cycles of period 4 (in blue) and 25 (in green). The unstable set of the saddle 25-cycle form a closed invariant curve of 'saddle-focus' type. The basin of attraction of the 4-cycle is bounded by the stable set of the saddle 4-cycle.

Let us increase the value of δ_R. At $\delta_R \approx 1.0874$ the first homoclinic bifurcation occurs for the saddle 4-cycle: Fig.15a shows an enlarged part of the phase space with one point of the attracting 4-cycle, one point of the saddle 4-cycle and the related branches of the stable and unstable sets. Homoclinic tangle at $\delta_R = 1.089$ is presented in Fig.15b, and Fig.15c shows the second homoclinic bifurcation occurring at $\delta_R \approx 1.0929$, which gives rise to four cyclic repelling closed invariant curves constituting the boundary of the basin of attraction of the attracting 4-cycle (see Fig.15d where $\delta_R = 1.1$). An analogous homoclinic structure is described in Kuznetsov [1995] in case of the Neimark-Sacker bifurcation (strong resonance 1 : 4) occurring for smooth maps.

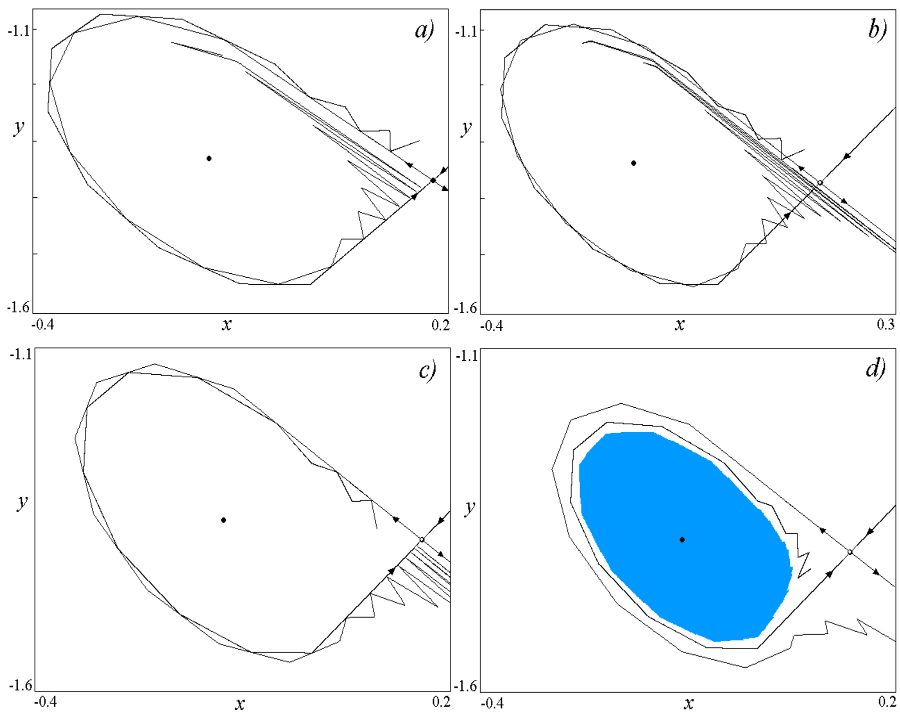

Figure 15: Enlarged part of the phase space with one point of the attracting 4-cycle and one point of the saddle 4-cycle with the related branches of stable and unstable sets at $\delta_L = 0.9$, $\tau_L = 0.7$, $\tau_R = -0.2$ and (a) $\delta_R = 1.0874$: The first homoclinic bifurcation of the saddle 4-cycle; (b) $\delta_R = 1.089$: Homoclinic tangle; (c) $\delta_R = 1.0929$: The last homoclinic bifurcation; (d) $\delta_R = 1.1$: One of the four closed invariant repelling curves bounds the basin of attraction of the attracting 4-cycle.

We continue to increase the value of δ_R. At $\delta_R = 1/\delta_L = 1.1111...$ the complex-conjugate eigenvalues of the former attracting 4-cycle reach the unit circle, that is, the 4-cycle undergoes a center bifurcation, but it is of subcritical type: Exactly at $\delta_R = 1/\delta_L$ in the phase space there are four cyclic repelling invariant regions filled with the invariant ellipses (see Fig.16); After the bifurcation the 4-cycle becomes unstable and nothing is born.

In terms of the BCB of the fixed point of the map F as μ varies through 0, the above case gives an example of a new kind of the BCB giving birth to a cyclic repelling closed invariant curve coexisting with one or several attractors (take, for example, the parameter values as in Fig.15d and let μ vary from a negative to a positive value).

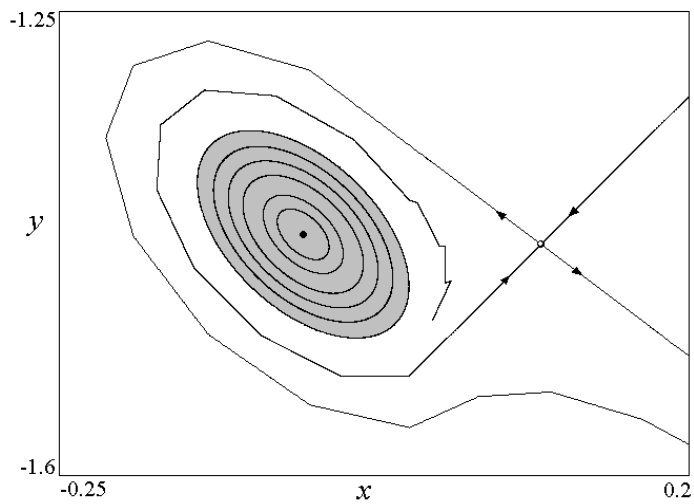

Figure 16: Enlarged part of the phase space with one of the 4-cyclic repelling invariant regions at the moment of the subcritical center bifurcation of the attracting 4-cycle; A point of the saddle 4-cycle is shown together with the related branches of the stable and unstable sets. Here $\delta_L = 0.9$, $\tau_L = 0.7$, $\tau_R = -0.2$ and $\delta_R = 1.11111111$.

6 Conclusion

In this work we have considered some of the bifurcation mechanisms which may be associated with the BCB in a two-dimensional piecewise linear map when one of the fixed points undergoes a center bifurcation, investigating a piecewise linear map which is considered as a normal form to study BCB in piecewise smooth two-dimensional maps. In Section 3 we have fully described the dynamics at the center bifurcation value, and the main results collected in Sections 4: Considering the dynamics 'after' the center bifurcation, which gives birth to an invariant attracting closed curve. We have analytically determined the BC-boundaries of the main periodicity regions issuing from the center bifurcation line in the parameter space, giving an example of the issuing point which is not a cusp, and described the so-called 'sausages' structure of the regions (mechanism typical in piecewise smooth or piecewise linear maps, which cannot occur in smooth maps). We have described the overlapping of the periodicity region, associated with the multistability phenomena, representing coexistence of several stable cycles both before and after the center bifurcation. In particular, we have given also examples of the dangerous BCB related to the case in which at the BCB value all the orbits with non zero initial conditions diverge to infinity, while bounded attracting sets exist (before and after). Also we have shown an example which may be considered as a piecewise-linear analogue of the *subcritical* Neimark-Sacker bifurcation for smooth maps, and still to be studied in detail. As stated in Section 2, we have considered only some of the possible signs of the parameters: Several other situations are still to be investigated. And also other mechanisms in the regions here analyzed (as for example the dynamic structure associated with quasiperiodic orbits) are left as open problems, for further works.

References

Banerjee, S. and C. Grebogi [1999] "Border-collision bifurcations in two-dimensional piecewise smooth maps", *Physical Review E*, 59, No. 4, 4052-4061.

Banerjee, S, M.S. Karthik, G. Yuan and J.A. Yorke [2000a] "Bifurcations in One-Dimensional Piecewise Smooth Maps - Theory and Applications in Switching Circuits"*, IEEE Trans. Circuits Syst.-I: Fund. Theory Appl.* 47 No. 3, 389-394.

Banerjee, S., Ranjan, P., and Grebogi, C. [2000b] "Bifurcations in 2D Piecewise Smooth Maps - Theory and Applications in Switching Circuits"*, IEEE Trans. Circuits Syst.-I: Fund. Theory Appl.* 47 No. 5, 633-643.

Di Bernardo, M., M.I. Feigen, S.J. Hogan and M.E. Homer [1999] "Local analysis of C-bifurcations in n-dimensional piecewise smooth dynamical systems", *Chaos, Solitons & Fractals,* 10(11), 1881-1908.

Dutta, M., Nusse, H. E., Ott, E., Yorke, J. A., and Yuan, G. H. [1999] "Multiple attractor bifurcations: A source of unpredictability in piecewise smooth systems", *Phys. Rev. Lett.* 83, 4281–4284.

Feely, O., D. Fournier-Prunaret, I. Taralova-Roux, and D. Fitzgerald [2000], "Nonlinear dynamics of bandpass sigma-delta modulation. An investigation by means of the critical lines tool", *International Journal of Bifurcation and Chaos*, 10(2), 303-327.

Fournier-Prunaret, D., O.Feely, I. Taralova-Roux [2001] "Lowpass sigma-delta modulation : an analysis by means of the critical lines tool", *Non Linear Analysis*, 47, 5343-5355.

Maistrenko, Y.L., V.L. Maistrenko and L.O. Chua [1993] "Cycles of chaotic intervals in a time-delayed Chua's circuit", *International Journal of Bifurcation and Chaos* 3(6), 1557-1572.

Maistrenko Y.L., V.L. Maistrenko, S.I. Vikul and L. Chua [1995] "Bifurcations of attracting cycles from time-delayed Chua's circuit", *International Journal of Bifurcation and Chaos* 5(3), 653-671.

Maistrenko, Y., Sushko, I., and Gardini, L. [1998] "About two mechanisms of reunion of chaotic attractors", *Chaos, Solitons & Fractals*, 9(8), 1373-1390.

Mira C. [1987] *Chaotic Dynamics*, World Scientific, Singapore.

Gallegati, M., Gardini, L., Puu, T. and Sushko, I. [2003], "Hicks' Trade Cycle Revisited: Cycles and Bifurcations", *Mathematics and Computers in Simulation, 63,* 505-527.

Ganguli, A. and S. Banerjee [2005] "Dangerous bifurcation at border collision: when does it occur? ", *Phys. Rev.E* 71 057202 (1-4).

Gardini, L., Puu, T. and Sushko, I.[2006a], "The Hicksian Model with Investment Floor and Income Ceiling". In: *Business Cycles Dynamics. Models and Tools* (T. Puu and I. Sushko Ed.s), Springer.Verlag, N.Y.

Gardini, L., Puu, T., Sushko, I. [2006b] "A Goodwin-type Model with a Piecewise Linear Investment Function". In: *Business Cycles Dynamics. Models and Tools* (T. Puu and I. Sushko Ed.s), Springer Verlag, N.Y.

Hao Bai-Lin [1989] *Elementary Symbolic Dynamics and Chaos in Dissipative Systems*. World Scientific, Singapore.

Hassouned, M.A., E.H. Abed and H.E. Nusse [2004] "Robust dangerous Bprder-Collision Bifurcations in piecewise smooth systems", *Phys. Rev. Letters*, 92(7) 070201 (1-4).

Hommes, C.H. [1991], *Chaotic dynamics in economic models.* Ph.D. Thesis, Wolters-Noodhoff, Groningen.

Hommes, C.H., Nusse, H. [1991] "Period three to period two bifurcations for piecewise linear models", *Journal of Economics* 54(2), 157-169.

Nusse, H.E., Yorke, J.A. [1992] "Border-collision bifurcations including "period two to period three" for piecewise smooth systems", *Physica D*, 57, 39-57.

Nusse, H.E., Yorke, J.A. [1995] "Border-collision bifurcations for piecewise smooth one-dimensional maps", *International Journal of Bifurcation and Chaos* 5(1) (1995), 189-207.

Kapitaniak, T. and Maistrenko, Yu. L. [1998] "Multiple choice bifurcations as a source of unpredictability in dynamical systems" *Phys. Rev. E 5*8(4), 5161–5163.

Kowalczyk, P. [2005] "Robust chaos and border-collision bifurcations in non-invertible piecewise-linear maps", *Nonlinearity* 18, 485-504.

Kuznetsov, Y. [1995] *Elements of applied bifurcation theory.* Springer-Verlag, N.Y.

Sushko, I., Puu, T. and Gardini, L. [2003] "The Hicksian floor-roof model for two regions linked by interregional trade", *Chaos Solitons & Fractals*, (18) 593-612.

Sushko, I., Agliari, A., and Gardini, L. [2005], "Bistability and border-collision bifurcations for a family of unimodal piecewise smooth maps", *Discrete and Continuous Dynamical Systems, Serie B*, 5(3), 881-897.

Sushko, I., Agliari, A., and Gardini, L. [2006], "Bifurcation Structure of Parameter Plane for a Family of Unimodal Piecewise Smooth Maps: Border-Collision Bifurcation Curves". In: *Dynamic Modelling in Economics and Finance in honour of Professor Carl Chiarella* (G.I. Bischi and I. Sushko Ed.s), *Chaos, Solitons & Fractals,* 29(3), 756-770.

Sushko, I., and Gardini, L. [2006] "Center Bifurcation for a 2D Piecewise Linear Map". In: *Business Cycles Dynamics. Models and Tools* (T. Puu and I. Sushko Ed.s), Springer Verlag, N.Y.

Taralova-Roux, I., D. Fournier-Prunaret [2002] "Dynamical study of a second order DPCM transmission system modeled by a piece-wise linear function", *IEEE Transactions of Circuits and Systems I*, 49(11), 1592-1609

Yuan, G.H. [1997] *Shipboard crane control, simulated data generation and border collision bifurcation*, Ph. D. dissertation, Univ. of Maryland, College Park, USA.

Zhusubaliyev, Z.T. and Mosekilde, E. [2003] *Bifurcations and Chaos in Piecewise-Smooth Dynamical Systems*, Singapore, World Scientific.

Zhusubaliyev, Z.T., Mosekilde, E., Maity, S., Mohanan, S., and Banerjee, S. [2006] "Border collision route to quasiperiodicity: Numerical investigation and experimental confirmation", *CHAOS* (16) 023122 (pp.1-11).

Zhusubaliyev, Z.T., E. Soukhoterin and E. Mosekilde [2007] "Quasiperiodicity and torus breakdown in a power electronic dc/dc converter" Mathematics and Computers in Simulation 73, 364-377.