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Center Bifurcation of a point on the Poincaré Equator

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Abstract

We describe the dynamics in a two-dimensional piecewise linear- fractional discontinuous map coming from an economic application. Our interest in this map is related to the fact that it gives the first example, to our knowledge, of a particular bifurcation: We show that its fixed point on the Poincaré equator undergoes a center bifurcation giving rise to a hyperbola-like attracting invariant set on which dynamics are either periodic or quasiperiodic.

1 Introduction

Dynamical systems defined by continuous *piecewise smooth* (PWS) functions are quite intensively studied nowadays, first of all due to numerous applied models defined by such systems, coming from different fields of science, such as electronics, mechanical engineering, economics, and others: See the books [23] and [5], in which there is a long list of related references. In the meantime, bifurcation theory of PWS dynamical systems remains still less developed then the theory of smooth systems. Peculiarity in the behavior of trajectories of PWS systems is related to the existence in the phase space of one or more boundaries separating the regions in which the function changes its definition. In general, collision with such a boundary of an invariant set can lead to abrupt changes in its structure and stability, so that one can observe, for example, the transition from an attracting cycle to a chaotic attractor, or to a chaotic attractor coexisting with an attracting cycle of some other period. Since Nusse and Yorke ([14], [15]) the related bifurcations are called *border-collision bifurcations* (BCB for short). It is worth to mention also that the first

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works devoted to the study of the bifurcation phenomena in PWS systems date back to the 60th due to Leonov [11], and to the 70th due to Feigen (see [4], where the related bifurcations are called *C-bifurcations*).

Among PWS dynamical systems essential attention has been paid to *piecewise linear* (PWL) systems, in particular, to one- and two-dimensional PWL maps, defined by two linear functions, which were proposed as a kind of *normal forms* to study BCB in PWS maps (see [14], [1], [2]). For a 1D PWS map such a normal form, allowing to classify all possible BCB occurring for this map, is represented by the so-called *skew-tent map* which is a PWL map defined by two linear functions depending on two parameters representing slopes of the related linear functions. Bifurcation phenomena occurring in the skew-tent map are well studied and described in details (see, [12], [15]). Thus, for example, in [8], [18] the skew-tent map was used to classify the BCB occurring in 1D PWS maps coming from economic applications.

For 2D PWS maps the normal form to study BCB is represented by a 2D PWL map (which we call as 2D BCB normal form), defined by two linear maps depending on four parameters which are traces and determinants of the Jacobian matrixes of the related linear maps. It is not an easy task to classify all possible BCB occurring in this map varying the parameters (see [2]), and indeed one can observe quite interesting and complicated bifurcation phenomena, among which it is worth to mention multistability and unpredictability of the number of coexisting attractors after the BCB (see, e.g., [24]) and the so-called dangerous BCB (see [7] and the references therein), related to the case in which a fixed point is attracting before and after the BCB, while at the bifurcation value the dynamics are divergent.

Our particular interest is connected with the birth of a closed invariant attracting set due to the BCB, which is closely related to the so-called *center bifurcation*, first described in [22] (see also [6], [19], [20], [17]). Such a bifurcation occurs, under suitable conditions, when a pair of complex-conjugate eigenvalues of an attracting fixed point crosses the unit circle, so that the fixed point becomes an unstable focus, giving rise to an attracting closed invariant set S. This bifurcation can be seen as a PWL analogue of the Neimark-Sacker bifurcation, occurring in smooth maps: Indeed, similar to the smooth case, the close invariant set S can be made by the saddle-node connection of a pair of cycles (a saddle and a node), but in this case S is not smooth but made up of finitely or infinitely many (depending on the type of noninvertibility of the map) segments and corner points. The trajectory on S can be also quasiperiodic. Differently from the smooth case, the set S appears not in the neighborhood of the fixed point, but at a certain distance from it (which depends on the distance of the fixed point from the border separating the definition regions of the two linear maps). Considering the bifurcation structure of the parameter space of the 2D BCB normal form, one can observe that the periodicity regions are organized in a way similar to the Arnold tongues in the smooth case, so that they can be classified with respect to the rotation numbers. However, they have a particular sausage-like structure ([24], [20]) and their boundaries issuing from the center bifurcation line at points associated with rational rotation numbers, are BCB curves, instead of saddle-node bifurcation curves issuing from the Neimark-Sacker bifurcation curve.

Among PWS maps one can distinguish a class of maps defined by linear-fractional (LF) functions which are those closest to the PWL maps in a sense of their dynamic properties. Indeed, as a simple observation, one can note that similarly to a PWL function which remains PWL under the iterations (and namely this fact allows to get many bifurcation conditions analytically), also a LF function under the iterations remains LF, that is, the degree of the system is not changed. As a simplest example of the similarity of the dynamics of these two classes of maps, we can compare the dynamics of a 1D linear map with the dynamics of a hyperbolic map reciprocal to the linear one: For example, if the linear map has a fixed point (repelling or attracting) one can consider also its second fixed point located at infinity (attracting or repelling, respectively), and then these two fixed points correspond to two fixed points (one attracting and another one repelling) of the hyperbolic map. So, studying dynamics of a PWLF map one can expect to observe bifurcation phenomena which are in a sense "reciprocal" to those observed in PWL systems. In particular, the subject of the present paper is to demonstrate an analogue of the center bifurcation occurring for a point on the Poincaré Equator (PE) for a 2D PWLF map coming from an economic application (see [16], [21]).

The paper is organized as follows. In the next section we describe the dynamic properties of the considered map F which is given by two LF maps F_1 and F_2 defined in the regions R_1 and R_2, respectively. The map F depends on three parameters a, c and r. We first present a 2D bifurcation diagram in the (a, c)-parameter plane with the periodicity regions issuing from the bifurcation line $a = 1$, whose structure is similar to the one obtained for a 2D PWL map in the case of a center bifurcation, but, as we show, the map has not a (finite) fixed point undergoing the center bifurcation, but instead, it is a fixed point on the Poincaré Equator which undergoes the center bifurcation. There exists also a vertical invariant line which at $a = 1$ changes its transverse stability. In sections 2.1 and 2.2 we describe the dynamics of each of the maps F_1 and F_2 separately, then in section 2.3 we show that at $a = 1$ in the phase plane of the map F there exist an invariant region bounded by either two branches of a hyperbola (analogues boundary in a center bifurcation is an ellipse), or by a "hyperbola-like" piecewise linear set being a broken line consisting of two branches (that corresponds to a polygon in a center bifurcation), such that any initial point of such a region is, respectively, either periodic, or quasiperiodic. In section 2.4 we present an example of an invariant attracting set C born due to the bifurcation, with attracting and saddle cycles on it, namely, we show that the set C is made up by the saddle-node connection of these cycles. But differently from the closed invariant attracting set S born at the center bifurcation, the set C is not closed: It has a "hyperbola-like" structure. Finally, in the section 3 we show that the map F indeed has a fixed point on the PE which at $a = 1$ undergoes the center bifurcation.

2 Dynamics of the map F : periodic and quasi-periodic trajectories

We consider a family of 2D maps $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by two linear-fractional functions F_1 and F_2 defined in the regions R_1 and R_2, respectively:

$$F : (x, y) \longmapsto \begin{cases} F_1(x, y), & if \ (x, y) \in R_1; \\ F_2(x, y), & if \ (x, y) \in R_2; \end{cases} \tag{1}$$

where

$$F_1 \ : \ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (x-a)/y + a \\ c + a - a/y \end{pmatrix}, \ R_1 = \{(x, y) : x(a(y-1) + rx) \geq 0\}\,; \tag{2}$$

$$F_2 \ : \ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (1-r)x/y \\ c - rx/y \end{pmatrix}, \ R_2 = \{(x, y) : x(a(y-1) + rx) < 0\}\,. \tag{3}$$

Here a, c and r are real parameters such that $a > 0$, $0 < c < 1$, $0 < r < 1$.

The map F comes from an economic application, namely, it represents the dynamics (in the so-called relative variables) of the multiplier-accelerator model with floor dependent on accumulated capital [16].

The (x, y)-plane of the map F is separated into four regions by the straight line $x = 0$, on which F is discontinuous, and by the line $y = 1 - rx/a$, on which F is continuous, so that the map F_1 is defined in

$$R_1 = \{x \geq 0, \ y \geq 1 - rx/a\} \cup \{x \leq 0, \ y \leq 1 - rx/a\}$$

and F_2 is defined in

$$R_2 = \{x < 0, \ y > 1 - rx/a\} \cup \{x > 0, \ y < 1 - rx/a\}\,.$$

We call these straight lines *critical lines* and denote them LC_{-1} and LC'_{-1} :

$$LC_{-1} = \{(x, y) : y = 1 - rx/a\}\,; \ LC'_{-1} = \{(x, y) : x = 0\}\,.$$

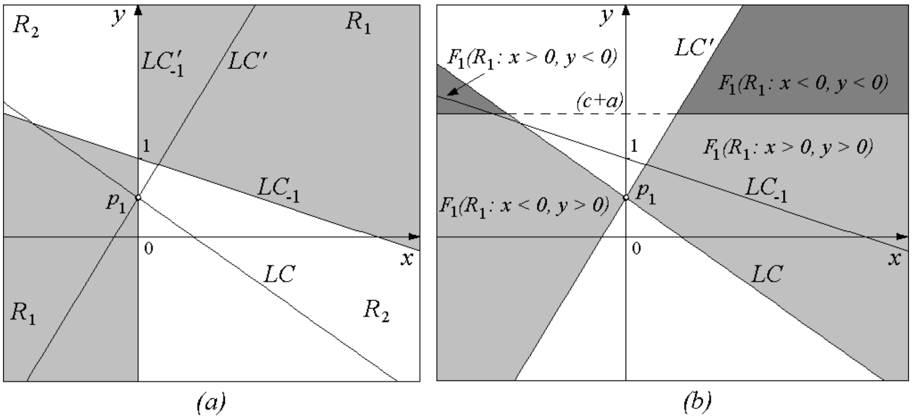

Figure 1: *(a): regions R_1 (in gray), R_2 and critical lines LC_{-1}, LC'_{-1}, LC, LC'; (b): the image of the region R_1 by the map F_1 is shown in light and dark gray.*

The image of LC_{-1} by F is a straight line denoted LC:

$$LC = \{(x, y) : y = c - rx/(1-r)\}\,.$$

The image of LC'_{-1} by F_1 is a straight line denoted LC' and by F_2 is a point denoted p_1 :

$$F_1(LC'_{-1}) = \{(x, y) : y = x + c\}\,,\ F_2(LC'_{-1}) = p_1,\ p_1 = (0, c).$$

The partition of the (x, y)-plane into the regions R_1 and R_2, as well as the above mentioned critical lines, are shown in Fig.1a (it is a qualitative picture for the case $a > 1$). In Fig.1b we present the image of the region R_1 by F_1 which is the union of the following sets:

$F_1(R_1\colon\ x > 0, y > 0) = \{(x, y) : c - rx/(1-r) < y < x + c,\ y < c + a, x > 0\}\,;$
$F_1(R_1\colon\ x < 0, y < 0) = \{(x, y) : c + a < y < x + c, x > 0\}\,;$
$F_1(R_1\colon\ x < 0, y > 0) = \{(x, y) : x + c < y < c - rx/(1-r), y < c + a, x < 0\}\,;$
$F_1(R_1\colon\ x > 0, y < 0) = \{(x, y) : c + a < y < c - rx/(1-r), x < 0\}\,.$

A general view on the dynamics of the map F is given by a 2D bifurcation diagram in the (a, c)-parameter plane at fixed $r = 0.2$ presented in Fig.2. Here different gray tonalities correspond to attracting cycles of different periods $n < 45$ (some regions are also marked by the numbers which are the periods of the related cycles). Details of this diagram will be discussed later in the present section, but one can see immediately that at $a = 1$ some bifurcation occurs which leads from divergent trajectories to attracting cycles of different periods. Indeed, the structure of the periodicity regions is similar to the one which is observed for 2D PWL maps

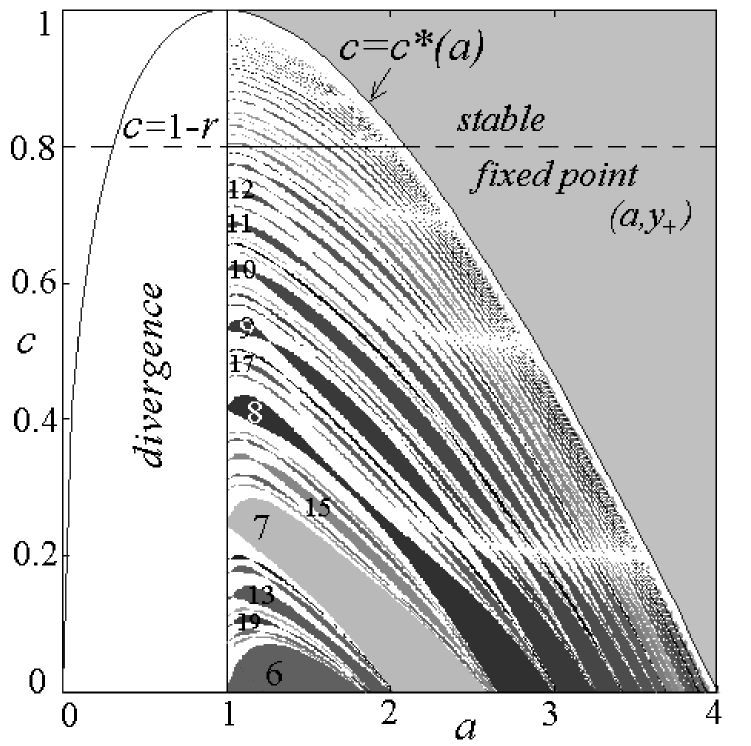

Figure 2: *2D bifurcation diagram of the map* F *in the* (a, c)*-parameter plane at* $r = 0.2$. *Regions of different gray tonalities correspond to attracting cycles of different periods* $n < 45$ *(some regions are marked by numbers which are the related periods).*

whose fixed point undergoes a center bifurcation (see, for example, [19], [20]). But for $a < 1$ the map F has no fixed point at final distance with complex-conjugate eigenvalues crossing the unit circle. The purpose of the present paper is to clarify what kind of bifurcation occurs at $a = 1$. In this section we present some standard analysis of the dynamics of F. Then, in Section 3 the results are interpreted in terms of the center bifurcation occurring for a fixed point of the map F which is located on the Poincaré Equator.

2.1 Triangular map F_1

To investigate the dynamics of the map F defined by the maps F_1 and F_2, let us first describe the behavior of trajectories in each of these maps separately. In this subsection we consider the map F_1 which is triangular: The variable y is mapped independently on x by a 1D map f:

$$f : y \mapsto f(y) = c + a - \frac{a}{y}. \tag{4}$$

The function $f(y)$ is a hyperbola with asymptotes $y = 0$ and $f(y) = c + a$. Using a usual convention, we define $f(\infty) = c + a$ and $f(0) = \infty$, so that the map f is defined for all $y \in \mathbb{R}$.

The map f has two fixed points denoted y_+ and y_-

$$y_\pm = \frac{c + a \pm \sqrt{(c+a)^2 - 4a}}{2}, \tag{5}$$

which have real values for

$$c \geq c^*(a) = 2\sqrt{a} - a. \tag{6}$$

(Note that $c > 0$, so, we don't consider the branch $c(a) = -2\sqrt{a} - a$). At $c = c^*(a)$ these fixed points appear due to a fold bifurcation.

The related fixed points of the map F_1 are (a, y_+) and (a, y_-). Obviously, F_1 has no other fixed points, but it has an invariant straight line $\{x = a\}$, and indeed, the fixed points, when they exist, belong to this line.

Let us check the stability of the fixed points (a, y_+) and (a, y_-). The eigenvalues of F_1 depend only on y: $\lambda_1(y) = a/y^2$, $\lambda_2(y) = 1/y$. For the parameter range considered $0 < \lambda_1(y_+) < 1$, $\lambda_1(y_-) > 1$. As for the eigenvalue $\lambda_2(y)$, the following inequalities hold: $\lambda_2(y_+) > 1$, $\lambda_2(y_-) > 1$ for $a < 1$, and $0 < \lambda_2(y_+) < 1$, $0 < \lambda_2(y_-) < 1$ for $a > 1$. Thus, the fixed points, when they exist, are such that for $a > 1$ the point (a, y_+) is attracting and (a, y_-) is a saddle, while for $a < 1$ the point (a, y_+) is a saddle and (a, y_-) is repelling.

Indeed, we are interested in the parameter range $0 < c < c^*(a)$ (see Fig.2 in which the curve $c = c^*(a)$ is shown), related to the case in which the map f (and, respectively, the map F_1) has no (finite) fixed points. The dynamics of the map f for such parameter range is described in the following proposition:

Proposition 1. *Let $0 < c < c^*(a)$. Then at $c = c_{m/n}(a)$, where*

$$c_{m/n}(a) = 2\sqrt{a}\cos(\pi m/n) - a, \tag{7}$$

any trajectory of the map f is n-periodic with rational rotation number m/n. At $c = c_\rho(a)$, where

$$c_\rho(a) = 2\sqrt{a}\cos(\pi\rho) - a \tag{8}$$

any trajectory of f is quasiperiodic with irrational rotation number ρ.

To see that the statement is true note that the map f can be written as a first order difference equation

$$y_{n+1} = \frac{(c+a)y_n - a}{y_n}, \tag{9}$$

which is a particular case of the so-called Riccati difference equation with constant coefficients[1], for which an analysis of the behavior of solutions is presented in [3], see also [10]. Namely, it is proved that if the roots of the equation $y^2 - (c+a)y + a = 0$, i.e., the values $y_\pm$, given in (5), are complex-conjugate, that is, $y_\pm = r(\cos\theta \pm i \sin\theta)$, where

$$r = \sqrt{a},$$

$$\cos\theta = (c+a)/2r, \tag{10}$$

$$\sin\theta = \sqrt{4a - (c+a)^2}/2r, \tag{11}$$

then

1) if $\theta/\pi = m/n$ for some integer $n > 0$ and m with $(m, n) = 1$, then every solution of (9) is periodic with minimal period n;

2) if $\theta/\pi = \rho$ is irrational, then every solution $\{y_i\}_{i=0}^{\infty}$ of (9) is aperiodic and the set of accumulation points of $\{y_i\}_{i=0}^{\infty}$ is the set of real numbers.

For the map f the first case occurs at parameter values given in (7) and the second case corresponds to the parameter values given in (8).

Note that the notion of rotation number for a trajectory $\{y_i\}_{i=0}^{\infty}$ of the map f when its fixed points are complex-conjugate, can be used in a quite natural way: Starting from some positive initial value $y_0 > 0$, the right branch of the hyperbola f is applied under the iterations, so that the points of the trajectory approach the discontinuity point $y = 0$, and after some number j of iterations we necessarily get a negative value $y_j < 0$, to which the left branch of f is applied giving a positive value again: $y_{j+1} > 0$. If the trajectory is n-periodic, it can in general make m such rounds before reaching the initial point. The number m can be considered as a counter of applications of the left branch of the hyperbola $f(y)$, or, equivalently, m is the number of periodic points on the left of the discontinuity point $y = 0$, and, respectively, the trajectory has $n - m$ points on the right of the discontinuity. We say that such a trajectory has the rational rotation number m/n. In Fig.3 we present the function f at $a = 1.5$, $c = c_{2/7}(a) \approx 0.027$: In this case any initial point y_0 is periodic of period 7 with the rotation number $m/n = 2/7$. As an example, two such cycles, with $y_0 = 5$ (gray circles) and $y_0 = 6$ (black circles), are also shown.

[1]The Riccati difference equation with constant coefficients is given by $x_{n+1} = \frac{ax_n+b}{cx_n+d}$, $n = 0, 1, ...$, where a, b, c, d are real numbers such that $ad - cd \neq 0$, $c \neq 0$ (see, e.g., [3]).

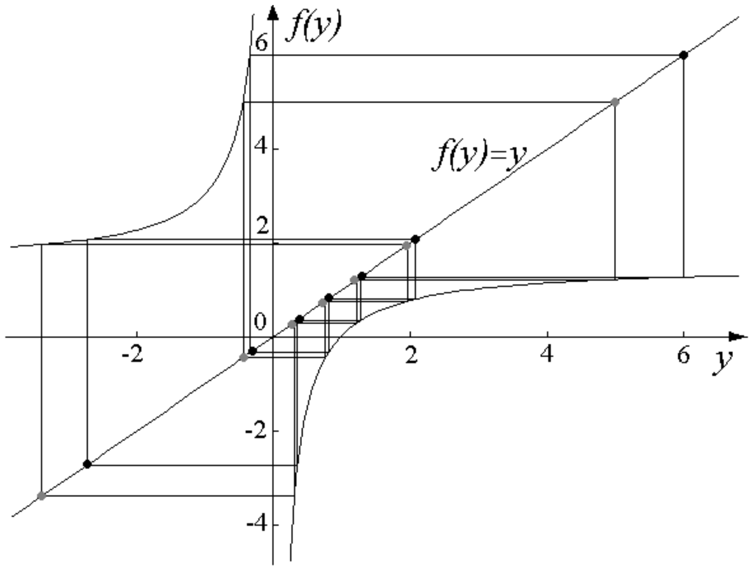

Figure 3: *The function* f *at* $a = 1.5$, $c = c_{2/7}(a) \approx 0.027$; *Any initial point* y_0 *is* 7*-periodic with rotation number* 2/7.

If a trajectory is quasiperiodic, it can be associated with a corresponding irrational rotation number.

In the (a, c)-parameter plane the curves (7) for different values of m/n fill densely the parameter region $0 < c < c^*$, approaching the limit curve $c^* = 2\sqrt{a} - a$ as $n \to \infty$ with fixed m (see Fig.4 on which the curves $c_{m/n}(a)$ are plotted for $n < 100$, $m = 1, 2$).

Let us come back now to the map F_1 for which the dynamics of its y variable for any initial value y_0 are either periodic (for $c = c_{m/n}(a)$) or quasiperiodic (for $c = c_\rho(a)$) independently on x. Below we show that the following proposition is true:

Proposition 2. *For the map* F_1 *the invariant line* $\{x = a\}$ *is globally attracting for* $a > 1$ *and globally repelling for* $a < 1$. *Any point* $(x_0, y_0) = (a, y_0)$ *is either periodic (for* $c = c_{m/n}(a)$ *), or quasiperiodic (for* $c = c_\rho(a)$*).*

The second part of the above statement follows immediately from Proposition 1: Given that the straight line $\{x = a\}$ is invariant under F_1, any point of this line is either periodic, or quasiperiodic. Thus, let us prove the first part of the proposition.

We consider first the case $c = c_{m/n}(a)$. Then for any (x_0, y_0) we have $(x_n, y_n) = (x_n, y_0)$, so that one can consider a 1D map of x on the line $\{y = y_0\}$. Let us define

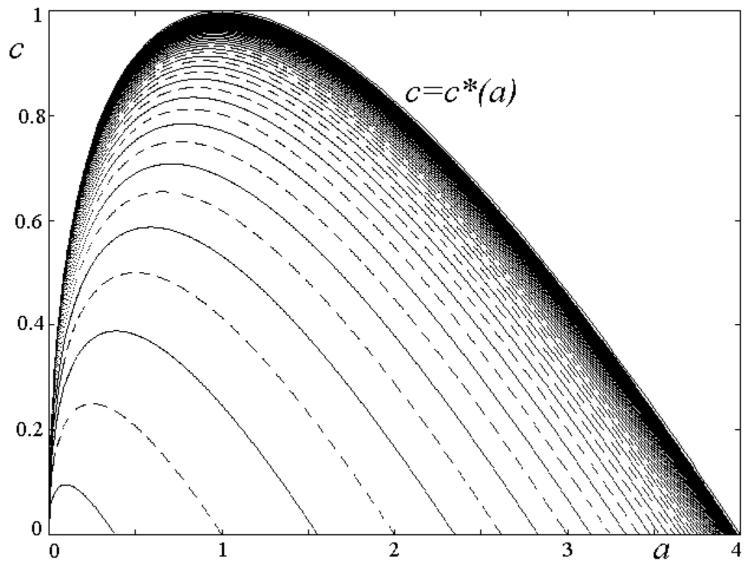

Figure 4: *The curves* $c = c_{m/n}(a)$ *given in (7) for* $n < 100$, $m = 1$ *(dashed lines) and* $m = 2$ *(solid lines).*

such a map. Indeed, the x variable is driven by the function $g(x, y) = (x - a)/y + a$. It is convenient to make the following shift of the variable x:

$$\widetilde{x} := x - a,$$

after which the function g gets the form $\widetilde{g}(\widetilde{x}, y) = \widetilde{x}/y$. The image of y_0 after n iterations can be obtained using the following equality:

$$y_n = \frac{b_{n+1}y_0 - ab_n}{b_n y_0 - ab_{n-1}}, \tag{12}$$

where b_j, $j > 1$, is defined by the second order linear difference equation

$$b_j = (c + a)b_{j-1} - ab_{j-2}, \ b_0 = 0, \ b_1 = 1. \tag{13}$$

The solution of the equation (13) can be written as

$$b_j = \frac{(\sqrt{a})^{j-1}\sin(\theta j)}{\sin\theta}. \tag{14}$$

Given that the point y_0 is n-periodic, that is, $y_n = y_0$ and $\theta = \pi m/n$, we get that

$$b_n = 0. \tag{15}$$

Now we can get the image of $\widetilde{x}_0$ after n iterations:

$$\widetilde{x}_n = \frac{\widetilde{x}_0}{\prod_{i=0}^{n-1} y_i} = \frac{\widetilde{x}_0}{b_n y_0 - a b_{n-1}} = \frac{\widetilde{x}_0}{-a b_{n-1}} = \begin{cases} \widetilde{x}_0/(\sqrt{a})^n, & \text{if } m \text{ is even;} \\ -\widetilde{x}_0/(\sqrt{a})^n, & \text{if } m \text{ is odd;} \end{cases} \tag{16}$$

where the equality $\prod_{i=0}^{n-1} y_i = b_n y_0 - a b_{n-1}$ follows from (12) and b_{n-1} is obtained from (14) with $j = n-1$, $\theta = \pi m/n$, using (10) and (11).

So, on the line $\{y = y_0\}$ the $\widetilde{x}$-variable is mapped by the 1D linear map given by the function $g_1(\widetilde{x}) = \widetilde{x}/(\sqrt{a})^n$ for even m, and by the function $g_2(\widetilde{x}) = -\widetilde{x}/(\sqrt{a})^n$ for odd m. In both cases the fixed point is $\widetilde{x}^* = 0$ (which corresponds to the point $x^* = a$ in the x variable). It is attracting for $a > 1$ and repelling for $a < 1$ (recall that for our model $a > 0$). At $a = 1$ any initial point $\widetilde{x}_0$ for the map g_1 is fixed, and is of period 2 for the map g_2.

Now, coming back to the variables (x, y), we conclude that the straight line $\{x = a\}$ is globally (transversely) attracting for $a > 1$ (the trajectory of any initial point (x_0, y_0), $x_0 \neq a$, tends to the m/n-cycle $\{(a, y_i)\}_0^{n-1}$); It is transversely repelling for $a < 1$ (the limit set of the trajectory is $\{(x_i, y_i)\}_0^{n-1}$, where x_i is either $+\infty$ or $-\infty$). At $a = 1$ any point (x_0, y_0) is n-periodic for even m and $2n$-periodic (for $x_0 \neq a$) if m is odd.

Let us consider now $c = c_\rho(a)$, then the trajectory $\{(x_i, y_i)\}_0^\infty$ of any initial point (x_0, y_0) is such that the sequence y_i is quasiperiodic as $i \to \infty$, while the sequence x_i for $a > 1$ tends to $x^* = a$ as $i \to \infty$, and it tends to $\pm\infty$ for $a < 1$. Thus, for $a > 1$ the limit set of the trajectory is the line $\{x = a\}$.

The following proposition describes the dynamics of F_1 at $a = 1$:

Proposition 3. *Let $a = 1$, $c = c_{m/n}(1)$ as given in (7). Then for the map F_1 any point (x_0, y_0), $x_0 \neq 1$, is n-periodic if m is even and $2n$-periodic if m is odd; any point $(1, y_0)$ is n-periodic. For $c = c_\rho(1)$ as given in (8) the trajectory of any point (x_0, y_0) is quasiperiodic. Moreover, the trajectory of any initial point (x_0, y_0) belongs to the invariant hyperbola with center $(x_c, y_c) = (1, (c+1)/2)$ of equation*

$$\frac{(x-1)^2}{(x_0-1)^2} = \frac{y^2 - (c+1)y + 1}{y_0^2 - (c+1)y_0 + 1}. \tag{17}$$

The equation (17) is obtained from the general equation of a hyperbola with center $(x_c, y_c) = (1, (c+1)/2)$, that is from

$$\frac{(x-1)^2}{k_1^2} - \frac{(y - (c+1)/2)^2}{k_2^2} = 1.$$

Substituting to this equation first $(x, y) = (x_0, y_0)$ and then $(x, y) = (x_1, y_1) = ((x_0 - 1)/y_0, c + 1 - 1/y_0)$, we get the coefficients k_1 and k_2 : $k_1^2 = (x_0 - 1)^2(1 - (c + 1)^2/4)/(y_0^2 - (c + 1)y_0 + 1)$ and $k_2^2 = 1 - (c + 1)^2/4$. After rearranging of the general equation of a hyperbola with such coefficients and center (x_c, y_c), we get the equation (17).

So, we have described the dynamics of the map F_1, showing that for $a > 1$ its invariant line $\{x = a\}$ is globally attracting (and any point of this line is either periodic or quasiperiodic) and globally repelling for $a < 1$. At $a = 1$ the trajectory of any initial point (x_0, y_0) is either periodic or quasiperiodic and belongs to the related invariant hyperbola (17).

2.2 Reduction of the map F_2 to a 1D map g on LC

Let us describe now the dynamics of the map F_2 which is quite simple. Note that the map F_2 doesn't depend on the parameter a. The eigenvalues of the Jacobian matrix of F_2 are $\eta_1 = 0$; $\eta_2 = ((1 - r)y + rx)/y^2$, so F_2 is reduced to a 1D map on the straight line LC. If x is the first coordinate of a point $(x, y) \in LC$, then its image by F_2 on LC is given by a 1D map g:

$$g : x \mapsto g(x) = \frac{x(1 - r)^2}{c(1 - r) - rx}. \tag{18}$$

The map g has two fixed points: $x_1^* = 0$ and $x_2^* = (1 - r)(c - 1 + r)/r$. If $c < 1 - r$, then x_1^* is repelling and $x_2^* < 0$ is attracting (see Fig.5b), while if $c > 1 - r$, then $x_2^* > 0$ is repelling and x_1^* is attracting (see Fig.6b). At $c = 1 - r$ these fixed points merge: $x_1^* = x_2^* = 0$.

The corresponding fixed points of the map F_2 are $p_1 = (0, c)$ and $p_2 = (x_2^*, (1-r))$.

2.3 Invariant regions of F at $a = 1$

In the following two sections we describe the dynamics of the map F, using the properties of F_1 and F_2 presented in the previous sections.

Recall that for $0 < c < c^*(a)$ the map F_1 has no fixed points, while the fixed points of F_2 are such that for $c < 1 - r$ the repelling fixed point $p_1 = (0, c)$ belongs to the border line separating the regions R_1 and R_2, and the attracting fixed point $p_2 \in R_1$. Thus, with respect to the map F we have that for $c < 1 - r$ the point p_2 is not a fixed point of F, while p_1 is right-side repelling along LC (which gives the eigendirection related to the eigenvalue η_2) and attracting in the vertical direction related to $\eta_1 = 0$ (see Fig.5, where $c = 0.5$, $r = 0.2$). For $c > 1 - r$ the point p_1

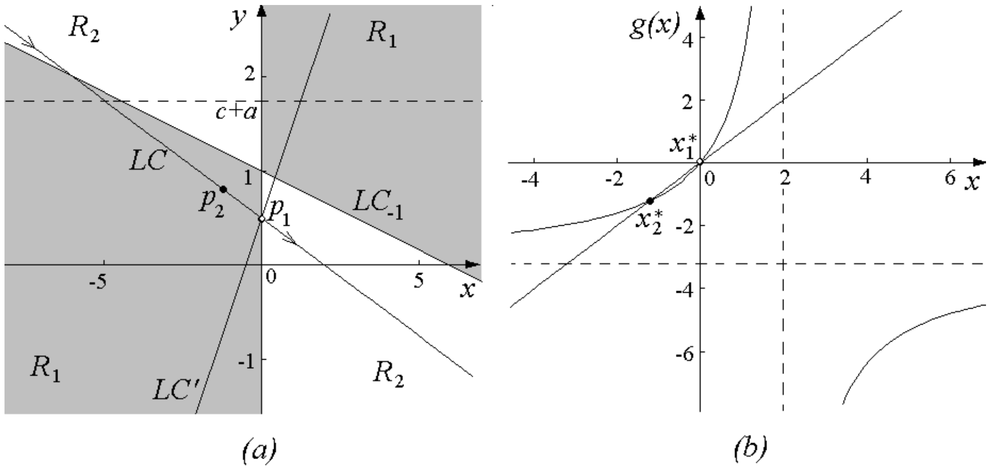

Figure 5: *The fixed points* p_1 *and* p_2 *of the map* F_2 *in the* (x, y)*-plane (a) and the related function* g *given in (18) (b) at* $c = 0.5$, $r = 0.2$ *(the case* $c < 1 - r$*).*

is right-side attracting and $p_2 \in R_2$ is a saddle fixed point of F (see Fig.6, where $c = 0.9$, $r = 0.5$). In Fig.6 it is also shown the eigendirection of F_2 at p_2 related to $\eta_1 = 0$, of equation $y = rx/(c - 1 + r)$, whose segment $[0, b]$ is one of the boundaries of the immediate basin of attraction of p_1, which is a triangle: Two other boundaries of this triangle are the segment $[0, 1]$ of the straight line $\{x = 0\}$ and the segment $[1, b]$ of LC_{-1}. The dashed region denoted B is the total basin of attraction of the point p_1 (its boundary is made by the segments which are preimages of $[0, b]$ by F_1 and the segment of $\{x = 0\}$).

Thus, for $c > 1 - r$ there is a set of initial points of the (x, y)-plane attracted by p_1, so that at the bifurcation diagram shown in Fig.2 the parameter region $c > 1 - r$ is related to the coexistence of the right-side attracting point p_1 (whose basin of attraction is relatively small) with other attractor (for $a > 1$).

Let us describe now the dynamics of F at the bifurcation value $a = 1$. As it was shown (see Proposition 3), at $a = 1$ any initial point (x_0, y_0) is either periodic, or quasiperiodic under the map F_1, moreover, each trajectory belongs to the related invariant hyperbola (17). We can find a subregion of R_1 denoted Q such that the trajectory of any initial point $(x_0, y_0) \in Q$ stays forever in R_1. To define the border of Q we need to find a hyperbola (17) tangent to LC_{-1} (and this hyperbola will be tangent to all consecutive images by F_1 of LC_{-1}). It is not difficult to get the

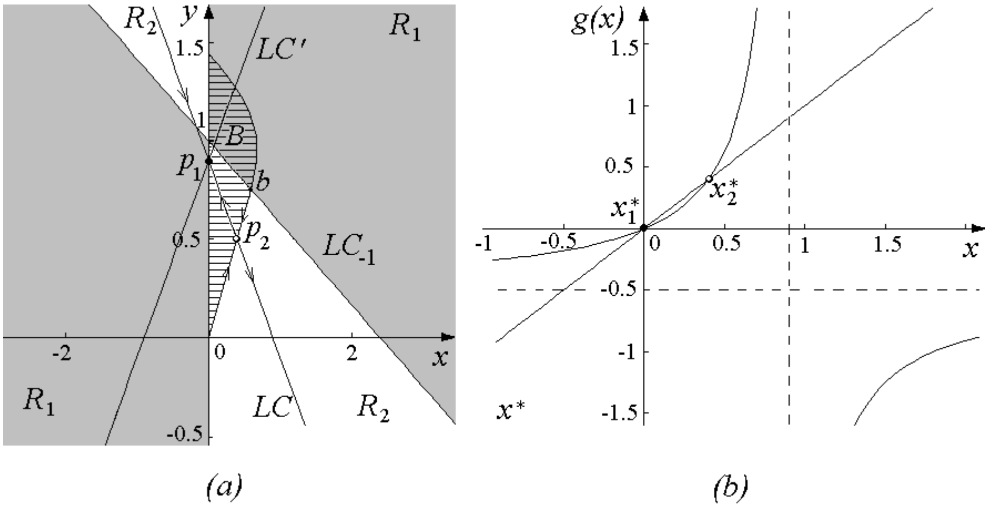

Figure 6: *The fixed points p_1 and p_2 of the map F_2 in the (x, y)-plane (a) and the related function g given in (18) (b) at $c = 0.9$, $r = 0.5$ (the case $c > 1 - r$).*

coordinates (x_t, y_t) of the tangent point of the hyperbola (17) with LC_{-1}:

$$x_t = \frac{(2-r)\,(1-c)}{(1-2r-c)\,r}, \; y_t = \frac{c-1-(c+1)r}{1-2r-c}, \tag{19}$$

so that the equation of the hyperbola, tangent to LC_{-1}, denoted by H, is

$$H: \qquad \frac{(x-1)^2}{(x_t-1)^2} = \frac{y^2-(c+1)y+1}{y_t^2-(c+1)y_t+1}. \tag{20}$$

Note that at $c = 1 - 2r$ the straight line LC_{-1} is one of two asymptotes of H, so that for $c > 1 - 2r$ we have $x_t < 0$, $y_t > 0$, while for $c < 1 - 2r$ the inequalities $x_t > 0$, $y_t < 0$ hold.

Two branches of the hyperbola H are the boundaries of the region Q, consisting of two disjoint parts. If $c = c_\rho(1)$, $a = 1$, then any initial point (x_0, y_0) is quasiperiodic under the map F_1, thus the trajectory of any initial point $(x_0, y_0) \in Q$ is quasiperiodic and belongs to the corresponding invariant hyperbola (17) through this point. Obviously, Q is invariant under the map F. Moreover, below we show that for $c < 1 - r$ the trajectory of any point $(x_0, y_0) \notin Q$ (where $(x_0, y_0) \neq (0, c)$) is attracted to a quasiperiodic trajectory on H, while for $c > 1 - r$ we have to exclude the initial points belonging to the basin of attraction of the point p_2 (see Fig.6 and the comments related to this figure), that is for $c > 1 - r$ the trajectory of any point $(x_0, y_0) \notin (Q \cup B)$ is attracted to H.

1) Let us first consider $1-2r<c<1-r$. To clarify the consideration, we will follow an example presented in Fig.7, in which the invariant region Q, bounded by the hyperbola (20), tangent to LC_{-1} at the point (x_t, y_t), $x_t<0$, $y_t>0$, is shown at $a=1$, $r=0.8$, $c=0.1$. As it was already mentioned, any point $(x_0,y_0)\in R_2$ is mapped in one step into a point of LC, on which the dynamics is defined by the map g, given in (18). For $c<1-r$ the attracting point p_2 of F_2 is in R_1, so the trajectory, approaching p_2, necessarily enters R_1 and then a number of consecutive points of the trajectory belongs to some invariant hyperbola, say, H_1. If such a hyperbola is not H, it intersects LC_{-1} and, thus, after a finite number of iterations we get a point belonging R_2. So, the trajectory comes back to LC and again begins to approach $p_2\in R_1$. Repeating the steps described above, to prove the statement we need to shown that the next hyperbola, which trajectory follows in R_1, say H_2, will be closer to H. We take the hyperbola most distant from H which a trajectory can follow after entering the region R_1, which is a hyperbola passing through the point $a_0=LC_{-1}\cap LC$, $a_0=((1-c)(r-1)/r^2,\ (1-c+cr))/r)$. Note that $x(a_0)<0$, $y(a_0)>0$ for any $0<c<1$, $0<r<1$.

So, let H_2 be the hyperbola through the point a_0, and consider as initial point of the trajectory exactly a_0. Then $a_1=F_2(a_0)=F_1(a_0)\in LC$; $a_1\in H_2$. Following the hyperbola H_2 the trajectory makes one "round" and comes back to the region R_2 (namely, to its part with $x<0$). Let it be a point $a_i\in H_2$ (in our example, it is the point $a_6\in H_2$, see Fig.7); $a_i\neq a_0$ because we consider the case $c=c_\rho(1)$ in which the trajectory on H_2 is quasiperiodic, thus a_i is either "above" a_0 on H_2, i.e., $y(a_i)>y(a_0)$, or "below", that is $y(a_i)<y(a_0)$. If $y(a_i)>y(a_0)$, then $F_2(a_i)=a_{i+1}\in LC$ is such that $c<y(a_{i+1})<c+1$ because $F_2(R_2:x<0)=\{LC:c<y<c+1\}$. So, $a_{i+1}\in(a_0,a_1)\subset R_1$ (in our example, $y(a_7)\in(0.1,1.1)$, $a_7\in(a_0,a_1)$, see Fig.7), and the next hyperbola (through the point a_{i+1}) which the trajectory will follow being in R_1, say, the hyperbola H_3, is located closer to H so that our statement is proved. If $y(a_i)<y(a_0)$, which means that $a_i\in R_1$, then we come to the same conclusion: Namely, after some number of iterations by F_1 (following the same hyperbola H_2) the point of the trajectory eventually will be mapped in a point a_j such that $y(a_j)>y(a_0)$, because the trajectory is quasiperiodic on H_2 and, thus, its points are dense on H_2.

2) If $c>1-r$ (and, thus, $c>1-2r$, so that $x_t<0$, $y_t>0$, as in the example presented above) the consideration can be repeated for the trajectory of any initial point $(x_0,y_0)\notin(Q\cup B)$. Note that in such a case the trajectory, once entered the region R_2, and thus is mapped into a point on LC, necessarily enters the region R_1 (namely, its part with $x<0$) approaching the point $p_1=(0,c)\in LC$ (which is attracting for F_2).

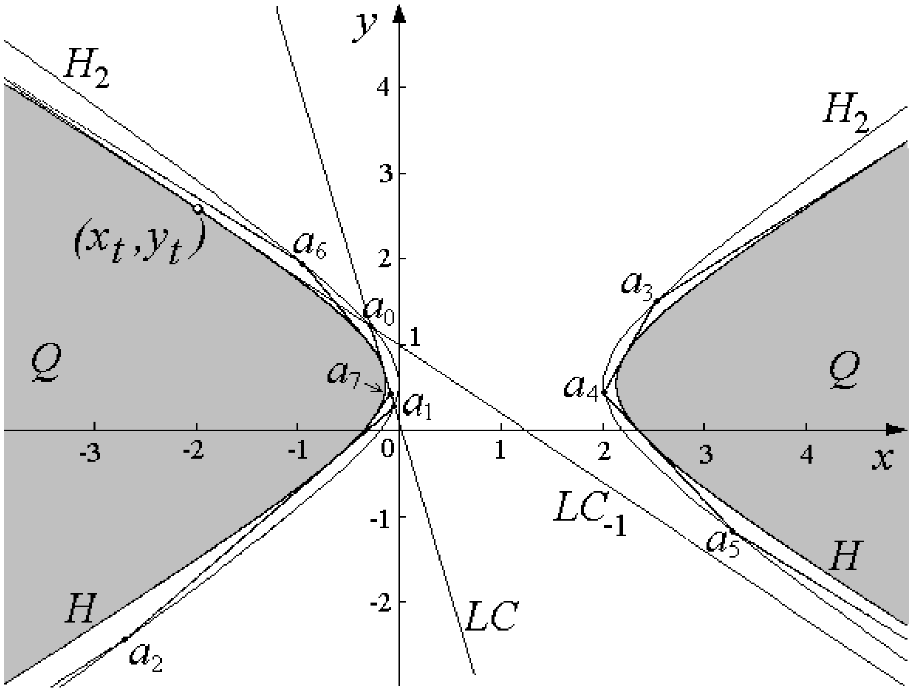

Figure 7: *The invariant region Q bounded by the hyperbola H at $a = 1$, $r = 0.8$, $c = 0.1$.*

3) As for the case $c < 1 - 2r$ (in which $x_t > 0$, $y_t < 0$) we only note that a trajectory starting from the initial point $a_0 = LC_{-1} \cap LC$, and then following under the map F_1 the hyperbola H_2 through a_0, after some number i of iterations we get a point $a_i \in \{R_2 : x > 0, y < 0\}$, and then, similarly to the first case, we can show that $F_2(a_i) = a_{i+1} \in (a_0, a_1) \in LC$, so that the next hyperbola touched by the trajectory is located close to the hyperbola H. Thus, also in this case any initial point $(x_0, y_0) \notin Q$ is attracted to the boundary of Q, i.e., to H.

Thus, the following statement is proved:

Proposition 4. *Let $a = 1$, $c = c_\rho(1)$. Then in the (x, y)-phase plane of the map F there exist an invariant region Q bounded by the hyperbola H of the equation (20) such that any point $(x_0, y_0) \in Q$ is quasiperiodic. Moreover, for $c < 1 - r$ any point $(x_0, y_0) \notin Q$ is attracted to H; and for $c > 1 - r$ any point $(x_0, y_0) \notin (Q \cup B)$ is attracted to H, where B is the attraction basin of the right-side attracting point $p_1 = (0, c)$.*

Now let us construct an invariant region in the case $c = c_{m/n}(1)$. Denote such a region by P. It obviously includes the region Q, bounded by H, but for $c = c_{m/n}(1)$

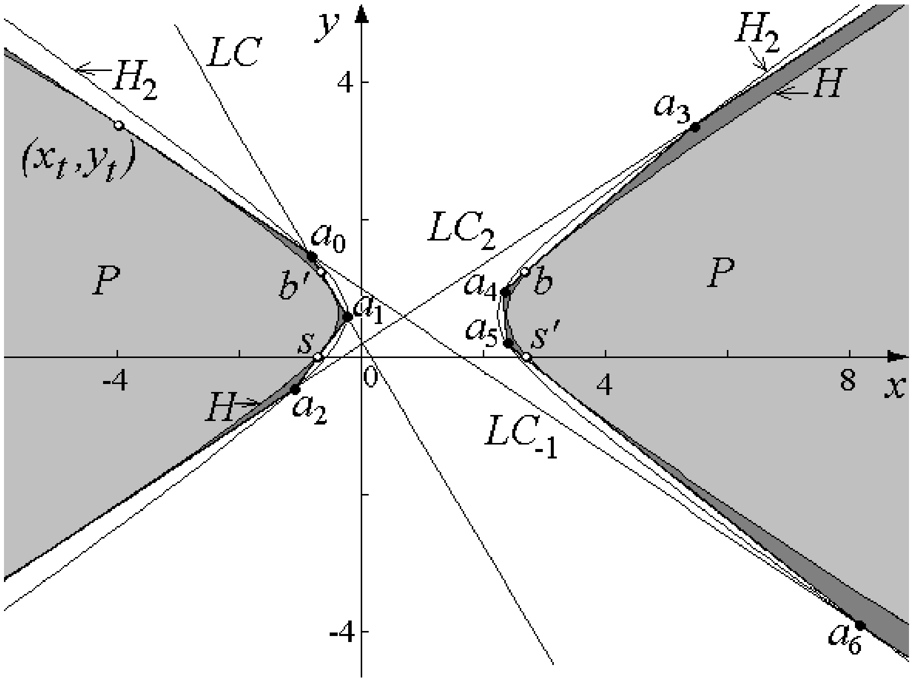

Figure 8: *The invariant region P bounded by the segment $[a_0, a_1] \subset LC$ and its* 6 *images by the map F_1 at $a = 1$, $c = c_{2/7}(1) \approx 0.247$, $r = 0.6$. Any point $(x_0, y_0) \in P$ is* 7*-periodic.*

the invariant region is wider. To see this, let us consider an example presented in Fig.8, where $a = 1$, $c = c_{2/7}(1) \approx 0.247$, $r = 0.6$. In such a case any point $(x_0, y_0) \in Q$ is periodic of period 7 (the region Q is indicated in light gray in Fig.8; The invariant region P includes, besides Q, also the "additional" dark gray regions).

Consider the hyperbola H_2 through the point $a_0 = LC_{-1} \cap LC$ and the segment $[a_0, a_1]$, called generating segment, where $a_1 = F_1(a_0) \in H_2$. The segment $[a_0, a_1] \in LC$ and its 6 images by F_1, belonging to corresponding images of LC by F_1, constitute the boundary of the region P such that any point $(x_0, y_0) \in P$ is periodic of period 7. To see this, first note that $F_1([a_1, a_2]) = [a_2, s_{-\infty}] \cup [s_{+\infty}, a_3]$ and $F_1([a_5, a_6]) = \left[a_6, s'_{-\infty}\right] \cup \left[s'_{+\infty}, a_7\right]$, where $a_7 = a_0$. Here $s_{-\infty} \in LC_2$ and $s_{+\infty} \in LC_2$ denote two points at infinity of LC_2, which are the images by F_1 of the point $s = [a_1, a_2] \cap \{y = 0\}$ at $y(s) \to 0_+$ and $y(s) \to 0_-$, respectively. Similarly, the points $s'_{-\infty} \in LC_{-1}$ and $s'_{+\infty} \in LC_{-1}$ denote two points at infinity of LC_{-1}, which are images by F_1 of the point $s' = [a_5, a_6] \cap \{y = 0\}$, at $y(s') \to 0_+$ and $y(s') \to 0_-$, respectively. Note also that $F_1([a_2, s_{-\infty}] \cup [s_{+\infty}, a_3]) = [a_3, a_4]$ and

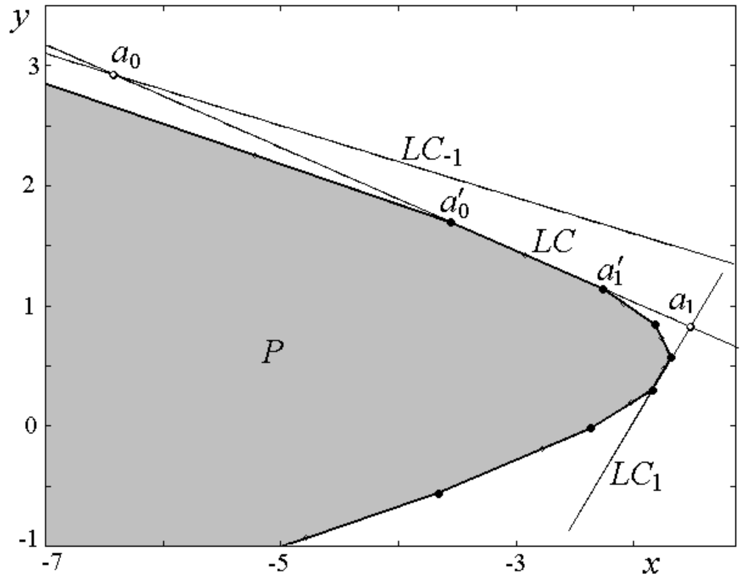

Figure 9: *The enlarged part of the invariant region* P *bounded by the segment* $[a_0', a_1'] \subset [a_0, a_1]$ *and its* 19 *images by the map* F_1*at* $a = 1$, $c = c_{3/10} \approx 0.176$, $r = 0.3$. *Any point* $(x_0, y_0) \in P$ *is periodic of period* 20.

$F_1([a_6, s'_{-\infty}] \cup [s'_{+\infty}, a_7]) = [a_0, a_1]$, where the point $F_1(s_{\pm\infty}) = b \in [a_3, a_4] \subset LC_3$ is such that $y(b) = f(\pm\infty) = c + 1$; and $F_1(s'_{\pm\infty}) = b' \in [a_0, a_1] \subset LC$ such that $y(b') = f(\pm\infty) = c + 1$. Given that all the points $a_i \in R_1$ and $a_i \in H_2$, $i = \overline{0, 6}$, they are periodic of period 7, that is $a_7 = a_0$. Moreover, all images of the segments $[a_0, a_1]$ are in R_1, thus any point of these segments is also 7-periodic. So, the region P bounded by the generating segment $[a_0, a_1]$ and its 6 images by F_1, is such that $P \subset R_1$, thus, any point of it is periodic of period 7.

In the example considered above $m = 2$ is even. As it was shown (see Proposition 3), if m is odd, for $c = c_{m/n}(1)$ any point (x_0, y_0) is $2n$-periodic under the map F_1, thus, any point $(x_0, y_0) \in Q$ is $2n$-periodic. So, it is clear that to construct the invariant region P for odd m we have to consider $2n - 1$ images of the generating segment of LC, which is the segment $[a_0, a_1]$ for $m = 1$ and $m = 2$, but for $m > 2$ the generating segment is $[a_0', a_1'] \subset [a_0, a_1]$. As an example, we show in Fig.9 an enlarged part of the (x, y)-plane with a part of the invariant region P in the case $a = 1$, $m/n = 3/10$, $c = c_{3/10} = 2\cos(3\pi/10) \approx 0.176$, $r = 0.3$: For such parameter values the region P is bounded by the segment $[a_0', a_1'] \subset [a_0, a_1]$ and its 19 images by F_1; Any point $(x_0, y_0) \in P$ is periodic of period 20.

Given that for $c = c_{m/n}(1)$ the boundary of P is made by the images of the generating segment of LC, it is obvious that the trajectory of any point $(x_0, y_0) \notin P$ (for $c < 1-r$), or $(x_0, y_0) \notin (P \cup B)$ (for $c > 1-r$) is mapped to the boundary of P. So, we can state

Proposition 5. *Let $a = 1$, $c = c_{m/n}(1)$. Then in the (x, y)-phase plane of the map F there exist an invariant region P, bounded by the generating segment $[a'_0, a'_1] \subset LC$ and its $n-1$ images by F_1 for even m, or $2n-1$ images for odd m. Any initial point $(x_0, y_0) \in P$ is n periodic for even m and $2n$-periodic for odd m. Any point $(x_0, y_0) \notin P$ (for $c < 1-r$), or $(x_0, y_0) \notin (P \cup B)$ (for $c > 1-r$), is mapped to the boundary of P, where B is the attraction basin of the right-side attracting point $p_1 = (0, c)$.*

Note that when m is odd, two branches of the boundary of P are symmetric with respect to $\{x = 1\}$.

2.4 Dynamics of F for $a > 1$

Recall that for the map F_1 the line $\{x = a\}$ is invariant and, as it was shown in subsection 2.1, for $a > 1$ it is transversely attracting. So, for $a > 1$ any trajectory by the map F with an initial point in R_1, tends towards this line. It is easy to show that the trajectory of any initial point $(x_0, y_0) = (a, y_0) \in R_1$ necessarily enters R_2 (Indeed, such a trajectory is either periodic, or quasiperiodic under the map F_1 (see Proposition 2), and at least one point of the trajectory is with $y < 0$, but $LC_{-1} \cap \{x = a\} = (a, 1-r)$, $1-r > 0$, thus, the trajectory has a point in R_2). As it was already mentioned, the image of any point of R_2 belongs to LC, on which the dynamics is defined by the map g given in (18). So, the trajectory comes back to R_1. Thus, there exists a kind of backward mechanism to generate cyclic behavior.

Indeed, as we have shown in the previous subsection, at $a = 1$ in the phase plane of the map F there exist an invariant region Q bounded by the hyperbola H for $c = c_\rho(1)$, and for $c = c_{m/n}(1)$ there exists the invariant region P, bounded by the generating segment of LC and its $n-1$ images (for even m), or its $2n-1$ images (for odd m). Increasing a only the boundary of such a region remains invariant, being transformed in an attracting invariant set denoted by C, with periodic or quasiperiodic trajectories on it. Namely, if the parameter values (for $a > 1$) belong to some n-periodicity region (see Fig.2), then a pair of cycles (one attracting and one saddle) of period n exist on the set C, which in such a case is made by the unstable set of the saddle cycle approaching the points of the attracting cycle. On other hand, the set C is formed by the generating segment of LC and its n images.

Without going into details we present one example of the invariant attracting set C of the map F existing for $a > 1$, $0 < c < c^*(a)$. Fig.10 shows the set C at $a = 1.3$,

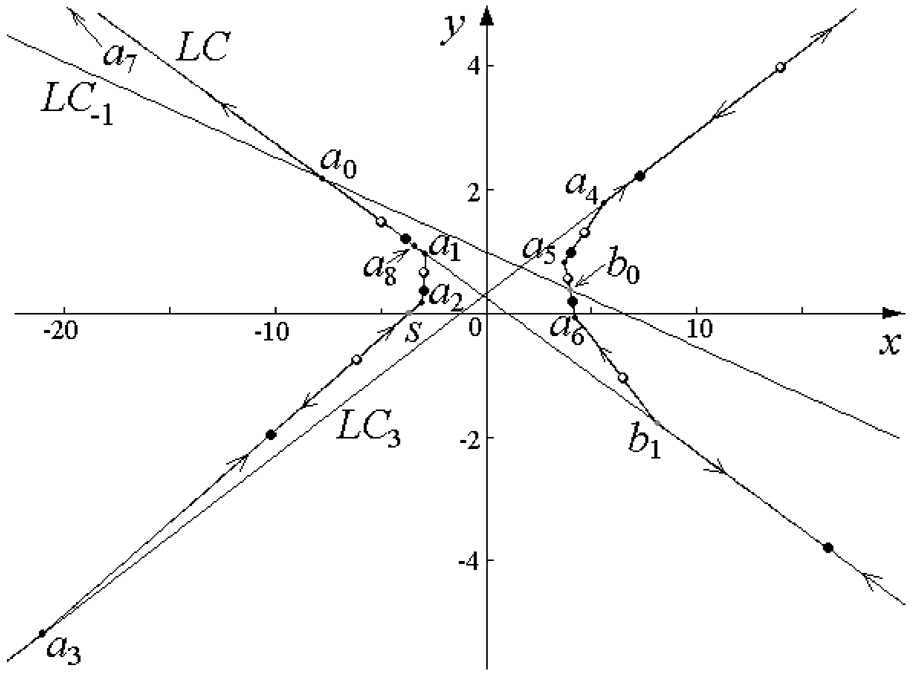

Figure 10: *The invariant attracting set* C *made up by* 7 *images of the segment* $[a_0, a_1]$ *at* $a = 1.3$, $c = 0.25$, $r = 0.2$.

$c = 0.25$, $r = 0.2$ (the (a, c)-parameter point belongs to the 7-periodicity region shown in Fig.2). In this figure the points of the attracting 7-cycle are indicated by black circles, and points of the saddle 7-cycle are marked by the white circles. The generating segment is $[a_0, a_1]$, where $a_0 = LC_{-1} \cap LC$, $a_i = F(a_{i-1})$, $i = \overline{1, 8}$. This segment and its 7 images constitute the set C. Note that as soon as an image of $[a_0, a_1]$ intersects LC_{-1} (such as the segment $[a_5, a_6]$ in our example), then the image of the intersection point, denoted b_0, gives a corner point on the set C (such as the point b_1 in Fig.10). And as soon as some an image of $[a_0, a_1]$ intersects the line $\{y = 0\}$ (such as, for example, the segment $[a_2, a_3]$ intersecting $\{y = 0\}$ in a point denoted s), then its next image is given by two half lines, belonging to the same image of LC (which are $(s_{-\infty}, a_3]$ and $[a_4, s_{+\infty})$, where $s_{-\infty}$ and $s_{+\infty}$ denote two points at infinity of LC_3). These two half lines are mapped to one segment (the segment $[a_4, a_5]$).

Indeed, the dynamics of the map F for $a > 1$ can be quite efficiently studied by means of a first return map on the segment $[a_0, a_1]$. Some results of such a study are presented in [21]. In particular, the first return map was used to show that the boundaries of the periodicity regions in the (a, c)-parameter plane are related to border-collision bifurcations occurring for the points of the cycles. Namely, one of the boundaries corresponds to the so-called "saddle-node" border-collision bifurcation at which points of the attracting and saddle cycles are pairwise merging at the related critical lines and disappear after the collision. Another boundary is related to a collision of the points of these cycle with the related images of the discontinuity point.

3 Center bifurcation of the fixed point of F on the Poincaré Equator

In this section we show that the bifurcation occurring for the map F at $a = 1$ described in the previous sections, can be seen as the center bifurcation of the fixed point of F located "at infinity". To study the behavior of the trajectories of the map F at infinity we can use the Poincaré sphere (see, e.g., [13])

$$S^2 = \left\{(X, Y, Z) \in R^3 : X^2 + Y^2 + Z^2 = 1\right\},$$

from the center of which a point $(X, Y, Z) \in S^2$ is projected onto the (x, y)-plane tangent to S^2 at north pole. The equations defining (x, y) in terms of (X, Y, Z) are given by

$$x = \frac{X}{Z},\ y = \frac{Y}{Z};$$

and the equations

$$X = \frac{x}{\sqrt{1 + x^2 + y^2}},\ Y = \frac{y}{\sqrt{1 + x^2 + y^2}},\ Z = \frac{1}{\sqrt{1 + x^2 + y^2}}$$

define (X, Y, Z) in terms of (x, y). In such a way the origin $(0, 0)$ of the (x, y) plane corresponds to the north pole $(0, 0, 1) \in S^2$; Points on the equator of S^2, i.e., points $(X, Y, 0) \in S^2$, called Poincaré Equator (PE), correspond to the "points at infinity" of the (x, y)-plane. Any two antipodal points on S^2 belong to the same point at infinity.

Indeed, the behavior of the trajectories at infinity can be investigated using the projection of S^2 on the planes $X = 1$ and $Y = 1$, so that to describe the behavior of the trajectories at $x \to \infty$ the following change of variables is used:

$$x = \frac{1}{Z},\ y = \frac{Y}{Z}; \tag{21}$$

and to study dynamics at $y \to \infty$ we can use the change of variables

$$x = \frac{X}{Z},\ y = \frac{1}{Z}. \tag{22}$$

Given that the map F_1 is triangular and its y variable is mapped by the 1D map f given in (4), for which we have shown that any initial point y_0 is either periodic, or quasiperiodic (see Proposition 1), it is clear that the map F_1 cannot have fixed points at $y \to \infty$. And, indeed, one can easy check that the change of variables (22) leads to a map with no fixed points at $Z = 0$.

So, let us make the change of variables (21). Then the map F_1 is transformed into

$$\widetilde{F}_1 : \begin{pmatrix} Z \\ Y \end{pmatrix} \mapsto \begin{pmatrix} \frac{Y}{1+a(Y-Z)} \\ \frac{cY+a(Y-Z)}{1+a(Y-Z)} \end{pmatrix}. \tag{23}$$

Fixed points of $\widetilde{F}_1$ are $(Z, Y) = (1/a, ((c + a) \pm \sqrt{(c + a)^2 - 4a})/2a)$ (which have complex-conjugate values and are not related to the fixed points on the PE of F_1, given that $Z = 1/a \neq 0$), and one more fixed point of $\widetilde{F}_1$ is $(Z, Y) = (0, 0)$, which corresponds to the fixed point of F_1 on the PE.

The map F_2, which is reduced to the 1D map g given in (18) obviously has no fixed points "at infinity". Thus, the only fixed point on the PE of the map F is the one of the map F_1. Note that $Y = 0$ means that all horizontal lines (i.e., the straight lines having the slope $s = y/x = 0$) of the (x, y)-phase plane have the same fixed point on the PE.

The stability of the fixed point $(Z, Y) = (0, 0)$ of the map $\widetilde{F}_1$ is defined by the eigenvalues of the Jacobian of $\widetilde{F}_1$ at this point, which are

$$\lambda_{1,2} = (c + a \pm \sqrt{(c + a)^2 - 4a})/2.$$

One can see that $\lambda_{1,2}$ are complex-conjugate for the parameter range considered, that is for $0 < c < c^*(a)$, $a > 0$, where $c^*(a)$ is given in (6). For $a < 1$ the inequality $|\lambda_{1,2}| < 1$ holds, so the fixed point $(Z, Y) = (0, 0)$ is an attracting focus; $|\lambda_{1,2}| > 1$ for $a > 1$, so the fixed point is a repelling focus. At $a = 1$ we have $|\lambda_{1,2}| = 1$ and the fixed point $(Z, Y) = (0, 0)$ is a center: The related invariant curve, denoted G, through any point (Z_0, Y_0) is obtained from (17) making the variable transformation (21):

$$G: \qquad \frac{(1 - Z)^2}{(1 - Z_0)^2} = \frac{Y^2 - (c + 1)YZ + Z^2}{Y_0^2 - (c + 1)Y_0Z_0 + Z_0^2}.$$

Some of these curves for different (Z_0, Y_0) are shown in Fig.11 at $c = 0.8$. For $(Z_0, Y_0) = (1, Y_0)$ the curve G becomes the straight line $Z = 1$. Similar to the fixed

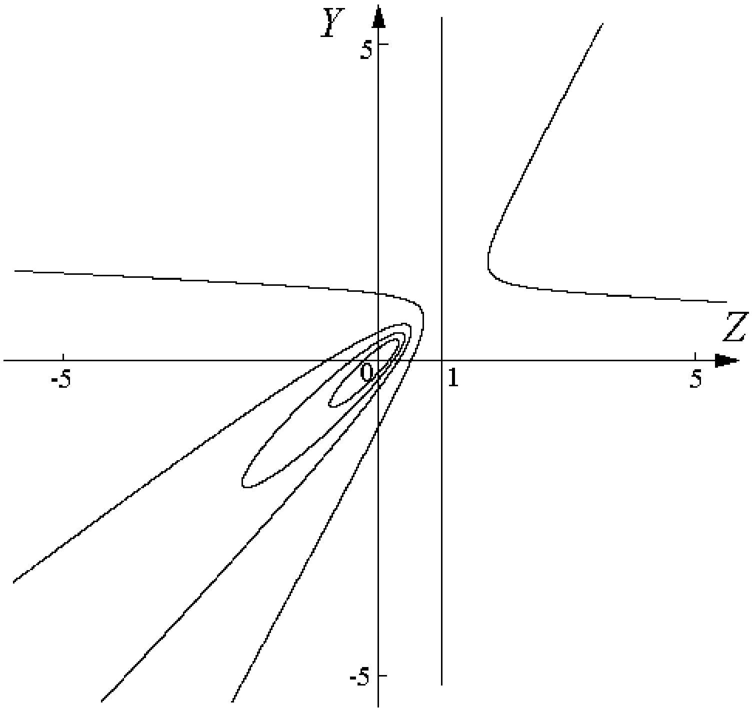

Figure 11: *Invariant curves G (through five different points (Z_0, Y_0)) of the map $\widetilde{F}_1$ at $a = 1$, $c = 0.8$.*

point of a 2D linear map, which is a center (see, e.g., [20]), any point on G is either n-periodic with rotation number m/n (for $c = 2\cos(2\pi m/n) - 1$, obtained from $Re\lambda_{1,2} = (c+1)/2 = \cos(2\pi m/n)$), or quasiperiodic (for $c = 2\cos(2\pi\rho) - 1$).

Thus, the bifurcation which occurs for the map F at $a = 1$ is related to the center bifurcation occurring for the fixed point of F on the PE.

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