



KNOT POINTS IN TWO-DIMENSIONAL MAPS AND RELATED PROPERTIES

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We consider the class of two-dimensional maps of the plane for which there exists a whole one-dimensional *singular set* (for example, a straight line) that is mapped into one point, called a “*knot point*” of the map. The special character of this kind of point has been already observed in maps of this class with at least one of the inverses having a vanishing denominator. In that framework, a knot is the so-called *focal point* of the inverse map (it is the same point). In this paper, we show that knots may also exist in other families of maps, not related to an inverse having values going to infinity. Some particular properties related to focal points persist, such as the existence of a “point to slope” correspondence between the points of the singular line and the slopes in the knot, lobes issuing from the knot point and loops in infinitely many points of an attracting set or in invariant stable and unstable sets.

Keywords: Chaotic attractors with knots; loops in attractors; loops in invariant sets; focal points.

1. Introduction

The study of the dynamic properties of two-dimensional maps having a vanishing denominator in at least one of their components began several years ago (see [Bischi *et al.*, 1999, 2003, 2005]). These papers have evidenced some new kinds of global bifurcations (due to the contacts of two singular sets of different nature), whose description requires the definition of new concepts which are specific to maps defined by functions having a vanishing denominator, like the *set of nondefinition*, the *focal points* and the *prefocal curves*. Roughly speaking, a *prefocal curve* is a set of points which are

mapped (or “focalized”) into a single point, called *focal point*, by the inverse function (if the map is invertible) or by at least one of the inverses (if the map is noninvertible). Such global bifurcations cause the creation of structures of the basins which are peculiar to maps with a vanishing denominator, called *lobes* and *crescents*, and have been explained in terms of contacts between basin boundaries and prefocal curves (see also [Mira, 1999; Bischi & Gardini, 1999; Gardini & Bischi, 1999; Bischi *et al.*, 2001a; Gardini *et al.*, 2007]). Fisher and Gillis [2006] considered the particular case in which a focal point belongs to the prefocal set. These structures have been observed in discrete dynamical systems

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of the plane arising in different contexts, see e.g. [Yee & Sweby, 1994; Billings & Curry, 1996; Bischi & Naimzada, 1997; Brock & Hommes, 1997; Billings *et al.*, 1997; Gardini *et al.*, 1999; Bischi *et al.*, 2001b; Gu & Huang, 2006; Gu & Hao, 2007; Gu, 2007].

In particular, the previous studies have emphasized that in the dynamic behavior of a smooth map T (also of class C^∞) it may occur that one or more of the inverse functions have the analytical expression with a denominator which may vanish in some set of the phase space. These maps in the two-dimensional phase space are characterized by the occurrence of a particular critical set. In fact, as shown in [Bischi *et al.*, 1999] (several examples can be seen in Sec. 3), when this occurs we can observe a whole set of points in the phase plane, say δ_K (often a straight line), which are mapped in a unique point K , i.e. $T(\delta_K) = K$. Thus at least at this point the map is “nonuniquely invertible”, not having a unique rank one preimage (as these are indeed infinitely many: $T^{-1}(K) = \delta_K$). The locus of points δ_K which are mapped into such a knot point K is obviously determined by looking for the locus of possible critical points of rank-0: The solutions of the equation $\det J_T(x, y) = 0$ where $J_T(x, y)$ denotes the Jacobian matrix of the map T evaluated at a point (x, y) of the phase plane. Up to now, such examples were always related to particular inverse functions (i.e. to maps having a vanishing denominator in at least one of the inverse functions) through which was proved the existence of a point-slope relationship between the points of the set δ_K and the slopes of arcs through the point K .

The object of the present paper is to show that this particular behavior may also occur in maps for which the inverses do not have a vanishing denominator. The properties persist with the only assumption that the map T possesses a set, called singular set δ_K , which is mapped into a point called knot K . From this assumption on the map T it is then possible to get the relation between the points of the singular set δ_K and the slopes of arcs through the knot point K . The specific dynamic behaviors, such as the properties due to loops issuing from the knot point K and loops belonging to invariant sets, persist in this class of maps.

The plan of the paper is as follows. In Sec. 2 we deduce the relation existing between points of the singular set and slopes in the knot point. We shall also recall the dynamic properties associated with such a point. In Sec. 3 we give an example of

map family having several singular sets mapped in the same knot point K , without having any inverse function with a vanishing denominator. Thus, in this case the knot point is not a focal point. This means that this kind of knot point is also regular for the inverse map. In particular, if a knot point belongs to the boundary of some basin of attraction, then its whole singular set also belongs to the boundary. In that example, we shall see how the knot point is relevant for the foliation of the map. In Sec. 4 we shall consider an example with a knot point which is also a focal point, showing that it is possible that the focal point belongs to the boundary of the basin of attraction, even if its whole stable set is not on the boundary. This behavior cannot occur when the knot point is not a focal point.

2. Relations Between a Singular Line and the Related Knot Point

Let us consider a generic two-dimensional map T , described by smooth functions:

$$T : \begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases} \quad (1)$$

and let us assume that there exists a set δ_K described by the equation

$$\delta_K : \phi(x, y) = 0 \quad (2)$$

such that for any point $(x_0, y_0) \in \delta_K$ we have $T(x_0, y_0) = K$, i.e. T maps this set into a point K :

$$T(\delta_K) = K \quad (3)$$

Then it is necessarily true that the Jacobian matrix of T is singular at all points $(x_0, y_0) \in \delta_K$. From the Jacobian matrix

$$J_T(x, y) = \begin{bmatrix} f_x(x, y) & f_y(x, y) \\ g_x(x, y) & g_y(x, y) \end{bmatrix} \quad (4)$$

$\forall (x_0, y_0) \in \delta_K$ we have:

$$\begin{aligned} \det J_T(x_0, y_0) &= f_x(x_0, y_0)g_y(x_0, y_0) \\ &\quad - g_x(x_0, y_0)f_y(x_0, y_0) = 0 \end{aligned} \quad (5)$$

that is:

$$f_x(x_0, y_0)g_y(x_0, y_0) = g_x(x_0, y_0)f_y(x_0, y_0) \quad (6)$$

One of the two eigenvalues of $J_T(x_0, y_0)$ is necessarily equal to zero and we assume that the second is different from zero, i.e.

$$\lambda_0 = 0, \quad \lambda_1(x_0, y_0) = f_x(x_0, y_0) + g_y(x_0, y_0) \neq 0 \quad (7)$$

Thus, whenever we consider a point $(x_0, y_0) \in \delta_K$ the linear approximation of the map T at this

point is the linear map characterized by the singular matrix $J_T(x_0, y_0)$, which maps the whole plane into a straight line: the eigenvector $r_1(x_0, y_0)$ associated with the eigenvalue $\lambda_1(x_0, y_0)$ different from zero. It follows that a generic arc ω which crosses the point $(x_0, y_0) \in \delta_K$ is mapped by the map T into an arc which crosses the knot point K and is tangent to the same eigenvector $r(x_0, y_0)$, whose slope is fixed and only depends on the point (x_0, y_0) , say $m(x_0, y_0)$, whichever is the slope of ω at the point $(x_0, y_0) \in \delta_K$. We can explicitly write the slope $m(x_0, y_0)$: From (4) we have that a vector (ξ, η) is mapped by $J_T(x_0, y_0)$ into the vector

$$\begin{bmatrix} f_x(x_0, y_0)\xi + f_y(x_0, y_0)\eta \\ g_x(x_0, y_0)\xi + g_y(x_0, y_0)\eta \end{bmatrix} \quad (8)$$

belonging to the eigenvector $r_1(x_0, y_0)$. So its slope is given by

$$m(x_0, y_0) = \frac{g_x(x_0, y_0)\xi + g_y(x_0, y_0)\eta}{f_x(x_0, y_0)\xi + f_y(x_0, y_0)\eta} \quad (9)$$

which is independent of the vector (ξ, η) due to condition (6). If $f_x(x_0, y_0) = 0$ then $g_y(x_0, y_0) \neq 0$ and from $f_x(x_0, y_0) = g_x(x_0, y_0)f_y(x_0, y_0)/g_y(x_0, y_0)$ we get $m(x_0, y_0) = g_y(x_0, y_0)/f_y(x_0, y_0)$; if $f_x(x_0, y_0) \neq 0$ then $g_y(x_0, y_0) = g_x(x_0, y_0)f_y(x_0, y_0)/f_x(x_0, y_0)$ and $m(x_0, y_0) = g_x(x_0, y_0)/f_x(x_0, y_0)$. So we have:

If $f_x(x_0, y_0) = 0$ then

$$m(x_0, y_0) = \frac{g_y(x_0, y_0)}{f_y(x_0, y_0)} \quad (10)$$

If $f_x(x_0, y_0) \neq 0$ then

$$m(x_0, y_0) = \frac{g_x(x_0, y_0)}{f_x(x_0, y_0)} \quad (11)$$

These properties come from the application of standard algebraic tools, and were already noticed in the paper by Bischi *et al.* [1999]. In that paper the relation between the slope m of an arc through the point K and the point $(x_0, y_0) \in \delta_K$ was described when the inverse map has a vanishing denominator. The object of this work is to remark that this is not a necessary condition. Thus we may have that the relation points–slopes in (10) or (11) occurs also in other classes of maps. The relation may be invertible, leading to the one-to-one correspondence between slopes through K and the points $(x_0, y_0) \in \delta_K$, but it is even possible to have relations points–slopes which are many-to-one. An example is given in the next section.

Let us first recall here the meaning of the relations given above. They will be used to explain the dynamic properties of the related map T . From (10), (11) we have that

- (i) Any arc ω through a point $(x_0, y_0) \in \delta_K$ with slope different from that of the eigenvector $r_0(x_0, y_0)$ associated with the eigenvalue $\lambda_0 = 0$ of $J_T(x_0, y_0)$ is mapped by T into an arc $\omega' = T(\omega)$ through the knot point K which is tangent in K to the straight line with slope $m(x_0, y_0)$, see the qualitative picture in Fig. 1(a);
- (ii) Any arc ω crossing δ_K at two different points (x_1, y_1) and (x_2, y_2) is mapped into a loop crossing the knot point K in two arcs generally with different slopes $m_1 = m(x_1, y_1)$ and $m_2 = m(x_2, y_2)$ [see Figs. 1(b) and 1(c)];
- (iii) In general, the knot point K is not a fixed point. However, it shares an important property with the periodic points: It may be crossed by infinitely many invariant curves.

In particular, the property described in (ii) is responsible for invariant sets having infinitely many loops. In fact, let us assume that an invariant set associated with some cycle (i.e. a stable or unstable set) crosses the singular set δ_K at two points, then the image of an arc of invariant curve is mapped into a loop issuing from K [see Fig. 1(c)]. This gives rise to invariant sets with an infinite number of loops (which are not necessarily related with chaotic dynamics). Similarly, let us assume that an invariant area A in the two-dimensional phase plane (as for example, an *absorbing area*) crosses the singular set δ_K , then the image of a portion of plane is a lobe issuing from K and a similar shape persists in its images of any rank.

In the next section we shall propose a family of maps without any vanishing denominator in the inverse functions, and all the properties (i)–(iii) described above will be shown to hold.

Moreover, another difference in terms of dynamic properties between knot points which are also focal points and knot points which are not focal points, will be shown in Sec. 4. There we shall recall a property which may occur when there is a focal point: When a focal point Q belongs to the boundary (or frontier) ∂B of a basin of attraction B (of some attracting set), then it is not necessarily true that *all its stable sets* also belong to the boundary. This may occur when the focal point is also a *singular fixed point* of the map, such that it is not a

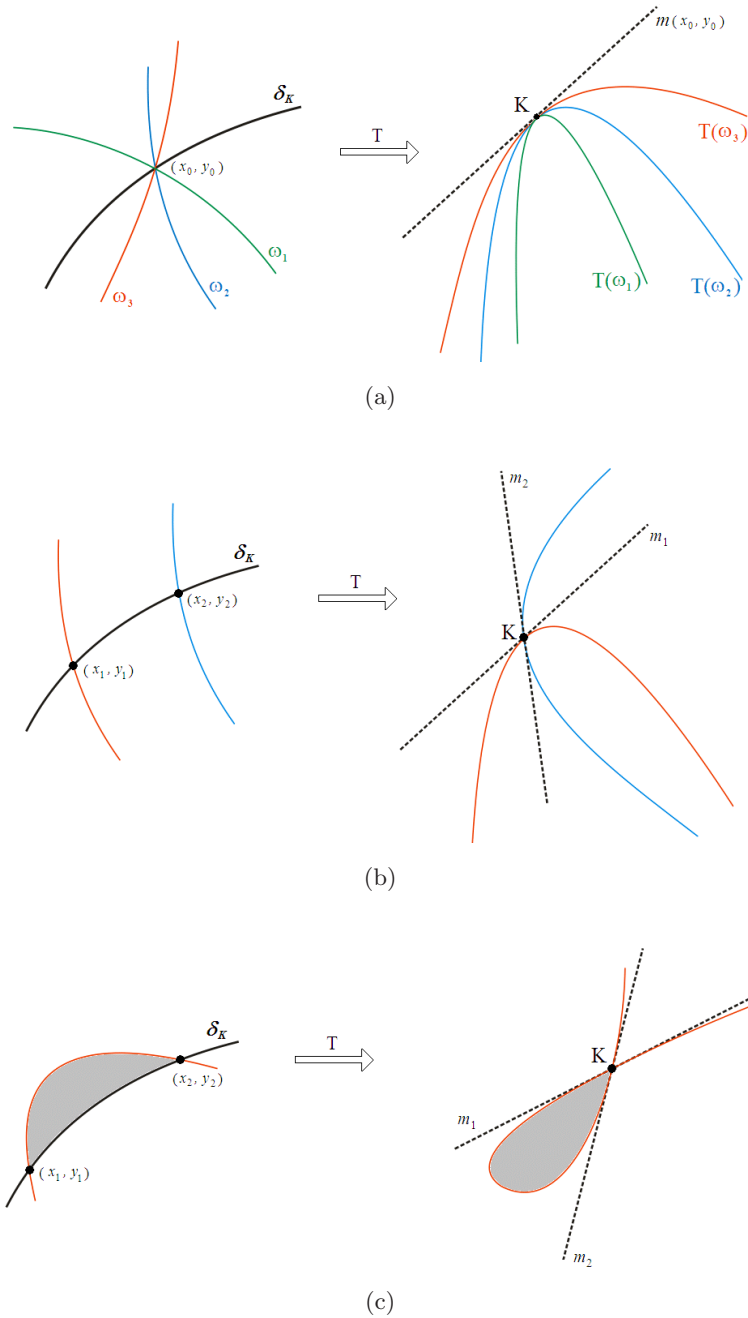


Fig. 1. (a) Three different arcs ω_i through a point $(x_0, y_0) \in \delta_K$ are mapped by T into arcs $\omega'_i = T(\omega_i)$ through the knot point K which are all tangent in K to the straight line with slope $m(x_0, y_0)$. (b) Two arcs, ω_1 and ω_2 , through two different points (x_1, y_1) and (x_2, y_2) respectively, are mapped into two different arcs crossing the knot point K with slopes $m_1 = m(x_1, y_1)$ and $m_2 = m(x_2, y_2)$. (c) One arc ω crosses the singular set δ_K in two different points (x_1, y_1) and (x_2, y_2) , and is mapped into a loop crossing the knot point K .

fixed point of the inverse (or of any of the inverses, as it occurs when all the inverses are not defined in the focal point). Thus it is possible to have:

$$Q \in \partial B \not\Rightarrow T^{-1}(Q) \in \partial B$$

and an example will be given. While in the case of a map with a knot point K , which is not also a focal

point (i.e. not related to the vanishing denominator of one inverse function), we have that K is a regular point for the map, and $T^{-1}(K)$ is well defined. Thus the dynamic behavior of K when it belongs to some frontier ∂B is the standard one, that is:

$$K \in \partial B \Rightarrow T^{-1}(K) \in \partial B$$

3. A Family of Maps with Knots Which are Not Focal Points

In order to show the properties of a singular set δ_K and its related knot point K , let us consider the following family of maps:

$$T : \begin{cases} x' = \sin(ax) + \cos(by) + u \\ y' = \cos(ax) + \sin(by) + v \end{cases} \quad (12)$$

where the parameters a, b, u, v are real numbers with the restrictions $a \neq 0$ and $b \neq 0$. As the functions $f(x, y)$ and $g(x, y)$ defining the map T are periodic with the same period, then T is also periodic. Moreover, in one iteration the phase plane is mapped into the square $\Pi = [u-2, u+2] \times [v-2, v+2]$, centered at the point (u, v) , which is mapped into itself: $T(\Pi) \subseteq \Pi$.

The Jacobian matrix of T is given by

$$J_T(x, y) = \begin{bmatrix} a \cos(ax) & -b \sin(by) \\ -a \sin(ax) & b \cos(by) \end{bmatrix} \quad (13)$$

and we have

$$\det J_T(x, y) = ab \cos(ax + by)$$

so that it vanishes on straight lines R_k of the phase space given by:

$$R_k : y = q_k - \frac{a}{b}x, \quad q_k = \frac{1}{b} \frac{(2k+1)\pi}{2}, \quad (14)$$

k integer

which constitute the critical set of rank-0 of the map. Some straight lines R_k are the critical lines associated with the usual “folding” of the Riemann plane, and denoted by LC_{-1} following the notation used in [Mira *et al.*, 1996], whose images are the critical lines LC of rank-1, which separate regions of points in the phase plane having a different number of rank-1 distinct preimages. These are the lines R_k associated with $k = 0$ and even values of k , say $k = 2n$. It is easy to see that their images belong to the circle C_2 of radius 2 and center at the point (u, v) :

$$LC_{-1}(k) : y = \frac{1}{b} \left[(2k+1) \frac{\pi}{2} - ax \right], \quad k = 0 \text{ and } k \text{ even} \quad (15)$$

$$LC = T(LC_{-1}) \subseteq C_2 = \{(x, y) | (x-u)^2 + (y-v)^2 = 4\} \quad (16)$$

while the straight lines R_k with k odd are all mapped into one point: $K = (u, v)$:

$$T(R_k) = K, \quad K = (u, v) \quad \text{for any } k = 2n + 1, \quad n \in \mathbb{Z}$$

Thus K is a knot for T , and all the straight lines R_k with k odd are singular sets. As in the previous section, let us denote by δ_k a straight line R_k for k odd.

It may be observed that all the straight lines R_k have the same slope, given by $-(a/b)$. The line R_0 is mapped into a *maximum circle*, as it is a circle centered in the knot K and radius $r(q_0) = 2$, which is the maximum compatible with the trapping square Π . The lines δ_{-1} and δ_1 are mapped into the knot K , which may be considered as the *minimum circle*, with center in K and radius $r(q_{\pm 1}) = 0$. Moreover, it is easy to see that any straight line of equation

$$S(q) : y = q - \frac{a}{b}x \quad (17)$$

(whichever is the real value q) is mapped into a circle C_q centered in the knot point K and of radius $r(q) \in [0, 2]$:

$$C_q = \{(x, y) | (x-u)^2 + (y-v)^2 = r(q)^2\} \quad (18)$$

where

$$r(q) = \sqrt{2 + 2 \sin(bq)} \quad (19)$$

In particular, when we increase the value of q from q_0 to q_1 then the circles C_q fill in the whole disk bounded by C_2 . Similarly, if we decrease the values of q from q_0 to q_{-1} then again the circles C_q fill in the whole disk bounded by C_2 . In other words: each circle C_q of radius $r(q) \in [0, 2]$ has the two distinct rank-1 preimages given by lines parallel to the critical curves, one above and one below $LC_{-1}(0)$ (associated with $k = 0$) (see $S(q)$ and $S(-q)$ in Fig. 2), and similarly, infinitely many straight line preimages exist in the strips in which the phase plane is separated by all the straight lines $LC_{-1}(k)$ for any even integer k . We have so proved that in one application of T the phase plane is mapped into the disk, say D , of radius 2, bounded by the *maximum circle* C_2 , which on its turn is mapped into itself, i.e. it is a trapping set: $T(D) \subseteq D$. Thus the disk D may be assumed to be the trapping region of interest in the study of the forward dynamic properties of T (as in one iteration any point is mapped in D , from which it will never escape under repeated iterations).

It is interesting to observe that even the particular knot point K may be considered a *critical point* of the map in the sense that it belongs to a boundary that separates regions of points having a different number of rank-1 preimages. Moreover, the knot itself has an *infinite number* of rank-1 preimages, which do not occur at nearby points (whichever is

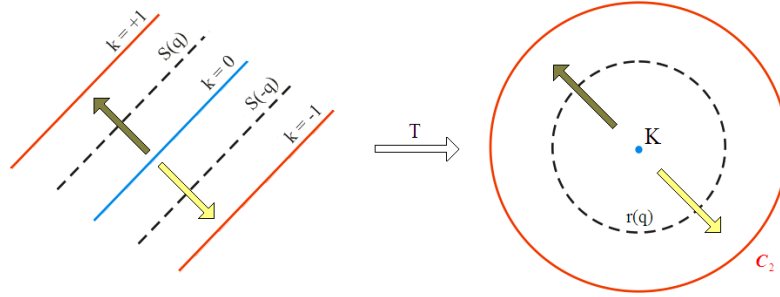


Fig. 2.

the number of related rank-1 preimages). Indeed it can be easily seen that the preimages of a point (x', y') belonging to D can be obtained as follows:

$$\bar{x} = y + \frac{1}{a} \arcsin \left(\frac{(x' - u)^2 + (y' - v)^2}{2} - 1 \right) \tag{20}$$

where y is a solution of the following equation:

$$\cos(2ay) + 2 \sin(a\bar{x} + ay) - \cos(2a\bar{x}) - (x' - u)^2 + (y' - v)^2 = 0 \tag{21}$$

And it is easy to see that $T^{-1}(K) = T^{-1}(u, v)$ includes all the straight lines R_k with k odd integer.

For the family of maps considered in this section, the point-slope relationship is expressed, by using (11), in the form:

$$m(x_0) = \tan(-ax_0) \tag{22}$$

Thus, considering a generic arc through a point $(x_0, q_k - (a/b)x_0) \in \delta_k$ we have that it is mapped into an arc through the knot K with tangent (in K) given by $m(x_0) = -\tan(ax_0)$. This relation is generally not one-to-one in the absorbing region of interest. In fact, considering one arc through K with slope m' in K , we may find one or more distinct points in D belonging to the singular lines, say $(x_i, q_k - (a/b)x_i)$ belonging to some δ_k with k odd, such that $m' = \tan(-ax_i)$ for all i . This depends on the value of the parameter a (and we shall see that when a is high then the relation is not one-to-one).

Moreover, the knot point K is also involved in the foliation of the Riemann plane due to the action of the map T , because there are portions of the phase space which are mapped into lobes issuing from K . As an example, let us consider the image of the disk D (a trapping set) looking at the foliation of the set $T(D)$. Several cases are shown in Fig. 3, where in each zone of $T(D)$ we evidence the number of distinct rank-1 preimages existing in the trapping region D , which depend on the values of

the parameters a and b . In Fig. 3(a), at low values of the parameters, we can see that the relation point-slope is one-to-one. The portion of $LC_{-1}(0)$ inside the disk D is mapped in a portion of the circle C_2 . There is also a portion of singular line δ_{-1} which is mapped in K . There are regions of points having 2, 1, 0 distinct preimages in the disk D . The portion of disk D above the singular line δ_{-1} which includes $LC_{-1}(0)$ is “folded” in one iteration by T giving the region above the knot K . This includes the region folded along LC with two preimages, and the region below it with only one preimage. While the portion of disk D below the singular line δ_{-1} is mapped into the circular region below K with only one preimage, and it is drawn with a different color because it is “turned” around the knot point K . In Fig. 3(b) we show the situation in which the two distinct points of intersection between the singular line δ_{k-1} and the circle C_2 have the same slope: $m = \tan(-ax_0) = \tan(-ax_1)$ in the relation points-slopes. In Fig. 3(c) the number of regions with two preimages increases. Here the relation points-slopes is many to one. To be more precise, any arc γ crossing the knot K between the cone with slopes $m_1 = m(x_1, y_1)$ and $m_2 = m(x_2, y_2)$, crossing regions with two preimages, has two distinct rank-1 preimages in D [Fig. 3(c)]. While any arc η crossing K in the complementary cone has only one preimage in D . In Fig. 3(d) we show the overlapping of zones, leading to a new zone with three distinct rank-1 preimages in D . This region is created due to a tangency between the upper and the lower zones, at a point which is a fixed point of the map. It may be seen that the straight lines $LC_{-1}(1)$ and δ_1 outside the disk D are close to a contact, after which they will enter D . The tangencies occur simultaneously. This will clearly increase the number of loops through the knot point K . As Fig. 3(e) shows, a new loop with two preimages appears between the two previously existing. Moreover, the new portion

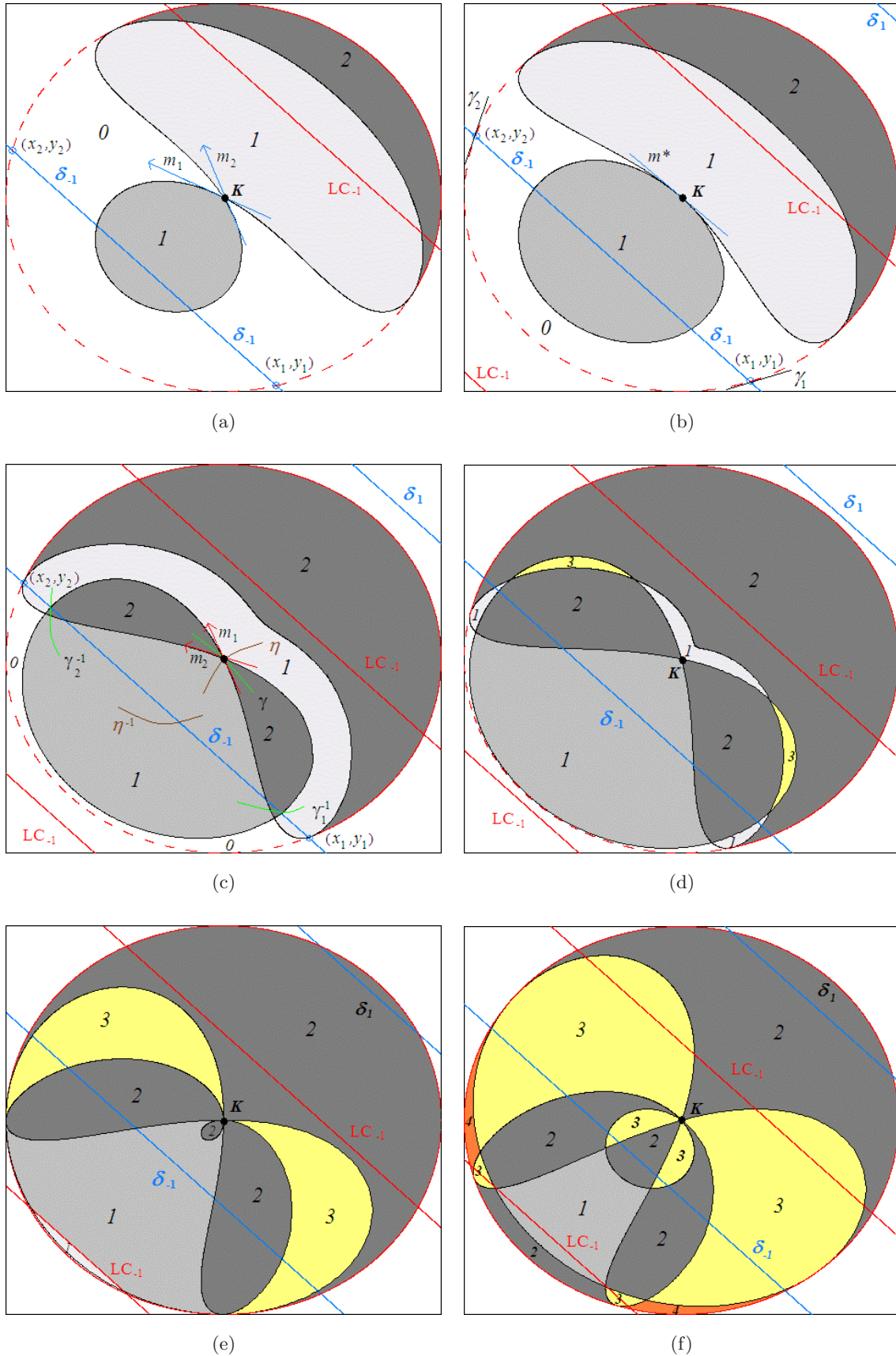


Fig. 3. Foliation of the disk D at $u = 0, v = 0$, and different values of a and b . (a) $a = b = 0.08$; (b) $a = b = 1.24$; (c) $a = b = 1.48$; (d) $a = b = 1.6$; (e) $a = b = 1.76$; (f) $a = b = 1.96$.

of critical line inside D gives one more piece of the boundary of the circle C_2 , and no region with zero preimages exists. In Fig. 3(f) only a small portion with one preimage is left. By increasing the parameters, the number of distinct preimages increases more and more, associated with bifurcations of the same kind as those described in Fig. 3, and due to the increasing number of straight lines $LC_{-1}(k)$ and δ_k entering the disk D .

In order to illustrate the dynamic behaviors related with the properties (i)–(iii) of the previous

section, let us consider the following example at fixed values of the parameters $u = 0, v = 0, b = 1.4$ and varying the parameter a in a small interval. Figure 4(a) shows that an invariant area is approaching the singular line. Chaotic areas crossing the singular lines δ_{-1} and δ_1 in one, two and three points are shown in Figs. 4(b)–4(d) respectively, at three different values of a . The related loops issuing from the knot point K progressively increase the complex structure of the invariant absorbing area. In all the four cases, the boundary

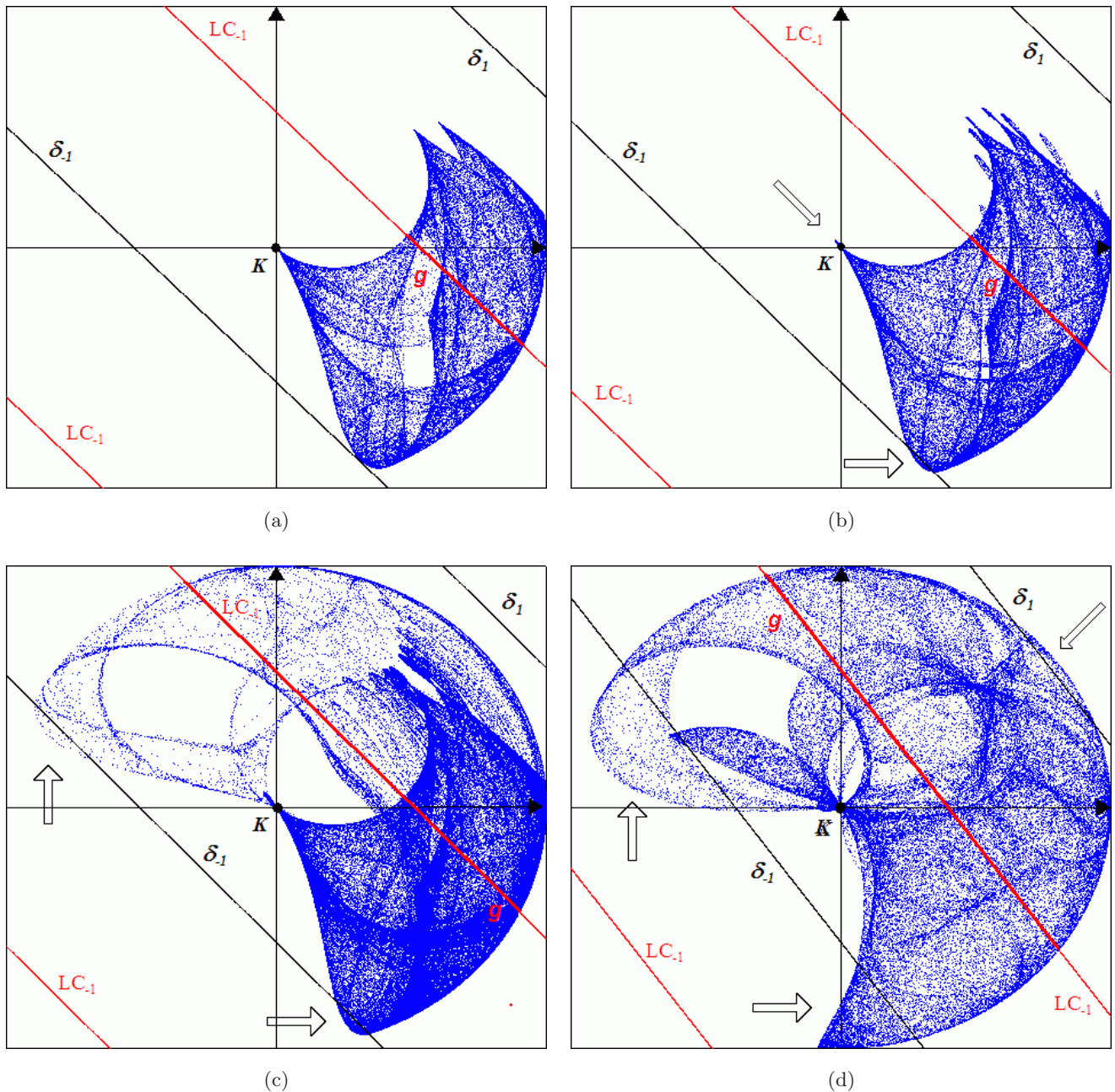


Fig. 4. Chaotic areas, at $u = 0, v = 0, b = 1.4$, and different values of a . (a) $a = 1.485$, the area is without loops; (b) $a = 1.52$, the area crosses δ_{-1} in one part; (c) $a = 1.548$, the area crosses δ_{-1} in two parts; (d) $a = 2$, the area crosses δ_{-1} in three parts.

of the chaotic area is obtained by a finite number of images of the arc (g) belonging to $LC_{-1}(0)$ (so called “generating arc” following the notation given in [Mira *et al.*, 1996]). It may be noticed the “explosion” of the invariant area occurring between Figs. 4(b) and 4(c): this is due to a tangential contact of the boundary of the invariant area with the stable set of a two-cycle saddle, existing outside the invariant area in Fig. 4(b), where the “tongues” of the invariant area are approaching the stable set of the two-cycle, while an ebranch of the unstable set of the two-cycle enters the invariant area. This “contact bifurcation” of the invariant area (also called “external crises”, see [Grebogi *et al.*, 1982] and [Sommerer & Grebogi, 1992]), is due to the first homoclinic bifurcation of the two-cycle saddle, and causes the explosion of the invariant set (which,

after the bifurcation, includes the two-cycle saddle and its whole stable and unstable sets).

Another example is shown in Fig. 5. An attracting set formed by four chaotic pieces is shown in Fig. 5(a), and it can be seen that one piece consists of one arc crossing the singular lines δ_{-1} at two points, so that its image is a chaotic set with a loop on the knot K , as the enlargements show. This example also shows that loops often appear on the unstable sets of a cycle. In fact, in Fig. 5(b) we show an unstable fixed point P^* and a small portion of an unstable set issuing from it, tangent to one of the eigenvectors, and loops are soon formed, and will persist in the forward images. We remark that the formation of this kind of loops may also occur in maps which are invertible (i.e. with only one inverse function) except for the knot point. While when

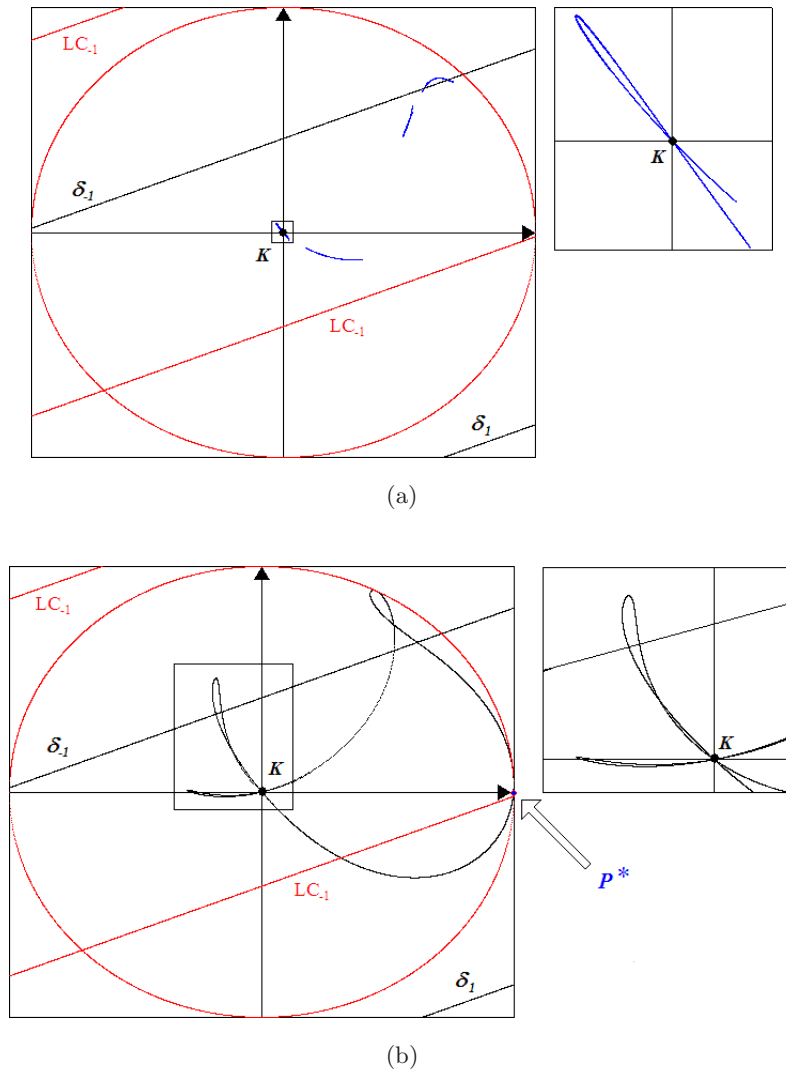


Fig. 5. (a) A chaotic set made up of four chaotic pieces is shown at $u = 0, v = 0, a = 0.75, b = -1.88$ crossing the singular line, with enlargement of the chaotic loop at the knot point K . In (b) it is reported a portion of the unstable set issuing from the unstable fixed point P^* with enlargement.

a map is nonuniquely invertible, the formation of loops on unstable sets may also be due to the standard foliation of the plane. In fact, in noninvertible maps it may occur that an arc crossing the critical set LC_{-1} includes two distinct preimages of a same point, thus forming a loop on the unstable set which also includes its infinitely many images (examples were given in the book [Gumowski & Mira, 1980; Mira et al., 1996], and a detailed analysis may be found in [Frouzakis et al., 1997], see also [Maistrenko et al., 2003]).

4. Singularity on the Basin's Boundary

In dynamical system theory, it is a common knowledge that if a singularity, as a cycle of any period, belongs to the frontier (or boundary) of some attracting set, then the whole stable set of the cycle also belongs to the boundary itself. Indeed, when this condition related to a cycle is not fulfilled, it means that some peculiarity exists, as it may occur at a bifurcation value. However, in the case of the singular sets described in this paper, such as straight lines mapped into one point, we shall show that this property may be *persistently not true*. We consider an example proposed in [Gu & Huang, 2006] (but a similar example can also be found in [Cathala & Barugola, 1999] and in [Cathala, 1999]).

Let M be the following family of maps

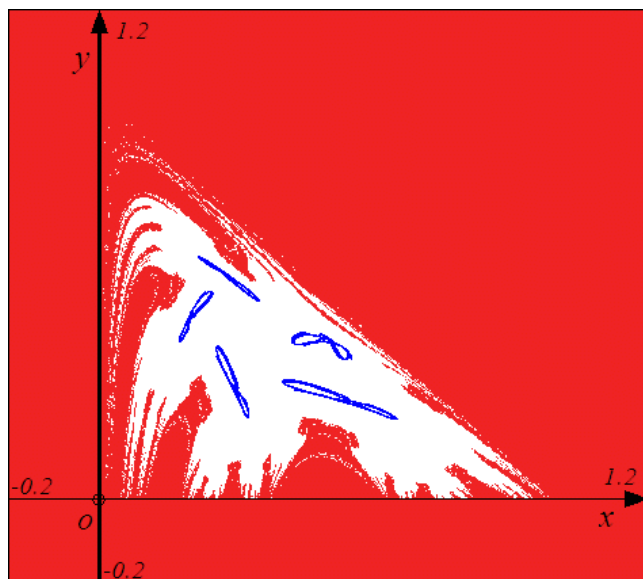
$$M : \begin{cases} x' = Ax(1 - x - y) \\ y' = Bxy \end{cases} \quad (23)$$

for which it is immediate to see that the y -axis $x = 0$ is mapped into the point $Q = (0, 0)$. In this case, the knot Q is also the *focal point* of the inverses of M (M has two distinct inverse functions, both with a vanishing denominator), for which the straight line $x = 0$ is the *prefocal set* (see [Bischi et al., 1999]).

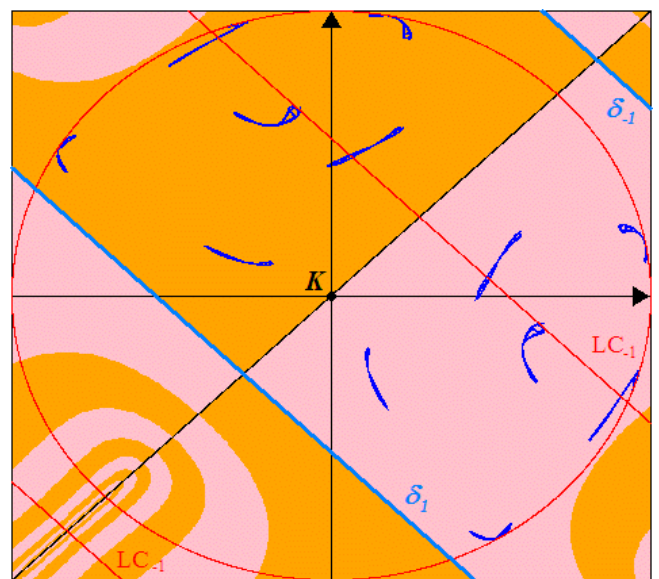
In this example, we can find intervals of parameter values in which the focal point Q belongs to the boundary ∂B of a basin of attraction (of some attracting set in the positive quadrant of the phase plane). An example is given in Fig. 6(a). We see that the point Q belongs to the immediate basin of the attractor. The boundary ∂B has a fractal structure, and infinitely many arcs of the boundary approach Q . Thus Q is a limit point of arcs of the frontier, and belongs to the frontier. However it is not true that all the points which are mapped in Q also belong to the frontier ∂B . In fact it is immediate to see that there are several points of the line $x = 0$ which are not on the frontier of the basin. Thus:

$$Q \in \partial B \not\Rightarrow M^{-1}(Q) \in \partial B$$

To be precise, we notice that Q is a *singular fixed point* of M because it is not a fixed point of the



(a)



(b)

Fig. 6. (a) A cyclic chaotic attractor of the map M in (23) at $A = 4.22$, $B = 3$ and the related basin of attraction. The focal point O is on the basin boundary, but the whole singular line $x = 0$ is not on the boundary. (b) Two chaotic attractors of the map T in (12) at $u = 0, v = 0, a = b = 1.424$ and the related basins of attraction. The knot point K is on the basin boundary, as well as the singular lines and related preimages.

two inverses of M (for which it is a *focal point*). This may explain the particular dynamic behavior at that point.

Instead, the behavior at a knot point K which is not a focal point is regular. An example with the family of maps considered in the previous section is shown in Fig. 6(b). The knot point K belongs to the main diagonal which is an invariant set separating two basins of attractions. K and all the related singular lines δ_{-1} and δ_1 also belong to the basin boundary (whose portion inside D is shown in the figure), as well as the related preimages of any rank.

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