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Use of homoclinic orbits and Iterated Function Systems in backward models

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Abstract

Several applied models (specially in Economics) appear formulated in the so called "backward dynamics", as discrete models in the form $x_t = F(x_{t+1})$, and the interest is in the behavior of the *forward values* of the state variable $(x_t, x_{t+1}, x_{t+2}...)$, in which the function $F(.)$ has not a unique inverse. We shall show that the knowledge of the properties of the dynamical system $x_{t+1} = F(x_t)$ when there are homoclinic orbits, coupled with the theory of Iterated Function Systems, may be used in the study also of the backward model $x_t = F(x_{t+1})$ to describe *fractal "attractors" in the forward states of the model,* by using an approach much simpler and immediate with respect to the Inverse Limit Theory, a different technique frequently used in this context.

1 Introduction

In the recent years several models, specially in Economics, appeared formulated in the so called "backward dynamics". That is, as discrete models in the form $x_t = F(x_{t+1})$, and the interest is in the behavior of the *forward values* of the state variable $(x_t, x_{t+1}, x_{t+2}...)$ (see [8], [9], [26], [27], [10], [22], [16]). There are no problems when the function $F(.)$ is invertible (as $x_{t+1} = F^{-1}(x_t)$ is a dynamical system), while difficulties arise in the cases in which the function $F(.)$ has not a unique inverse, and difficulties may also arise in the interpretation of the models.

Mathematically, this kind of models have been investigated considering the space of all possible sequences, which is a space of infinite dimension (the so-called Hilbert Cube), and is known as Inverse Limit Theory (for the interested reader we refer to

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[14], [15] and the references therein). As applications to economic models see [25], [17], [18]).

However, the inverse limit approach is rather abstract (as it always considers infinitely many states all together at once, without a real selection of the states step by step), so we prefer to follow a different approach, which is based on the theory of Iterated Function Systems, as described in [2], [3]. The theory of Iterated Function Systems, IFS for short, is a generic tool to obtain fractal sets by iterating contraction functions. This subject is clearly related with chaotic systems and expanding functions, although the connection is not so widely known. We shall see how the *homoclinic orbits* existing in the standard dynamical system $x_{t+1} = F(x_t)$, which are the basic tools to prove the existence of chaotic behavior (see for example in [20], [5]), are also the basic tools to get the *contraction functions* giving us fractal sets in the IFS context. That is, we show a kind of "bridge" between the theory of Dynamical Systems and the theory of IFS, which is useful to describe *fractal "attractors" in the forward states of backward models.*

The plan of the work is as follows. In Section 2 we introduce the definition of Cantor sets, and their relations with the theory of IFS, giving examples of chaos game and of Random IFS. In Section 3 we recall the homoclinic theorem of expanding periodic points, and its relation with invariant Cantor sets of points, giving a new proof in Theorem 2, by using the IFS theory. In Sections 4 we illustrate the results by using both a 1-dimensional example and a 2-dimensional one, coming from economic models (of overlapping generations).

2 Iterated Function System (IFS)

As stated in the introduction, the homoclinic orbits give us a kind of "bridge" between the theory of Dynamical Systems and the theory of Iterated Function Systems (IFS). To see this connection let us recall the basic tools: the properties of Cantor sets, and its relation with the IFS. That is, such kind of sets are the limit sets, or invariant sets, occurring in the IFS.

2.1 Invariant Cantor sets

Let us start recalling the definition of a *Cantor set*, which plays an important role as a repelling invariant set in chaotic systems and as a fractal attractor for an IFS. By definition a set Λ is a Cantor set if it is *closed, totally disconnected and perfect*[1]. The

[1]Totally disconnected means that it contains no intervals (i.e. no subset $[a, b]$ with $a \neq b$) and perfect means that every point is a limit point of other points of the set.

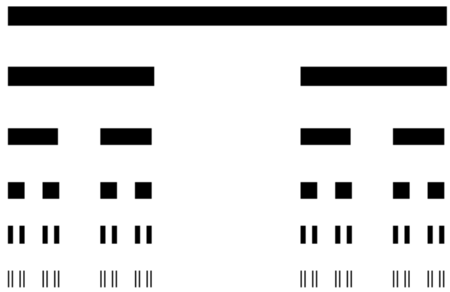

Figure 1: *Middle-third Cantor set.*

simplest example is the celebrated "Middle-third Cantor set": start with a closed interval I and remove the open "middle third" of the interval (see Fig.1). Next, from each of the two remaining closed intervals, say I_0 and I_1, remove again the open "middle thirds", and so on. After n iterations, we have 2^n closed intervals inside the two intervals I_0 and I_1. The Cantor set is obtained as the limiting set of the process, as $n \to \infty$.

Cantor sets are obtained in a natural way as chaotic invariant sets in dynamical systems (see e.g. in [5] or in [19]). To illustrate this point, consider the well known logistic map:

$$x_t = f_\mu(x_{t+1}) = \mu x_{t+1}(1 - x_{t+1}). \tag{1}$$

Notice that we have written this quadratic difference equation as a backward dynamical system, as it is used within the theory of Overlapping Generation Models (OLG for short) by Medio and Raines [25] (and it will be used as example also here, in Section 4). Since f_μ is non-monotonic, its inverse map is not unique. For any point ξ belonging to the interval $I = [0, 1]$ there are two distinct inverse functions[2], say

$$f_\mu^{-1}(.) = f_0^{-1}(.) \cup f_1^{-1}(.) \tag{2}$$

where

$$f_1^{-1}(\xi) = \frac{\mu - \sqrt{\mu^2 - 4\mu\xi}}{2\mu} \quad , \quad f_0^{-1}(\xi) = \frac{\mu + \sqrt{\mu^2 - 4\mu\xi}}{2\mu}. \tag{3}$$

Hence for the OLG-model, the forward state is not uniquely defined. To clarify the notation (and to avoid confusion) we shall call by "*dynamics*" only the repeated application of the function f, which is a real dynamical systems, independently on the fact that it is used as a forward or backward model, i.e. as $x_{t+1} = f(x_t)$ or

[2]Instead of the two symbols 0 and 1, one may use L and R, respectively.

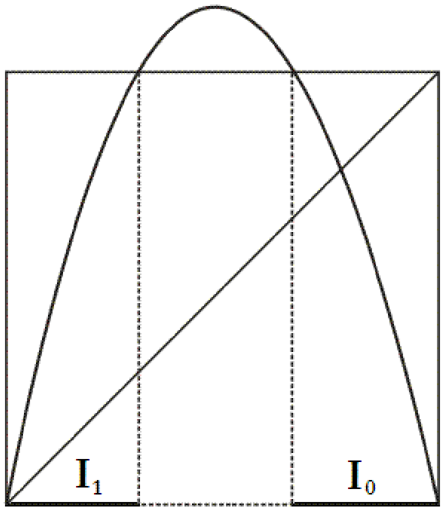

Figure 2: *Logistic map for* $\mu > 2+\sqrt{5}$ *and the two disjoint intervals* $f_\mu^{-1}(I) = I_0 \cup I_1$.

$x_t = f(x_{t+1})$, while in the second case, within a backward model, the states obtained by applications of one of the inverses give a "sequence of forward states".

It is well known that for $\mu > 4$, the standard map $x_{t+1} = f_\mu(x_t)$ has a chaotic invariant Cantor set Λ. Moreover, as we shall see, the same Cantor set is an attractor of the Iterated Function System formed by the interval I and the two inverses f_0^{-1} and f_1^{-1}. To simplify the exposition let us consider the case $\mu > 2 + \sqrt{5}$ (although the result is true for $\mu > 4$). For $\mu > 2 + \sqrt{5}$ the two inverses f_0^{-1} and f_1^{-1} are contractions in I.[3]

The set of points whose dynamics are bounded forever in the interval I can be obtained removing from the interval all the points which exit the interval after n iterations, for $n = 1, 2, ...$ Thus let us start with the two closed disjoint intervals

$$f_\mu^{-1}(I) = f_0^{-1}(I) \cup f_1^{-1}(I) = I_0 \cup I_1, \tag{4}$$

as shown in Fig.2: i.e. by applying (2) to the interval I we have removed the points leaving I after one iteration. Next we remove the points exiting after two iterations obtaining four closed disjoint intervals

$$f_\mu^{-2}(I) = I_{00} \cup I_{10} \cup I_{01} \cup I_{11},$$

[3]We recall that a function f is a contraction in a space X if $||f(x) - f(y)|| < k||x - y||$ for any pair of points in X, with a fixed constant of contraction $0 < k < 1$. If f is a contraction in a space X then there exists a unique fixed point in X, globally attracting.

defining in a natural way $f_\mu^{-1}(I_0) = f_0^{-1}(I_0) \cup f_1^{-1}(I_0) = I_{00} \cup I_{10}$ and $f_\mu^{-1}(I_1) = f_0^{-1}(I_1) \cup f_1^{-1}(I_1) = I_{01} \cup I_{11}$. Note that if a point x belongs to I_{01} (or to I_{11}) then $f_\mu(x)$ belongs to I_1 (i.e. one iteration means dropping the first symbol of the index). Continuing the elimination process we have that $f_\mu^{-n}(I)$ consists of 2^n disjoint closed intervals (satisfying ${f_\mu}^{-(n+1)}(I) \subset f_\mu^{-n}(I)$), and in the limit we get

$$\Lambda = \bigcap_{n=0}^{\infty} f_\mu^{-n}(I) = \lim_{n\to\infty} f_\mu^{-n}(I). \tag{5}$$

The set Λ is closed (as intersection of closed intervals), invariant by construction (as $f^{-1}(\Lambda) = f^{-1}(\cap_{n=0}^{\infty} f_\mu^{-n}(I)) = \cap_{n=0}^{\infty} f_\mu^{-n}(I) = \Lambda$) and it cannot include any interval (because otherwise, since f_μ is expanding, after finitely many applications of f_μ to an interval, we ought to cover the whole set $[0, 1]$). Thus Λ is a Cantor set.

Moreover, by construction, to any element $x \in \Lambda$ we can associate a symbolic sequence, called *itinerary*, or *address*, of x: $S_x = (s_0 s_1 s_2 s_3 ...)$ with $s_i \in \{0, 1\}$, i.e. S_x belongs to the set of all one-sided infinite sequences of two symbols $\sum_2$. S_x results from the symbols that we put as indices to the intervals in the construction process, and there exists a one-to-one correspondence between the points $x \in \Lambda$ and the elements $S_x \in \sum_2$. Moreover, from the construction process we have that if x belongs to the interval $I_{s_0 s_1 ... s_n}$ then $f_\mu(x)$ belongs to $I_{s_1 ... s_n}$. Thus the action of the function f_μ on the points of Λ corresponds to the application of the "shift map σ" to the itinerary S_x in the code space $\sum_2$:

$$\begin{aligned} if \quad x &\in \Lambda \; has \; S_x = (s_0 s_1 s_2 s_3 ...) \\ & \quad then \\ f_\mu(x) &\in \Lambda \; has \; S_{f(x)} = (s_1 s_2 s_3 ...) = \sigma(s_0 s_1 s_2 s_3 ...) = \sigma(S_x) \end{aligned}$$

Given a point $x \in \Lambda$, to construct its itinerary S_x we proceed in the obvious way: we put $s_0 = 0$ if $x \in I_0$ or $s_0 = 1$ if $x \in I_1$, then we consider $f_\mu(x)$ and we put $s_1 = 0$ if $f_\mu(x) \in I_0$ or $s_1 = 1$ if $f_\mu(x) \in I_1$, and so on. It is easy to see that each periodic sequence of symbols of period k represents a periodic orbit with k distinct points, and thus a so-called $k-$cycle. Since the elements of $\sum_2$ can be put in one-to-one correspondence with the real numbers[4], we have that the periodic sequences are dense in the space, thus the periodic orbits are dense in Λ. Also there are infinitely many aperiodic sequences (i.e. trajectories) which are dense in Λ and we also have sensitivity with respect to the initial conditions. It follows that the (standard) dynamics of f_μ is chaotic in Λ. But more, we also have nice properties in the forward process we are interested in (within a backward model). In fact, for any

[4]We can think for example of the representation of the numbers in binary form.

initial condition $x_0 \in I$, whichever is the sequence that we construct, by applying at each iteration any one of the two inverses of f_μ, the forward states $(x_0, x_1, x_2, ...$ where $x_{i+1} = f_j^{-1}(x_i)$ with $j = 0$ or $j = 1$ for any i) tend to Λ (i.e. the distance from Λ of the points so obtained tends to zero). Thus the Cantor set Λ also describes the asymptotic states of all the possible forward sequences.

2.2 Iterated Function System (IFS)

The construction process with the two contraction functions in (4) leading to the Cantor set in (5) can be repeated with any number of contraction functions defined in a complete metric space D of any dimension[5], as it is well known since the pioneering work by Barnsley [3]. Let us recall the definition of an IFS:

Definition 1. *An Iterated Function System (IFS)* $\{D; H_1, ...H_m\}$ *is a collection of* m *mappings* H_i *of a compact metric space* D *into itself.*

We can so define $W = H_1 \cup ... \cup H_m$. Denoting by s_i the contractivity factor of H_i then the contractivity factor of W is $s = max\,\{s_1, ...s_m\}$, and for any point or set $X \subseteq D$ we define

$$W(X) = H_1(X) \cup ... \cup H_m(X).$$

The main property of this definition is given in the following theorem:

Theorem 1. (Barnsley [3], p. 82) *Let* $\{D; H_1, ...H_m\}$ *be an IFS. If the* H_i *are contraction functions then there exists a "unique attractor"* Λ *such that* $\Lambda = W(\Lambda)$ *and* $\Lambda = \lim_{n\to\infty} W^n(X)$ *for any non-empty set* $X \subseteq D$.

The existence and uniqueness of the set Λ is guaranteed by the theorem and it is also true that given any point or set $X \subseteq D$ by applying each time one of the m functions H_i the sequence tends to the same set Λ.

In the case described above (in Section 2.1) with the logistic map we have $D = I = [0, 1]$, $H_1 = f_0^{-1}$, $H_2 = f_1^{-1}$, so that $W(X) = f_\mu^{-1}(X) = f_0^{-1}(X) \cup f_1^{-1}(X)$ (as it has been used in (2)), and the set $\Lambda = \lim_{n\to\infty} W^n(X) = \lim_{n\to\infty} f_\mu^{-n}(X)$ for any $X \subseteq D$ is the set already described above and obtained in (5).

In general, if the sets $D_i = H_i(D)$, $i \in \{1, ..., m\}$, are disjoint, we can put the elements of Λ in one-to-one correspondence with the elements of the code space on m symbols $\sum_m$. The construction is the generalization of the process described

[5] or better (D, d) where d denotes the function distance.

above for the two inverses of the logistic function. Let $U_0 = D$ and define

$$\begin{aligned} U_1 &= W(U_0) = H_1(D) \cup ... \cup H_m(D) = D_1 \cup ... \cup D_m \subset U_0 \\ U_2 &= W(U_1) = W^2(U_0) = H_1(U_1) \cup ... \cup H_m(U_1) = D_{11} \cup ... \cup D_{mm} \subset U_1 \\ & ... \\ U_n &= W(U_{n-1}) = W^n(U_0) \subset U_{n-1} \end{aligned}$$

i.e. all the disjoint sets of U_1 are identified with one symbol belonging to $\{1, ..., m\}$, all the disjoint sets of U_2 are identified with two symbols belonging to $\{1, ..., m\}$ (m^2 in number) and so on, all the disjoint sets of U_n are identified with n symbols belonging to $\{1, ..., m\}$ (m^n in number). And in the limit, as $\Lambda = \lim_{n\to\infty} U_n = \lim_{n\to\infty} W^n(U_0) = \cap_{n=0}^{\infty} W^n(U_0)$, each element $x \in \Lambda$ is in one-to-one correspondence with the elements $S_x \in \sum_m$, where $S_x = (s_0 s_1 s_2 s_3 ...)$, $s_i \in \{1, ..., m\}$.

Moreover, for any element $x \in \Lambda$ we can define a transformation (or map) F on the elements of Λ by using the inverses of the functions H_i (the so called *shift transformation* or *shift dynamical system* in Barsnley [3], p. 144):

$$if \quad x \in H_i(D) \quad then \quad F(x) = H_i^{-1}(D)$$

so that we can also associate an induced dynamic to the points belonging to Λ, and the rule described above holds for F, i.e. if $x \in \Lambda$ has itinerary $S_x = (s_0 s_1 s_2 s_3 ...)$ then $F(x) \in \Lambda$ has itinerary $S_{F(x)} = (s_1 s_2 s_3 ...) = \sigma(s_0 s_1 s_2 s_3 ...) = \sigma(S_x)$. *Clearly, when the functions H_i are distinct inverses of a unique function f then the induced dynamic system is the same, as $F = f$.*

2.3 The chaos game and Random IFS

As a second relevant example let us consider another well known IFS with three functions, the so-called *chaos game*. Choose three different points A_i, $i = 1, 2, 3$, in the plane, not lying on a straight line. Let D be the closed set bounded by the triangle with vertices given by the three points A_i, and consider the system $\{D; H_1, H_2, H_3\}$ where the H_i are linear contractions in D with center A_i and contractivity factor 0.5. Then choose an arbitrary initial state x_0 in D. An orbit of the system is obtained by applying one of the three maps H_i, after throwing a dice. More precisely, $x_{n+1} = H_i(x_n)$ with $i = 1$ after throwing 1 or 2, $i = 2$ after throwing 3 or 4, $i = 3$ after throwing 5 or 6. For any initial state $x_0 \in D$, plotting the points of this orbit after a short transient gives Fig.3. This fractal shape is called the Sierpinski triangle and it is the unique attractor of the chaos game. Almost all the orbits generated in the chaos game are dense in the Sierpinski triangle.

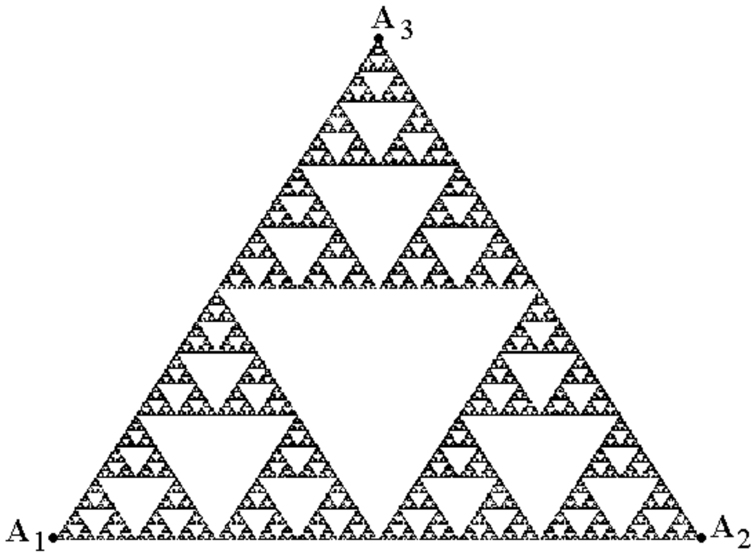

Figure 3: *Sierpinski triangle, unique attractor* Λ *of the IFS* $\{D; H_1, H_2, H_3\}$.

Moreover, in Barnsley ([3], p. 335) it is also shown how, besides the standard IFS, we can consider a Random IFS (RIFS for short, or IFS with probabilities) by associating a probability $p_i > 0$ to each function H_i, such that $\sum_{i=1}^{m} p_i = 1$. Considering a point $x_0 \in D$ then we choose recursively

$$x_{n+1} \in \{H_1(x_n), ..., H_m(x_n)\}$$

and the probability of the event $x_{n+1} = H_i(x_n)$ is p_i. The iterated points always converge to the unique attractor Λ of the standard IFS, but the density of the points over the set Λ reflects in some way the chosen probabilities p_i. However, we note that if the probabilities in the RIFS are strictly positive, $p_i > 0$, then the unique attractor does not change, and the iterated points are dense in Λ.

This may be very useful and convenient when using IFS theory applied to backward dynamic models. Using an approach similar to the Random IFS, we can define a Restricted IFS (or IFS with restrictions) imposing that, depending on the position of a point $x \in D$ not all the maps H_i can be applied but only some of them. Stated differently, we can impose some restrictions on the order in which the functions can be applied. As an example let us consider the chaos game described above, but now with some restrictions, that is: The order in which the three different maps H_i are applied is not completely random, but subject to certain restrictions. Suppose for example that the map H_1 is never applied twice consecutively, i.e. whenever H_1 is

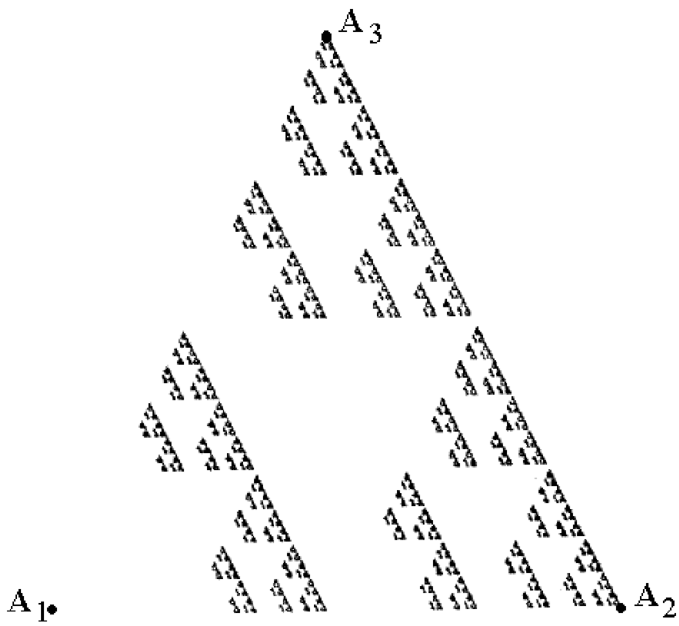

Figure 4: *A subset of the Sierpinski triangle.* Λ^* *is the unique attractor of the RIFS* $\{D; H_1, H_2, H_3\}$ *with the restriction that whenever* H_1 *is never applied twice consecutively.*

applied then the next map to be applied is either H_2 or H_3. Let $\sum_3$ be the code space on three symbols, and let $\sum^* \subset \sum_3$ be the subset of all sequences which do not have two consecutive 1's. The chaos game $\{D; H_1, H_2, H_3\}$ with the restriction so described has a unique attractor Λ^* whose points are in one-to-one correspondence with the restricted space $\sum^*$. A typical orbit of this chaos game with restrictions, after a short transient, is shown in Fig.4. The unique attractor of the chaos game with restrictions is a subset of the Sierpinski triangle, the attractor of the chaos game. In fact, the attractor contains precisely those points of the Sierpinski triangle whose itinerary, or addresses, do not have two consecutive 1's.

This example shows that when some restrictions upon the order in which the maps are applied is imposed, then a unique fractal attractor can arise, which is some subset of the unique attractor of the IFS.

In the following sections we shall see how IFS are related in a natural way to non-uniquely defined forward sequences within a backward model. We will also see that the forward states can be described by IFS, whenever the uniquely defined dynamics has homoclinic trajectories due to the existence of a snap-back repellor.

3 Homoclinic theorem for expanding periodic points

Let us now recall how chaotic behaviors exist in a dynamical system whenever we have transverse (or non critical) homoclinic orbits of expanding cycles, also called *snap-back repellors* by Marotto [20]. Without loss of generality we can deal with an *expanding fixed point* x^* of a $\mathcal{C}^{(1)}$ map T from a space X into itself, $X \subset \mathbf{R}^n$ with $n \geq 1$, as for a cycle of period k we can consider the map T^k (k-th iterate of T).

We recall that a fixed point x^* is hyperbolic if all the eigenvalues of $J_T(x^*)$ are different from 1 in modulus, when all are less then 1 in modulus then x^* is attracting, when all are higher then 1 in modulus, then x^* is expanding (or repelling). Also, a homoclinic trajectory of a fixed point x^* is called non degenerate (or non critical, or transverse) if $\det J_T(.) \neq 0$ in all the points of the homoclinic trajectory.

Definition 2. *A fixed point x^* of a smooth dynamical system is called a snap-back repellor if it possesses a neighborhood U such that the Jacobian matrix has all the eigenvalues higher than 1 in modulus in all the points of U, and in U there exist a homoclinic point of x^*.*

It is well known that in any neighborhood of a nondegenerate homoclinic trajectory we can find an invariant set Λ in which a suitable iterate of T, and thus T, is chaotic in the sense of Li and Yorke. For the proof we refer to [5], [20], [21] (see also [6]), but it is convenient to reformulate it for our purposes, showing its connection with the IFS.

The proof consists in showing that in any neighborhood U of x^* we can find two disjoint compact sets, U_0 and U_1, $U_0 \cap U_1 = \varnothing$, such that for a suitable m we have

$$T^m(U_0) \supset U_0 \cup U_1 \text{ and } T^m(U_1) \supset U_0 \cup U_1 \tag{6}$$

thus for the map T^m there exists an invariant chaotic set $\Lambda \subset U_0 \cup U_1$. In the following we illustrate:

(I) how the set property in (6) is used to construct an invariant Cantor set $\Lambda \subset U_0 \cup U_1$, on which T^m, and thus T, is chaotic;

(II) in Theorem 2 we show how the set property in (6) can be found associated with a given homoclinic trajectory.

(I) Let us consider $\widetilde{T} = T^m$. As, from (6), $\widetilde{T}(U_0) \supset U_0$ then a suitable inverse, say $F = \widetilde{T}_0^{-1}$, exists such that $F(\widetilde{T}(U_0)) = U_0$, and as $\widetilde{T}(U_1) \supset U_1$ (from (6) as well) then a suitable inverse, say $G = \widetilde{T}_1^{-1}$, exists such that $G(\widetilde{T}(U_1)) = U_1$.

Let $S = U_0 \cup U_1$ then $F(S)$ is made up of two disjoint pieces $U_{00} \subset U_0$ and $U_{10} \subset U_0$, and the action of the map $\widetilde{T}$ on such sets may be read on the symbols

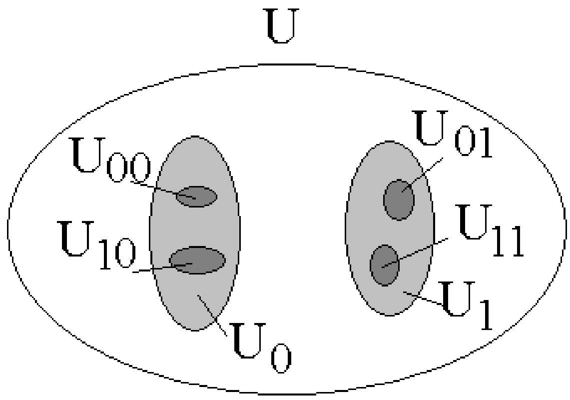

Figure 5: *Qualitative picture showing the application of F and G on the sets U_0 and U_1.*

which label the set, dropping the first symbol: $\widetilde{T}(U_{00}) = U_0$ and $\widetilde{T}(U_{10}) = U_0$ (see the qualitative picture in Fig.5). Similarly $G(S)$ is made up of two disjoint pieces $U_{01} \subset U_1$ and $U_{11} \subset U_1$, and the action of the map $\widetilde{T}$ on such sets may be read on the symbols which label the set, dropping the first symbol: $\widetilde{T}(U_{01}) = U_1$ and $\widetilde{T}(U_{11}) = U_1$. And so on, by repeating this mechanism we construct, in the limit process, a set $\Lambda \subset S = U_0 \cup U_1$, $\Lambda = \cap_{n=0}^{\infty}(F \cup G)^n(S)$. The elements (or sets) V_s of Λ are in $1-1$ correspondence with the elements $s = (s_0 s_1 s_2 s_3 ...)$ $(s_i \in \{0,1\})$ of the space $\sum_2$ of (one sided) infinite sequences on two symbols. Moreover the action of the map $\widetilde{T}$ in Λ corresponds to the action of the shift map σ to elements of $\sum_2$, that is: if x is a point of Λ and $x \in V_s$ then $\widetilde{T}(x) \in V_{\sigma(s)}$ (when $s = (s_0 s_1 s_2 s_3 ...)$ the shift map drops the first symbol $\sigma(s) = (s_1 s_2 s_3 ...)$).

This set Λ constructed up to now, without any other information on the map $\widetilde{T}$, is *invariant* ($\widetilde{T}(\Lambda) = \Lambda$), and its elements satisfy $V_s \neq \varnothing$ for any s, and $V_s \cap V_{s'} = \varnothing$ for $s \neq s'$: It is what we call *a set with Cantor like structure*, and its elements V_s are closed and compact (and thus Λ is closed and compact) and simply connected if so are the starting sets U_0 and U_1.

When F and G are "contraction mappings" then Λ is a classical *Cantor set of points*. In fact, if the inverses F and G of $\widetilde{T}$ are contractions in U (or in $S = U_0 \cup U_1$), then we can apply the IFS theory which states that $\{U; F, G\}$ is *an Iterated Function System (IFS)* (or $\{S; F, G\}$ is a IFS) which has a unique attractor $\Lambda \subset U$ or (S): an invariant Cantor set on which the shift automorphism $\widetilde{T}$ is chaotic. Let us show when we are in this situation (with contractions). We distinguish between the cases

$n = 1$ and $n \geq 2$.

If $\widetilde{T}$ is a *one dimensional map*, i.e. $n = 1$, and U_0 and U_1 intervals, then the condition on the first derivative $|\widetilde{T}'(x)| > 1$ in any point $x \in S$ is enough to state that the inverses F and G of $\widetilde{T}$ are one-dimensional functions such that $|F'(x)| < 1$ for all $x \in S$ and $|G'(x)| < 1$ for all $x \in S$. In the one-dimensional case this is enough to state that F and G are contractions (as contractivity constant for F we can take the maximum of $|F'(x)|$ in S, and similarly for G). In this case the set $\Lambda \subset S$ constructed above is a classical Cantor set *of points* (i.e. any element V_s of Λ is a point, and the periodic orbits of $\widetilde{T}$, all those associated with a periodic symbol sequence, are dense in Λ).

When U_0 and U_1 are subsets of $\mathbf{R}^n$, $n \geq 2$, then (a) the condition on the Jacobian determinant $|J_{\widetilde{T}}(x)| > 1$ in any point $x \in S$ is not enough to state that the inverses F and G of $\widetilde{T}$ are contractions (we may have saddle points), and (b) also assuming that in any point $x \in S$ the two eigenvalues of $J_{\widetilde{T}}(x)$ are greater than 1 in absolute value, we have not a sufficient condition for F and G to be contractions. However, when in any point $x \in S$ all the eigenvalues of $J_{\widetilde{T}}(x)$ are greater than 1 in absolute value, then for the local inverse the eigenvalues are less than 1 in absolute value, and the map *is locally homeomorphic to a contraction mapping* in a neighborhood of the fixed point.

In fact, in Hirsch and Smale [13] pp.278-281, it is proved that if a $\mathcal{C}^{(1)}$ map f in $X \subseteq \mathbf{R}^n$ has a hyperbolic attracting fixed point x^* then it is a *local contraction in a suitable norm*; it is locally invertible and for the local inverse x^* is a hyperbolic repellor, and the local inverse is a *local expansion in a suitable norm.*

Thus, from a mathematical point of view the above "imperfection" of the Cantor like set, existing for $n \geq 2$, can be overcome, leading to a Cantor set of points. In fact, an expanding fixed point implies the existence of *a suitable norm* d_1 for which the inequality $d_1(f(x), f(y)) < s\ d_1(x, y)$ holds in a neighborhood W of x^*. But all the norms in $\mathbf{R}^n$ are equivalent. So an homeomorphism h exists such that, for any pair of points, $d_1(x, y) = d_2(h(x), h(y))$, where d_2 denotes the Euclidean distance. It follows that we have $d_2(h \circ f(x), h \circ f(y)) < s\ d_2(h(x), h(y))$, and for any $(\xi, \eta) \in U = h(W)$ we have that $d_2(h \circ f \circ h^{-1}(\xi), h \circ f \circ h^{-1}(\eta)) < s\ d_2(\xi, \eta)$. That is: in the Euclidean norm the function f, in a suitable neighborhood U, is *locally topologically conjugated* with a contraction (and thus, qualitatively, the dynamics are the same). We have so proved the following property:

Property 1. *If a $k-$cycle is a hyperbolic attractor (resp. repellor) for a $\mathcal{C}^{(1)}$ map f, then f is a local contraction (resp. expansion) in a suitable norm, and f is locally topologically conjugated with a contraction (resp. expansion) in the Euclidean norm.*

The functions F and G used above may be constructed as contractions in the euclidean norm, so that to the above properties of the set with Cantor like structure we can add that any set V_s associated with a (periodic or aperiodic) symbol sequence s is really a single point, and all the cycles in Λ are unstable, so we may say that Λ is unstable for $\widetilde{T}$. In fact, assume that F is not a contraction in U, then we can consider the function $\widetilde{F} = F^p$ where p is a suitable integer such that $\widetilde{F}$ is a contraction in U (which necessarily exists). Similarly for G, if it is not a contraction then we consider $\widetilde{G} = F^q \circ G$ where q is a suitable integer such that $\widetilde{G}$ is a contraction in U (which necessarily exists). Then $\{U; \widetilde{F}, \widetilde{G}\}$ is *an Iterated Function System (IFS)* which has a unique attractor $\Lambda \subset U$: an invariant Cantor set on which the shift automorphism $\widetilde{T}$ is chaotic.

(II) Well, these conditions are satisfied when we have a repelling fixed point (or cycle), unstable node or unstable focus, and a non degenerate homoclinic trajectory, which means that the preimages of the fixed point belonging to the considered homoclinic orbit are not on the critical curves (while degenerate homoclinic trajectories denote homoclinic explosions). We prove now the following:

Theorem 2. *If a fixed point x^* is expanding for a $\mathcal{C}^{(1)}$ map T in $X \subseteq \mathbf{R}^n$ with a non degenerate homoclinic orbit, then in any neighborhood of the homoclinic orbit there exist an invariant set Λ on which T is chaotic.*

Proof. Consider a compact neighborhood U of x^* in which T is expanding (i.e. all the eigenvalues of $J_T(x)$ are higher then 1 in modulus for all the points x in U). Let us first show that under the assumptions of the theorem we can always find two disjoint compact sets in U, U_0 and U_1, $U_0 \cap U_1 = \varnothing$, such that for a suitable m we have $T^m(U_0) \supset U_0 \cup U_1$ and $T^m(U_1) \supset U_0 \cup U_1$. Then we show that two suitable inverses are contractions, so that the result comes from the properties described in (I).

Let $O(x^*) = \{x^*, x_1, x_2, ...x_p, ...\}$ be the homoclinic orbit, and let T_0^{-1} be the local inverse, satisfying $T_0^{-1}(x^*) = x^*$ and T_1^{-1} the inverse such that $T_1^{-1}(x^*) = x_1$, while the point x_p is such that the repeated applications of T_0^{-1} to x_p converge to x^*. Notice that $T_1^{-1}(U) \cap U = \varnothing$. The nondegeneracy implies that $Det J_T(x_i) \neq 0$ in all the points of the homoclinic orbit. The expansivity in a neighborhood implies that T_0^{-1} is a contraction in U or locally homeomorphic to a contraction, but we can choose a suitable integer p such that T_0^{-p} is a contraction in U. Define $F = T_0^{-p}$, and $U_0 = F(U)$. Then we apply to U the sequence of inverses which follow the homoclinic orbit until we have again points located inside U (see the qualitative picture in Fig.6). Define $G = T_0^{-s} \circ ... \circ T_1^{-1}$ where the integer s is such that $G(U) \subset U$ and G is a contraction in U. Define $U_1 = G(U)$. Obviously $x^* \in U_0$, U_1 and U_0 are disjoint (because $T_1^{-1}(U)$ and $T_0^{-1}(U)$ are disjoint by construction), and thus all the

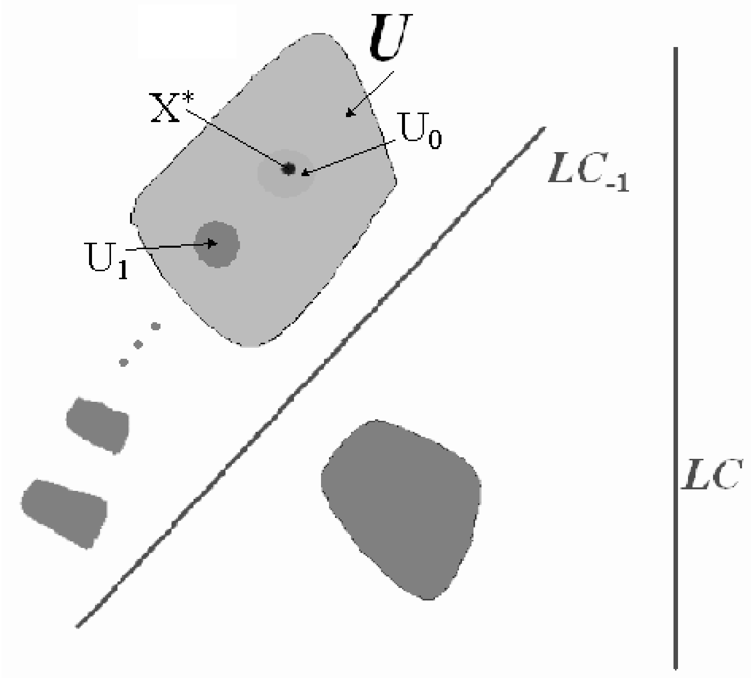

Figure 6: *Qualitative description of the construction of an IFS.*

applications of the inverses give disjoint sets, and by properly choosing the integers p and s (number of local inverses with T_0^{-1}) in the construction of G and F we can assume $m = p$ and such that $T^m(U_0) = U \supset U_0 \cup U_1$ and $T^m(U_1) = U \supset U_0 \cup U_1$. Now we have that F and G are contractions in U, so that $\{U; F, G\}$ is *an Iterated Function System (IFS)* which has a unique attractor $\Lambda \subset U$: an invariant cantor set on which the shift automorphism T^m, and thus T, is chaotic, which ends the proof.

In the next section we shall show applications of the above theorem to examples in 1 and 2 dimensional phase spaces in backward dynamic models coming from economics.

4 Examples

Let us now introduce suitable applications of the theory recalled in the previous sections, to models in the economic context, the first of which is in a 1-dimensional space, while the second refers to a 2-dimensional map.

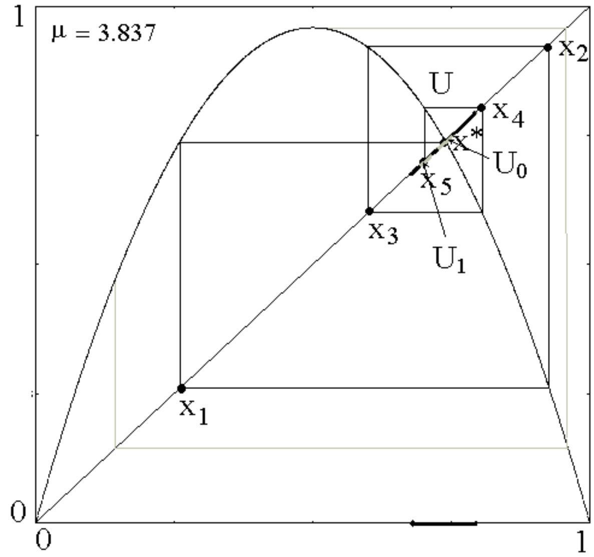

Figure 7: *Example of a homoclinic trajectory of x^* at $\mu = 3.837$.*

4.1 Example in $\mathbf{R}^1$

In the work by Medio and Raines [25] it is proposed an overlapping generation model represented by the backward model with the one-dimensional logistic map:

$$x_t = f_\mu(x_{t+1}) = \mu x_{t+1}(1 - x_{t+1})$$

already used in Section 2, whose two inverses are given in (3). It is clear that when $\mu > 4$ then for any initial condition x_0 belonging to the unit interval $I = [0, 1]$, no matter which one of the two inverses we apply at each step to get the sequence $x_0, x_1, x_2, ...$ we can never escape from the interval I, and we can say that we have a unique chaotic set $\Lambda \subset I$ as attractor for the forward states of the model. However it is not necessary to take $\mu > 4$ in order to have forward states in a bounded chaotic set. Whenever we have a cycle with homoclinic trajectories we can construct suitable *Iterated Function System* or *Random Iterated Function System.* For example, let $\mu^* < 4$ be the bifurcation value of the parameter such that the positive fixed point x^* of the function $f_\mu(.)$ has a critical homoclinic orbit. Then for any $\mu > \mu^*$ there are noncritical (i.e. nondegenerate) homoclinic orbits of x^*. Let us consider such a case, and let $O(x^*) = \{x^*, x_1, x_2, ...x_p, ...\}$ be the homoclinic orbit (see Fig.7), such that $x_1 = f_1^{-1}(x^*)$ (while $x^* = f_0^{-1}(x^*)$), and $x_i = f_0^{-1}(x_{i-1})$ for any $i > 1$.

Let U be a neighborhood of x^* in which $f_\mu(.)$ is expanding and such that $U_1 =$

$f_0^{-4} \circ f_1^{-1}(U) \subset U$, $U_0 = f_0^{-5}(U)$ (clearly $U_0 \cap U_1 = \varnothing$), $G = f_0^{-5}(.)$ and $F = f_0^{-4} \circ f_1^{-1}(.)$ are contractions in $S = U_0 \cup U_1$. Thus $\{S; F, G\}$ is *an Iterated Function System (IFS)* which has a unique attractor $\Lambda \subset S$: an invariant Cantor set on which f_μ is chaotic.

Then, to find some particular sequences in the forward process, for any initial condition $x_0 \in S$ let us consider the following rule: whenever we apply the left inverse f_1^{-1} then we apply the right inverse f_0^{-1} for at least 4 times consecutively, i.e. any number q of times with the only restriction $q \geq 4$. It is clear that the sequence of forward states of the backward model always belongs to the set $A = \bigcup_{i=0}^{4} S_i$ where $S_0 = S$, $S_1 = f_1^{-1}(S)$, $S_i = f_0^{-1}(S_{i-1})$ for $i = 2, 3, 4$, and the points have a kind of chaotic behavior in this set.

The "rules" which we may construct leading to bounded forward sequences (which seem chaotic) are infinitely many. Thus it depends on the applied meaning of the model to have meaningful rules or not. In the economic context such rules have been associated to "sunspot" dynamics ([4], [27], [1]).

4.2 Example in $\mathbf{R}^2$

The example we propose here is similar to the two-dimensional model presented in [7]. It is still related with an overlapping generation model (we refer to that paper for its deduction). Here it is enough to say that it reduces to a map T of the plane into itself of so-called $Z_0 - Z_2$ type (for more details see [24], and the pioneering works [11], [12], [23]). In our case there exists a critical line LC_{-1} in which $DetJ_T(X) = 0$ for any $X \in LC_{-1}$, which is mapped into a line $LC = T(LC_{-1})$ which separates the phase plane in two regions: Z_0 whose points have no rank-1 preimages and Z_2 whose points have two distinct rank-1 preimages, $T_R^{-1}(.)$ and $T_L^{-1}(.)$ giving one point on the right and one point on the left of LC_{-1}, respectively. Explicitly we have that the backward dynamics is described by the two-dimensional backward map

$$(x_t, y_t) = T(x_{t+1}, y_{t+1}) = (f[a(1 - \delta + \frac{1}{a})x_{t+1} - ay_{t+1}], y_{t+1}).$$

where the function f is a unimodal $\mathcal{C}^{(1)}$ function with a unique critical point, a local maximum, and two inverses, say f_L^{-1} and f_R^{-1}. Thus for the map T the two inverses are given by

$$T_i^{-1} : \begin{cases} x_{t+1} = y_t \\ y_{t+1} = (1 - \delta + \frac{1}{a})y_t - \frac{1}{a} f_i^{-1}(x_t) \end{cases}$$

where $i = L, R$. For the shake of simplicity we consider for $f(.)$ the standard logistic function already introduced in the previous sections, that is: $f(x) = \mu x(1 - x)$.

At suitable values of the parameters we have that the fixed point X^* of the map T, which belongs to the L side with respect to LC_{-1}, becomes an unstable node, with homoclinic points, i.e. it becomes a snap-back repellor. Then we may consider forward states of the backward model as described below.

For a suitable neighborhood U of X^* we have that $U_1 = G(U) = T_L^{-13} \circ T_R^{-1}(U) \subset U$ is disjoint from $U_0 = F(U) = T_L^{-14}(U) \subset U$. Thus $T^{14}(U_0) = U \supset U_0 \cup U_1$ and $T^{14}(U_1) = U \supset U_0 \cup U_1$. Either the functions F and G are contractions in U or we choose a suitable integer s, and $p = s + 1$, such that $G = T_L^{-s} \circ T_R^{-1}$ and $F = T_L^{-p}$ are contractions in U. In our numerical example, at the parameters' values $\delta = 0.226$, $a = 3$ and $\mu = 5.16$, we have $s = 13$ and $p = 14$, and $\{U; F, G\}$ is *an Iterated Function System (IFS)*, leading to an invariant Cantor set Λ in U.

Then we conclude that the forward states obtained as follows: "whenever a point belongs to the R region then we apply the function T_L^{-1} at least 13 time consecutively before the application of T_R^{-1}" give a kind of random sequence of points in the bounded region obtained by the starting interval U and its images with the functions which are involved in the definition of the contractions of the IFS. In our example all the states belong to the set

$$A = U \cup T_R^{-1}(U) \cup T_{LR}^{-2}(U) \cup \ldots \cup T_{L\ldots LR}^{-(14)}.$$

The sequence of points is trapped in A, i.e. the forward states cannot escape, and the qualitative shape of the asymptotic points has the set Λ and its images by T as limit set.

Moreover, as discussed in Section 2, we can also consider the IFS with probabilities, or Random Iteration Function System (RIFS) $\{U; F, G; p_1, p_2\}$, $p_i > 0$, $p_1 + p_2 = 1$, which means that given a point $x \in U$ we consider the trajectory obtained by applying the function F with probability p_1 or the function G with probability p_2, that is, one of the functions is selected at random, with the given probability. Also in such a case the unique attractor of the RIFS is the same Cantor set, although some points in Λ are visited more often than others, that is, the typical trajectory may be described by an invariant measure with support on the fractal set Λ.

It is clear that once that we have a homoclinic fixed point, we can find infinitely many different IFS. For example, it is not always necessary to apply the function T_R^{-1} only once in a row, as we may construct an IFS in which two consecutive applications of T_R^{-1} occur. In our example, with the same parameter values $\delta = 0.226$, $a = 3$ and $\mu = 5.16$, we have that $U_1 = \widetilde{G}(U) = T_L^{-14} \circ T_{RR}^{-2}(U) \subset U$ is disjoint from $U_0 = \widetilde{F}(U) = T_L^{-16}(U) \subset U$ (thus $T^{16}(U_0) = U \supset U_0 \cup U_1$ and $T^{16}(U_1) = U \supset U_0 \cup U_1$), and the functions $\widetilde{F}$ and $\widetilde{G}$ are contractions in U, so that $\{U; \widetilde{F}, \widetilde{G}\}$ is *an Iterated Function System (IFS)*, leading to an invariant Cantor set in U.

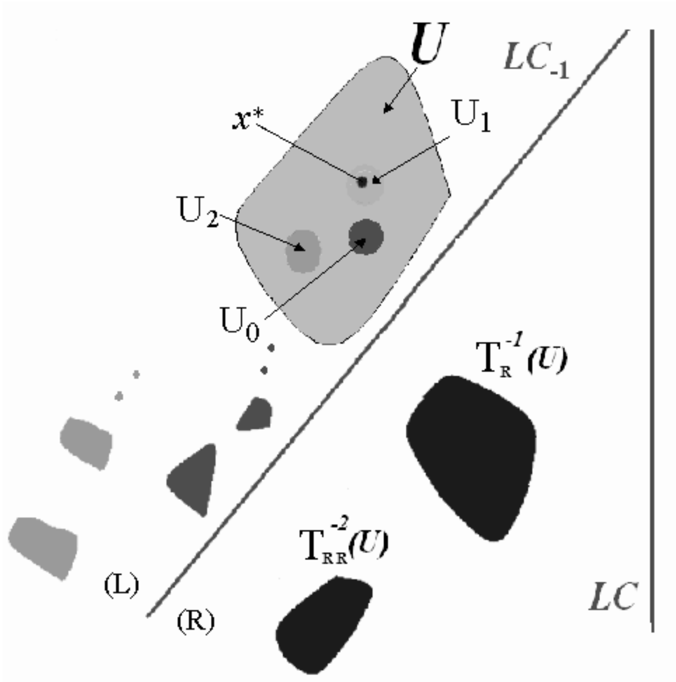

Figure 8: *Qualitative description of the construction of the different sets involved in the IFS similar to the "chaos game" associated with the two-dimensional model.*

As a third example, we follow a different approach in order to get invariant sets by applying the iterated function systems technique to three functions, combining the two examples given above. In this way we obtain an IFS similar to the chaos game, with which we can construct forward sequences.

Consider the following functions: $H_1 = \widetilde{F} = T_L^{-(16)}$, $H_2 = \widetilde{G} = T_{L\ldots LRR}^{-(16)}$, and $H_3 = T_{RL\ldots L}^{-(16)}$ (note that in the definition of H_3 the inverse T_L^{-1} is applied 15 times, while in the definition of H_2 the inverse T_L^{-1} is applied 14 times). As the first two iterates $T_{LL}^{-2}(U)$, $T_{RR}^{-2}(U)$ and $T_{RL}^{-2}(U)$ are disjoint sets, it follows that also $U_1 = H_1(U)$, $U_2 = H_2(U)$, and $U_3 = H_3(U)$ are disjoint sets, all belonging to U. See the qualitative picture in Fig.8, where the dark gray neighborhoods denote the preimages by T_L^{-1} of $T_R^{-1}(U)$, while the black neighborhoods denote the preimages by T_L^{-1} of $T_{RR}^{-2}(U)$. In our numerical example we have three contractions (in general, as stated above, it is always possible to reach this situation by applying the inverse map T_L^{-1} as much as necessary). Thus we can consider the set associated with the IFS $\{U; H_1, H_2, H_3\}$ (invariant for the backward dynamics of $T^{(16)}$), defining

$W = H_1 \cup H_2 \cup H_3$

$$U_1 = W(U) = H_1(U) \cup H_2(U) \cup H_3(U) \subset U \ (= U_0)$$

and iteratively

$$U_{n+1} = H_1(U_n) \cup H_2(U_n) \cup H_3(U_n) \subset U_n,$$

for $n \geq 1$ each U_n consists of 3^n elements, and we have

$$\Lambda_3 = \bigcap_{n\geq 0} U_n = \bigcap_{n\geq 0} W^n(U) = \lim_{n\to\infty} U_n,$$

which is an invariant chaotic set (repelling for the backward map T), and the unique attractor in U for the iterated function system. Notice that the contraction H_1 has a unique fixed point in U, belonging to U_1, which is the fixed point X^* of the map T, while the contraction H_2 has a unique fixed point P^* in U, belonging to U_2, which is a cycle of period 16 for T, and similarly the contraction H_3 has a unique fixed point Q^* in U, belonging to U_3, which is a cycle of period 16 for T.

Moreover, as shown for the 1-D case, we may consider the Random Iteration Function Systems, say RIFS $\{U; H_1, H_2, H_3; p_1, p_2, p_3\}$, $p_i > 0$, $p_1+p_2+p_3 = 1$, which means that given a point $x \in U$ we consider the trajectory obtained by applying the function H_i with probability p_i, that is, at each step one of the functions is selected at random, with the given probability. Then the random sequence of points is trapped in U, i.e. the forward states cannot escape, and the asymptotic orbit is always dense in the set Λ_3. The distribution of points of the fractal set Λ_3 may be uneven, as some regions may be visited more often than others depending on the magnitude of the probabilities.

In our example, in order to have forward sequence for the model, we may choose to follow some specific rule, for example as follows: whenever we apply the function T_R^{-1} it is once or at most two time consecutively, immediately thereafter we must the function T_L^{-1} as many times as needed, but at least 13 times (if the right T_R^{-1} was applied once) or at least 14 times (if the right inverse T_R^{-1} was applied twice). In this way we can conclude that "the generic forward states" can be obtained as a random sequence of points with an initial state in the set U and the states always belong to U and its images with the functions which are involved in the definition of the contractions of the IFS. For example, in the chaos game given above the states always belongs to the set

$$A = U \cup T_R^{-1}(U) \cup T_{LR}^{-2}(U) \cup ... \cup T_{L...LR}^{-(14)} \cup T_{RR}^{-2}(U) \cup T_{LRR}^{-3}(U) \cup ... \cup T_{L...LRR}^{-16}(U)$$

An example of forward states numerically obtained in our 2-dimensional model with this last rule is shown in Fig.9.

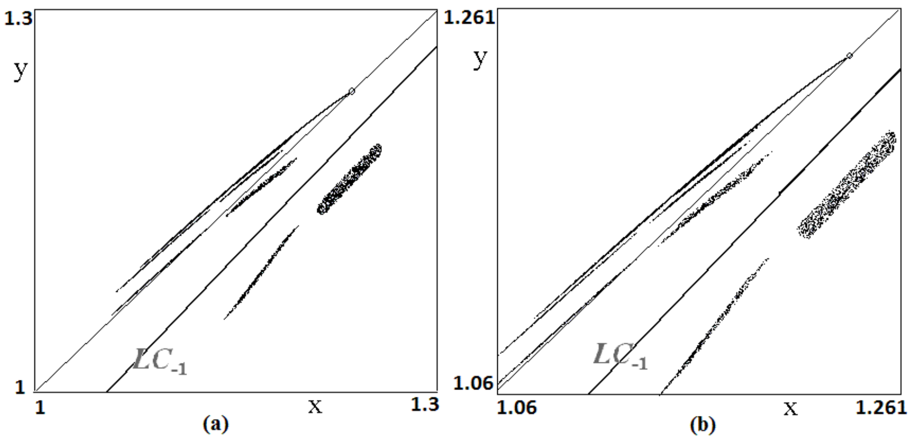

Figure 9: *Examples of numerically calculated forward states belonging to the set A, in the example similar to the "chaos game" for our model. (b) shows an enlargement of (a).*

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